Bipartite Measurements in Minkowski Spacetime

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We study the measurements which Alice and Bob can perform on a bipartite quantum system, where Alice and Bob are spacelike separated. For a measurement to be possible, it must be causal i.e. non-signalling. Within causal measurements, we define four notions of 'localisability'. Each of the four classes of measurement restricts the actions of Alice and Bob in different ways, and we study their relative power. We end with a discussion of the difficulties posed by non-local measurements for the idea of wavefunction collapse.

I. INTRODUCTION

In non-relativistic quantum mechanics and quantum information theory, a fundamental assumption is that we are able to perform any measurement instantaneously. However, if a quantum system is distributed over space, then the assumption that all measurements can be achieved instantaneously is at odds with special relativity. Indeed, this discussion began with the EPR thought experiment [1]. The first difficulty is that the notion of instantaneity is frame-dependent: what is instantaneous for Romeo will not be instantaneous for Juliet. The second difficulty is the possibility that the projection postulate will allow superluminal signalling, in contradiction with the relativistic principle of causality. We can see an intuitive example of this second issue in the case of a single free particle. Suppose the particle is initially localised in some compact region of space, and the momentum of the particle is measured. The position of the particle will then become instantaneously smeared over all space, which will produce a positive probability of locating the particle in some far away region of space, at an arbitrarily soon instant of time.

Of course, the fully consistent theory of quantum mechanics in Minkowski spacetime is quantum field theory. However, we will investigate these issues in the context of a certain toy model. We will consider Alice and Bob, separated in space by a distance d, possessing point-like quantum subsystems A and B respectively. The aim is to investigate measurements on their joint bipartite system AB; measurements which are instantaneous in some fixed Lorentz frame. We will assume that Alice and Bob have complete instantaneous control over their subsystems and any apparatus systems local to them.

The reader may notice some hypocrisy in this model. We began by criticising the assumption of complete instantaneous control of a quantum system, only to bestow Alice and Bob individually with such power over their respective subsystems. However, this can be justified as follows. In reality, A and B will occupy smeared regions in space with some small width ϵ. We then allow measurements to take a small time $\delta t \simeq \epsilon/c$. We take ϵ large enough to ignore quantum field theoretic effects on the smallest scales, and we take $d \gg \epsilon$. In this way, we can assume that non-relativistic quantum mechanics applies to A and B individually, and the relevant notion of instantaneity is $\delta t \ll d/c$. This Alice and Bob toy model is embodied by the slogan 'put a bipartite non-relativistic quantum system in Minkowski spacetime'. See Fig. 1.

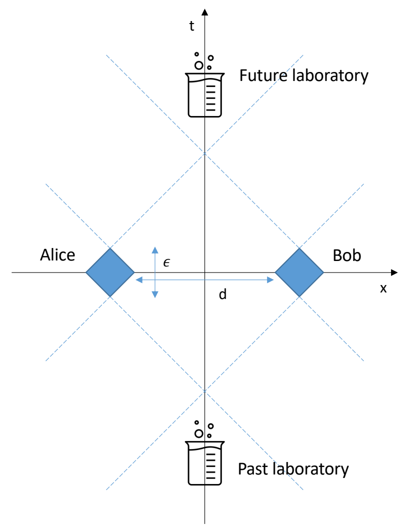

FIG. 1

In quantum mechanics, measurement plays a dual role: it extracts classical information about the quantum system, and it also prepares the quantum system in an appropriate post-selected state. In [2], non-selective (i.e. trace-preserving) measurements are considered in a similar context to ours. On the other hand, other authors have focused on the observational part of measurement, without regard to the post-measurement state of the system [3]. In this essay, we focus explicitly on selective measurements; in other words, we are studying how to make sense of the projection postulate in Minkowski spacetime.

For our purposes, a general measurement on a single quantum system takes the form: (a) introduce some apparatus (or ancilla) quantum degrees of freedom, (b) apply a unitary to the combined system and apparatus, (c) perform a complete *measurement* of the apparatus system in an orthonormal basis. For a bipartite system AB, the apparatus system will also be bipartite. Let Alice's apparatus subsystem be R, and Bob's S. Throughout this essay, A, B, R, S will all be finite-dimensional quantum systems.

There are various restrictions we could make on the measurement procedure of Alice and Bob. In step (a), we could allow Alice and Bob to have prepared R and S at some *past laboratory* in an initial entangled state. Alternatively, we could require the apparatus R and S to initially be separable. In step (b), we require the unitary to be localisable i.e. $U_{AR} \otimes U_{BS}$. In step (c), we could allow Alice and Bob to coherently transport R and S to a *future laboratory* to perform a complete *measurement* in a general orthonormal basis. Alternatively, in step (c), we could require them to immediately *measure* R and S in some tensor product basis and communicate their results to the future laboratory classically.

These possible restrictions give rise to a variety of classes of measurement on AB which we will be interested in.

- SLC – in (a) the apparatus is **Separable**; in (b) the unitary is **Localisable**; in (c) Alice and Bob *measure* R and S independently and communicate their results **Classically**.
- SLQ – in (a) the apparatus is **Separable**; in (b) the unitary is **Localisable**; in (c) Alice and Bob have access to **Quantum** communication prior to *measuring* the apparatus.
- ELC – in (a) the apparatus is possibly **Entangled**; in (b) the unitary is **Localisable**; in (c) Alice and Bob *measure* R and S independently and communicate their results **Classically**.
- ELQ – in (a) the apparatus is possibly **Entangled**; in (b) the unitary is **Localisable**; in (c) Alice and Bob have access to **Quantum** communication prior to *measuring* the apparatus.
- CAUSAL – measurements which do not allow superluminal signalling.
- ALL – no restrictions placed on the measurements.

SLC is the class of 'local' measurements; i.e. measurements which are a tensor product of independent measurements on A and B. We can immediately notice the series of inclusions shown in Fig. 2. They are all tautological, except for ELQ $\subseteq$ CAUSAL, which is a consequence of the no-signalling principle.

There have been several attempts to characterise classes of measurements similar to these [2] [4] [5] [7] [8] [9], although I do not believe there exist completely satisfactory general characterisations. I believe such general characterisations are an important direction for future work; however it will not be addressed here. Indeed, this essay will ask more questions than it answers.

Our main results can be summarised as follows. See also Fig. 3.

$$\begin{aligned} &\text{SLC} \neq \text{ELC} \approx \text{ELQ} \\ &\text{SLC} \neq \text{SLQ} \neq \text{ELQ} \\ &\text{ELQ} \neq \text{CAUSAL} \neq \text{ALL} \end{aligned} \tag{1}$$

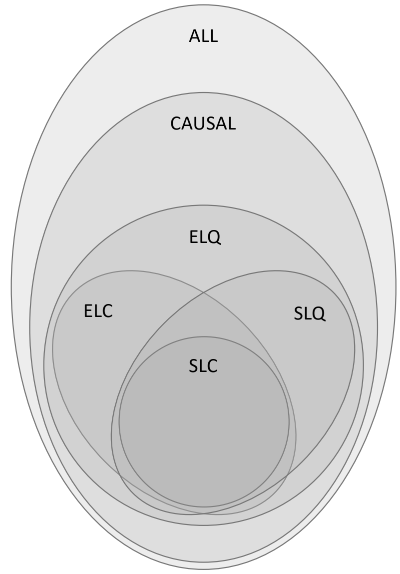

FIG. 2

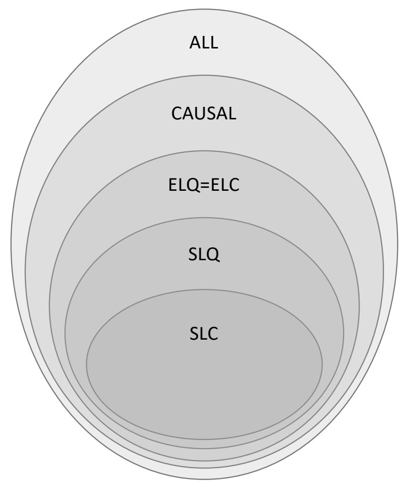

FIG. 3

For CAUSAL $\neq$ ALL, we have already glimpsed an acausal measurement above in our discussion of momentum measurement. This was made explicit by Sorkin [10].

Our next guess would perhaps be that, in relativistic quantum mechanics, local measurements are the only ones that are able to occur instantaneously. We will see that this is not true: we can design explicit localisable experiments which perform non-local measurements. This is possible using shared entangled apparatus and local *measurements* [2] [4] [5] [7]; that is SLC $\neq$ ELC. It is also possible without prior entanglement, using quantum post-processing [2]; that is SLC $\neq$ SLQ.

We have ELC $\approx$ ELQ in the following sense: every measurement in ELQ can be performed in ELC to arbitrarily high fidelity, as the size of the apparatus systems increase. This will follow from a technique introduced by Vaidman [3]. This can be interpreted as the statement that quantum post-processing offers no advantage if Alice and Bob already have access to prior entanglement.

We then ask this question the other way around: does prior shared entanglement offer any advantage in addition to quantum post-processing? The answer in this case is yes, and SLQ $\neq$ ELQ.

The final surprise is that ELQ $\neq$ CAUSAL. This means that a measurement being non-signalling is not enough to guarantee that there is a way to implement that measurement using only local actions of Alice and Bob [2].

Although special relativity is the motivation for introducing and studying these classes of measurement, it should be noted that we can interpret these results from a purely quantum information theoretic point of view. We are studying the measurements achievable by two parties whose communication is restricted in appropriate ways.

There are non-local measurements which can be performed instantaneously. This has a disturbing implication for the conventional view of wavefunction collapse in the setting of Minkowski spacetime [4] [5]. We argue that, in relativistic quantum mechanics, we are forced to abandon the traditional notion of an objective wavefunction collapse. One cannot assign quantum states to spacetime points (t, x); rather, quantum states can be defined only for entire spacelike hypersurfaces [6].

Section II will discuss measurement more formally. In Section III we then use this formalism to define the above classes of measurement for a bipartite system. In Section IV, we state and prove two distinct versions of the no-signalling principle. I believe the difference in content between these two versions is important, and often overlooked. Sections V, VI, VII, VIII, IX and X are devoted to showing the main results given in Eq. 1, Fig. 3. The systems of Alice and Bob in these sections will always be finite-dimensional and consist of qubits. In Section XI, we discuss the implications of non-local measurements for the ontology of the wavefunction. Section XII concludes.

II. OPERATIONS AND MEASUREMENT

We would like to define exactly what we mean by a measurement of a quantum system. To motivate our definition of measurement, it will be helpful to first study the form of a general (trace-preserving) operation on a quantum system.

A quantum system is described by a Hilbert space $\mathcal{H}$. Let $\mathcal{B}(\mathcal{H})$ denote the set of bounded linear operators on $\mathcal{H}$. The state of the system is a density matrix $\rho \in \mathcal{B}(\mathcal{H})$ which we require to be (i) positive semi-definite (ii) $\operatorname{Tr} \rho = 1$. Let $\Theta(\mathcal{H})$ denote the set of linear maps $\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, known as *superoperators*. The most general form of an operation on a quantum system is a superoperator $\Lambda \in \Theta(\mathcal{H})$ which is *CPTP*; that is, (i) *completely positive* (see Appendix XIII A) (ii) trace preserving. We will refer to CPTP superoperators as *quantum operations* or *operations*.

The reader is reminded that we will take all Hilbert spaces to be finite-dimensional.

Theorem 1. *Kraus Representation Theorem*
Let $\Lambda \in \Theta(\mathcal{H})$ be CPTP. Then there is a set $\{A_k\}_{k\in K} \subset \mathcal{B}(\mathcal{H})$ of linear operators (called Kraus matrices*) satisfying $\sum_k A_k^\dagger A_k = \mathbb{I}$ such that*

$$\Lambda(\rho) = \sum_k A_k \rho A_k^\dagger \tag{2}$$

Conversely, if $\{A_k\}_{k\in K} \subset \mathcal{B}(\mathcal{H})$ satisfy $\sum_k A_k^\dagger A_k = \mathbb{I}$, then $\Lambda(\rho) = \sum_k A_k \rho A_k^\dagger$ is CPTP.

