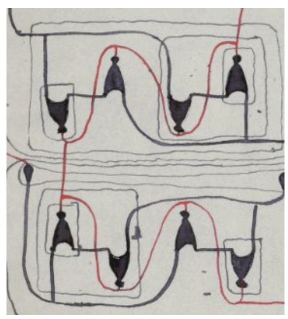
CUADERNOS DE SISTEMÁTICA PEIRCEANA



Número 7 – 2015

CENTRO DE SISTEMÁTICA PEIRCEANA
CSP

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Diseño de cubierta
AZ Estudio (www.azestudio.com)

ISBN 978-958-46-4548-7

Impreso por Editorial Nomos
Impreso en Colombia

NOTA BIBLIOGRÁFICA.

A lo largo del número, las referencias usuales a escritos de Peirce se denotan con las siguientes siglas:

[CP] Charles S. Peirce, *Collected Papers* (8 vols.), Harvard: Harvard University Press, 1931-1958.

[EP] Charles S. Peirce, *Essential Peirce* (2 vols.), Bloomington: Indiana University Press, 1992-1998.

[N] Charles S. Peirce, *Contributions to The Nation* (4 vols.), Lubbock: Texas Tech University Press, 1975-1987.

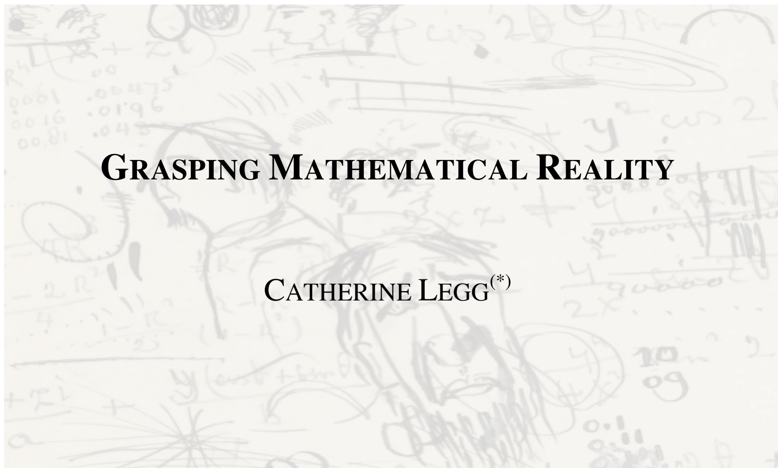
[NEM] Charles S. Peirce, *The New Elements of Mathematics* (4 vols., en 5), The Hague: Mouton, 1976.

[W] Charles S. Peirce, *Writings. A Chronological Edition* (7 vols. hasta la fecha), Bloomington: Indiana University Press, 1981-.

[MS/L] Charles S. Peirce, *The Charles S. Peirce Papers* (32 rollos de microfilms de los manuscritos conservados en la Houghton Library), Cambridge: Harvard University Library, 1967-1971. La numeración corresponde a Richard Robin, *Annotated Catalogue of the Papers of Charles S. Peirce*, Amherst: University of Massachusetts Press, 1967, y/o Richard Robin, "The Peirce Papers: A Supplementary Catalogue", *Transactions of the Charles S. Peirce Society* 7 (1971): pp. 37–57. "MS" se refiere a los manuscritos y "L" a las cartas.

Una referencia del tipo [A *b.pqr*; xyzt] en las fuentes *publicadas* envía a la colección [A], volumen *b*. En el caso [CP], *pqr* envía al *párrafo pqr*. En los demás casos, *pqr* envía a la *página pqr*. El dato xyzt (cuando incluido por los autores) indica fecha de escritura del texto.

Otras referencias específicas a escritos puntuales de Peirce se incluyen en cada artículo por separado.



GRASPING MATHEMATICAL REALITY

CATHERINE LEGG[(*)]

1. KNOWING HOW TO GO ON IN MATHEMATICS[1]

Mathematical knowledge is generally understood to be a special kind of knowledge, with a peculiarly impressive epistemic authority. This is sometimes expressed by stating that mathematics is *necessarily true*, or as some philosophers like to say, *true in all possible worlds*. But why do we believe this of mathematics, and not other sciences, such as physics? It has been suggested that this is because mathematics studies special kinds of objects (which live somehow 'beyond' the world we inhabit).[2] But what makes these objects special? One might be tempted here to posit so-called "necessary objects", drawing on a venerable history leading back to Plato, who suggested that mathematical objects were endowed by their necessity with a special edifying character, so that a sign purportedly hung over the entrance to his Academy stating: "Let no-one ignorant of geometry enter

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1 For discussions which threw helpful light on this paper I'm grateful to Jeff Downard, Bill McCurdy, Matthew Moore and of course Fernando Zalamea.

