States of Ignorance and Ignorance of States: Examining the Quantum Principal Principle*

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Abstract

Earman (2018) has recently argued that the Principal Principle, a principle of rationality connecting objective chance and credence, is a theorem of quantum probability theory. This paper critiques Earman's argument, while also offering a positive proposal for how to understand the status of the Principal Principle in quantum probability theory.

1 Introduction

A particle is approaching a Stern-Gerlach magnet oriented in the x direction. You know the particle is in the spin-up-z state, $\uparrow_z$. How confident should you be that an 'UP' outcome will be registered? Answer: you should be 50% confident. Here is one story for why. Knowing the particle is in the $\uparrow_z$ state tells you that the chance of 'UP' is $|\langle\uparrow_x|\uparrow_z\rangle|^2 = 1/2$. And epistemic rationality demands you conform your credences to known chances.

The last step in this reasoning is an application of Lewis's Principal Principle, which says (roughly) that if an agent knows the objective chance of a proposition E is x, then she should set her degree of belief in E to x (Lewis, 1980). It is controversial what quantum chances really are, and if they are objective. But if they are objective, then this looks like a paradigm application of the Principal Principle.

On the standard way of understanding the Principal Principle, it is a normative requirement that goes above and beyond other more background constraints on rational credence, like probabilism and conditionalization. While authors have disagreed about whether the principle can be

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independently justified and if so how (Strevens, 1999; Ward, 2005; Pettigrew, 2012, 2013; Childers, 2012; Schwarz, 2014), most agree that an agent can be probabilistically coherent and yet still violate it. However, Earman (2018) has recently suggested that in the quantum context, the principle:

Quantum Principal Principle (QPP), rough version: if an agent learns that the state of the system is ψ, then she should set her credence in E to $\psi(E)$, the chance ψ assigns to E,

follows as a "theorem of quantum probability theory" from the usual coherence requirements on rational credence. Thus, there is no need to seek a further justification of the principle in the quantum setting, or agonize over which variant of it is correct.[1]

This paper assesses Earman's proposal. I first note that on some interpretations, quantum chances are modeled by ordinary 'classical' (Kolmogorovian) probability theory, not quantum probability theory, and so there is no immediate need for a new treatment of the QPP on these interpretations. On other interpretations, Earman is right to point out that QPP requires its own special formulation, and that this is an unaddressed issue in the literature. However, I argue, Earman's proposed formulation is not the correct one. I lay out what I take to be the correct formulation, and show that although intuitively true, it does not follow as a theorem.

Before proceeding, it is worth mentioning a further reason that philosophers and physicists may be interested in the status of the QPP. An important practice within physics is the reconstruction of unknown quantum states through measurement. This process of *quantum state tomography* is one of several methods used to learn and estimate quantum states (Paris and Rehacek, 2004). To the extent we think of the physicists in these scenarios as rational agents with credences, we may be interested in modeling their learning experiences as they home in on the unknown state. Here QPP plays a crucial role. Suppose ψ is the true state of the system and E is some experimental

[1] Earman's claim in some ways parallels a claim of Wallace (2012, pp. 150-1) that, specifically in Everettian quantum mechanics, the Principal Principle can be derived from background constraints on rational credence. One difference is that Earman's claim seems intended more generally (though for discussion of the ultimate scope of Earman's argument, see Section 3). Another difference is that Wallace assumes further constraints on rational credence beyond probabilism and conditionalization, related to betting, symmetry and branching (Wallace, 2012, Part II). For reasons that will become clearer later in the paper, questions of the Everett interpretation will ultimately fall beyond the scope of our discussion, which focuses on quantum probability theory quite narrowly construed. For discussion of Wallace's derivation of the Everettian Principal Principle and the status of the Deutsch-Wallace Born rule theorem, see e.g. Adlam (2014), Dawid and Thébault (2015), Jansson (2016), Read (2018), Brown and Porath (2020), Wilson (2020, Ch. 3) and Saunders (forthcoming).

data. By Bayes' formula:

$$C(\text{state is } \psi | E) = \frac{C(E| \text{ state is } \psi) \cdot C(\text{state is } \psi)}{C(E)}. \tag{1}$$

Crucially, the QPP fixes $C(E| \text{ state is } \psi)$ to $\psi(E)$. This ensures that the higher the chance ψ assigns to the observation E, the stronger the true hypothesis is supported. The QPP also helps fix the value of the prior term $C(E)$: letting $\{\psi_i\}$ denote the set of candidate states, we have

$$C(E) = \sum_i C(E| \text{ state is } \psi_i) \cdot C(\text{state is } \psi_i) \tag{2}$$

$$= \sum_i \psi_i(E) \cdot C(\text{state is } \psi_i), \tag{3}$$

in other words, the agent's credence in the experimental outcome E is equal to her expectation of its chance. The overall result is that the agent's credence in the state hypotheses evolves in the way familiar from Bayesian statistical inference (cf. Lewis (1980, pp. 285–7)). One of my criticisms of Earman will be that his formulation of the QPP can't do the normative work required for this application to quantum state tomography, and statistical inference more generally. At the end of the paper, I will show that my formulation can.

2 Earman on quantum chance

In *quantum probability theory*,[2] the objects of quantum chance—what I shall call, for lack of a better term, the quantum propositions—correspond mathematically to the set of all projection operators on the Hilbert space of the system in question. *Quantum chance functions* are then defined as functions on this set of quantum propositions. They output real numbers between 0 and 1, and obey similar axioms to classical (Kolmogorovian) probability functions.

In more detail, let $\mathcal{H}$ be a separable Hilbert space and let $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$ denote the von Neumann algebra of all bounded operators acting on $\mathcal{H}$. A *projection* $E \in \mathfrak{N}$ is a self-adjoint element satisfying $E^2 = E$. Let $\mathcal{P}(\mathfrak{N})$ denote the set of projections on $\mathfrak{N}$. Then a *quantum chance function* is a map $\omega : \mathcal{P}(\mathfrak{N}) \to [0, 1]$ satisfying:

[2] Here I follow the presentation of Earman (2018). Cf. also von Neumann (1955), Bub (1977), Hughes (1989, Ch. 8), Pitowsky (2006), Rédei and Summers (2007), Strocchi (2008, Ch. 2.4), and Hemmo and Shenker (2020).

1. $\omega(I) = 1$, where $I \in \mathcal{P}(\mathfrak{N})$ is the identity operator (analogue of the necessity), and

2. $\omega(\bigvee_i E_i) = \sum_i \omega(E_i)$ for any countable collection $\{E_i\}$ of mutually orthogonal projections.[3]

Example 1. In the Stern Gerlach case, we can consider the projection $P_{\uparrow_x} = |\uparrow_x\rangle\langle\uparrow_x|$ corresponding to the proposition that the 'UP' outcome will occur. Since the particle is in state $\uparrow_z$, the quantum chance function is given by $\omega(E) = \langle\uparrow_z|E|\uparrow_z\rangle$ for all projections $E \in \mathcal{P}(\mathfrak{N})$. One can verify that ω satisfies both axioms. In addition, $\omega(P_{\uparrow_x}) = \langle\uparrow_z|\,(|\uparrow_x\rangle\langle\uparrow_x|)\,|\uparrow_z\rangle = |\langle\uparrow_z|\uparrow_x\rangle|^2 = 1/2$ as expected.