Proof. See Appendix XIII A. □

Kraus representation gives us an elegant and convenient characterisation of CPTP operations. We can make contact with some familiar special cases. If $\{A_k\}_{k\in K} = \{\Pi_k\}_{k\in K}$ are a complete set of orthogonal projectors $\Pi_k^2 = \Pi_k = \Pi_k^\dagger \ \forall k,\ \sum_k \Pi_k = \mathbb{I}$, we recover vanilla non-selective measurement. If $\{A_k\}_{k\in K} = \{U\}$ for a single unitary U, we recover a unitary transformation.

We will now use the Kraus representation to prove another characterisation of quantum operations. We will go through the proof here, as the ideas will be important for when we finally define measurement.

Theorem 2. *Stinespring Dilation Theorem*
Let $\Lambda \in \Theta(\mathcal{H})$ be CPTP. Then there exists a Hilbert space $\mathcal{H}'$, a pure state $|\psi\rangle \in \mathcal{H}'$, and a unitary transformation U on $\mathcal{H} \otimes \mathcal{H}'$ such that

$$\Lambda(\rho) = \operatorname{Tr}_{\mathcal{H}'}(U(\rho \otimes (|\psi\rangle\langle\psi|))U^\dagger) \tag{3}$$

for all states $\rho \in \mathcal{B}(\mathcal{H})$, where $\operatorname{Tr}_{\mathcal{H}'}$ denotes partial trace over $\mathcal{H}'$.

Conversely, if U is a unitary transformation on $\mathcal{H}\otimes\mathcal{H}'$, and $|\psi\rangle \in \mathcal{H}'$ is a pure state, then $\Lambda(\rho) = \operatorname{Tr}_{\mathcal{H}'}(U(\rho \otimes (|\psi\rangle\langle\psi|))U^\dagger)$ is CPTP on $\mathcal{H}$.

Proof. We start with the forwards direction. Let $\Lambda \in \Theta(\mathcal{H})$ be CPTP, with Kraus decomposition $\Lambda(\rho) = \sum_{k\in K} A_k \rho A_k^\dagger$. Attach to $\mathcal{H}$ an ancilla space $\mathcal{H}'$ with basis $\{|0\rangle\} \cup \{|k\rangle : k \in K\}$. Define a unitary transformation U on $\mathcal{H} \otimes \mathcal{H}'$ by requiring first:

$$U|\chi\rangle|0\rangle = \sum_k A_k|\chi\rangle|k\rangle \quad \forall\ |\chi\rangle \in \mathcal{H}$$

Since $\sum_k A_k^\dagger A_k = \mathbb{I}$, U preserves the inner product between any $(\langle\chi'|\langle 0|)(|\chi\rangle|0\rangle)$. Thus we can extend it to a full unitary U on $\mathcal{H} \otimes \mathcal{H}'$.

Now we have

$$\begin{aligned} U(\rho \otimes (|0\rangle\langle 0|))U^\dagger &= \sum_{k,k'} A_k(\rho \otimes (|k\rangle\langle k'|))A_{k'}^\dagger \\ \operatorname{Tr}_{\mathcal{H}'}(U(\rho \otimes (|0\rangle\langle 0|))U^\dagger) &= \sum_k A_k \rho A_k^\dagger \\ &= \Lambda(\rho) \end{aligned}$$

This completes the forwards direction of the proof.

For the converse, let $\Lambda(\rho) = \mathrm{Tr}_{\mathcal{H}'}(U(\rho \otimes (|\psi\rangle\langle\psi|))U^\dagger)$ for some ancilla space $\mathcal{H}'$, pure state $|\psi\rangle \in \mathcal{H}'$, and unitary transformation U on $\mathcal{H} \otimes \mathcal{H}'$. Let $\{|k\rangle : k \in K\}$ be a basis of $\mathcal{H}'$. Then

$$\begin{aligned}\Lambda(\rho) &= \mathrm{Tr}_{\mathcal{H}'}(U(\rho \otimes (|\psi\rangle\langle\psi|))U^\dagger) \\ &= \sum_k \langle k|U(\rho \otimes (|\psi\rangle\langle\psi|))U^\dagger|k\rangle \\ &= \sum_k A_k \rho A_k^\dagger\end{aligned}$$

where $A_k = \langle k|U|\psi\rangle$. We have $\sum_k A_k^\dagger A_k = \sum_k \langle\psi|U^\dagger|k\rangle\langle k|U|\psi\rangle = \langle\psi|\mathbb{I}|\psi\rangle = \mathbb{I}$, so this is indeed a valid Kraus representation. □

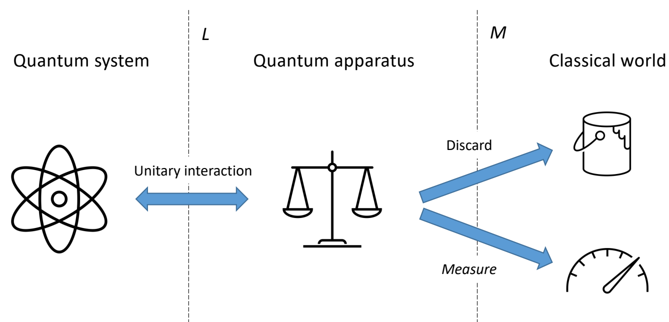

FIG. 4

Stinespring dilation has a pleasing philosophical interpretation, illustrated by Fig. 4. It states that any physical operation can be achieved by the following three steps: (a) introduce some apparatus (or ancilla) quantum system in a pure state (b) apply a unitary to the combined system (c) discard the apparatus system. We will refer to this process as the *church protocol* for an operation, following John Smolin who described the process as 'going to the church of the larger Hilbert space'.

The church characterisation of quantum operations motivates the following definition of a quantum measurement: instead of discarding the apparatus system in (c), we *measure* it in a complete orthonormal basis. The italic font of *measurement* reminds us that this is the (only) place where we invoke the Born rule and the mysterious wavefunction collapse. In this way, all of the philosophical controversy of *quantum measurement* is relegated to step (c), at Line M in Fig. 4. The process of decoherence in step (b) is well-defined and Born-rule-free. Moreover, for an instantaneous measurement in relativity, we do not require any individual to know instantly the *measurement* outcome. Rather we only require that the decoherence in step (b) is instantaneous. The apparatus systems can be gathered together and *measured* at a leisurely pace after the interaction is finished. For the remainder of the present section, we will refer to this as the *church protocol* for measurement: (a) introduce apparatus, (b) interact apparatus with system to decohere, then (c) *measure* the apparatus in a complete orthonormal basis. In subsequent sections, this is what we will understand the term 'measurement' to mean.

The Kraus formalism gives us an elegant description of church measurement, and we adopt this formalism for the remainder of the essay. A measurement is given by a set of Kraus matrices $\{A_k\}_{k\in K}$. The outcome of the *measurement* in step (c) of the church protocol will be an element $k \in K$. The Born rule takes the form $\mathbb{P}(k) = \mathrm{Tr}\, A_k \rho A_k^\dagger$. The state of the system after step (b) of the church protocol is the result of applying the usual operator $\Lambda(\rho) = \sum_k A_k \rho A_k^\dagger$. This is known as *non-selective* measurement. The state of the system after step (c), if we observe outcome k, is $\rho' = (A_k \rho A_k^\dagger)/\mathbb{P}(k)$. This is known as *selective* measurement. If we define $E_k = A_k^\dagger A_k$, then the Born rule becomes $\mathbb{P}(k) = \mathrm{Tr}\, E_k \rho$. $\{E_k\}_{k\in K}$ is known as a *positive operator valued measure* (POVM); each E_k is a positive semi-definite operator, and they satisfy $\sum_k E_k = \mathbb{I}$. Although the POVM specifies the statistics of the *measurement*, it does not specify the state of the system after non-selective or selective measurement.

To justify this description of measurement, we would like two things to be true. Firstly, given any set $\{A_k\}_{k\in K}$ of Kraus matrices, we can perform the corresponding measurement with a church protocol. Secondly, any measurement given as a church protocol can be described by a set $\{A_k\}_{k\in K}$ of Kraus matrices. To see why these two statements are true, we can use the ideas from the proof of Theorem 2. In fact, not only does the proof of Theorem 2 imply that these statements are true, but it gives an explicit algorithm for moving between the church model of measurement and the Kraus representation of measurement.

For the first statement, consider the proof of the forwards direction in Theorem 2. Instead of tracing out $\mathcal{H}'$, we can perform a complete *measurement* in the basis $\{|k\rangle : k \in K\}$. In the pure case $\rho = |\chi\rangle\langle\chi|$, the Born rule tells us we will observe k with probability $\mathbb{P}(k) = \langle\chi|A_k^\dagger A_k|\chi\rangle = \mathrm{Tr}(A_k \rho A_k^\dagger)$. This extends to mixed states ρ. Post-selecting the $\mathcal{H}'$ to $|k\rangle$, we are indeed left with the unnormalized state $\rho' = A_k \rho A_k^\dagger$.

For the second statement, consider the proof of the converse in Theorem 2. Say we measure $U(\rho \otimes (|\psi\rangle\langle\psi|))U^\dagger$ in the arbitrary basis $\{|k\rangle : k \in K\}$. The Born rule tells us we will observe k with probability $\mathbb{P}(k) = \mathrm{Tr}(A_k \rho A_k^\dagger)$. Post-selecting the $\mathcal{H}'$ to $|k\rangle$, we are indeed left with the unnormalized state $\rho' = A_k \rho A_k^\dagger$.

It will be useful to point out a counterintuitive consequence of our definition of measurement: in the degenerate case where the ancilla Hilbert space has dimension 1, a measurement of the system is precisely the application of an arbitrary unitary to the system.

There is also a word of warning to be mentioned. Although every operation has a Kraus representation, this representation is far from unique. As a consequence, a church protocol for an operation with Kraus decomposition $\{A_k\}_{k\in K}$ will not necessarily extend to a church protocol for the measurement with Kraus decomposition $\{A_k\}_{k\in K}$. That is, given an ancilla Hilbert space, pure state and unitary which implements the operation with Kraus matrices $\{A_k\}_{k\in K}$, there may not exist an orthonormal basis of the ancilla space such that complete *measurement* in this basis induces measurement in the Kraus matrices $\{A_k\}_{k\in K}$.

As a final note, one may wonder whether we could achieve a greater generality by allowing the state we attach in step (a) of the church protocol to be a mixed state $\varphi \in \mathcal{B}(\mathcal{H}')$ rather than a pure state. The technique of *purification* shows us that no extra generality is achieved. Let φ have diagonal form $\varphi = \sum_{i=1}^{m} p_i |\psi_i\rangle\langle\psi_i|$. If we attach an extra ancilla space $\mathcal{H}''$ with basis $\{|i\rangle : i = 1, \dots m\}$, then the pure state $|\Psi\rangle = \sum_{i=1}^{m} \sqrt{p_i}|\psi_i\rangle|i\rangle \in \mathcal{H}' \otimes \mathcal{H}''$ has $\mathrm{Tr}_{\mathcal{H}''}(|\Psi\rangle\langle\Psi|) = \varphi$. Alternatively, attaching φ can be interpreted as using classical randomness to attach the pure state $|\psi_i\rangle$ with probability p_i.

III. ALICE AND BOB

We are now in a position to formally define the classes of measurement discussed in the introduction, in the case of a bipartite system AB. A is Alice's system, and B is Bob's. Alice's ancilla system will be R, and Bob's S. Recall that a measurement is specified by a collection of Kraus matrices $\{A_k\}$, so the measurement classes will consist of collections of Kraus matrices.