2 See for instance [Balaguer 1998].

here".[3] Yet what sort of thing could a "necessary object" be? In what does its necessity consist? The idea received stern rebuke in the mid-20th century as a holdover of alleged medieval mysticism from W.V.O. Quine, who stressed that necessity pertains not to things, but only the truth of sentences,[4] producing a fear of "Platonism" that lingers in the minds of many analytic philosophers today.[5] One reason many philosophers view mathematics as necessarily true is that mathematics seems paradigmatically *certain*. This appears undeniable insofar as mathematicians do not change their minds on specific questions nearly as often as, say, physicists. Once a mathematical proof is produced and rigorously checked, mathematicians tend to accept it from then on. But an interesting question now arises: *in virtue of what* does mathematics enjoy this remarkable level of certainty and agreement?

One common, though usually unarticulated, answer to this question will be the focus of this paper. This answer is that mathematics is developed by *defining and then following rules*. Thus "+2" has been presented (in discussions of rule-following initiated by the later Wittgenstein), as an example of a mathematical concept which demonstrates this rule-following character.[6] John McDowell [McDowell 1994] and Crispin Wright [Wright 1989, 2007], following Wittgenstein, have called this understanding of rule-following as certain and binding: "*rules as rails*". And this seems like a useful metaphor. How could 2 + 2 be anything but 4? If I were to sincerely assert, for instance, "2 + 2 = 5", wouldn't I be "going off the rails"?

This picture of mathematics as defined and justified by "rules as rails" is perhaps overly encouraged by the way the subject is taught in elementary schools, as a set of algorithms with clear right and wrong answers. Yet excessive codification of mathematics into rules can directly block mathematical insight. This is nicely demonstrated by the

[3] The sign must have been heeded to at least some degree, as Plato became known as the "maker of mathematicians".
[4] "Reference and Modality", in [Quine 1953].
[5] See for instance [Benacerraf 1973].
[6] Here it must be admitted that this simple concept represents the lowest level of mathematical development. The fact that philosophers have taken it as representative of mathematics as a whole is part of a problematic which will be noted in this paper, and is incisively diagnosed in [Zalamea 2012b].

example in *Fig. 1* of a third-grade mathematics teacher marking down a student's test, which was shared and discussed by concerned parents on the *Reddit* website.[7]

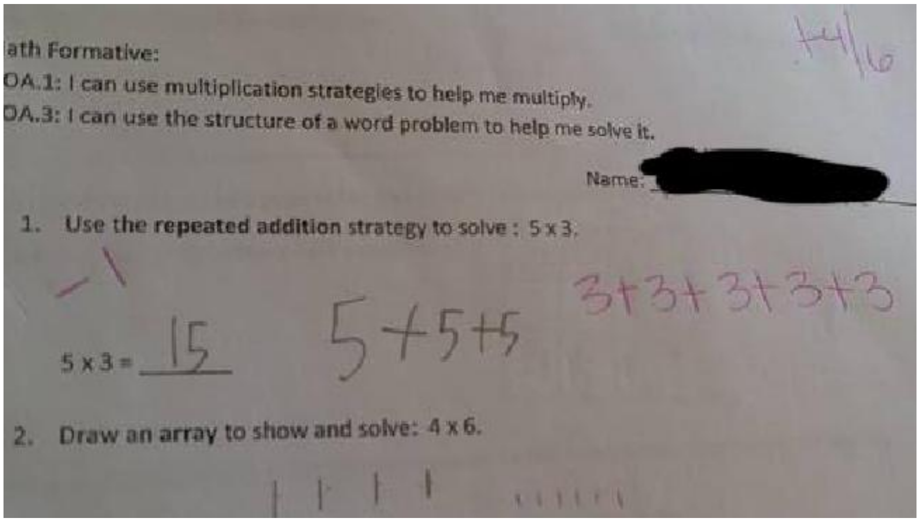

Figure 1
A pupil is rebuked for utilising a mathematical identity that his/her teacher does not recognise

Here the student is asked to use "the" (sic) repeated addition strategy to solve the multiplication problem: "5 x 3". It is true that all multiplication problems in elementary arithmetic reduce to such concatenated addings, and that this is an important mathematical insight to grasp at that level. Yet the 'repeated addition' relationship cuts both ways – it may be constructed from either multiplicand. Whilst this student appears to have in fact chosen the more elegant (shorter) version to calculate, the teacher seems wedded to the left-hand multiplicand representing the number of additions and the right-hand representing the value that is repeatedly added, despite the fact that this distinction is arbitrary and that (at least in elementary mathematics) these two ways of calculating always give the same answer. So the teacher 'corrects' the student on that basis, even though one might argue that the student here has an understanding that the teacher lacks.