We just saw that specifying the quantum state fixes the quantum chance function. It turns out that when the dimension of $\mathcal{H}$ is greater than two, the converse also holds: specifying the quantum chance function ω uniquely fixes a (normal) quantum state $\omega : \mathfrak{N} \to \mathbb{C}$, which we also denote ω.[4] If ω is a *vector state*, in other words if there exists a vector $\psi \in \mathcal{H}$ such that $\omega(A) = \langle\psi|A|\psi\rangle$ for all $A \in \mathfrak{N}$, then we simply write ω as ψ.

What about credence functions?[5] Earman assumes that in the quantum setting, rational initial credence functions C are also defined over the quantum propositions $\mathcal{P}(\mathfrak{N})$. Furthermore, Earman imposes two basic coherence requirements on these functions:

Probabilism (quantum analogue): rational C satisfy the axioms listed above (and hence can be extended to a normal state $C : \mathfrak{N} \to \mathbb{C}$, which we also denote C), and

Conditionalization (quantum analogue): rational C obey an analogue of Bayesian updating, called *Lüders updating*, which specifies that for all $E, F \in \mathcal{P}(\mathfrak{N})$:[6]

$$C(F|E) = \frac{C(EFE)}{C(E)}, \qquad \text{provided } C(E) > 0.$$

[3]Here $\vee$ is the 'join' operator, the analogue of disjunction: if E_1 and E_2 are projection operators, then $E_1 \vee E_2$ is the projection corresponding to the closure of $\mathrm{Ran}(E_1) \cup \mathrm{Ran}(E_2)$. Two projection operators are E_1 and E_2 are *orthogonal* when $E_1E_2 = \mathbf{0}$. A collection is *mutually orthogonal* if every pair is orthogonal.

[4]This follows from Gleason's theorem and the assumption that ω is countably additive. Here a *quantum state* is a normed positive linear functional $\omega : \mathfrak{N} \to \mathbb{C}$ on $\mathfrak{N}$. A quantum state ω on $\mathfrak{N}$ is *normal* if there exists a unique density operator $\rho_\omega \in \mathfrak{N}$ such that $\omega(A) = \mathrm{Tr}[A\rho_\omega]$ for all $A \in \mathfrak{N}$. When $\mathcal{H}$ is finite, all states are normal, but this may not hold in infinite settings (Ruetsche, 2011). When ω is normal, it is common to use ω and ρ_ω interchangeably.

[5] I follow Earman in assuming a common *Lewisian dualist* view of probability, according to which objective and subjective probabilities both exist and are distinct. In particular, in the quantum setting, we can consider not just the chances assigned to propositions by the Born rule (which at least appear to be objective), but also an agent's subjective credences in those propositions. Of course, as mentioned earlier, it is controversial whether Born rule probabilities are ultimately objective chances (at least in Lewis's sense), and arguably the answer depends on which interpretation of QM we assume. For further discussion of this dualist view in the context of quantum theory, see e.g. Wallace (2012), Steeger (2019), Earman and Ruetsche (2020) and Myrvold (2021, Ch. 9) (cf. also footnote 11).

[6]Cf. Bub (1977). Here it is important that C is extended to a state $C : \mathfrak{N} \to \mathbb{C}$, since for $F \in \mathcal{P}(\mathfrak{N})$ that do not commute with E, EFE may not be a projection operator, thus leaving the numerator $C(EFE)$ undefined otherwise.

Say an initial credence function $C : \mathcal{P}(\mathfrak{N}) \to [0,1]$ is *coherent* if it satisfies these quantum analogues of probabilism and conditionalization. Earman's main claim is that any such coherent credence function must automatically satisfy the QPP, thanks to the following result:

Theorem 1 (Ruetsche and Earman (2011) Fact 1)**.** *Set-up as above, where we suppose* $\dim(\mathcal{H}) > 2$*. Let* $C : \mathcal{P}(\mathfrak{N}) \to [0,1]$ *be any coherent initial credence function. Let* ψ *be any vector state and let* $P_\psi \in \mathcal{P}(\mathfrak{N})$ *denote the projection onto the vector.*[7] *Then, provided* $C(P_\psi) > 0$*,*

$$C(\cdot|P_\psi) = \psi(\cdot). \tag{4}$$

In other words, conditional on P_ψ*, the credences* C *must align with the chances* ψ*.*

Based on this result, Earman concludes that the truth of the QPP is not a matter "to be submitted to the intuitions of wise analytical metaphysicians" but rather one to be proved: "credence and chance have been brought into alignment without the cudgel of any extra normative principle" (Earman, 2018, p. 10). Crucially, Earman cannot be accused of "trying to derive an 'ought' from an 'is'" here. His claim is not that a normative principle follows from mathematical axioms, despite what his slogan "Lewis's Principal Principle is a theorem of quantum probability theory" may suggest. Rather, his claim is that a normative principle, the QPP, follows from *other* normative principles, Probabilism and Conditionalization, and we can see this entailment clearly once we appropriately formulate these principles in the setting of quantum probability theory.

Of course, while Earman's proposal may be safe from Hume's guillotine, this does not mean it is safe from all present (philosophical) dangers. Let us now examine Earman's argument.

3 Examining Earman's argument

Note that in order for Earman's result to establish QPP, it is crucial that the projection P_ψ represent the proposition $\langle$the state of the system is $\psi\rangle$, or $\langle$the chances are given by $\psi\rangle$. My main criticism of Earman's argument is that this claim is false, and so Equation (4) does not establish the Principal Principle.

[7] Here it crucial that ψ is a vector state, in particular that it is *normal* (cf. footnote 4) and *pure*; in ordinary QM, where $\mathfrak{N}$ is the von Neumann algebra, the pure normal states coincide with the vector states. Notably, the result cannot be extended to entangled ω, for instance; see Section 4 for discussion.

Before elaborating on this criticism, I should clarify an important issue regarding the scope of Earman's argument.[8] Earman's choice to model quantum chances using quantum probability theory—and in particular to define chances over the set $\mathcal{P}(\mathfrak{N})$ of *all* projection operators—is not uncontroversial. Some quantum theories end up privileging a certain subset of commuting operators. This often brings them into the setting of ordinary 'classical' Kolmogorovian probability theory (Loewer, 1994). For example, in GRW theory, position is given a privileged role. The objective chances in this theory are the chances attached to spontaneous collapses of the position wave function. At a given time, the possible states of the world at the next time form the relevant set of possibilities. Chance is then more naturally represented by a classical probability measure over this set: it selects which of these disjoint possibilities obtain, with the specific probabilities determined by the current spatial wave function and the given collapse constant. (Of course, spin and other degrees of freedom are still subject to chancy jumps, but only insofar as they are entangled with the spatial wave function.)

Similar remarks apply to Bohmian and Everettian mechanics. Bohmian mechanics is deterministic and so does not involve objective chances in Lewis's original sense. But insofar as it does feature a kind of objective chance, it is represented by an ordinary classical measure over initial Bohmian particle configurations. Again, this is related to the privileging of position in the theory. Similarly, the status of chance in Everettian mechanics is a fraught issue, but to the extent it features in the theory, objective chance is more naturally represented as a classical probability measure, this time over disjoint branching possibilities. What makes this possible is the selection through decoherence of a preferred 'quasi-classical' basis over which branching occurs.[9]

The upshot is that on many quantum theories, standard formulations of the Principal Principle will carry over, and Earman's discussion will not apply.[10] That said, on other interpretations, including but not limited to quantum logic interpretations (Stairs, 1983) and information-theoretic

[8]For similar remarks about the scope of Earman's argument, see also Brown and Porath (2020, Sect. 5.2).

[9]For a detailed treatment of Bayesianism and the PP in the Everettian setting, see Wallace (2012, Part II).