Definition 1. SLC – *Separable Localisable Classical*
$\{A_{lm}\}_{(l,m)\in L\times M} \in \mathrm{SLC}(A,B)$ *if there exist ancilla systems R, S, pure states $|\psi\rangle \in \mathcal{H}_R$ and $|\varphi\rangle \in \mathcal{H}_S$, unitaries $U_{AR} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_R)$, $U_{BS} \in \mathcal{B}(\mathcal{H}_B \otimes \mathcal{H}_S)$, and bases $\{|l\rangle\}_{l\in L}$ of $\mathcal{H}_R$ and $\{|m\rangle\}_{m\in M}$ of $\mathcal{H}_S$ such that*

$$A_{lm} = \langle l|\langle m|U_{AR} \otimes U_{BS}|\psi\rangle|\varphi\rangle \tag{4}$$

Definition 2. SLQ – *Separable Localisable Quantum*
$\{A_k\}_{k\in K} \in \mathrm{SLQ}(A,B)$ *if there exist ancilla systems R, S, pure states $|\psi\rangle \in \mathcal{H}_R$ and $|\varphi\rangle \in \mathcal{H}_S$, unitaries $U_{AR} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_R)$, $U_{BS} \in \mathcal{B}(\mathcal{H}_B \otimes \mathcal{H}_S)$, and a basis $\{|k\rangle\}_{k\in K}$ of $\mathcal{H}_R \otimes \mathcal{H}_S$ such that*

$$A_k = \langle k|U_{AR} \otimes U_{BS}|\psi\rangle|\varphi\rangle \tag{5}$$

Definition 3. ELC – *Entangled Localisable Classical*
$\{A_{lm}\}_{(l,m)\in L\times M} \in \mathrm{ELC}(A,B)$ *if there exist ancilla systems R, S, a pure state $|\Psi\rangle \in \mathcal{H}_R \otimes \mathcal{H}_S$, unitaries $U_{AR} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_R)$, $U_{BS} \in \mathcal{B}(\mathcal{H}_B \otimes \mathcal{H}_S)$, and bases $\{|l\rangle\}_{l\in L}$ of $\mathcal{H}_R$ and $\{|m\rangle\}_{m\in M}$ of $\mathcal{H}_S$ such that*

$$A_{lm} = \langle l|\langle m|U_{AR} \otimes U_{BS}|\Psi\rangle \tag{6}$$

Definition 4. ELQ – *Entangled Localisable Quantum*
$\{A_k\}_{k\in K} \in \mathrm{ELQ}(A,B)$ *if there exist ancilla systems R, S, a pure state $|\Psi\rangle \in \mathcal{H}_R \otimes \mathcal{H}_S$, unitaries $U_{AR} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_R)$, $U_{BS} \in \mathcal{B}(\mathcal{H}_B \otimes \mathcal{H}_S)$, and a basis $\{|k\rangle\}_{k\in K}$ of $\mathcal{H}_R \otimes \mathcal{H}_S$ such that*

$$A_k = \langle k|U_{AR} \otimes U_{BS}|\Psi\rangle \tag{7}$$

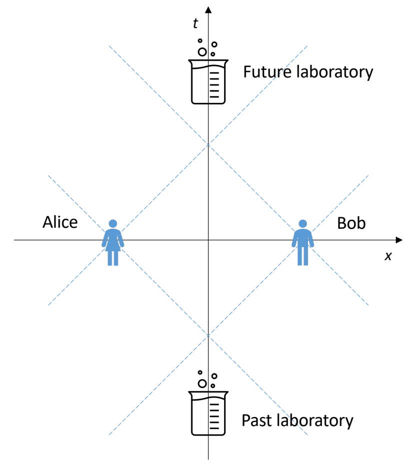

FIG. 5

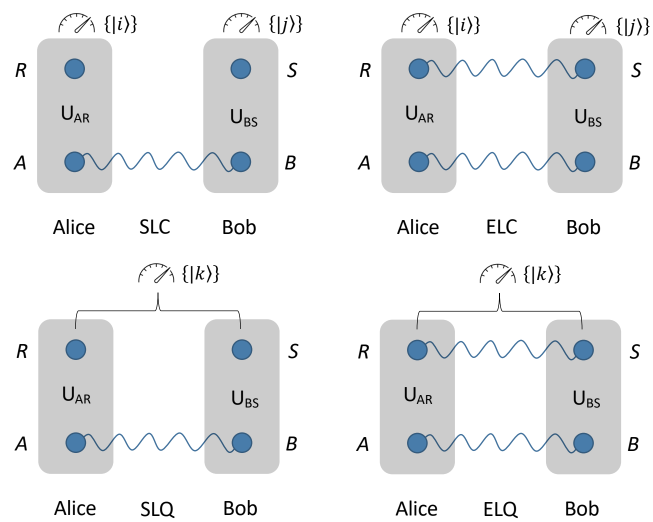

FIG. 6

These four situations have practical interpretations for Alice and Bob, illustrated by Fig. 5 and Fig. 6. For ELC and ELQ, we can imagine that Alice and Bob have been planning this measurement for a long time. At some *past laboratory* in the intersection of their past light cones, they prepared an entangled bipartite apparatus system RS. Alice then kept with her subsystem R to eventually interact with A, and likewise Bob kept S to interact with B. On the other hand, for SLC and SLQ, Alice and Bob must independently produce their own apparatus systems, without the ability to bring them together and entangle them before the measurement.

The protocols SLC and ELC are truly instantaneous measurements. The *measurements* of the apparatus take place on the $t = 0$ slice of Minkowski spacetime, and the result of the measurement is instantaneously written irreversibly into some classical information. Although neither Alice nor Bob knows the entire measurement outcome, we can morally claim that the post-selection too is instantaneous. The no-signalling principle tells us that Alice's density matrix will be independent of Bob's outcome, but we can take the view that this is due to Alice's ignorance of Bob's measurement rather than quantum uncertainty. On the other hand, for SLQ and ELQ, the apparatus-system interaction is indeed instantaneous, yet Alice and Bob do not immediately *measure* their apparatus. Rather, they must maintain the quantum coherence of their apparatus systems for enough time to transport them to a *future laboratory* for further quantum processing, at some point in the intersection of their future light cones. It is when they finally *measure* the apparatus in this central laboratory that the post-selection takes place.

We can immediately characterise SLC as the class of local measurements, encapsulated in the following lemma.

Lemma 1. $\{A_k\}_{k\in K} \in \mathrm{SLC}(A, B)$ *if and only if there exist Kraus matrices* $\{B_l\}_{l\in L} \subset \mathcal{B}(\mathcal{H}_A)$ *and* $\{C_m\}_{m\in M} \subset \mathcal{B}(\mathcal{H}_B)$ *such that* $\{A_k\}_{k\in K} = \{B_l \otimes C_m\}_{(l,m)\in L\times M}$.

Proof. Given $A_{lm} = \langle l|\langle m|U_{AR}\otimes U_{BS}|\psi\rangle|\varphi\rangle$, define $B_l = \langle l|U_{AR}|\psi\rangle$ and $C_m = \langle m|U_{BS}|\varphi\rangle$. Then $A_{lm} = B_l \otimes C_m$.

Conversely, we can always write $B_l = \langle l|U_{AR}|\psi\rangle$ and $C_m = \langle m|U_{BS}|\varphi\rangle$. Then $A_{lm} = B_l \otimes C_m = \langle l|\langle m|U_{AR} \otimes U_{BS}|\psi\rangle|\varphi\rangle$. □

IV. NO-SIGNALLING PRINCIPLES

The no-signalling principle is the statement that it is impossible to harness the nonlocality of quantum entanglement to send a signal containing information. We will now state and prove two distinct no-signalling principles.

Theorem 3. *No-Signalling Principle Version 1*

Consider two independent quantum subsystems A and B, initially in a possibly entangled state $\rho_{AB} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$*. A local action on A will not affect the B reduced density matrix* $\rho_B = \mathrm{Tr}_{\mathcal{H}_A}\,\rho_{AB}$.

Proof. After performing the local action $\Lambda_A \in \Theta(\mathcal{H}_A)$ on $\mathcal{H}_A$, the new joint state is $\rho'_{AB} = \Lambda_A\otimes \mathrm{id}_B(\rho_{AB})$, and the new reduced density matrix on $\mathcal{H}_B$ is $\rho'_B = \mathrm{Tr}_{\mathcal{H}_A}\,\rho'_{AB}$. We must show that $\rho'_B = \rho_B$.

We can write ρ_{AB} as a sum of separable states, $\rho_{AB} = \sum_i \varphi_i \otimes \sigma_i$ with $\varphi_i \in \mathcal{B}(\mathcal{H}_A)$, $\sigma_i \in \mathcal{B}(\mathcal{H}_B)$ for each i.

$$
\begin{aligned}
\rho'_B &= \mathrm{Tr}_{\mathcal{H}_A}(\Lambda_A \otimes \mathrm{id}_B(\sum_i \varphi_i \otimes \sigma_i)) \\
&= \sum_i \mathrm{Tr}_{\mathcal{H}_A}(\Lambda_A(\varphi_i) \otimes \sigma_i) \\
&= \sum_i \mathrm{Tr}(\varphi_i)\sigma_i \\
&= \mathrm{Tr}_{\mathcal{H}_A}(\sum_i \varphi_i \otimes \sigma_i) \\
&= \rho_B
\end{aligned}
$$

where in going from line 2 to line 3, we have used that Λ_A is trace preserving. □

Theorem 4. *No-Signalling Principle Version 2*

Consider a quantum system with Hilbert space $\mathcal{H}$*, initially in state* $\rho \in \mathcal{B}(\mathcal{H})$*. Let* Λ *be the operation with Kraus matrices* $\{A_k\}_{k\in K}$*, and consider a POVM* $\{E_l\}_{l\in L}$*. Suppose* E_l *and* A_k *commute* $[E_l, A_k] = 0$ *for all* $l \in L$*,* $k \in K$*. Then the outcome distribution of the POVM for* $\Lambda(\rho)$ *is the same as for* ρ*.*

Proof. The POVM on ρ has outcome distribution

$$\mathbb{P}(l) = \mathrm{Tr}(E_l\rho)$$

The POVM on $\Lambda(\rho)$ has outcome distribution

$$
\begin{aligned}
\mathbb{P}(l) &= \mathrm{Tr}(E_l(\sum_k A_k\rho A_k^\dagger)) \\
&= \sum_k \mathrm{Tr}(E_l A_k \rho A_k^\dagger) \\
&= \sum_k \mathrm{Tr}(A_k E_l \rho A_k^\dagger) \\
&= \mathrm{Tr}((\sum_k A_k^\dagger A_k)E_l\rho) \\
&= \mathrm{Tr}(E_l\rho)
\end{aligned}
$$

□

The two versions are subtly different in content. Version 1 has stronger conditions; it assumes the Hilbert space decomposes as a tensor product. Certainly if we have an operation and a *measurement* on distinct factors in a tensor product Hilbert space, they will commute as required for Version 2. Correspondingly, Version 1 has a stronger statement. Whilst Version 2 merely demonstrates the invariance of the *measurement* statistics, Version 1 demonstrates the invariance of the entire quantum state, namely the reduced density matrix. Another crucial difference between the two is that Version 2 relies on the Born rule, whilst Version 1 is true independent of the Born rule.

To see that ELQ ⊆ CAUSAL, Version 1 of the no-signalling principle is the relevant statement. In the language of Definition 4, we would like to consider AR as one subsystem in Theorem 3, and BS as the other. This establishes the hierarchy shown in Eq. 1 and Fig. 3.

On the other hand, to see that superluminal signalling is impossible in QFT, we must invoke Version 2 of the no-signalling principle. This follows from the QFT postulate that two operators localised in spacelike separated regions of spacetime always commute. (Two regions A and B are spacelike separated if x and y are spacelike separated for all $x \in A$, $y \in B$.)

V. WHAT'S THE ISSUE? CAUSAL $\neq$ ALL

I claimed in the introduction that arbitrary measurements of a quantum system distributed through space would contradict relativistic causality. Our first job is to justify this explicitly. However, recalling that a degenerate case of measurement is the application of a unitary, this conclusion becomes *entirely obvious*. Thinking about measuring the momentum of a particle is overcomplicating the point, and actually we do not need to be in the quantum world to observe the contradiction.

Suppose Alice and Bob each have a bit, initially both 0. At $t = 0$ the CNOT operation is instantaneously applied to Bob's bit with Alice's bit as the control, so both bits remain 0. If instead Alice had applied the X (NOT) gate at $t = -\epsilon$, Bob's bit would read 1 at time $t = +\epsilon$ rather than 0. Thus Alice can signal to Bob, and the CNOT operation is acausal.

