On the relationship between geometric objects and figures in Euclidean geometry[1]

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Abstract: In this paper, we will make explicit the relationship that exists between geometric objects and geometric figures in planar Euclidean geometry. That will enable us to determine basic features regarding the role of geometric figures and diagrams when used in the context of pure and applied planar Euclidean geometry, arising due to this relationship. By taking into account pure geometry, as developed in Euclid's *Elements*, and practical geometry, we will establish a relation between geometric objects and figures. Geometric objects are defined in terms of idealizations of the corresponding figures of practical geometry. We name the relationship between them as a relation of idealization. This relation, existing between objects and figures, is what enables figures to have a role in pure and applied geometry. That is, we can use a figure or diagram as a representation of geometric objects or composite geometric objects because the relation of idealization corresponds to a resemblance-like relationship between objects and figures. Moving beyond pure geometry, we will defend that there are two other 'layers' of representation at play in applied geometry. To show that, we will consider Euclid's *Optics*.

1 Introduction

The role of diagrams in geometry has been the subject of many philosophical inquires, and different views have been proposed. Here, we endeavor to determine what kind of relationship exists between geometric objects and geometric figures in planar Euclidean geometry. In this work, geometric figure stands for drawings that we commonly name as segments, circles, and so on. We reserve the term diagrams for composite figures. One well-known example is that of the diagram accompanying proposition 1 of book 1 of Euclid's *Elements*. There, we have a drawing of two circles that intersect and three segments that form an equilateral triangle.

The rationale behind this work is the following. If there is a clear relation existing between geometric objects and geometric figures, then, this might condition or even determine what role geometric figures and diagrams can have when used in the context of pure or applied geometry. In this work, we will show what kind of relation there is between objects and figures and what it enables regarding the use of figures or diagrams in the context of pure and applied planar Euclidean geometry.

The work unfolds as follows. First, in part 2, we will address geometric figures. For that purpose, we will consider a historical example of practical geometry that is previous to the arising of pure geometry. This will enable two things: 1) address geometric figures in a context of a concrete geometrical practice – not some fictional account that might be twisted according to our interests (even if inadvertently); 2) address geometric figures independently of geometric objects, since during this historical period there was still no pure geometry. After addressing geometric figures in the context of a practical geometrical practice, we will consider, in part 3, the treatment of geometric objects in Euclid's *Elements*. This will enable us to bring to light the relationship that geometric

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objects have with geometric figures. We will keep from the first part only those aspects of practical geometry that are taken into account in the articulation of geometric objects in pure geometry. Having established this, we will, in part 4, determine basic features that geometric figures or diagrams have due to this relationship when used in the context of pure or applied planar Euclidean geometry. For this purpose, we will consider how figures and diagrams are used in propositions of pure and applied geometry in the light of the established relationship.

2 Geometric figures in practical geometry

We can have geometric figures even without a clear indication of how they are conceptualized. In fact, e.g., there is evidence that there was a notion of circular shape already at play in Mesopotamia before the mid-third millennium. This is attested in early Mesopotamian visual culture. We see it in sketchy drawings or in the form given to some artifacts (see, e.g., Frankfort 1970, pp. 17-37). One very beautiful example is a game board decorated with symmetrical drawings some of which have a circular form. The game's pieces are also circular (see, e.g., Robson 2008, pp. 45-6).

A conceptualization proper of geometric figures arises in the context of a practical geometry where there are clear geometrical practices, and, importantly, the figures are named. This is already the case during the Old Babylonian period (Robson 2004).

A good example is that of the rectangle. Each side is given a name. They are named the 'long side' and the 'front'. This naming refers to agricultural field plots that, for practical reasons, were given rectangular forms. The side called the 'front' is one of the small sides and is parallel to an irrigation channel (Høyrup 2002, p. 34).

That Mesopotamian practical geometry arises in the context of field measurements has important implications regarding how the geometric figures were conceptualized. The rectangle, be it an actual field plot or a drawing (for example, a field plan), is conceptualized in terms of the boundary that establishes an inner space separated from the outside by it. The figure proper is what is inside the boundary; however, due to the importance of this boundary, in many cases, the names for the figure and the boundary are the same (Robson 2008, 64). In Eleanor Robson's terms, we have a "fundamentally boundary-oriented conceptualization of two-dimensional space" (Robson 2008, 64). This can be seen in the land surveyors' practices of the time.