[10]Another interpretation which resists the move to quantum probability theory is Healey's pragmatist view (Healey, 2017, 2020). On this view, quantum probabilities are relativized to particular physical situations; this relativization has the effect of singling out a particular Boolean event space, corresponding to a Boolean sub-algebra of $\mathcal{P}(\mathfrak{N})$, over which probabilities are defined. Crucially, there is no combined probability over the entire space $\mathcal{P}(\mathfrak{N})$ (though, as Healey acknowledges, measures on $\mathcal{P}(\mathfrak{N})$ are still "convenient devices" for generating objective probabilities on the Boolean sub-algebras). This resistance to combining is related to the issue of non-contextuality, and in particular the issue of whether the probability of a projection must be independent of the measurement context. As Brown and Porath (2020) point out, the quantum probability theory framework (as Earman and others construe it) seems premised on this assumption, which can be denied.

interpretations (Bub, 2005, 2016), quantum chances are understood both as objective (or at least as yielding objective rational prescriptions) and as ranging over all projection operators.[11] In these settings and others, Earman is correct to point out that we need to resituate QPP within quantum probability theory. Thus, for the rest of this paper, I will assume we are in such a setting.

With these caveats in mind, let us return to my main criticism. In order for Earman's result (4) to establish the QPP, it is crucial that P_ψ represent the proposition $\langle$the state of the system is $\psi\rangle$. But there is strong reason to reject this premise. For example, consider a spin-1/2 particle whose state is entirely unknown. Then the proposition $\langle$the state is either $\uparrow_z$ or $\downarrow_z\rangle$ is contingent, since it is possible that the true initial state is neither $\uparrow_z$ nor $\downarrow_z$ but rather some superposition of the two. However, $P_{\uparrow_z} \vee P_{\downarrow_z}$ is a necessity; it coincides with the identity operator. More generally, for any orthonormal basis $\{\psi_i\}$, even though $\bigvee_i\langle$the state is $\psi_i\rangle$ may be contingent because the true state may be a superposition of the basis elements, $\bigvee_i P_{\psi_i}$ is always a necessity. This suggests that the projections P_{ψ_i} do not represent propositions like $\langle$the state is $\psi_i\rangle$ because they do not have the right logical structure. To press the point, note that if ψ_1 and ψ_2 are distinct, we would expect $C(\text{state is } \psi_1 \text{ or } \psi_2) = C(\text{state is } \psi_1) + C(\text{state is } \psi_2)$, since either the true initial state is ψ_1, or it is ψ_2, or it is neither. However, it is well known that we may have $C(P_{\psi_1} \vee P_{\psi_2}) > C(P_{\psi_1}) + C(P_{\psi_2})$ when ψ_1 and ψ_2 are non-orthogonal.[12]

In response, one might object that just because $\langle$the state is $\psi_i\rangle$ is represented by P_{ψ_i} for all i, it does not follow that the disjunction $\langle$the state is ψ_1 or $\psi_2\rangle$ must be represented by $P_{\psi_1} \vee P_{\psi_2}$, or more generally that $\bigvee_i\langle$the state is $\psi_i\rangle$ must be represented by $\bigvee_i P_{\psi_i}$. In particular, this step assumes that the disjunction in the state hypothesis must be interpreted as the join operator '$\vee$' on the projections (which yields the closure of the union of their ranges), and this assumption could be denied. Of course, this line of response immediately faces the question of what operation, if not join, we are supposed to take the 'or' to correspond to. At least at first blush, there is no clear candidate operation, or alternative way of representing these disjunctive state hypotheses, yet we surely want to say that agents can entertain and assign credences to them.

[11] There are also views, most notably that of Pitowsky (2006), on which quantum probabilities are understood as ranging over all projection operators, however they are given a purely subjective interpretation. Since my interest is in the Quantum Principal Principle, I focus on dualist views of quantum probability (cf. footnote 5). For discussion of why one might be attracted to a dualist view in this context, see Earman and Ruetsche (2020).

[12] Cf. Strocchi (2008, p. 51). For instance, again consider a spin-1/2 particle (examples in higher dimensions can also be given) and let C be the coherent credence function corresponding to the vector state $\uparrow_z$. Suppose $\psi_1 = \uparrow_x$ and $\psi_2 = \downarrow_z$. Then $P_{\psi_1} \vee P_{\psi_2} = I$ so $C(P_{\psi_1} \vee P_{\psi_2}) = 1$, but $C(P_{\psi_1}) + C(P_{\psi_2}) = |\langle\uparrow_z \mid \uparrow_x\rangle|^2 + |\langle\uparrow_z \mid \downarrow_z\rangle|^2 = 1/2 + 0 < 1$.

But there is a bigger obstacle for this line of response, which is that the problems with trying to represent state hypotheses as projections P_ψ extend beyond cases involving the join operator $\vee$. Imagine that the agent is not sure which of some collection of states $\{\psi_i\}$ the system is in, but she is certain the system is in one of them, so $\sum_i C(\text{state is } \psi_i) = 1$. For simplicity, suppose this collection $\{\psi_i\}$ is finite and forms an orthonormal basis. Recalling the discussion of quantum state tomography from the introduction (cf. (2)) we would expect $C(E) = \sum_i C(E|\text{ state is } \psi_i) \cdot C(\text{state is } \psi_i)$, i.e. the agent's credence in E equals her expectation taken over the mutually exclusive and exhaustive hypotheses about the state. However, the example below shows that we may have

$$C(E) > \sum_i C(E|P_{\psi_i}) \cdot C(P_{\psi_i}), \tag{5}$$

even though $\sum_i C(P_{\psi_i}) = 1$ and the $\{\psi_i\}$ are an ON basis, and hence the $\{P_{\psi_i}\}$ are partitional. This is further indication that the projections P_{ψ_i} do not represent state hypotheses. (For another similar cautionary tale about the interpretation of projection operators, see also Weinstein (1996).)

Example 2. Consider a spin-1/2 particle (the example can be extended to higher dimensions) and let the collection $\{\psi_i\}$ consist of $\uparrow_z$ and $\downarrow_z$. Let C be the coherent credence function corresponding to the vector state $\uparrow_x$; note that $\sum_i C(P_{\psi_i}) = C(P_{\uparrow_z}) + C(P_{\downarrow_z}) = 1/2 + 1/2 = 1$, as required. Now let $E = P_{\uparrow_x}$. Then $C(E) = 1$, but $C(E|P_{\uparrow_z}) \cdot C(P_{\uparrow_z}) + C(E|P_{\downarrow_z}) \cdot C(P_{\downarrow_z}) = 1/2 \cdot 1/2 + 1/2 \cdot 1/2 < 1$.[13]

An important consequence of this last result is that Earman's alleged Principal Principle, even if interpreted as such, does not do the normative work that it is supposed to do. (4) and (5) imply:

$$C(E) \neq \sum_i \psi_i(E) \cdot C(P_{\psi_i}) \tag{6}$$

even though $\sum_i C(P_{\psi_i}) = 1$. If we do interpret P_{ψ_i} as $\langle$the state is $\psi_i\rangle$, as Earman's formulation of QPP suggests, then we are led to conclude that $C(E) \neq \sum_i \psi_i(E) \cdot C(\text{state is } \psi_i)$, even though $\sum_i C(\text{state is } \psi_i) = 1$. In other words, the agent's credence in E need not match her expectation of the chance of E.[14] But as we saw in the discussion of quantum state tomography (cf. (3)), this

[13]Here we apply (4) to obtain $C(E|P_{\uparrow_z}) = \uparrow_z(E) = 1/2$ and $C(E|P_{\downarrow_z}) = \downarrow_z(E) = 1/2$. More generally, let $d = \dim(\mathcal{H})$ where $1 < d < \infty$. Let $\{\psi_i\}$ be an orthonormal basis and define $\varphi = \sum_i \sqrt{\frac{1}{d}}\, \psi_i$. Set $C(\cdot) = \varphi(\cdot)$ and $E = P_\varphi$. Then $C(E) = 1$ but $\sum_i \psi_i(E) \cdot C(P_{\psi_i}) = \frac{1}{d} \cdot \sum_i C(P_{\psi_i}) = \frac{1}{d} < 1$.