We can directly lift this example to the quantum case to find an example of an acausal quantum unitary. Alice and Bob each have a qubit, initially in state $|00\rangle$. The CNOT and X operations become the respective unitaries on the qubits. Thus the CNOT unitary is an acausal degenerate quantum measurement.

Although we should have already hammered home the point, we will mention an example of an acausal measurement due to Sorkin; see Section 3 of [10]. Let Alice and Bob each possess one qubit. Consider the incomplete measurement given by the orthogonal projectors (Kraus matrices) $\{\Pi, (\mathbb{I} - \Pi)\}$, where $\Pi = |\phi^+\rangle\langle\phi^+|$, $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. The corresponding operation for the non-selective measurement is

$$\Lambda(\rho) = \Pi\rho\Pi + (\mathbb{I} - \Pi)\rho(\mathbb{I} - \Pi) \tag{8}$$

Suppose Alice and Bob start with the state $|00\rangle$. At $t = 0$, Λ is applied to the joint system. In the absence of action from Alice, Bob's reduced density matrix is $\mathbb{I}/2$ i.e. completely mixed. If Alice applied the X gate at $t = -\epsilon$, Bob's density matrix at $t = +\epsilon$ would instead be $|0\rangle\langle 0|$ pure. Thus Λ is acausal.

VI. CAN WE MEASURE ANYTHING NON-LOCAL? PART I: SLC $\neq$ ELC

Now we come to our first result about the quantum world. It is possible to perform a non-local measurement using only local actions. The example in this section will rely crucially on shared entanglement between Alice and Bob.

Let Alice and Bob each have a one-qubit system. The Bell basis for the joint two-qubit Hilbert space is Bell $= \{|\phi^+\rangle, |\phi^-\rangle, |\psi^+\rangle, |\psi^-\rangle\}$ where $|\phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$, $|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$. We will show that we are able to perform complete measurement in the Bell basis, with Kraus matrices $\{|\chi\rangle\langle\chi| : |\chi\rangle \in \text{Bell}\}$. The corresponding non-selective measurement is the operation $\Lambda(\rho) = \sum_{|\chi\rangle\in\text{Bell}} |\chi\rangle\langle\chi|\rho|\chi\rangle\langle\chi|$.

First notice that, since the Bell basis consists of (maximally) entangled states, measurement in the Bell basis is not a local measurement i.e. it is not in SLC. To see this rigorously, suppose $|\phi^+\rangle\langle\phi^+| = P \otimes Q$ for some $P \in \mathcal{B}(\mathcal{H}_A)$, $Q \in \mathcal{B}(\mathcal{H}_B)$. Taking Tr_B of both sides gives $P = \mathbb{I}/2$, and similarly Tr_A gives $Q = \mathbb{I}/2$, clearly a contradiction.

Non-selective measurement in the Bell basis is causal, so at this point we can be satisfied that we have found an example showing SLC $\neq$ CAUSAL. To see this, recall that every member of the Bell basis is a maximally entangled state. Thus if the state $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is decohered in the Bell basis to get the state $\rho' = \Lambda(\rho)$, we will have $\text{Tr}_A\, \rho' = \text{Tr}_B\, \rho' = \mathbb{I}/2$. In particular, the reduced density matrices are independent of ρ. The maximal entanglement ensures that all information in the pre-decoherence state is lost if we trace out either $\mathcal{H}_A$ or $\mathcal{H}_B$.

Measurement in the Bell basis is not only causal but it is in ELC. The localisability of this measurement has been noticed by several authors [2] [4] [5] [7] [8]; we will follow the approach from Section II C of [2].

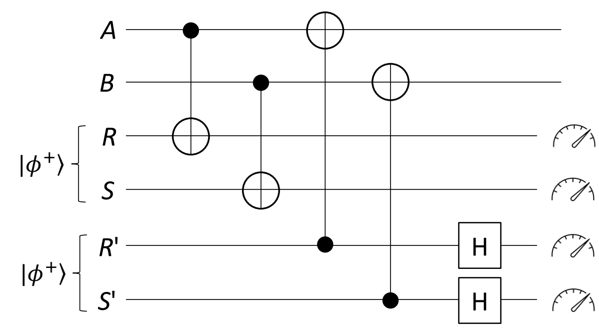

FIG. 7

Alice has apparatus qubits R and R', and Bob S, S'. $\mathcal{H}_R \otimes \mathcal{H}_S$ and $\mathcal{H}_{R'} \otimes \mathcal{H}_{S'}$ are each prepared in the entangled Bell state $|\phi^+\rangle$. Let the unitary U be given by the circuit in Fig. 7. U visibly consists only of local actions: all gates are either applied to Alice's system $\mathcal{H}_A \otimes \mathcal{H}_R \otimes \mathcal{H}_{R'}$ or Bob's system $\mathcal{H}_B \otimes \mathcal{H}_S \otimes \mathcal{H}_{S'}$. After performing U, the qubits R, S, R', S' are *measured* in the standard basis. This is a local *measurement*, which Alice and Bob can perform separately and immediately after the interaction. We can follow the algorithm described in Section II to derive the Kraus matrices for this measurement: they are $\{A_{i_1 i_2 i_3 i_4}\}_{(i_1,i_2,i_3,i_4)\in\{0,1\}^4} \subset \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$, where

$$A_{i_1 i_2 i_3 i_4} = \langle i_1 i_2 i_3 i_4|_{RSR'S'} U |\phi^+\rangle_{RS} |\phi^+\rangle_{R'S'} \tag{9}$$

By direct calculation, we get

$$A_{0000} = A_{0011} = A_{1100} = A_{1111} = \frac{1}{2}|\phi^+\rangle\langle\phi^+|$$
$$A_{0001} = A_{0010} = A_{1101} = A_{1110} = \frac{1}{2}|\phi^-\rangle\langle\phi^-|$$
$$A_{0100} = A_{0111} = A_{1000} = A_{1011} = \frac{1}{2}|\psi^+\rangle\langle\psi^+|$$
$$A_{0101} = A_{0110} = A_{1001} = A_{1010} = \frac{1}{2}|\psi^-\rangle\langle\psi^-|$$

This is a measurement in the Bell basis. The non-selective measurement induces decoherence in the Bell basis, and a *measurement* of the apparatus in the standard basis allows us to post-select a particular Bell state. Thus we have succeeded in constructing a measurement in ELC which is not in SLC.

VII. CAN WE MEASURE ANYTHING NON-LOCAL? PART II: SLC $\neq$ SLQ

Similarly we can achieve a non-local measurement without prior entanglement, but rather by maintaining the coherence of Alice and Bob's apparatus after the interaction, and measuring the joint apparatus in an arbitrary basis at some point in the intersection of the light cones of Alice and Bob. This example is again lifted from Section II C of [2].

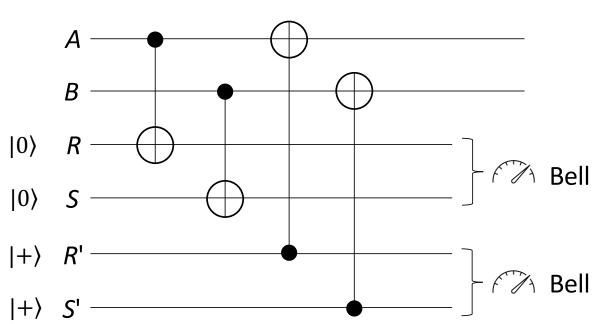

FIG. 8

As in the previous section, Alice has one system qubit A and two apparatus qubits R, R'; Bob has one system qubit B and two apparatus qubits S, S'. R and S are initially in the $|0\rangle$ state and R', S' initially in the $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ state. In particular, the apparatus is initially in a separable state. Let U be the unitary given by the circuit in Fig. 8, which again visibly consists of actions localisable to Alice and Bob. Instead of measuring R, R', S, S' in the standard basis, we now transport them to a central laboratory where we measure them in an entangled basis. Namely, we measure RS in the Bell basis, and also $R'S'$ in the Bell basis. The Kraus matrices for this measurement are given by

$$A_{\chi\chi'} = \langle\chi|_{RS}\langle\chi'|_{R'S'} U |0\rangle_R |0\rangle_S |+\rangle_{R'} |+\rangle_{S'} \ , \ |\chi\rangle, |\chi'\rangle \in \text{Bell} \tag{10}$$

Again if we explicitly calculate these matrices, we get

	$\lvert\phi^+\rangle$	$\lvert\phi^-\rangle$	$\lvert\psi^+\rangle$	$\lvert\psi^-\rangle$	RS
$\lvert\phi^+\rangle$	$\frac{1}{2}\lvert\phi^+\rangle\langle\phi^+\rvert$	$\frac{1}{2}\lvert\phi^-\rangle\langle\phi^-\rvert$	$\frac{1}{2}\lvert\psi^+\rangle\langle\phi^+\rvert$	$\frac{1}{2}\lvert\psi^-\rangle\langle\phi^-\rvert$	
$\lvert\phi^-\rangle$	$\frac{1}{2}\lvert\phi^+\rangle\langle\phi^-\rvert$	$\frac{1}{2}\lvert\phi^-\rangle\langle\phi^+\rvert$	$\frac{1}{2}\lvert\psi^+\rangle\langle\phi^-\rvert$	$\frac{1}{2}\lvert\psi^-\rangle\langle\phi^+\rvert$	
$\lvert\psi^+\rangle$	$\frac{1}{2}\lvert\psi^+\rangle\langle\psi^+\rvert$	$\frac{1}{2}\lvert\psi^-\rangle\langle\psi^-\rvert$	$\frac{1}{2}\lvert\phi^+\rangle\langle\psi^+\rvert$	$\frac{1}{2}\lvert\phi^-\rangle\langle\psi^-\rvert$	
$\lvert\psi^-\rangle$	$\frac{1}{2}\lvert\psi^+\rangle\langle\psi^-\rvert$	$\frac{1}{2}\lvert\psi^-\rangle\langle\psi^+\rvert$	$\frac{1}{2}\lvert\phi^+\rangle\langle\psi^-\rvert$	$\frac{1}{2}\lvert\phi^-\rangle\langle\psi^+\rvert$	
$R'S'$					

These Kraus matrices are not in SLC. We can see this by using same argument from the previous section. Intuitively, the above Kraus matrices have the ability to act on a separable state $|\alpha\rangle|\beta\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ to produce an entangled state. The ability to introduce entanglement between the systems of Alice and Bob is a non-local feat.

VIII. DOES QUANTUM POST-PROCESSING HELP? ELC $\approx$ ELQ

We saw in Section VI that Alice and Bob can achieve non-local measurements by using prior shared entanglement and immediate local measurements. In addition to Alice and Bob having access to unlimited initial entanglement, we could allow them to preserve the quantum coherence of their apparatus systems and *measure* them in an arbitrary non-product basis in some future laboratory. The question is: does this increase the set of (non-local) measurements Alice and Bob are able to perform? In other words, does ELC equal ELQ? We will see in this section that the answer is subtle: ELC is '*dense in*' ELQ. *Dense* is in quotation marks, since we do not technically have a topology on the set of possible measurements. Every measurement in ELQ can be approximated by measurements in ELC in the following sense: if $\{A_k\}_{k\in K} \in$ ELQ, then for any $\epsilon > 0$, $\{\sqrt{1-\epsilon}A_k\}_{k\in K} \cup \{B_l\}_{l\in L} \in$ ELC for some Kraus matrices $\{B_l\}_{l\in L}$ with $\sum_{l\in L} B_l^\dagger B_l = \epsilon$. That is, there is an ELC measurement which implements $\{A_k\}_{k\in K}$ with probability at least $1-\epsilon$. As ϵ gets smaller, the dimension of the required apparatus systems will diverge to infinity. The question of whether or not we have exact equality ELC = ELQ we leave as an open problem.

We are given an ELQ measurement, and we are required to find an ELC protocol which reproduces the same Kraus matrices with high probability. Suppose the ELQ measurement introduces apparatus R and S for Alice and Bob respectively, in the possibly entangled state $|\Psi\rangle \in \mathcal{H}_R \otimes \mathcal{H}_S$. Alice applies U_{AR} and Bob U_{BS}. They then transport R and S to the future laboratory and *measure* RS in the (possibly non-product) basis $\{|k\rangle\}$ of $\mathcal{H}_R \otimes \mathcal{H}_S$.