Not only was it fundamental to have a clear demarcation between field plots belonging to different people but also to measure the areas of plots for personal and administrative control (Baker 2011). In Mesopotamia, the area of a field plot was calculated from the measurement of its boundary. Surveyors could only rely on length measurements. For that purpose, they could use, e.g., ropes whose lengths were given in terms of a metrological length unit (Baker 2011, 296-7).[2] To calculate the area of quadrilateral field plots, surveyors applied the so-called surveyors' formula. This formula enables us to calculate what for us is the approximate value of a quadrilateral figure. For field plots not deviating too much from a rectangular, the formula gives a close approximation to the area of the plot. According to Peter Damerow, "the ancient surveyors apparently assumed that the area of a field remains equal if they subtract some part from one side of a field and add a part of the same size to the opposite side" (Damerow 2016, 107). This view leads

[2] In the Old Babylonian period, an important unit of length was the *rod*, corresponding approximately to 6 meters (Cooper 2013, 403).

straightforwardly to the surveyors' formula in which the area is calculated by the multiplication of the mean values of the two sets of two opposite sides.[3]

This boundary-oriented conceptualization of space lasted in Mesopotamian mathematics. In geometrical problems from the Old Babylonian period, one still finds "the assumption that the area of a quadrilateral is determined by the surveyors' formula" (Damerow 2016, 117).

We can say that the notion of area of geometric figures derives from the notion of practical geometry (Damerow 2016, 115-7). We see an example of this in the conception of circle in Mesopotamian mathematics. It can be hardly the case that surveyors dealt with circular fields. But the existence of circular objects or figures is well attested. It might well be the case that one needed at some point to determine the area of a circular object or figure. For example, considering a circular oven (Cooper 2013, 404), we might for some reason need to calculate its area. Like in the case of a rectangular figure, a circle is conceptualized in terms of its boundary (what we call the circumference). Accordingly, "a circle was the shape contained within an equidistant circumference" (Robson 2004, 20). The circle and its boundary were given the same name, something like "thing that curves" (Robson 2004, 20). Like in the case of quadrilateral figures the area of the circle is calculated from the length of its boundary (which can be measured). The area is calculated using a formula. It is given by the square of the length of the circumference divided by 12 (Robson 2004, 18).

Circle figures are well-attested in ancient Mesopotamian mathematical problems. Following Damerow's view, one should address these figures in light of their practical geometry. The drawings can be very sketchy but also quite precise. One example of this can be found in the tentative solution of mathematical assignment made in a clay tablet. Here, there is a drawing of an equilateral triangle inscribed in a circle (Friberg 2007, 207 and 488). Not only the sides of the triangle are quite rectilinear, which can be achieved by using some sort of straightedge, but also the circle is very precise. This results from the fact that the circle was drawn using a compass (Høyrup 2002, 105; Friberg 2007, 207). In summary: during the Old Babylonian period, a circle is still conceptualized in terms of the practical geometrical practice of land surveyors, i.e., in terms of a boundary. The circle is 'individuated' as a geometrical figure by being given a name, by the existence of a specific formula to calculate its area from the measurement of its boundary, and by the possibility of being drawn in a very precise way using a specific instrument – a compass.

3 Relating geometric objects to geometric figures

In pure planar geometry as developed in Euclid's *Elements,* the geometric object called circle, like other geometric objects, is explicitly defined in the definitions. This is done in definitions 15 and 16 of book 1. Taking them together, the geometric object called circle is defined as follows:

[3] Let l_1, l_2, l_3, and l_4 be the sides of a quadrilateral field plot, in which l_1 and l_3 are opposite sides as is the case with l_2 and l_4. The surveyors' formula gives for the area of the field plot the value $(l_1 + l_3)/2 \times (l_2 + l_4)/2$. As mentioned, the sides of the field plot are measured by the surveyors using, e.g., ropes with given measures.

A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another; and the point is called the center of the circle. (Euclid 1956, pp. 153-4)

But what warrants that there is a geometric object corresponding to this definition? This warranty is given by the third postulate of the *Elements*: "Let the following be postulated: [...] To describe a circle with any center and distance" (Euclid 1956, p. 154).

At this point, it might be useful to make a silly question. Why do we name this geometric object with the same name as that of a figure? This is at the crux of the relationship between geometric objects and figures.