[14]It is true that given a C, it can always be expressed as $C(\cdot) = \sum_i \psi_i(\cdot) \cdot C(P_{\psi_i})$ for *some* orthonormal $\{\psi_i\}$.

is supposed to be a crucial consequence of QPP.

4 Preparation, projective measurement, and a classical analogue

If Earman's theorem does not establish QPP, then what does it establish?

The theorem establishes that if the system was knowingly prepared in a state ψ by a projective measurement with outcome P_ψ, then rational credence must align with ψ. Indeed this is the gloss that Earman offers in several places (2018, p. 10, notation adapted):

> The intended interpretation of this [result] should be obvious. Suppose that an agent whose initial credence function on $\mathcal{P}(\mathfrak{N})$ is C learns that a Yes-No experiment for P_ψ has been performed and that a Yes answer has been returned. Since she is rational she Lüders updates her credence function to $C(\cdot|P_\psi)$. On the objectivist interpretation of quantum probabilities, the returning of a Yes answer implies that the normal pure state ψ has been prepared, and, hence, that the objective chance of an event $E \in \mathcal{P}(\mathfrak{N})$ is $\psi(E)$, which is the same as the agent's updated credence, $C(E|P_\psi)$.

Although interestingly related to QPP, this result is not sufficient for it. There are two main ways to see that it falls short.

First, note that many preparations do not involve projective measurements (Fröhlich and Schubnel, 2016). For example, a scientist can prepare a crystal in its ground state ψ by leaving it in a cold room. QPP still requires she conform her credences to ψ, even though no projective measurement took place. Furthermore, it seems that even if no explicit preparation of the system has taken place—for instance, if a scientist is going to measure some cosmic rays incident on her laboratory that nobody has interacted with—QPP should still require she conform her credence in accordance with the expected state. Yet, in these cases, Earman's result won't apply.

Second, the result only applies to vector states ψ. But suppose that the system S is entangled with another S'. If the agent knows S's state, it seems she should still be required to set her credences in line with that state. But since the state won't be pure, the result won't apply.

However, the issue was whether given any collection of mutually exclusive and exhaustive chance hypotheses, the agent's credences equal her expectations over those chance hypotheses. (At the very least, we would expect that if all the $\{\psi_i\}$ assign the same chance α to E, then $C(E)$ equals α. But if the proposition $\langle$state is $\psi_i\rangle$ is represented by the projection P_{ψ_i} then this also fails: note that in the previous example, $\uparrow_z(E) = \downarrow_z(E) = 1/2$ yet $C(E) \neq 1/2$.)

(Indeed, one can check that if ω is not pure, then there does not exist any non-zero projection $P \in \mathcal{P}(\mathfrak{N})$ such that $C(\cdot|P) = \omega(\cdot)$ for all coherent C with $C(P) > 0$ (Ruetsche and Earman, 2011, Fact 1, adapted).) This is a serious restriction considering the ubiquity of entanglement in quantum theory.[15]

In fact, focusing on this restriction to pure states, we can see that an exact analogue of Earman's Theorem 1 also holds in classical probability theory. Some preliminaries: Let $(S, \mathcal{S})$ be a classical outcome space, with S a set and $\mathcal{S}$ a sigma-algebra on S. Define a *state* μ as a probability measure on $(S, \mathcal{S})$. A state μ is *pure* if it cannot be expressed as a non-trivial convex combination of other states. One can check that in this classical context, a state is pure if and only if it is a point measure concentrated at some $s \in S$, which I will write as μ_s. It follows that the pure states μ_s are in one-to-one correspondence with the elements of S. Next, define a *projection operator* as a random variable $X : S \to \mathbb{R}$ that satisfies $X^2 = X$. One can check that every projection can be expressed as an indicator $X = 1_J$ for some $J \in \mathcal{S}$, in other words a $\{0, 1\}$-valued function:

$$1_J(s) = \mu_s(J) = \begin{cases} 1 & \text{if } s \in J \\ 0 & \text{if } s \notin J. \end{cases}$$

For some pure state $\psi = \mu_s$ we can consider the projection $P_\psi = 1_{\{s\}}$ which tests whether $s' \in S$ belongs to $\{s\}$. Like in the quantum case, we can also associate P_ψ with the claim that a Yes-No experiment for P_ψ has been performed, and that a Yes answer has been returned: $P_\psi = \{s' \in S : 1_{\{s\}}(s') = 1\} = \{s\}$. Now, with these preliminaries in hand, we can obtain a classical analogue of Theorem 1:

Theorem 2 (Classical analogue)**.** *Let $C : \mathcal{S} \to [0, 1]$ be any coherent initial credence function. Let ψ be any pure state and let P_ψ denote the projection defined above. Then, provided $C(P_\psi) > 0$,*

$$C(\cdot|P_\psi) = \psi(\cdot).$$

[15]Here one could respond that we should move to the composite system $\mathcal{H}_c = \mathcal{H} \otimes \mathcal{H}'$ and consider pure states ψ_c of it. But I am considering a case where the agent is only probing S, so there is no "Yes-No experiment for P_{ψ_c}" being performed, only Yes-No experiments for propositions of the form $P \otimes I$, where $P \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ and $I \in \mathcal{P}(\mathfrak{B}(\mathcal{H}'))$ the identity. Notably, it follows from the result cited above that if ω is not pure, then there does not exist any non-zero $P \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ such that $C(\cdot \otimes I | P \otimes I) = \omega(\cdot)$ for all coherent C on $\mathfrak{B}(\mathcal{H} \otimes \mathcal{H}')$ for which $C(P \otimes I) > 0$.

For proof, recall we can write $\psi = \mu_s$ for some $s \in S$ and $P_\psi = \{s' \in S : 1_{\{s\}}(s') = 1\} = s$ where we abuse notation and write $\{s\}$ as s. Now if $C(s) \neq 0$, then for all $J \in \mathcal{S}$,

$$C(J|P_\psi) = C(J|s) = \frac{C(Js)}{C(s)} = \begin{cases} 1 & \text{if } s \in J \\ 0 & \text{if } s \notin J, \end{cases}$$

and so $C(J|s) = \mu_s(J) = \psi(J)$ as desired.

Intuitively, what is going on here is that since ψ is pure, P_ψ is an *atomic proposition* that specifies the exact outcome. Conditionalizing on this outcome, one is forced to align one's credence with the post-experiment chance of the outcome, which is either 0 or 1. And so, in this very restricted sense, "credence and chance have been brought into alignment without the cudgel of any normative principle."

Needless to say, while a notable result, this does not constitute a proof of the original Principal Principle. And as I have suggested here, although its quantum analogue is far more interesting, it is not a proof of the Quantum Principal Principle either.

This discussion also makes evident the reliance, in Earman's result, on the choice to focus on the von Neumann algebra $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$, which has atoms. In some quantum field theoretic settings, such as in the setting of Type-III factor algebras, there are no atoms in the relevant algebra (Ruetsche and Earman, 2011). The prospect of extending Earman's treatment to such settings looks dim.[16]

5 Formulating the QPP in quantum probability theory

I now present my positive proposal for how to formulate the Quantum Principal Principle within the framework of quantum probability theory. As we'll see, while the resulting principle is highly intuitive, it does not follow as a theorem.