To construct our ELC protocol, we are free to introduce larger entangled ancilla systems for Alice and Bob. In fact in our construction, we will introduce the same R and S as in the ELQ protocol, and we will additionally introduce extra apparatus systems R' and S' which consist of a large supply of Bell states $|\phi^+\rangle^{\otimes M} \in \mathcal{H}_{R'} \otimes \mathcal{H}_{S'}$ entangled between R' and S'.

The alert reader will object: how can we possibly reproduce the same Kraus matrices as the ELQ measurement if the apparatus systems in the ELC protocol are larger? Is the number of Kraus matrices not equal to the dimension of the joint apparatus Hilbert space? The answer is: multiple basis vectors of the apparatus *measurement* and thus multiple *measurement* outcomes may be associated to the same Kraus matrix. Indeed we saw an example of this with the Bell measurement in Section VI. If we are concerned only with the effect of the measurement on the system, we can declare two measurements to be the same if their Kraus matrices are the same after appropriately summing over degeneracies.

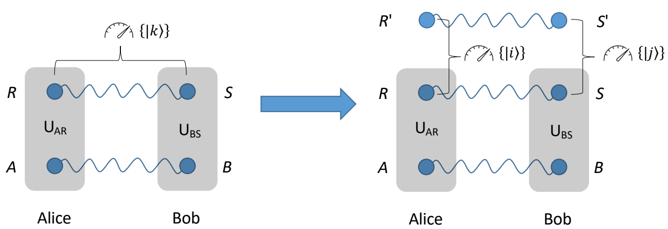

FIG. 9

As mentioned, for our ELC protocol we will introduce the same R and S as in the ELQ protocol. In addition, we will introduce further apparatus systems R' and S' which consist of a large supply of Bell states $|\phi^+\rangle^{\otimes M} \in \mathcal{H}_{R'} \otimes \mathcal{H}_{S'}$ entangled between R' and S'. For the unitary interaction between apparatus and system, we will exactly copy the ELQ measurement: Alice applies U_{AR} and Bob U_{BS}. We must then cook up a way to measure RS approximately in the (possibly non-product) basis $\{|k\rangle\}$ by locally measuring RR' and SS', exploiting the entanglement between R' and S'. For this, we will use a scheme proposed by Vaidman [3]. This is shown in Fig. 9. Since Vaidman's scheme relies heavily on the technique of quantum teleportation, we will briefly review teleportation here.

A. Review of quantum teleportation

Bob can teleport one qubit of quantum information to Alice, consuming one Bell state of entanglement. Let us recall how this is done: suppose Bob possesses qubits 1 and 2, and Alice qubit 3. Qubits 2 and 3 are in the

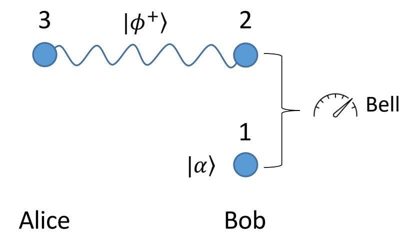

FIG. 10

entangled state $|\phi^+\rangle$, and Bob wishes to teleport qubit 1 in state $|\alpha\rangle$ to Alice. Bob does this by *measuring* his qubits 1 and 2 in the Bell basis. This is shown in Fig. 10. Since $|\alpha\rangle|\phi^+\rangle = \frac{1}{2}(|\phi^+\rangle|\alpha\rangle + |\phi^-\rangle Z|\alpha\rangle + |\psi^+\rangle X|\alpha\rangle + |\psi^-\rangle XZ|\alpha\rangle)$, we have the following

Bob's *mmt*	Alice's qubit 3
$\|\phi^+\rangle$	$\|\alpha\rangle$
$\|\phi^-\rangle$	$Z\|\alpha\rangle$
$\|\psi^+\rangle$	$X\|\alpha\rangle$
$\|\psi^-\rangle$	$XZ\|\alpha\rangle$

If Bob's *mmt* happens to give $|\phi^+\rangle$, which happens with probability $1/4$, the teleportation works perfectly. We will refer to this scenario as *direct*. The rest of the time, $|\alpha\rangle$ is teleported up to a unitary $\{Z, X, XZ\}$, which we will refer to as the teleportation *error*. After the Bell *mmt*, Bob can communicate the *mmt* result to Alice, allowing her to correct the error and complete the teleportation with certainty. However, this takes a time at least that which light takes to get from Bob to Alice. We are interested in instantaneous processes, and we will take *teleportation* to refer to just the first stage consisting of Bob's Bell *mmt*.

There are two remarks to be made which will be important for the application of teleportation to Vaidman's scheme. Firstly, if Bob's qubit 1 is initially entangled with some other quantum system, then after teleportation Alice's qubit 3 will be entangled to the same system in exactly the same way. In other words, any entanglement of qubit 1 gets teleported to qubit 3 along with the state of the qubit (possibly up to some unitary error). This essentially follows from the linearity of the teleportation process. The second remark is that the unitary errors $\{Z, X, XZ\}$ all map the Z-basis to itself.

B. Vaidman's scheme

We will now proceed to describe Vaidman's scheme for measuring RS approximately in any given basis $\{|k\rangle\}$ by locally measuring RR' and SS'. Without loss of generality, the systems R and S consist of qubits. If not, then Alice and Bob can apply suitable SWAP operations to write the quantum information in R and S into systems of qubits. Again without loss of generality, let R and S each consist of N qubits.

For conceptual clarity, we will describe Vaidman's scheme as an interactive procedure between Alice and Bob. However, the only operations carried out by Alice and Bob will be to *measure* locally each qubit at most once. At some points in the description, Alice will apply a unitary before *measuring*, but this can simply be absorbed into the *mmt* by appropriately transforming the *mmt* basis. Thus the entire process can be collapsed into single local *mmts* for Alice and Bob.

We will now describe Vaidman's scheme:

1. Bob teleports the whole of S to Alice. Bob's *mmt* will have 4^N possible outcomes, indexed by $\hat{n} \in \{1, \dots 4^N\}$. Only $\hat{n} = 1$ gives a direct teleportation.

2. Alice applies a unitary U_0 to RS which, under the assumption of direct teleportation in (1), would rotate the $\{|k\rangle\}$ basis to the Z-basis.

3. Alice teleports RS to Bob. Note that, under the assumption of direct teleportation in (1), the $\{|k\rangle\}$ basis of RS will correspond to the computational basis of Bob's received system, independent of teleportation error from Alice.

4. (a) If the teleportation in (1) was direct, which occurs with probability $1/4^N$, then Bob *measures* RS in the Z-basis. This successfully completes the *mmt*.

   (b) If (1) was *not* direct, then Bob teleports RS back to Alice to a cluster of Alice's qubits labelled $n = \hat{n}$. Bob's *mmt* will have 4^{2N} possible outcomes, indexed by $\hat{m}_1 \in \{1, \dots 4^{2N}\}$, with $\hat{m}_1 = 1$ giving direct teleportation. This can be interpreted as Bob 'telling' Alice that the teleportation in (1) failed, and specifying the error to her.

5. Alice performs unitaries $U_1^{(n)}$ on each of the clusters labelled $n \in \{1, \dots 4^N\}$. Under the assumption of direct teleportation in (4b), the unitary $U_1^{(n)}$ must correct Alice's teleportation error from (3), undo U_0, correct the teleportation error corresponding to n from (1), and re-apply U_0.

6. Alice teleports all of her clusters $n \in \{1, \dots 4^N\}$ to Bob.

7. (a) If (4b) was direct, which occurs with probability $1/4^{2N}$, then Bob *measures* RS in the Z-basis. This successfully completes the *mmt*.

   (b) If not, then Bob teleports RS back to Alice to a cluster of Alice's qubits labelled $(n, m_1) = (\hat{n}, \hat{m}_1)$. Bob's *mmt* will again have 4^{2N} outcomes, indexed by $\hat{m}_2$.

8. Alice performs unitaries $U_2^{(n,m_1)}$ on each of the clusters labelled $(n, m_1) \in \{1, \dots 4^N\} \times \{1, \dots 4^{2N}\}$. Under the assumption of direct teleportation in (7b), these unitaries are supposed to rotate the original $\{|k\rangle\}$ basis to the Z-basis, taking into account all previous errors.

9. Alice teleports all clusters $(n, m_1) \in \{1, \dots 4^N\} \times \{1, \dots 4^{2N}\}$ back to Bob.

10. (a) If (7b) was direct, which occurs with probability $1/4^{2N}$, then Bob *measures* RS in the Z-basis. This successfully completes the *mmt*.

    (b) If not, then Bob teleports RS back to Alice to a cluster of Alice's qubits labelled $(n, m_1, m_2) = (\hat{n}, \hat{m}_1, \hat{m}_2)$. Bob's *mmt* will yet again have 4^{2N} outcomes, indexed by $\hat{m}_3$.

11. Continue steps 8, 9, 10 recursively.

Each round has constant success probability $1/4^{2N}$, and there is no limit to how many rounds of the procedure one can perform. Thus the *measurement* can be performed with an arbitrarily small probability of failure. Since the entanglement of R and S with A and B will be carried through the teleportations throughout the procedure, this completes the construction that ELC can replicate any ELQ measurement $\{A_k\}_{k\in K}$ with arbitrarily high probability of success.

Formally, for any $\epsilon > 0$, we can perform enough rounds so that RS is *measured* in basis $\{|k\rangle\}$ with probability at least $1-\epsilon$. The Kraus matrices of the resulting ELC measurement will then be $\{\sqrt{1-\epsilon}A_k\}_{k\in K} \cup \{B_l\}_{l\in L}$ with $\sum_{l\in L} B_l^\dagger B_l = \epsilon$, and where the outcomes $l \in L$ correspond to the various ways in which Vaidman's scheme can fail.

C. Discussion

Vaidman's scheme makes no attempt to be efficient with the entanglement consumption ie. the dimension of the apparatus systems. Indeed, for fixed error probability ϵ, the number of Bell states M required scales doubly exponentially in the number of qubits N. This entanglement consumption was improved to a single exponential by Beigi, Konig [11] using the technique of port-based teleportation [12] [13]. (We exhibited Vaidman's scheme rather than the port-based teleportation scheme for simplicity.) However, both schemes have in common that the entanglement consumption is required to diverge to infinity as the probability of error ϵ goes to zero.

This leads us to pose two open questions.

Open Question 1. *Do we have exact equality ELC = ELQ?*

Open Question 2. *Can we implement all POVMs on a bipartite system AB perfectly using local measurements with a strictly finite amount of prior shared entanglement between Alice and Bob?*

Let us remark on Open Question 2. The vanilla quantum mechanical projection postulate says that measurement has a dual role: it both allows the observer to gain information about the system, and prepares the system in the observed eigenstate. Authors have argued that these roles should be separated [3]. We can ask the question: what is measurable by Alice and Bob if we relax the preparation role, and are only concerned with gaining information about the system?

It turns out that even the SLQ model is sufficient to render absolutely every observable measurable in this relaxed sense. To see this, Alice and Bob can introduce ancilla systems R and S of dimension equal to those of A and B. They can then each perform a SWAP unitary, which instantaneously swaps the states of the systems with the apparatus: $\mathrm{SWAP}_{AR}|\psi\rangle_A|\varphi\rangle_R = |\varphi\rangle_A|\psi\rangle_R$, $\mathrm{SWAP}_{BS}|\psi\rangle_B|\varphi\rangle_S = |\varphi\rangle_B|\psi\rangle_S$. Alice and Bob are then free to 'freeze' their apparatus, transport the quantum information coherently to a future laboratory, and perform whatever measurement they like. Vaidman's scheme (and the port-based teleportation improvement) give us that every observable is approximately measurable in this relaxed sense also in the ELC model. Indeed, destructive measurements were the original purpose of Vaidman's scheme; we have somewhat repurposed it in this work to have implications for selective measurement.

With this in mind, we can see that the two open questions above are in fact equivalent.