As it is, the definition of geometric circle in the *Elements* is rather distinct from the conceptualization of circle as a figure in ancient Mesopotamian mathematics. But this is not a real problem. If there is a relationship between a notion of circle from practical geometry and pure geometry this must be made in the same cultural context. That is, taking into account Greek practical geometry. We will come to this just next.

To help to answer our silly question we will first consider another silly question. Why not take the above definition of a circle as a geometric object as a definition of a circle figure? As it is the definition of circle in the *Elements* seems to correspond to the practice of drawing a circle figure using a compass, something that was already present in Mesopotamian mathematics. Taking this definition outside the context of the *Elements*, it could very well be the recipe for drawing a circle using a compass. The center of the circle is the needle point of the compass, and all points of the circumference drawn with the compass lead are at the same distance from this point as measured using, e.g., a ruler. In fact, we could even dispense with this measurement. We can imagine having a cord stretched between the needle point and the lead of the compass. In practical terms, while drawing the circle this cord is 'measuring' its radius and all points of the circumference are at the same distance from the center of the circle as measured by this cord. This definition/conceptualization of practical circle is certainly different from that we have seen in the previous section but there is a crucial aspect in common. In fact, this aspect is the one that is fundamental for our present purpose. When we say that the radii of a circle figure have all the same length, we are saying this in the context of a practical geometrical practice in which we talk about measurable lengths. This is so independent of the details of how a figure is conceptualized. For practical purposes, or due to the limitations in the precision of a length measurement using, e.g., a measuring rod, the radii are equal. Whatever small differences are there, these are negligible. For practical purposes the radii are equal. And this is so independent of how we conceptualize the circle figure. As mentioned by Eleanor Robson:

If plane figures were conceptualised, named, and defined from the inside out, then the centre of the circle and the idea of the rotating radius could not have played an important part in Mesopotamian mathematics. (Robson 2004, 20)

However:

[This does] not mean that the ancient Mesopotamians did not know that circles could be generated by rotating radii. There is a great deal of visual evidence to show that they did. For example, BM 15285, a compilation of plane geometry problems from Larsa, depicts several circles whose deeply impressed centres reveal that they were drawn by means of rotating compasses. (Robson 2004, 20)

In this way, we can conceive of a conceptualization of a circle figure in the exact terms of the definition of geometric circle in Euclid's *Elements*. The crucial aspects of a circle figure are that it is drawn with a compass, that the lengths of the circumference and radii are measurable using, e.g., measuring rods or ropes, and that the area is calculated indirectly from these measures. The few elements we have regarding Greek practical geometry and its conception of circle are compatible with this reduced list of crucial aspects of a circle figure.[4]

Let us return to Euclid's definition of circle as a geometric object. We have been using it as a definition of a circle figure. Who can this be possible, having the same definition for a drawn figure and for a geometric object? For us, it comes down to semantics; in particular to the meaning of a few terms, which must be understood in the context of a particular mathematical practice. The definition of circle as a geometric object starts by mentioning that it is a plane figure. What does it mean for a geometric object to be a plane figure? That a circle figure is taken to be a plane figure has, certainly, a quite different meaning. In practical geometry what this means is that it is drawn in a practically planar surface, e.g., a dusted surface or a wax tablet (Netz 1999, pp. 14-6).

As mentioned above, what warrants that there is a geometric circle is the third postulate of the *Elements*. This postulate mentions the possibility of 'describing' a circle with any center and radius. Where is the circle described and why can it have any center and radius? A geometric object is described or instantiated in what we might call the Euclidean plane. This plane is not mentioned in the *Elements,* however, as Mueller noticed, "normally, when plane geometry is developed as an independent subject, it is taken for granted that all objects considered lie in a single plane, which never has to be mentioned" (Mueller 1981, p. 208). We can say that the Euclidean plane is "an abstraction from physical boards" (Taisbak 2003, p. 19). It is in pure geometry the counterpart of, e.g., a wax tablet.

A geometric object like the circle is, implicitly, described or instantiated in the Euclidean plane. Like the case of a wax tablet where we might put the needle point of the compass in different positions and draw a circle with a chosen radius, we take the same to be feasible in the Euclidean plane. This is why in the third postulate it is mentioned that the circle as a geometric object can have any center and radius.