We begin by dividing the agent's credence function C into two components, x and μ. The first component, x, is what I previously denoted C: it is a function on the set of projections $\mathcal{P}(\mathfrak{N})$ that, at minimum, satisfies the coherence requirements from Section 2. The second component, μ, is the key ingredient that was missing from Earman's approach. It is a classical (Kolmogorovian) credence

[16]Crucially, Type-III algebras do not admit any pure normal states. As Ruetsche and Earman (2011, p. 273) point out, it then follows that *no* normal states on these algebras have "filters" in the sense described by Theorem 1.

distribution over the set of possible initial quantum states of the system. Roughly speaking, whereas x represents the agent's uncertainty about experimental outcomes associated with the system, μ represents her uncertainty about the true initial state of the system. If C is coherent, then x and μ will satisfy an analogue of the law of total probability, that is, an analogue of Equation (2):

$$x(E) = \sum_i x_{\omega_i}(E) \cdot \mu(\omega_i) \qquad \forall E \in \mathcal{P}(\mathfrak{N}), \tag{7}$$

where we have assumed for now that μ is a discrete distribution over finitely many state hypotheses $\{\omega_i\}_{i=1}^N$, and that for each ω_i, there is a constant $x_{\omega_i}(E)$ which represents the agent's credence in outcome E given this hypothesis. The QPP then requires that this credence match the given state:

$$x_{\omega_i}(E) = \omega_i(E) \qquad \forall i,\ \forall E \in \mathcal{P}(\mathfrak{N}). \tag{8}$$

One consequence is an analogue of Equation (3):

$$x(E) = \sum_i \omega_i(E) \cdot \mu(\omega_i) \qquad \forall E \in \mathcal{P}(\mathfrak{N}), \tag{9}$$

in other words, the agent's credence in an experimental outcome E equals her expectation of its chance. Extending x to a state by Gleason's theorem, this is equivalent to the condition that x is the *barycenter* for μ (Alfsen, 2012), that is, $x = \sum_i \omega_i \cdot \mu(\omega_i)$.

Even in this discrete case, we can already see that the QPP is not a theorem of quantum probability theory, since there are many pairs (x, μ) which are coherent yet violate it. For instance, fix $\mathcal{H}$ with dimension $1 < d < \infty$. Suppose x corresponds to a vector state ψ and μ assigns all its weight to the maximally mixed state $\omega = I/d$. Then the choice $x_\omega(E) = x(E)$ for all $E \in \mathcal{P}(\mathfrak{N})$ ensures (7). But we have a violation of (8) and (9), since $x(P_\psi) > \omega(P_\psi) = \sum_i \omega_i(P_\psi) \cdot \mu(\omega_i)$. As I understand the QPP, then, it is a norm that goes beyond usual coherence requirements. Let us now formulate this norm in full detail.

Fix a separable Hilbert space $\mathcal{H}$ and associated algebra of bounded operators $\mathfrak{N}$. Let K denote the set of quantum states on $\mathfrak{N}$. This set comes naturally equipped with the weak* topology. Let $\mathcal{K}$ denote the sigma-algebra on K generated by this topology. Now we can consider Kolmogorovian probability measures μ on $(K, \mathcal{K})$. As in Section 2, we can also consider quantum probability functions x on $\mathcal{P}(\mathfrak{N})$. A credence function C is then modeled as a pair (x, μ).

Example 3. In the case of a spin-1/2 particle, μ can be thought of as a probability density over the Bloch sphere of spin states, and x a point in the sphere. If (x, μ) obeys QPP, then x will be the 'weighted average' of the points supported by μ.

We are interested in how a rational $C = (x, \mu)$ changes with new evidence. If the evidence is a projective outcome $E \in \mathcal{P}(\mathfrak{N})$ with $x(E) > 0$, then we know x updates by Lüders conditionalization:

$$x_E(F) = \frac{x(EFE)}{x(E)} \qquad \forall F \in \mathcal{P}(\mathfrak{N}). \tag{10}$$

However, to know how C updates, we also need to know how μ updates: after learning E, how should the agent's credences in hypotheses $H \in \mathcal{K}$ about the (initial) state of the state of the system change? A coherent agent obeys Bayes' formula, so (whenever $x(E) > 0$) we set:

$$\mu_E(H) = \frac{x_H(E) \cdot \mu(H)}{x(E)} \qquad \forall H \in \mathcal{K}. \tag{11}$$

In the case $H = \{\psi\}$, this is just a more elaborate way of stating Equation (1).

What is the agent's credence $x_H(E)$ in data E given H? Focusing first on the case $H = \{\omega\}$, our law of total probability constraint (7) (generalized to the continuous case) implies

$$x(E) = \int_K x_\omega(E)\, d\mu(\omega) \qquad \forall E \in \mathcal{P}(\mathfrak{N}), \tag{12}$$

where we have abused notation by writing $x_H(E) = x_{\{\omega\}}(E)$ as $x_\omega(E)$. In fact, the generalized law of total probability also requires that we can express $x_H(E)$ for any $H \subseteq K$ as the appropriate μ-weighted average of these conditional $x_\omega(E)$ terms, in particular,

$$x_H(E) = \frac{1}{\mu(H)} \int_H x_\omega(E)\, d\mu(\omega) \qquad \forall E \in \mathcal{P}(\mathfrak{N}) \tag{13}$$

for all $H \in \mathcal{K}$ such that $\mu(H) > 0$.

Given these coherence requirements, we can rewrite (11) as:

$$\mu_E(H) = \frac{\int_H x_\omega(E)\, d\mu(\omega)}{\int_K x_\omega(E)\, d\mu(\omega)} \qquad \forall H \in \mathcal{K}. \tag{14}$$

Note that μ_E is determined entirely by the prior distribution μ and terms of the form $x_\omega(E)$. Since μ is already fixed, the crucial question is, what determines $x_\omega(E)$?

Enter the QPP:

Quantum Principal Principle (QPP): If $C = (x, \mu)$ is a rational initial credence function, then

$$x_\omega(E) = \omega(E) \qquad \forall \omega \in K,\, E \in \mathcal{P}(\mathfrak{N}). \tag{15}$$

> In other words, a rational agent's credence in an outcome E given the state hypothesis ω equals the chance of E according to ω.

Importantly, unlike the formulation (4), this formulation applies both when ω is a vector state ψ and when ω is an entangled state.

Substituting (15) into the expressions above, we obtain several results of interest. For instance, substituting (15) into (12), we find that the agent's unconditional credence in E equals her expectation of the chance of E,

$$x(E) = \int_K \omega(E)\, d\mu(\omega) \qquad \forall E \in \mathcal{P}(\mathfrak{N}), \tag{16}$$

which is just the continuous generalization of (9). Substituting (15) into (14), we also obtain:

$$\mu_E(H) = \frac{\int_H \omega(E)\, d\mu(\omega)}{\int_K \omega(E)\, d\mu(\omega)} \qquad \forall H \in \mathcal{K}. \tag{17}$$

In the special case where $H = \{\psi\}$ and μ is a discrete distribution over finitely many vector states $\{\psi_i\}$, this simplifies to:

$$\mu_E(\psi) = \frac{\psi(E) \cdot \mu(\psi)}{\sum_i \psi_i(E) \cdot \mu(\psi_i)} \tag{18}$$

which is just a more precise way of spelling out Equations (1)–(3), as desired.