Theorem 5. *Open Question 1* $\iff$ *Open Question 2.*

Proof. Note that the SWAP measurement described above demonstrates that all POVMs can be implemented in SLQ, and thus certainly in ELQ. If ELC = ELQ, it follows that all POVMs can be implemented in ELC, implying Open Question 2.

On the other hand, suppose all POVMs can be perfectly implemented using local measurements with a strictly finite amount of prior shared entanglement. Then, in the argument above that ELC $\approx$ ELQ, we can replace Vaidman's scheme by a scheme which measures RS in the basis $\{|k\rangle\}$ with zero probability of error, giving us exact equality ELC = ELQ. □

IX. DOES PRIOR ENTANGLEMENT HELP? SLQ ≠ ELQ

In the previous section, we saw a surprising result: If Alice and Bob share entanglement, then allowing them to measure the apparatus in a non-local basis does not increase the set of measurements they can perform. We currently have

$$\begin{aligned} &\mathrm{SLC} \neq \mathrm{ELC} \approx \mathrm{ELQ} \\ &\mathrm{SLC} \neq \mathrm{SLQ} \end{aligned} \tag{11}$$

In this section, we will investigate SLQ $\subseteq$ ELQ. This can be viewed as a converse question to that of the previous section: If Alice and Bob have the ability to measure their apparatus in a non-local basis, then does giving them access to prior shared entanglement increase the set of measurements they can perform? Contrary to the previous section, the answer here is yes. In fact, the Bell measurement from Section VI will suffice as an example of a measurement in ELQ which is not in SLQ. To see this, we will make use of an elegant characterisation of SLQ measurements. To my knowledge, the following lemma is a new result.

Lemma 2. $\{A_k\}_{k\in K} \in \mathrm{SLQ}(A,B)$ *if and only if there exist Kraus matrices* $\{B_l\}_{l\in L} \subset \mathcal{B}(\mathcal{H}_A)$ *and* $\{C_m\}_{m\in M} \subset \mathcal{B}(\mathcal{H}_B)$ *with* $|L||M| = |K| = N$*, and a* $N\times N$ *unitary matrix* $W_{k,(l,m)}$*, such that*

$$A_k = \sum_{l,m} W_{k,(l,m)} B_l \otimes C_m \tag{12}$$

Proof. See Appendix XIII B. □

Lemma 2 says that SLQ measurements are unitary mixings of local SLC measurements. This is not the case for the Bell measurement from Section VI. If we consider the operator subspace $\mathrm{span}\{|\chi\rangle\langle\chi| : |\chi\rangle \in \mathrm{Bell}\} \leq \mathcal{B}(\mathcal{H}_A\otimes\mathcal{H}_B)$, this contains only two separable operators of the form $B\otimes C$; that is, $\mathbb{I}_{AB} = \mathbb{I}_A\otimes\mathbb{I}_B$ and $X_A\otimes X_B$. Thus it is impossible for the Bell measurement Kraus matrices to satisfy Lemma 2.

X. ARE ALL CAUSAL MEASUREMENTS LOCALISABLE? ELQ ≠ CAUSAL

Lastly we address the question: are all causal measurements localisable? The answer is surprisingly no. Here we will exhibit a counterexample, following the arguments from Section V of [2]. Our first step is to state a lemma which contains a necessary condition for a measurement to be in ELQ. This gives us the machinery to demonstrate that a given measurement is *not* in ELQ.

Lemma 3. *Let* $\{A_k\}_{k\in K}$ *be a measurement on* $\mathcal{H}_A\otimes\mathcal{H}_B$*, with corresponding non-selective operation* $\Lambda(\rho) = \sum_k A_k\rho A_k^\dagger$*. We say a state* $|\psi\rangle\in\mathcal{H}_A\otimes\mathcal{H}_B$ *is an* eigenstate *of* Λ *if* $\Lambda(|\psi\rangle\langle\psi|) = |\psi\rangle\langle\psi|$*. Suppose* $|\psi\rangle$*,* $P\otimes\mathbb{I}|\psi\rangle$*,* $\mathbb{I}\otimes Q|\psi\rangle$ *are all eigenstates of* Λ*, where* P *is a unitary on* $\mathcal{H}_A$ *and* Q *a unitary on* $\mathcal{H}_B$*. If* $\{A_k\}_{k\in K}\in\mathrm{ELQ}(A,B)$*, then* $P\otimes Q|\psi\rangle$ *must also be an eigenstate of* Λ*.*

Proof. See Appendix XIII C. □

We are now ready to show the counterexample, which was named the *twisted partition* in Section V A of [2]. Alice and Bob each have two qubits, which we will call $A1$, $A2$, $B1$, $B2$. Recall the Bell basis for the two-qubit Hilbert space is $\mathrm{Bell} = \{|\phi^+\rangle, |\phi^-\rangle, |\psi^+\rangle, |\psi^-\rangle\}$ where $|\phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle\pm|11\rangle)$, $|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle\pm|10\rangle)$. Let $S = \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}$. We can apply to $\mathbb{I}\otimes S$ to get a new basis $\mathbb{I}\otimes S(\mathrm{Bell}) = \{\mathbb{I}\otimes S|\phi^+\rangle, \mathbb{I}\otimes S|\phi^-\rangle, \mathbb{I}\otimes S|\psi^+\rangle, \mathbb{I}\otimes S|\psi^-\rangle\}$.

	$B2 = \|0\rangle$	$B2 = \|1\rangle$
$A2 = \|0\rangle$	Bell	Bell
$A2 = \|1\rangle$	Bell	$\mathbb{I} \otimes S(\text{Bell})$

Consider the basis of the 4-qubit Hilbert space $\mathcal{H}_{A1} \otimes \mathcal{H}_{A2} \otimes \mathcal{H}_{B1} \otimes \mathcal{H}_{B2}$ which consists of the Bell basis on $\mathcal{H}_{A1} \otimes \mathcal{H}_{B1}$ tensored with $\{|00\rangle, |01\rangle, |10\rangle\} \subset \mathcal{H}_{A2} \otimes \mathcal{H}_{B2}$, completed with the $\mathbb{I} \otimes S(\text{Bell})$ basis on $\mathcal{H}_{A1} \otimes \mathcal{H}_{B1}$ tensored with $|11\rangle \in \mathcal{H}_{A2} \otimes \mathcal{H}_{B2}$. This is shown in the table. We will look at complete measurement in this basis.

The complete measurement in this basis is *not* in ELQ, as it does not satisfy the necessary condition of Lemma 3. To see this, let X_{A2} and X_{B2} be the Pauli X operators on qubits $A2$, $B2$. $|\phi^+\rangle_{A1,B1}|00\rangle_{A2,B2}$, $|\phi^+\rangle_{A1,B1}|10\rangle_{A2,B2} = X_{A2} \otimes \mathbb{I}(|\phi^+\rangle_{A1,B1}|00\rangle_{A2,B2})$ and $|\phi^+\rangle_{A1,B1}|01\rangle_{A2,B2} = \mathbb{I} \otimes X_{B2}(|\phi^+\rangle_{A1,B1}|00\rangle_{A2,B2})$ are all elements of the basis, and so are eigenstates of the non-selective measurement Λ. However, $|\phi^+\rangle_{A1,B1}|11\rangle_{A2,B2} = X_{A2} \otimes X_{B2}(|\phi^+\rangle_{A1,B1}|00\rangle_{A2,B2})$ is not an element of the basis, and is not an eigenstate of the non-selective measurement Λ. Thus the condition in Lemma 3 is broken, and the measurement is not in ELQ.

We will now show that non-selective measurement in the above basis is causal. First notice that all elements of both the Bell basis and the $\mathbb{I} \otimes S(\text{Bell})$ basis are maximally entangled. Suppose the initial state of the 4-qubit system is $\rho \in \mathcal{B}(\mathcal{H}_{A1} \otimes \mathcal{H}_{A2} \otimes \mathcal{H}_{B1} \otimes \mathcal{H}_{B2})$. The state after decoherence into the basis is $\rho' = \Lambda(\rho)$. Imagine now tracing out Alice's system to get $\rho'_B = \mathrm{Tr}_{\mathcal{H}_{A1} \otimes \mathcal{H}_{A2}} \rho'$. The state of the $B1$ qubit will be completely mixed, so Bob's density matrix will take the form $\rho'_B = \mathbb{I}/2 \otimes (b_0|0\rangle\langle 0| + b_1|1\rangle\langle 1|)$. The coefficients b_i are unaffected if we replace ρ with $\Lambda_A \otimes \mathbb{I}(\rho)$ for some local action Λ_A on Alice's system, as shown by the following argument:

$$\begin{aligned} b_i &= \mathrm{Tr}_{\mathcal{H}_{B1} \otimes \mathcal{H}_{B2}}(|i\rangle\langle i|_{B2}\rho'_B) \\ &= \mathrm{Tr}(|i\rangle\langle i|_{B2}\rho') \\ &= \mathrm{Tr}(|i\rangle\langle i|_{B2}\Lambda(\rho)) \\ &= \mathrm{Tr}(\Lambda(|i\rangle\langle i|_{B2}\rho)) \\ &= \mathrm{Tr}(|i\rangle\langle i|_{B2}\rho) \end{aligned}$$

We have used that $|i\rangle\langle i|_{B2}$ commutes with Λ, and that Λ is trace-preserving. Now if we replaced ρ with $\Lambda_A \otimes \mathbb{I}(\rho)$, crucially $|i\rangle\langle i|_{B2}$ commutes with $\Lambda_A \otimes \mathbb{I}$, so b_i is unaffected, using that $\Lambda_A \otimes \mathbb{I}$ is trace-preserving. Thus Alice cannot signal to Bob by performing a local operation on her system. The same argument shows that Bob cannot signal to Alice. Thus measurement in the above basis is causal.

This section shows that there appears to be a 'perplexing gap' [2] between what is possible with localised quantum actions, and what is consistent with the principle of causality. These conclusions echo that of the Tsirelson inequality, which bounds non-local correlation between observables in quantum mechanics. The Tsirelson inequality is similarly stricter than what is required by the principle of causality alone. Indeed, in Section VI of [2], the Tsirelson inequality was used to find a further example of a causal measurement which is not localisable. These results raise the question: do there exist further physical principles, in addition to relativistic causality, which restrict quantum non-locality and non-local measurements to the true quantum bounds? Such principles would confirm the belief that the structure of quantum mechanics is natural, and would give insight as to why. Various such principles have been suggested, including no advantage for non-local computation [19] and information causality [20].

XI. WAVEFUNCTION COLLAPSE

The vanilla non-relativistic quantum mechanical projection postulate says that measurement has a dual role: it both allows the observer to gain information about the system, and prepares the system in the observed eigenstate. This gives us something that neither role alone gives: if we are able to both verify the state of the system and leave it undisturbed, this allows us to *monitor* the system state.

There is another side to the coin. The postulates of non-relativistic quantum mechanics give us an objective, albeit possibly stochastic, history for the state of the system. The system evolves mostly unitarily, and occasionally stochastically collapses into an eigenstate when an observable is measured. It is metaphysically satisfactory that, although we cannot attribute to the system well-defined values of (for example) position and momentum, we can nevertheless attribute a well-defined quantum state. Such an objective history has metaphysical content, as it provides answers to the *counterfactual* question: what would have happened if I had monitored the state?

In Galilean spacetime, there is no issue with declaring that the wavefunction collapse occurs on the $t = 0$ slice. However, if we try to import this collapse postulate into Minkowski spacetime, we are confronted with the question: where does the collapse occur? Suppose we make a pointlike measurement at the origin of Minkowski spacetime. The $t = 0$ slice in some given Lorentz frame is not a Lorentz covariant location. In view of this, there are two natural choices: the future light cone of the measurement, and the past light cone.

Suppose for the moment that only local (pointlike) variables are instantaneously measurable. Indeed, this was roughly the conclusion of Landau and Peierls [14]. Under this assumption, the future and past light cone prescriptions are both tenable. We have an intact objective wavefunction history, where now the wavefunction is well-defined for each point in Minkowski space. Hellwig and Kraus preferred the past light cone prescription for purely aesthetic reasons [15].