At this moment we are very close to understanding the relationship between circle as a geometric object and circle as a drawn figure. A circle as a geometric object is instantiated in an idealized plane – an abstraction from a real physical plane. As defined,

[4] To the best of our knowledge, there is no extant text containing a practical definition of circle corresponding to that of geometric circle in the *Elements*. However, there are records of a conceptualization that approaches that in terms of radius. It is a conceptualization of circle whose main element is the diameter that is conceived as rotating about its center (Robson 2004, 20). That this conceptualization can be ascribed to circle figures independently of having been adopted in the context of pure geometry is suggested, e.g., by a Greek third-century BC papyrus containing practical geometrical problems among others. Here, the key measure associated with a circle is the diameter (Cuomo 2001, 70-2), and not the circumference like in the Old Babylonian case.

the circle as a geometric object has all radii equal to one another. Here, 'equal' does not mean the same as 'equal' in practical geometry. In the latter case, the equality of different radii is a practical one; we simply neglect whatever small difference in lengths there are. In the case of the geometric object, equality is absolute – is of an ideal type. The radii of the geometric circle instantiated in the Euclidean plane are absolutely equal to one another. This is an idealization of the practice of practical geometry of taking measured lengths as equal by disregarding what we might call the uncertainty in the measurement. In practical geometry, when we say that the length of one of the radii of a circle is, e.g., 20 cm, this is a shortcut to say that the measurement gave the result 20 ± the uncertainty, where the uncertainty depends on the precision of the measuring instrument (Hughes and Hase 2010, pp. 2-6).

In pure geometry, it is made an idealization of this practical approach and instead of conceiving of practically equal radii, these are conceived as exactly equal. We have what we might call an exactification of the equality of lengths. The relationship between the geometric circle and the circle figure is what we might call a relation of idealization: the abstract object is defined in terms of an idealization of the concrete figure.

This relation of idealization is made clearer by taking into account similar relations for lines and points (Valente 2020a). They are all taken into account, implicitly, in the definition of circle. As defined in the *Elements*, "a point is that which has no part. A line is breadthless length" (Euclid 1956, p. 153). A geometric line can be seen as an idealization of a concrete line. Some idealizations that are manifested in the geometric line are its lack of depth, that it is breadthless, and the exactification of length. The lack of depth can be seen as a property of the Euclidean plane and because of this of geometric objects instantiate in the plane. That a line is defined as breadthless length and a point as that which has no part can be seen as arising from an idealization of what is done in the practice of practical geometry. As mentioned by Harari, "a point is characterized as a non-measurable entity, as it has no parts that can measure it" (Harari, 2003, p. 18). This is what happens with concrete points like the needle point of a compass; it is simply meaningless to conceive of a practical measurement made on this concrete point. In the same way, in practical geometry, lines have small breadths (this is the case, e.g., of a circumference drawn using a compass). The lengths of these breadths are disregarded in practical terms (even if we might conceive of measuring them with some precision instrument). We idealize the concrete line as a geometric object that has an exact length and is breadthless. We can say that both the geometric line and the geometric point are in a relation of idealization with lines and points from practical geometry.

The relationship between geometric objects and figures is manifested very clearly in the definition of geometric line. We can see that the definition of geometric line is made by reference to an idealization at play. The geometric line is defined in relation to what is being implicitly idealized: a line from practical geometry. It only makes sense a definition in terms of breadthless length, in relation to something that has breadth. That is, we can only have a notion of breadthless when having first a notion of breadth. And this is the case with a line figure. The definitions of geometric objects are dependent on the figures and result from an idealization of these. This enables us to say that between geometric objects and figures we have a relation of idealization.

4 Basic features of the role of diagrams in pure and applied geometry

In the previous section, we have determined what we called the relation of idealization between geometric objects and figures. As mentioned in the introduction, we expect that this relationship determines basic features regarding the role of figures or diagrams when used in the context of pure or applied geometry.[5] To show this, in this section, we will consider two propositions, one from pure geometry and another from applied geometry. We will start with pure geometry.