Example 4. A physicist has two identically prepared electrons. She knows they are either in state $\uparrow_x$, $\uparrow_y$, or $\uparrow_z$, and she is equally confident in these three hypotheses. She then measures the first electron in the spin-z direction and obtains an 'UP' outcome. How should her confidences over the hypotheses about the (initial) spin state change? Here we have

$$\mu(\uparrow_x \otimes \uparrow_x) = \mu(\uparrow_y \otimes \uparrow_y) = \mu(\uparrow_z \otimes \uparrow_z) = \frac{1}{3}.$$

The data the agent received is represented by $E = P_{\uparrow_z} \otimes I$. Assuming she obeys QPP, her posterior credence in the hypothesis $\uparrow_z \otimes \uparrow_z$ is determined by (18):

$$\mu_E(\uparrow_z \otimes \uparrow_z) = \frac{(\uparrow_z \otimes \uparrow_z)(E) \cdot \frac{1}{3}}{\frac{1}{3}\left[(\uparrow_x \otimes \uparrow_x)(E) + (\uparrow_y \otimes \uparrow_y)(E) + (\uparrow_z \otimes \uparrow_z)(E))\right]} = \frac{1 \cdot \frac{1}{3}}{\frac{1}{3}\left[\frac{1}{2} + \frac{1}{2} + 1\right]} = \frac{1}{2}.$$

As expected, the agent's credence in the $\uparrow_z$ hypothesis increases. A similar calculation shows that, as expected, her credence in the $\uparrow_x$ and $\uparrow_y$ hypotheses decreases:

$$\mu_E(\uparrow_x \otimes \uparrow_x) = \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{3}\left[\frac{1}{2} + \frac{1}{2} + 1\right]} = \frac{1}{4}, \qquad \mu_E(\uparrow_y \otimes \uparrow_y) = \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{3}\left[\frac{1}{2} + \frac{1}{2} + 1\right]} = \frac{1}{4}.$$

So the agent begins with credence function (x, μ) and ends with credence function (x_E, μ_E), where $x_E(\cdot) = x(E \cdot E)/x(E)$ and μ_E is specified by the above three equations.

What if the agent then obtains a second piece of experimental evidence $F \in \mathcal{P}(\mathfrak{N})$, say, through measuring the spin of the second electron? How should her credences over the various state hypotheses change? To answer this question, we first need to generalize (17) to obtain an expression for $(\mu_E)_F$. In particular, we show that for any pair $E, F \in \mathcal{P}(\mathfrak{N})$ with $x(E) > 0$ and $x_E(F) > 0$:[17]

$$(\mu_E)_F(H) = \frac{\int_{H \cap \{\omega : \omega(E) > 0\}} \omega_E(F)\, d\mu_E(\omega)}{\int_{\{\omega : \omega(E) > 0\}} \omega_E(F)\, d\mu_E(\omega)} \qquad \forall H \in \mathcal{K}. \tag{19}$$

[17]First we note that, by (11), $(\mu_E)_F(H) = \frac{x_{EH}(F) \cdot \mu_E(H)}{x_E(F)}$. So to it suffices to prove that, for all $H \in \mathcal{K}$ and $E, F \in \mathcal{P}(\mathfrak{N})$ such that $x(E) > 0$, $x_{EH}(F) \cdot \mu_E(H) = \int_{H \cap D} (x_E)_\omega(F)\, d\mu_E(\omega)$ where $D = \{\omega \in K : x_\omega(E) > 0\}$. (For then the QPP implies $(x_E)_\omega(F) = \omega_E(F)$ and $D = \{\omega : \omega(E) > 0\}$.) Since the right-hand-side of this expression is zero when $\mu_E(H)$ is zero, it suffices to show that whenever $x(E) > 0$ and $\mu_E(H) > 0$, $x_{EH}(F) = \frac{1}{\mu_E(H)} \int_{H \cap D} (x_E)_\omega(F)\, d\mu_E(\omega)$. We can obtain this last expression by inspection from (13). However, for consistency we check that conditioning x first on H, and then on E, gives the same result. In particular, we check that:

$$\begin{aligned} x_{HE}(F) &= \frac{x_H(EFE)}{x_H(E)} = \frac{1}{x_H(E)} \cdot \frac{1}{\mu(H)} \int_H x_\omega(EFE)\, d\mu(\omega) = \frac{1}{x_H(E)} \cdot \frac{1}{\mu(H)} \int_{H \cap D} x_\omega(EFE) \cdot \frac{x(E)}{x_\omega(E)}\, d\mu_E(\omega) \\ &= \frac{1}{\mu_E(H)} \cdot \frac{\mu_E(H)\, x(E)}{x_H(E)} \cdot \frac{1}{\mu(H)} \int_{H \cap D} (x_E)_\omega(F)\, d\mu_E(\omega) = \frac{1}{\mu_E(H)} \int_{H \cap D} (x_E)_\omega(F)\, d\mu_E(\omega) \end{aligned}$$

as desired, where the second equality follows from (13) and the remaining equalities follow from the fact that $x_\omega(EFE) = 0$ for $\omega \notin D$ and (11) implies $\mu(H) = \frac{\mu_E(H)\, x(E)}{x_H(E)}$ and so $d\mu(\omega) = \frac{x(E)}{x_\omega(E)}\, d\mu_E(\omega)$ for all $\omega \in D$.

In the case where $H = \{\psi\}$ and μ is a discrete distribution over vector states, this simplifies to

$$(\mu_E)_F(\psi) = \frac{\psi_E(F) \cdot \mu_E(\psi)}{\sum_{i:\, \psi_i(E)>0} (\psi_i)_E(F) \cdot \mu_E(\psi_i)}, \tag{20}$$

which we can apply directly to our question:

Example 4 (*continued*)**.** The physicist then measures the second electron in the spin-y direction and obtains a 'DOWN' outcome, represented by $F = I \otimes P_{\downarrow_y}$. How should her credences over the three hypotheses about the initial state change now? Applying (20), we obtain:

$$(\mu_E)_F(\uparrow_z \otimes \uparrow_z) = \frac{(\uparrow_z \otimes \uparrow_z)_E(F) \cdot \frac{1}{2}}{\frac{1}{4}\left[(\uparrow_x \otimes \uparrow_x)_E(F) + (\uparrow_y \otimes \uparrow_y)_E(F)\right] + (\uparrow_z \otimes \uparrow_z)_E(F) \cdot \frac{1}{2}} = \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{4}\left[\frac{1}{2} + 0\right] + \frac{1}{2} \cdot \frac{1}{2}} = \frac{2}{3}.$$

By similar reasoning, $(\mu_E)_F(\uparrow_x \otimes \uparrow_x) = \frac{1}{3}$ and $(\mu_E)_F(\uparrow_y \otimes \uparrow_y) = 0$. As expected, the $\downarrow_y$ result ruled out the hypothesis that the electrons were initially prepared in the $\uparrow_y$ state, but did not change the relative likelihood of the $\uparrow_z$ and $\uparrow_x$ hypotheses.[18]

One notable feature of the current framework is that it can also be applied to cases where E and F fail to commute. While $(\mu_E)_F = (\mu_F)_E = \mu_{(EF)}$ when E and F commute, this does not hold in general. An interesting case to consider is one in which a rational agent measures a system and obtains the atomic outcome $E = P_\psi$, updating to $(x_{P_\psi}, \mu_{P_\psi})$. She then measures it again in a different basis and obtains the atomic outcome P_φ, where $\varphi \in \mathcal{H}$ is non-orthogonal to $\psi \in \mathcal{H}$. What is her final credal state $((x_{P_\psi})_{P_\varphi}, (\mu_{P_\psi})_{P_\varphi})$? Intuitively, one would not expect the agent's knowledge of the initial state of the system to improve, since by "disturbing" the system with her first projective measurement, she has already extracted all the information about the initial state that she possibly can. Indeed, the QPP entails that the agent's credences in the initial state hypotheses do not change upon this second learning experience: $(\mu_{P_\psi})_{P_\varphi} = \mu_{P_\psi}$.[19] So, her final