The issue is: we have seen that it is possible to monitor a non-local state with only local interactions. Specifically, in Section VI we saw that it is possible to perform measurement (in the two-role sense) in the non-local Bell basis for two qubits. We will see that this destroys all hope of defining an objective spacetime hypersurface where we can take the collapse to occur. These arguments were first presented by Aharonov and Albert [4] [5].

As a thought experiment, consider Alice and Bob, each possessing a qubit. In some fixed Lorentz frame, Alice and Bob are stationary, with Alice at $x = 0$ and Bob at $x = d$. Let the observer Romeo inhabit this fixed frame. Initially (i.e. for $t < -d/c$), the qubits of Alice and Bob are in the $|\phi^+\rangle$ state i.e. they constitute an EPR pair. At $(t, x) = (0, 0)$, Alice measures her qubit in the Z-basis. The causes the joint state to collapse to $|00\rangle$ or $|11\rangle$, each with probability 1/2.

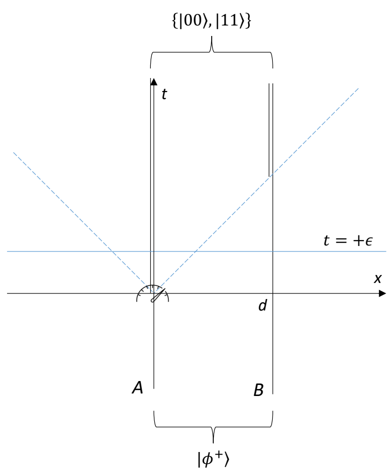

FIG. 11

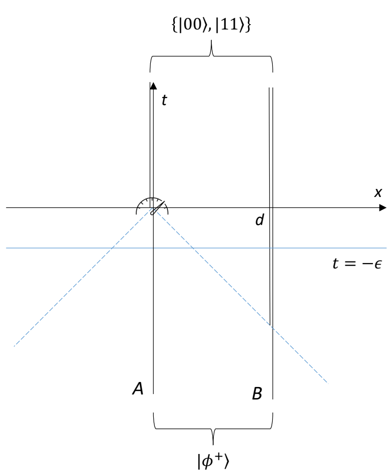

FIG. 12

Fig. 11 and Fig. 12 show us what goes wrong with the future and past light cone prescriptions respectively once we consider the measurements Romeo could have made. In the figures throughout this section, the double line denotes a qubit post-selected for the measurement at the origin $(t, x) = (0, 0)$.

Let $0 < \epsilon < d/c$. If the state collapses along the future light cone, then what would Romeo's Bell measurement at $t = +\epsilon$ observe? If the state collapses along the past light cone, then what would Romeo's Bell measurement at $t = -\epsilon$ observe? It is not that these questions do not have answers; it is just that the future/past light cone collapse prescriptions break down when we ask these questions. Romeo's measurement at $t = -\epsilon$ would observe $|\phi^+\rangle$ with certainty, and Romeo's measurement at $t = +\epsilon$ would observe $|\phi^+\rangle$ or $|\phi^-\rangle$ each with probability 1/2. Crucially, in the case of the past light cone, Romeo's counterfactual measurement is in particular a monitoring. Thus it cannot be argued that, if Romeo's measurement had been performed, it would have disrupted the evolution of the system.

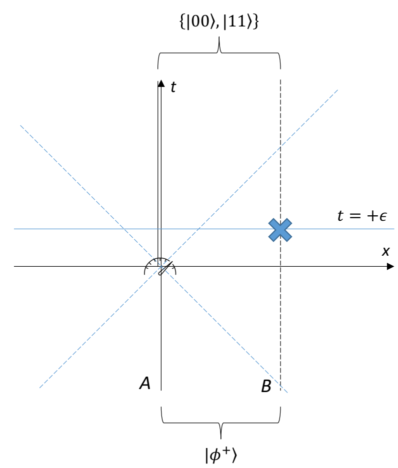

FIG. 13

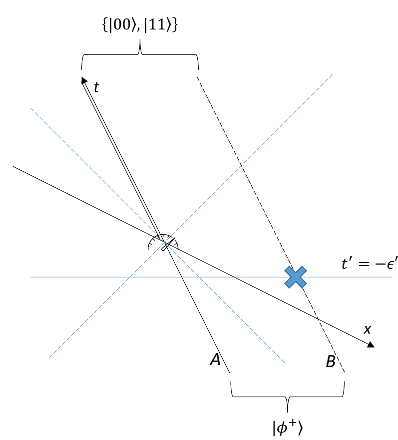

FIG. 14

Romeo's Bell measurement on the joint state of Alice and Bob at $t = +\epsilon < d/c$ would observe $|\phi^+\rangle$ or $|\phi^-\rangle$ each with probability 1/2. Thus, if an objective state history exists and is consistent with Romeo's counterfactual measurement, it cannot assign to the point $(t, x) = (+\epsilon, d)$ the state $|\phi^+\rangle$.

However, consider now another observer Juliet, whose frame has origin coinciding with Romeo's, but is boosted in the $+x$ direction. Romeo and Juliet's frames are depicted in Fig. 13 and Fig. 14 respectively. Romeo's point $(t, x) = (+\epsilon, d)$ becomes Juliet's point $(t', x') = (-\epsilon', d')$ for some $0 < \epsilon' < d'/c$. If Juliet measured at $t' = -\epsilon'$, she would observe $|\phi^+\rangle$ with certainty. Moreover, since this measurement is a monitoring, she would be able to perform this measurement without disrupting the evolution of the state. Thus an objective state history consistent with Juliet's counterfactual measurement would need to assign $(t', x') = (-\epsilon', d')$ the state $|\phi^+\rangle$. We have reached a paradox, and we are forced to conclude that the possibility of monitoring non-local states in some fixed Lorentz frame rules out the notion of an objective hypersurface where the collapse occurs.

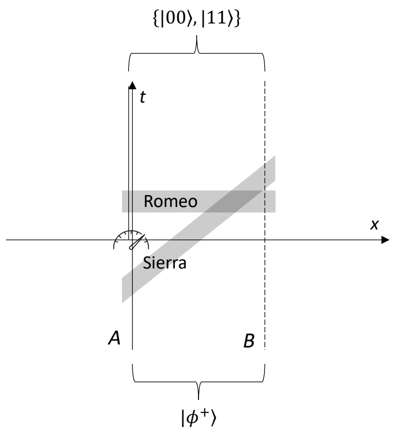

FIG. 15

The paradox lies with the notion of an objective collapse location. How do we reconcile these conclusions to escape a contradiction in the physical world? The key observation is that, although Romeo and Juliet's counterfactual conclusions individually hold true in the absence of the other, if Romeo and Juliet *both* tried to perform their measurements in a way that would contradict one another, then the measurements of Romeo and Juliet would end up frustrating each other. See Fig. 15.

In fact we can argue that, as long as we stick to procedures at least contained in ELQ, quantum mechanics will never contradict itself in the prediction of experimental statistics. This is because the way in which we measure non-local variables is to use local interactions which correlate the non-local variables with local variables of the apparatus, before measuring the local variables of the apparatus. Thus, taking a quantum mechanical description of the system and apparatus together, all measurements are ultimately local in this sense. This means that our no-signalling principles apply, promising no violations of causality and no predictive contradictions.

What can we conclude from these arguments? Suppose there is a quantum system at (t, x), possibly entangled with other systems at other locations. We have learnt that we cannot consistently associate a quantum state to the system at (t, x). Rather, as was realised by Aharonov and Albert in [6], the state of the system depends on the spacetime hypersurface to which we are considering the point (t, x) to belong.

XII. CONCLUSION

We have studied in detail the measurements which can be made by two distant parties on their joint system. Although special relativity was the motivation behind the definitions of SLC, ELC, SLQ, ELQ for Alice and Bob, it is possible to interpret our results in a purely information theoretic perspective. Independent of special relativity, the result ELC $\approx$ ELQ says that any post-selective measurement performed by Alice and Bob using prior shared entanglement and quantum post-processing, can be performed with use of prior shared entanglement alone. By contrast, SLQ $\neq$ ELQ says that some non-local measurements performed by Alice and Bob do indeed rely crucially on prior shared entanglement.

We could imagine several directions for extending this work, in addition to the open questions from Section VIII C. We could extend the discussion to infinite-dimensional systems, and/or we could allow infinite-dimensional apparatus systems. We could also relax the notion of locality from a bipartite system with a tensor product Hilbert space, to a single Hilbert space where Alice's and Bob's operations are required to commute. This is known as the *commuting operator framework*, as opposed to the *tensor product framework*. We could model the system and/or the apparatus in this relaxed version of locality.

Do the results we have derived in the Alice and Bob toy model apply to the real world? For an account of quantum measurement fully consistent with relativity, we would need to use quantum field theory. Measurements should take place in smeared regions of spacetime. We would need to model both the system and the apparatus with quantum fields, and model the interaction between the system and apparatus with a Lorentz covariant interaction Hamiltonian. Here, both the system and apparatus would be infinite-dimensional, and locality will be encoded in the commutation of local operators. Such a framework has been developed recently by Fewster and Verch [16] [17], using the language of algebraic quantum field theory.

That said, the Alice and Bob model can be recovered from quantum field theory in a particular limit, as described in the introduction. Moreover, I would argue that the conceptual conclusions we have made should hold true in the general setting. Namely, there are measurements which are impossible because they would allow superluminal signalling; there are localisable protocols which enable instantaneous measurement of non-local observables; and there are measurements which are causal but nevertheless cannot be implemented by a combination of local actions.

Acknowledgements This work is in response to an essay title proposed by Jeremy Butterfield as coursework for the Part III Mathematics Tripos at Cambridge University. The author would like to thank Jeremy, Richard Jozsa, Sergii Strelchuk, Damian Pitalua-Garcia, Thomas Vidick, Daniel Gottesman, Salman Beigi, Robert Konig, Wayne Myrvold, Philip Pearle and Gordon Fleming for helpful correspondence.

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XIII. APPENDIX

A. Proof of Kraus representation

The objective of this section is to prove Theorem 1. Our first port of call is to write down what we mean by *completely positive*, a property of CPTP superoperators $\Lambda \in \Theta(\mathcal{H})$.

Definition 5. *A superoperator* $\Lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ *is* positive *if* $\Lambda(\rho)$ *is positive semi-definite for every* ρ *positive semi-definite.*

A superoperator $\Lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ *is* completely positive *if, for every ancilla Hilbert space* $\mathcal{H}'$, $\Lambda \otimes \mathrm{id} : \mathcal{B}(\mathcal{H} \otimes \mathcal{H}') \to \mathcal{B}(\mathcal{H} \otimes \mathcal{H}')$ *is positive.*

We are now in a position to prove the Kraus representation theorem.

Theorem 1. *Kraus Representation Theorem*

Let $\Lambda \in \Theta(\mathcal{H})$ *be CPTP. Then there is a set* $\{A_k\}_{k \in K} \subset \mathcal{B}(\mathcal{H})$ *of linear operators (called* Kraus matrices*) satisfying* $\sum_k A_k^\dagger A_k = \mathbb{I}$ *such that*

$$\Lambda(\rho) = \sum_k A_k \rho A_k^\dagger \tag{13}$$

Conversely, if $\{A_k\}_{k \in K} \subset \mathcal{B}(\mathcal{H})$ *satisfy* $\sum_k A_k^\dagger A_k = \mathbb{I}$, *then* $\Lambda(\rho) = \sum_k A_k \rho A_k^\dagger$ *is CPTP.*

Proof. We start with the forwards direction. Let $\Lambda \in \Theta(\mathcal{H})$ be CPTP, and $\dim \mathcal{H} = n$ with basis $\{|i\rangle : i = 1, \ldots n\}$. For the purposes of this proof, attach an ancilla Hilbert space $\mathcal{H}'$ with the same dimension $\dim \mathcal{H}' = n$ and basis $\{|i'\rangle : i = 1, \ldots n\}$. Let $|\Omega\rangle \in \mathcal{H} \otimes \mathcal{H}'$ be the non-normalised maximally entangled state

$$|\Omega\rangle = \sum_{i=1}^{n} |i\rangle |i'\rangle$$

Define the *Choi matrix* of Λ to be

$$\begin{aligned} J(\Lambda) &= \Lambda \otimes \mathrm{id}(|\Omega\rangle\langle\Omega|) \\ &= \sum_{i,j=1}^{n} \Lambda(|i\rangle\langle j|) \otimes |i'\rangle\langle j'| \end{aligned}$$

By complete positivity of Λ, the Choi matrix $J(\Lambda)$ is positive semi-definite. Thus we can diagonalise $J(\Lambda)$ as

$$J(\Lambda) = \sum_k |\phi_k\rangle\langle\phi_k|$$

with $\{|\phi_k\rangle\}$ unnormalised.