In proposition 1 of book 1 of Euclid's *Elements* (proposition I.1), one constructs a geometric object – an equilateral triangle – by a particular procedure where one uses two circles that intersect each other. We can see this proposition as containing two aspects: 1) the procedure to instantiate the geometric object; 2) the proof that the procedure enables, in fact, this construction. The text is accompanied by a lettered diagram (see figure 1), and with the letters one refers in the text to parts of the diagram. For example, one mentions the segment AB, the circles BCD and ACE, the point C, and so on.[6]

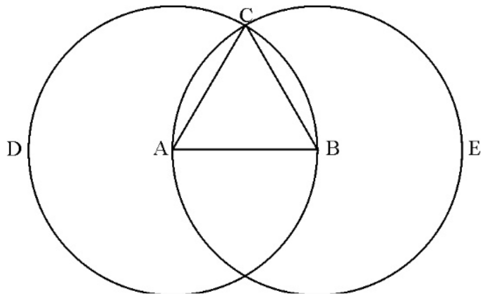

Figure 1. The diagram in proposition I.1 of the *Elements*

The basic point we want to make here is to question how come that in the demonstration of a result in pure geometry we use a diagram which is a drawing consisting of several figures? The evident answer is that we take the diagram to represent the geometric objects. But what justifies using figures from practical geometry as a representation of geometric objects? More generally, we have to know what enables something to be a representation of something else. There are two features related to representation that are relevant here: intentionality and resemblance (see, e.g., Abell 2009; Kulvicki 2006; Blumson 2014). According to John Kulvicki:

There are at least two senses in which one object might be judged to resemble another, and the differences between them need to be kept clear. Objects may be judged to look alike, in the sense of being apparently similar or experienced as similar to one another, or

[5] These features should be general enough to be compatible with different views on the role of diagrams in Euclidean geometry. We will not address this issue here. For that, we need a detailed analysis of different approaches to the role of diagrams in pure and applied geometry while taking into account the view proposed here.

[6] Proposition I.1 is as follows: *On a given finite straight line to construct an equilateral triangle.* Let *AB* be the given finite straight line. Thus it is required to construct an equilateral triangle on the straight line *AB.* With center *A* and distance *AB* let the circle *BCD* be described; [Post. 3] again, with center *B* and distance *BA* let the circle *ACE* be described; [Post. 3] and from the point C, in which the circles cut one another, to the points *A, B* let the straight lines *CA, CB* be joined. [Post. I] Now, since the point *A* is the center of the circle *CDB, AC* is equal to *AB.* [Def. I5] Again, since the point *B* is the center of the circle *CAE, BC* is equal to *BA.* [Def. I5] But *CA* was also proved equal to *AB;* therefore each of the straight lines *CA, CB* is equal to *AB.* And things which are equal to the same thing are also equal to one another; [*C. N.* I] therefore *CA* is also equal to *CB.* Therefore the three straight lines *CA, AB, BC* are equal to one another. Therefore the triangle *ABC* is equilateral; and it has been constructed on the given finite straight line *AB.* (Being) what it was required to do (Euclid 1956, 241-42).

they may be judged to be genuinely similar in that they share specified properties. (Kulvicki 2006, 82)

For our purpose, it is not necessary to distinguish between these two possibilities. Regarding the intentionality in the adoption of a representation, what is relevant for us here is not so much that the intention of the author that adopts a particular representation is usually relevant in the interpretation of the representation, but that 'intentionality' underlies the possibility of choosing quite freely what we take to be the representation of something else. For example, we might decide that a hand-drawn line represents a segment drawn using a straightedge. That is, we intentionally take the sketchy line to represent the practical segment or segment figure.

The intentionality enables us to choose whatever we want as a symbol (representation) of something else. With an ad hoc representation, we would not go very far in the case under consideration. If we choose an abstract Pollock-like image as our representation of a geometric object it would not be useful neither for the 'construction' of the equilateral triangle neither for the proof that this construction actually instantiates a triangle that is equilateral. So, we rely on another concept related to representation. That of resemblance. We would go from just a symbol to a symbol that has iconic properties. That is, to a symbol that in some way resembles what it is symbolizing. But here we face a major problem. A geometric object is not something that we can see. It is instantiated in an abstract space – the Euclidean space – not in the space of our experience. There is no way in which we might say that a circle figure resembles a geometric circle. How do we overcome this difficulty?