[18] The other component of the agent's final credence function is $(x_E)_F$. If we assume that $\mathcal{H}$ is exhausted by the spin degrees of freedom of the two electrons, then EF is an atomic proposition P_ψ for $\psi = \uparrow_z \otimes \downarrow_y \in \mathcal{H}$, so it follows from Fact 1 that $(x_E)_F = \psi$. Here it is worth emphasizing that while the agent's posterior over outcomes is a pure state ψ, her posterior over state hypotheses remains spread out, and in fact does not assign any weight to the hypothesis $\psi \in K$. That is because $\psi \in K$ is a hypothesis about the *initial* state. As a sanity check, one can verify that the combination of $x_{P_\psi} = \psi$ and $\mu_{P_\psi}(\psi) < 1$ does not contradict (9): although (9) implies $\frac{x(P_\psi(\cdot)P_\psi)}{x(P_\psi)} = \frac{\sum_i \psi_i(P_\psi(\cdot)P_\psi)\,\mu(\psi_i)}{\sum_i \psi_i(P_\psi)\,\mu(\psi_i)} = \frac{\sum_j \psi_j(P_\psi(\cdot)P_\psi)/\psi_j(P_\psi)\,\mu_{P_\psi}(\psi_j)}{\sum_j \psi_j/\psi_j\,\mu_{P_\psi}(\psi_j)} = \sum_j \frac{\psi_j(P_\psi(\cdot)P_\psi)}{\psi_j(P_\psi)}\,\mu_{P_\psi}(\psi_j)$, the terms $\frac{\psi_j(P_\psi(\cdot)P_\psi)}{\psi_j(P_\psi)}$ all reduce to $\psi(\cdot)$. So even if μ_{P_ψ}'s weight is spread out over various $\psi_j \neq \psi$, the $\psi(\cdot)$ will factor out and we will obtain $x_{P_\psi} = \psi(\cdot)$ a pure state, as desired. (Here j ranges over all i such that $\psi_i(P_\psi) > 0$.)

[19] Here we assume μ is supported by the set $N \subset K$ of normal states, that is $\mu(N) = 1$. Then, by (19), $(\mu_{P_\psi})_{P_\varphi} =$

credal state is $((x_{P_\psi})_{P_\varphi}, (\mu_{P_\psi})_{P_\varphi}) = (\varphi, \mu_{P_\psi})$. Note that if the order of the learning experiences were reversed, the final credal state would be (ψ, μ_{P_φ}) instead.

To sum up: I have proposed dividing an agent's credal state into two components, the first of which, x, encodes her uncertainty about outcomes (her state of ignorance), and the second of which, μ, encodes her uncertainty about the initial state of the system (her ignorance of states). I have imposed three main constraints on how x and μ relate. Two of these constraints, the analogue of Bayes' formula (11) and the analogue of the law of total probability (13), correspond to familiar coherence requirements, and do not impose any alignment between the conditional credence $x_\omega(E)$ and the chance $\omega(E)$. The third constraint, however, the Quantum Principal Principle (15), does impose such an alignment. Like in the classical case, it is not a logical consequence of the Bayesian coherence requirements, but rather has the status of an additional norm.

What should we make of this alternative conclusion about the status of the QPP? Should we be moved to pessimism about its validity, now that Earman's original hope of deriving it looks dashed? Of course, just because the QPP cannot be derived "as a theorem" from basic coherence norms does not mean it is false. However, the worry is that if we cannot appeal to other accepted norms to derive it, then we need some independent motivation for accepting it. Indeed, this worry may appear even more pressing in the case of the QPP, which looks at first blush like a rather esoteric norm, involving quantum states and projection operators, that is several steps removed from our everyday chance reasoning.

I am more optimistic. While far from a full justification, I take it that, even if not part of our folk chance reasoning, actual epistemic and scientific practice does pay heed to quantum chances in the way the QPP describes, which lends it some initial motivation. In particular, I have been emphasizing the connection between the QPP and the practice of quantum state tomography, which involves the inference of unknown quantum states from experimental data. It is difficult, if not impossible, to reconstruct those inferences without invoking the principle that the probabilities given a state hypothesis match the probabilities assigned by the state (remaining neutral, for the moment, on the metaphysical status of those probabilities). Characteristically Bayesian approaches become useful when the tomographer has prior information about the state and wants to incorporate

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$\frac{\int_{(\cdot)\cap\{\omega:\,\omega(P_\psi)>0\}} \omega_{P_\psi}(P_\varphi)\, d\mu_{P_\psi}(\omega)}{\int_{\{\omega:\,\omega(P_\psi)>0\}} \omega_{P_\psi}(P_\varphi)\, d\mu_{P_\psi}(\omega)} = \frac{\int_{(\cdot)\cap\{\omega:\,\omega(P_\psi)>0\}\cap N} \psi(P_\varphi)\, d\mu_{P_\psi}(\omega)}{\int_{\{\omega:\,\omega(P_\psi)>0\}\cap N} \psi(P_\varphi)\, d\mu_{P_\psi}(\omega)} = \frac{\psi(P_\varphi)\int_{(\cdot)} d\mu_{P_\psi}(\omega)}{\psi(P_\varphi)} = \frac{\psi(P_\varphi)\mu_{P_\psi}}{\psi(P_\varphi)} = \mu_{P_\psi}$ as desired, where we use the fact that $\mu_{P_\psi}(\{\omega : \omega(P_\psi) > 0\} \cap N) = 1$ and if $\omega \in \{\omega : \omega(P_\psi) > 0\} \cap N$ then $\omega_{P_\psi}(\cdot) = \psi(\cdot)$.

that information in their inference. Indeed, there is a rich practice of Bayesian approaches to quantum state tomography in which the QPP makes a nearly explicit appearance. In this sense, the first formulation of the QPP cannot be attributed to me or Earman, but dates back at least to Jones (1991) (with important precursors by Sýkora (1974), Helstrom (1976), and Hovelo (1982)), whose approach received further development by Bužek et al. (1998) and Schack et al. (2001), and enjoys many practical applications today (Granade et al., 2016). (A necessary caveat: while I am giving a reading in terms of the QPP, some authors who followed and developed this tradition, such as QBists (Fuchs and Schack, 2004), took quantum states to be subjective, and would object to the interpretation in terms of QPP and objective chance.) In these works, a version of the QPP emerges as part of a "natural inversion procedure" that parallels ordinary Bayesian statistical inference. In fact, my formulation of QPP is continuous with the version that emerges there. To close the section, I briefly present the framework introduced by Jones (1991), and explain how it relates to the quantum-probability-theoretic framework developed here.

The Bayesian-tomographic inversion procedure begins by specifying which measurement will be performed. In particular, we fix a *POVM measurement*, represented by a collection $\{A_i\}_{i=1}^N$ of positive self-adjoint operators in $\mathfrak{N}$ summing to the identity I (for simplicity I focus on the finite case). Note that we can associate with this POVM a Kolmogorovian outcome space $(S, \mathcal{S})$ whose elements $S = \{1, 2, \ldots, N\}$ represent the different possible outcomes of the measurement. The inversion procedure then consists of three steps:

1. Define conditional probability values $p(i|\omega) \equiv \omega(A_i)$ for all $i \in S$ and $\omega \in K$.

2. Specify a prior distribution μ on the space of possible states $(K, \mathcal{K})$. Then use this to construct a joint distribution $dp(i, \omega) = p(i|\omega) \cdot d\mu(\omega)$ over outcomes and states.