For each $|\phi_k\rangle \in \mathcal{H} \otimes \mathcal{H}'$, we can associate an operator $A_k \in \mathcal{B}(\mathcal{H})$ by

$$\begin{aligned} |\phi_k\rangle &= \sum_{i,j} \alpha_{i,j} |i\rangle |j'\rangle \\ \longrightarrow \ A_k &= \sum_{i,j} \alpha_{i,j} |i\rangle\langle j| \end{aligned}$$

These satisfy

$$(A_k \otimes \mathbb{I})|\Omega\rangle = |\phi_k\rangle$$
$$\langle i'|\phi_k\rangle = A_k|i\rangle$$

Now

$$\begin{aligned}\Lambda(|i\rangle\langle j|) &= \langle i'|J(\Lambda)|j'\rangle \\ &= \sum_k \langle i'|\phi_k\rangle\langle\phi_k|j'\rangle \\ &= \sum_k A_k|i\rangle\langle j|A_k^\dagger\end{aligned}$$

Extending by linearity, we have for all ρ

$$\Lambda(\rho) = \sum_k A_k \rho A_k^\dagger$$

Taking the trace of both sides, using that Λ is trace preserving, we get

$$\sum_k A_k^\dagger A_k = \mathbb{I}$$

This completes the forwards direction of the proof. Note that we can always take the number of Kraus matrices $|\{A_k\}_{k\in K}|$ to be at most n^2.

For the converse, let $\Lambda(\rho) = \sum_k A_k \rho A_k^\dagger$ for some set of Kraus matrices $\{A_k\}_{k\in K} \subset \mathcal{B}(\mathcal{H})$ satisfying $\sum_k A_k^\dagger A_k = \mathbb{I}$. Linearity of Λ is clear, so it is certainly a superoperator. We must check (i) complete positivity and (ii) trace preservation. For any ancilla Hilbert space $\mathcal{H}'$ and $X \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}')$, we have

$$\Lambda \otimes \mathrm{id}(X) = \sum_k (A_k \otimes \mathbb{I})X(A_k \otimes \mathbb{I})^\dagger$$

When X is positive semi-definite, $(A_k \otimes \mathbb{I})X(A_k \otimes \mathbb{I})^\dagger$ is positive semi-definite for each k, thus the sum is positive semi-definite. This shows complete positivity.

For trace preservation, just note that

$$\begin{aligned}\mathrm{Tr}(\Lambda(\rho)) &= \mathrm{Tr}(\sum_k A_k \rho A_k^\dagger) \\ &= \mathrm{Tr}((\sum_k A_k^\dagger A_k)\rho) \\ &= \mathrm{Tr}(\rho)\end{aligned}$$

□

B. Proof of Lemma 2

Here we prove Lemma 2, which gives a characterisation of SLQ measurements.

Lemma 2. $\{A_k\}_{k\in K} \in \mathrm{SLQ}(A, B)$ *if and only if there exist Kraus matrices* $\{B_l\}_{l\in L} \subset \mathcal{B}(\mathcal{H}_A)$ *and* $\{C_m\}_{m\in M} \subset \mathcal{B}(\mathcal{H}_B)$ *with* $|L||M| = |K| = N$*, and a* $N \times N$ *unitary matrix* $W_{k,(l,m)}$*, such that*

$$A_k = \sum_{l,m} W_{k,(l,m)} B_l \otimes C_m \qquad (14)$$

Proof. First suppose we are given $A_k = \langle k|U_{AR} \otimes U_{BS}|\psi\rangle|\varphi\rangle$. Choose bases $\{|l\rangle\}_{l\in L}$ of $\mathcal{H}_R$ and $\{|m\rangle\}_{m\in M}$ of $\mathcal{H}_S$. Then $|L||M| = |K| = N$ and there is a $N\times N$ unitary matrix $W_{k,(l,m)}$ taking the basis $\{|l\rangle|m\rangle\}$ to $\{|k\rangle\}$. That is,

$$|k\rangle = \sum_{l,m} W_{k,(l,m)}|l\rangle|m\rangle$$

Then if we define $B_l = \langle l|U_{AR}|\psi\rangle$ and $C_m = \langle m|U_{BS}|\varphi\rangle$, we have

$$A_k = \sum_{l,m} W_{k,(l,m)} B_l \otimes C_m$$

Conversely, say we are given $B_l = \langle l|U_{AR}|\psi\rangle$ and $C_m = \langle m|U_{BS}|\varphi\rangle$ with $|L||M| = N$, and $N \otimes N$ unitary matrix $W_{k,(l,m)}$. Then define the basis $\{|k\rangle\}_{k\in K}$ as the image of $\{|l\rangle|m\rangle\}$ under $W_{k,(l,m)}$. Consider measuring $R \otimes S$ in the $\{|k\rangle\}$ basis. The corresponding Kraus matrices are

$$\begin{aligned}A_k &= \langle k|U_{AR} \otimes U_{BS}|\psi\rangle|\varphi\rangle \\ &= \sum_{l,m} W_{k,(l,m)} B_l \otimes C_m\end{aligned}$$

□

C. Proof of Lemma 3

Here we prove Lemma 3, which gives a necessary condition for a measurement to be in ELQ. We follow the proof in Appendix B of [2].

Lemma 3. *Let* $\{A_k\}_{k\in K}$ *be a measurement on* $\mathcal{H}_A \otimes \mathcal{H}_B$*, with corresponding non-selective operation* $\Lambda(\rho) = \sum_k A_k \rho A_k^\dagger$*. We say a state* $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ *is an* eigenstate *of* Λ *if* $\Lambda(|\psi\rangle\langle\psi|) = |\psi\rangle\langle\psi|$*. Suppose* $|\psi\rangle$*,* $P \otimes \mathbb{I}|\psi\rangle$*,* $\mathbb{I} \otimes Q|\psi\rangle$ *are all eigenstates of* Λ*, where* P *is a unitary on* $\mathcal{H}_A$ *and* Q *a unitary on* $\mathcal{H}_B$*. If* $\{A_k\}_{k\in K} \in \mathrm{ELQ}(A, B)$*, then* $P \otimes Q|\psi\rangle$ *must also be an eigenstate of* Λ*.*

Proof. Suppose $\{A_k\}_{k\in K} \in \mathrm{ELQ}(A, B)$. Then there are ancilla spaces $\mathcal{H}_R$ and $\mathcal{H}_S$, a pure state $|\varphi\rangle \in \mathcal{H}_R \otimes \mathcal{H}_S$, and unitaries U on $\mathcal{H}_A \otimes \mathcal{H}_R$ and V on $\mathcal{H}_B \otimes \mathcal{H}_S$ such that

$$\Lambda(\rho) = \mathrm{Tr}_{RS}(U \otimes V(\rho \otimes (|\varphi\rangle\langle\varphi|))U^\dagger \otimes V^\dagger)$$

The system $\mathcal{H}_A$ and ancilla $\mathcal{H}_R$ belong to Alice, who applies the unitary U. $\mathcal{H}_B$ and $\mathcal{H}_S$ belong to Bob, who applies the unitary V. $|\varphi\rangle$ is the initial state of the joint (possibly entangled) apparatus.

By hypothesis, $|\psi\rangle$, $P\otimes\mathbb{I}|\psi\rangle$, $\mathbb{I}\otimes Q|\psi\rangle$ are eigenstates of Λ. Thus

$$\begin{aligned} U\otimes V|\psi\rangle|\varphi\rangle &= |\psi\rangle|\varphi_0\rangle \\ U\otimes V(P\otimes\mathbb{I})|\psi\rangle|\varphi\rangle &= |\psi\rangle|\varphi_P\rangle \\ U\otimes V(\mathbb{I}\otimes Q)|\psi\rangle|\varphi\rangle &= |\psi\rangle|\varphi_Q\rangle \end{aligned}$$

for some states $|\varphi_0\rangle, |\varphi_P\rangle, |\varphi_Q\rangle \in \mathcal{H}_R\otimes\mathcal{H}_S$.

Now consider $\Delta = UPU^{-1}P^{-1}$, which acts on Alice's system $\mathcal{H}_A\otimes\mathcal{H}_R$.

$$\begin{aligned} &(\Delta\otimes\mathbb{I})(P\otimes\mathbb{I})|\psi\rangle|\varphi_0\rangle \\ &= (\Delta\otimes\mathbb{I})(P\otimes\mathbb{I})(U\otimes V)|\psi\rangle|\varphi\rangle \\ &= (U\otimes V)(P\otimes I)|\psi\rangle|\varphi\rangle \\ &= (P\otimes I)|\psi\rangle|\varphi_P\rangle \end{aligned}$$

Thus acting on $(P\otimes\mathbb{I})|\psi\rangle|\varphi_0\rangle$, we can replace $\Delta\otimes\mathbb{I}$ by $R\otimes\mathbb{I}$, where R is a unitary *acting on* $\mathcal{H}_R$ *alone* which sends $|\varphi_0\rangle$ to $|\varphi_P\rangle$.

Then

$$\begin{aligned} &(UP\otimes V)|\psi\rangle|\varphi\rangle \\ &= (\Delta\otimes\mathbb{I})(PU\otimes V)|\psi\rangle|\varphi\rangle \\ &= (\Delta\otimes\mathbb{I})(P\otimes\mathbb{I})|\psi\rangle|\varphi_0\rangle \\ &= (R\otimes\mathbb{I})(P\otimes\mathbb{I})|\psi\rangle|\varphi_0\rangle \\ &= (R\otimes\mathbb{I})(PU\otimes V)|\psi\rangle|\varphi\rangle \end{aligned}$$

Multiplying by $\mathbb{I}\otimes V^{-1}$,

$$(UP\otimes\mathbb{I})|\psi\rangle|\varphi\rangle = (R\otimes\mathbb{I})(PU\otimes\mathbb{I})|\psi\rangle|\varphi\rangle$$

That is, acting on $|\psi\rangle|\varphi\rangle$, we can replace UP by RPU. Similarly,

$$(\mathbb{I}\otimes VQ)|\psi\rangle|\varphi\rangle = (\mathbb{I}\otimes S)(\mathbb{I}\otimes QV)|\psi\rangle|\varphi\rangle$$

where S is a unitary *acting on* $\mathcal{H}_S$ *alone.*

We can view the two above equations as commutation relations $[U,P]=R$, $[V,Q]=S$ when acting on $|\psi\rangle|\varphi\rangle$. We can use these to examine how Λ will act on $P\otimes Q|\psi\rangle|\varphi\rangle$.

$$\begin{aligned} &(U\otimes V)(P\otimes Q)|\psi\rangle|\varphi\rangle \\ &= (\mathbb{I}\otimes VQ)(UP\otimes\mathbb{I})|\psi\rangle|\varphi\rangle \\ &= (R\otimes\mathbb{I})(\mathbb{I}\otimes VQ)(PU\otimes\mathbb{I})|\psi\rangle|\varphi\rangle \\ &= (R\otimes\mathbb{I})(PU\otimes\mathbb{I})(\mathbb{I}\otimes VQ)|\psi\rangle|\varphi\rangle \\ &= (R\otimes S)(PU\otimes\mathbb{I})(\mathbb{I}\otimes QV)|\psi\rangle|\varphi\rangle \\ &= (R\otimes S)(P\otimes Q)(U\otimes V)|\psi\rangle|\varphi\rangle \\ &= (R\otimes S)(P\otimes Q)|\psi\rangle|\varphi_0\rangle \\ &= ((P\otimes Q)|\psi\rangle)((R\otimes S)|\varphi_0\rangle) \end{aligned}$$

That is, $(P\otimes Q)|\psi\rangle$ is an eigenstate of Λ. $\square$