A circle figure does not resemble a geometric circle; this simply has no meaning, unless we twist considerably the semantics of the word 'resemblance'. However, we have another kind of relationship between geometric objects and figures. We can adopt the relation of idealization, e.g., between a geometric circle and a circle figure to take the second as a representation of the first. The relation of idealization works as a resemblance-like relation. While the circle figure does not resemble the geometric figure, we can nevertheless establish a simulacrum of a resemblance between them. The circle figure has as its center the needle point of the compass. To this concrete point corresponds the geometric point as the center of the geometric circle. To the drawn circumference corresponds a breadthless line. While the radii of the circle figure are equal only within a particular practice of practical geometry where we neglect small measurement differences, the radii of a geometric circle are equal exactly as if corresponding to an idealized measurement in which all lengths are exactly equal.

The relation of idealization is a sort of resemblance-like relation that enables us to take the circle figure to be a representation of the geometric circle. Since we have one-to-one resemblance-like relations between all relevant elements of the geometric circle and the circle figure (center, circumference, radii, diameter, etc.), the circle figure works as an avatar of the geometric object in the diagram. When in the text we refer to aspects of the diagram we can take these as referring to the corresponding aspects of geometric objects.

This is a very basic characteristic of the use of diagrams in pure geometry. We suggest that any account of the role of diagrams in pure geometry should be compatible with this feature and how it arises.

Let us now address the issue of the role of diagrams in applied geometry. We will consider proposition 1 of Euclid's *Optics*. What we want to determine here is in what way, if any, do we move beyond the representational role that a figure has in pure geometry. In that case, as we have just seen, we can establish a resemblance-like relationship between geometric object and figure.

When applying geometry like in the *Optics*, we take geometric objects to represent physical phenomena. As noticed by Stephen Toulmin, the development of geometric optics corresponds to «the application of new modes of representation» (Toulmin 1953, p. 43); it consists of a «geometrical method of representing optical phenomena» (Toulmin 1953, p. 26). This view applies equally to the case of Euclid's *Optics*. Let us see why.

The basic idea developed in Euclid's *Optics* is that the eyes emit 'visual fire'. It is the 'visual fire' that enables us to see the world around us. For example, the incidence of 'visual fire' in objects is what enables us to see them. 'Visual fire' is represented in the *Optics* by geometric segments (Darrigol 2012, p. 8; see also Burton 1945). Some of the assumptions of the *Optics* are the following:

The straight lines drawn from the eye diverge to embrace the magnitudes seen. The figure contained by a set of visual rays is a cone of which the apex is in the eye and the base at the limits of the magnitudes seen. Those magnitudes are seen upon which visual rays fall, and those magnitudes are not seen upon which visual rays do not fall. (cited in Darrigol 2012, p. 9)

The statement of proposition 1 of the *Optics* is as follows:

Prop. 1: No observed magnitude is seen simultaneously as a whole.
Call AD the observed magnitude, and B the eye from which the visual rays BA, BG, BK, BD fall. Since the visual rays diverge, they do not fall on the magnitude AD in a contiguous manner; so that there are intervals of this magnitude on which the visual rays do not fall. Consequently, the entire magnitude is not seen simultaneously. However, as the visual rays move rapidly, it is as if we saw [the entire magnitude] simultaneously. (cited in Darrigol 2012, p. 10)

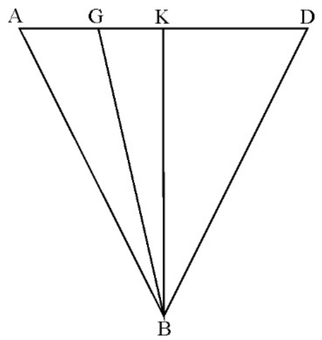

Figure 2. The diagram in proposition 1 of the *Optics*.

Here, we are going to make some magic. We will use diagram 2 to help us to interpret the text. We will take advantage of the representational roles of the diagram even if we have not clarified what these are. The geometric segment AD represents the physical object that we see. The geometric segments BA, BG, BK, BD represent the physical 'visual fire' emitted by an eye of the observer. The geometric point B represents an eye. Here, we are using the diagram to help us clarify the representational role of geometric figures. In fact, we are ascribing to the diagram these features. When we look, for example, at the drawn line BA, we take it to represent the 'visual fire'. The point is that since we have a resemblance-like relation established between the figure and its corresponding geometric object, and we take the geometric object to represent a physical entity, we can intentionally ascribe to the figure the representational role of its corresponding geometric object. We put another 'layer' of representation on top of the first one.