3. Apply Bayes' rule to $p(\cdot, \cdot)$ to obtain, for all $i \in S$ (cf. (17)):

$$p(H|i) = \frac{p(i|H) \cdot p(H)}{p(i)} = \frac{\int_H p(i|\omega)\, d\mu(\omega)}{\int_K p(i|\omega)\, d\mu(\omega)} = \frac{\int_H \omega(A_i)\, d\mu(\omega)}{\int_K \omega(A_i)\, d\mu(\omega)} \qquad \forall H \in \mathcal{K}.$$

It should be emphasized that these probability spaces are all Kolmogorovian; the final object $p(\cdot, \cdot)$ is a probability measure over the product sigma-algebra $(S, \mathcal{S}) \otimes (K, \mathcal{K})$. In that sense, this Bayesian model is different from the quantum-probability-theoretic model presented above. Nevertheless, it shares many features, such as the analogue of QPP in step one and the final expression for $p(H|i)$ in step three. Indeed there is a precise sense in which the two frameworks, and the two versions of QPP, are related. First, note that while we have been focused on projective outcomes $E \in \mathcal{P}(\mathfrak{N})$, if $C = (x, \mu)$ is coherent and $\dim(\mathcal{H}) > 2$, then by Gleason's theorem, $x(\cdot)$ and $x_H(\cdot)$ uniquely extend to states on $\mathfrak{N}$.[20] Thus, terms like $x(A_i)$ and $x_H(A_i)$ are well-defined even when A_i is not a projection.[21] Furthermore, the properties of a POVM imply that the maps

$$x(A_{(\cdot)}) : S \to \mathbb{C}, \qquad \text{and} \qquad x_H(A_{(\cdot)}) : S \to \mathbb{C} \tag{21}$$

are classical, real-valued probability distributions on $(S, \mathcal{S})$. It then follows that each coherent (x, μ) gives rise to a unique classical probability measure $p(\cdot, \cdot)$ on $(S, \mathcal{S}) \otimes (K, \mathcal{K})$, defined by:[22]

$$p(H) = \mu(H), \qquad p(i) = x(A_i), \qquad p(i|H) = x_H(A_i) \text{ when } \mu(H) > 0. \tag{22}$$

Crucially, this measure coincides with the Bayesian-tomographic distribution defined above when (and only when) $p(i|\omega) = \omega(A_i)$ for all $i \in S$ and $\omega \in K$. And this condition holds, in turn, when and only when $x_\omega(A_i) = \omega(A_i)$ for all $i \in S$ and $\omega \in K$.[23]

So: the version of the QPP implicit in Bayesian-tomographic methods is equivalent to a particular consequence of the version of the QPP proposed here. Whereas the version of the QPP proposed here requires (by Gleason's theorem) the alignment of a quantum credence function with

[20] Here we assume by coherence that, at least when $\mu(H) > 0$, $x_H : \mathcal{P}(\mathfrak{N}) \to [0, 1]$ is finitely additive and so can be extended to a state. In the case of an infinite POVM and thus infinite $(S, \mathcal{S})$, we need that for μ-almost every $\omega \in K$, x_ω is countably additive, so that composing with the POVM yields a measure on $(S, \mathcal{S})$ that is countably additive. If (x, μ) satisfies the QPP, then this is equivalent to the requirement that μ is supported by the set of normal states.

[21] In contrast, a term like $x_{A_i}(\cdot)$, where x is conditionalizing on a non-projective outcome $A_i \in \mathfrak{N}$, is *not* well-defined in my and Earman's framework, because Lüders conditionalization, as we have construed it (10), is only applicable to projective outcomes $E \in \mathcal{P}(\mathfrak{N})$. This immediately raises the question of how and whether to extend Lüders conditionalization to POVM outcomes—and what the implications are for the quantum-probability-theoretic framework if this cannot (or should not) be done. I leave this important issue as a question for future investigation.

[22] We check that the $p(\cdot, \cdot)$ defined in (22) is indeed a probability measure. (Note that since (22) specifies $p(i \times H) = p(i|H) \cdot p(H) = x_H(A_i) \cdot \mu(H)$ and rectangles of the form $i \times H$ generate $(S, \mathcal{S}) \otimes (K, \mathcal{K})$, this $p(\cdot, \cdot)$ must then also be unique.) First, note $p(S \times K) = p(S|K) \cdot p(K) = \left(x_H(A_{(\cdot)})\right)(S) \cdot \mu(K) = 1$ using the fact that $x_H(A_{(\cdot)})$ is a probability distribution. Next, note that the map $m_i : H \mapsto x_H(A_i) \cdot \mu(H)$ is countably additive by (13) and the fact that μ is countably additive. So $p(\cup_{ik_i} i \times H_{k_i}) = \sum_i p(i|\cup_{k_i} H_{k_i}) \cdot p(\cup_{k_i} H_{k_i}) = \sum_i m_i(\cup_{k_i} H_{k_i}) = \sum_i \sum_{k_i} m_i(H_{k_i}) = \sum_{ik_i} p(i|H_{k_i}) \cdot p(H_{k_i}) = \sum_{ik_i} p(i \times H_{k_i})$ which, since $(S, \mathcal{S})$ is finite, implies $p(\cdot, \cdot)$ is countably additive, as desired.

[23] Strictly speaking, these conditions are only required to hold for μ-almost every $\omega \in K$. To prove that $p(i|\omega) = \omega(A_i)$ for μ-almost every $\omega \in K$ if and only if $x_\omega(A_i) = \omega(A_i)$ for μ-almost every $\omega \in K$, note that (22) and (13) imply that for any $i \in S$, $p(iH) = p(i|H) \cdot p(H) = x_H(A_i) \cdot \mu(H) = \int_H x_\omega(A_i)\, d\mu(\omega) = \int_H x_\omega(A_i)\, dp|_{\mathcal{K}}(\omega)$ for all $H \in \mathcal{K}$. It then follows that the map $\omega \mapsto x_\omega(A_i)$ is a *version* of the (classical) conditional expectation for i given $\mathcal{K}$ (Billingsley, 1986, Chapter 6.33), that is, it agrees μ-almost surely with $\omega \mapsto p(i|\omega)$, as claimed.

a quantum state,

$$x_\omega = \omega, \tag{23}$$

the Bayesian-tomographic QPP requires the alignment of a *classical* credence function with a quantum state *composed by a choice of POVM* $f : i \mapsto A_i$:

$$x_\omega \circ f = \omega \circ f. \tag{24}$$

While the conditions are distinct, there is a clear logical relation, as (23) entails (24). If we extend (24) to quantify over all possible choices of f, then the two coincide.

6 Conclusion

How do quantum chances constrain credence? At least some interpretations of quantum mechanics demand that we recast the Principal Principle within the framework of *quantum probability theory*. Earman (2018) sought to do so, and came to a surprising conclusion: the principle follows as a theorem. In this paper, I started down the same path as Earman, but came to a different conclusion. The Quantum Principal Principle, appropriately formulated in quantum probability theory, is not a theorem; its justification does not come so straightforwardly. However, I still found that bringing the Principal Principle into the domain of quantum probability theory is a worthwhile endeavor. It not only yields new insight into the principle itself, as Earman showed, it also allows us to bring the framework to bear on important applications, like the learning of quantum states.

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