At this point, we can say that in applied geometry, the geometric figure has a double representational role. The geometric figure (or diagram) represents the geometric object, and this represents a physical entity. In this way, the geometric figure represents the physical entity, via the geometric object represented by the figure.[7]

There is in our view a third 'layer' of representation in the diagrams of Euclid's *Optics*. As we have seen, the geometric objects are given a representational character in the context of several assumptions. For example, it is assumed that the straight lines drawn from the eye diverge to embrace the magnitudes seen. How the visual rays 'diverge to embrace' is further specified in proposition 1. There, the visual rays are taken to "move rapidly" (Darrigol 2012, p. 10). This corresponds to ascribing to the diagram a new 'layer' of representation of the optical phenomena. We have the assumption that there is a sort of scanning of magnitudes by emitting successively the visual rays BA, BG, BK, and BD. The diagram as a whole is a static representation of a dynamic situation (see also Valente 2020b).

While this layer of representation relates, as the second one, to the geometric objects, it is only meaningful when taking into account the whole diagram. Like with the second layer of representation (where we can regard it as implemented directly on the figures), we can see this further 'layer' of representation as implemented directly on the diagram. An important difference with the second 'layer' is that it is not implemented so much on the figures that form the diagram but on the diagram as a whole.

For applied geometry, the situation is then as follows. The figures represent geometric objects due to the relation of idealization existing between them. Since we take the geometric objects to represent physical phenomena like, e.g., 'visual rays', we take the corresponding figures to represent the physical phenomena. This is a second 'layer' of representation that we ascribe to the figures. Besides this, the geometric objects on a whole have a dynamic relationship between them since they represent not only physical

[7] One might ask what justifies taking a geometric object to represent physical phenomena in the first place. Again, it is due to the relation of idealization that we have between geometric objects and concrete objects. For instance, a geometric segment is in a relation of idealization not only with, e.g., a practically drawn segment but also, e.g., with a rod, a stretched rope, or with the 'visual fire' taken to be a sort of light beam (Valente 2020a).

entities but also their dynamics. We must take into account that the visual rays 'move rapidly'. This corresponds to ascribing a third 'layer' of representation not to each figure individually but to the diagram as a whole since it is only at this 'level' that we can represent the dynamics. With this third 'layer' of representation, the diagram represents the dynamics of the physical phenomena.

These are basic aspects of the role of diagrams in applied geometry that follow from the relation of idealization that exists between geometric objects and figures and from taking geometric objects to represent physical phenomena. We suggest that any account of the role of diagrams in applied geometry should be compatible with this view.

5 Conclusions

In this work, we have tried to determine what kind of relationship there is between geometric figures of practical geometry and geometric objects of pure geometry. Since not much is known about Greek practical geometry we have focused on the more ancient Mesopotamian practical geometry and mathematics. From this particular case, we have extracted some key elements of practical figures that are relevant for our discussion and are compatible with the little we know about Greek practical geometry. When considering pure geometry as developed in Euclid's *Elements* and practical geometry together, we have established that there is a relationship between geometric objects and figures. Geometric objects in the *Elements* are defined in terms of idealizations of the corresponding figures of practical geometry. We have named the relationship between them as a relation of idealization.

This relation existing between objects and figures is what in our view enables figures to have a role in pure and applied geometry. That is, we can use a figure or diagram as a representation of geometric objects or composite geometric objects because the relation of idealization corresponds to a resemblance-like relationship between objects and figures. It is not simply that we intentionally decide to adopt a figure as a representation of a geometric object. It goes beyond this. We might as well choose a Pollock-like figure as a representation, but this would not be very fruitful in the practices of pure and applied geometry as manifested in Euclid's *Elements* and *Optics*. The geometric figures and diagrams can have a role in pure and applied geometry because the relation of idealization works as a resemblance-like relationship between objects and figures.

Moving beyond pure geometry we have defended that there are two other 'layers' of representation at play in applied geometry: 1) geometric figures can be ascribed as representing physical phenomena when we give this representational role to their corresponding geometric objects due to the relation of idealization existing between them; 2) The diagram as a whole can be taken to represent dynamical features of the physical phenomena also for the same reason. For example, the diagram in proposition 1 of the *Optics* represents a sequence of 'emissions' of visual rays; the diagram is a static rendering of a dynamical situation. We suggest that any consistent view on the role of diagrams in pure and applied geometry should be compatible with these basic features and the way in which they are established.

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