I believe that this example nicely encapsulates why we need a richer account of mathematical practice and insight than "rules as rails". We need an account which

[7] https://www.reddit.com/r/pics/comments/3pmyh3/teachers_logic_in_grading_math/, downloaded October 28, 2016.

acknowledges and explains how mathematicians do not merely follow definitions laid out in advance, but often make discoveries which are radically new, although (what is perhaps most intriguing) those discoveries can always be demonstratively proven in hindsight. This paper will make some preliminary moves towards such an account, proceeding as follows. I will first discuss some failures of the "rules as rails" model which have been noted in discussions of the later Wittgenstein's so-called "rule following problem" (§2). I will then draw on the distinctive resources of Charles Peirce's philosophy to diagnose the rule-following problem and introduce some conceptual tools for resolving it (§3), then critically reflect on this solution. At this point some further insights will emerge: that a certain 'basic empiricism' very much taken for granted in analytic epistemology is in fact incorrect (§4), not least because of the important role in gaining mathematical knowledge played by hypostatic abstraction (§5). The paper will close with some reflections on realism in mathematics (§6).

This paper would have been impossible without the wonderful intellectual stimulation of the 2015 "Peirce's Mathematics" conference at the Universidad Nacional de Colombia, and reading Zalamea's groundbreaking books bringing mathematical ideas into philosophy [Zalamea 2012a, 2012b]. Writing this paper has also given me the opportunity to draw together in new ways ideas that I have published in a number of past papers on a wide variety of topics, thereby serving as a pivot for reflection that I have found very illuminating, and hope will be of some benefit to others. In writing on these topics I must however apologise for my profound lack of knowledge of real debates in mathematics as the discipline has moved from its late 19th century preoccupation with set theoretic 'foundations' into its 'modern' and 'contemporary' phases, as described in [Zalamea 2012b]. I also apologise for being still entangled in ways that I am sure I am still unable to fully understand in ways of thinking drawn from analytic philosophy according to which, sadly, "…the range of mathematics…is reduced to a lattice of logics and classical set theory" [Zalamea 2012b, p. 102], so that "analytic philosophy…has turned its back on and has abandoned high mathematical creativity…thereby estranging itself from the real center of the discipline that helped it to emerge" [Zalamea 2012b, p. 87].

2. THE "RULE-FOLLOWING" PROBLEM[8]

The naïve understanding of mathematical epistemology as "rules as rails" has faced some upheaval in discussions of the so-called *rule-following problem*, which draws specifically on mathematical examples to conjure an apparently severe scepticism. I will now outline the problem. [Kripke 1982], citing [Wittgenstein 1951], invites us to imagine some number larger than we have ever added before. (For simplicity's sake, he calls it 57). He then invites us to imagine a "bizarre sceptic", who declares that the answer to the sum "68 plus 57" should be 5! When we protest that this strange answer is obviously not following the rule for adding, the sceptic claims that what we mean by addition is in fact the *'quus'* function:

$$x \text{ quus } y \quad = x + y, \text{ if } x \text{ and } y \text{ are less than 57}$$
$$= 5 \text{ otherwise.}$$

If I have never added numbers greater than 57, how can I prove to the sceptic that that is not what I meant by addition, in all previous cases? Kripke insists that the rule-following problem involves both epistemological and metaphysical dimensions. The epistemological dimension is that I *cannot justify* deriving the answer 125 – not 5 – to the sum 68 + 57, given that I have never added numbers over 57 before. The metaphysical dimension is that there is *no fact about me* which determines whether I mean *plus* or *quus*. Kripke concludes that, as incredible as it may seem, we really have no justification for extending our meanings in the way we do, if we consider ourselves in isolation. He claims this is "the most radical form of scepticism philosophy has seen to date", which in 1962-3 struck him "with the force of a revelation" [Kripke 1982, p. 1].

Kripke considers and rejects a number of purported solutions to his "paradox". I will discuss two. The first is that *our 'adding dispositions' determine how we should go on*. Just as delicate crystal has the disposition to shatter if knocked, we have the disposition to produce "125" as the answer to "68 + 57". This is often offered as a 'naturalistic' answer to the rule-following problem. But the problem with this answer is that our dispositions are *finite* (as we are finite creatures), while our mathematical practice is potentially *infinite*. So

[8] This section is adapted from section 1 in [Legg 2003].

the answer seems incomplete at best. A further problem with 'naturalistic' solutions to the paradox is that what we want is not a *descriptive* theory of what we *do* do, but a *normative* theory of what we *should* do (which, ideally, gives us the reason why we should do that thing). We also want to leave room for it to be true that we make mistakes in mathematics. Wittgenstein broaches this issue by talking about how we think of rules in terms of a *"machine-as-symbol*". This is a confused picture, he suggests, because for instance we imagine that the machine-as-symbol (unlike a real machine) "never breaks down" [Wittgenstein 1953, §193].

The second purported solution to the rule-following problem I shall discuss is that *"a special feeling" determines how we should go on*. Kripke gently mocks this idea:

> [S]uppose I do in fact feel a certain headache with a very special quality whenever I think of the '+' sign. How on earth would this headache help me figure out whether I ought to answer '125' or '5' when asked about '68 + 57'?... If there were a special experience of 'meaning' addition by 'plus', analogous to a headache, it would not have the properties that a state of meaning addition by 'plus' ought to have – it would not tell me what to do in new cases [Kripke 1982, pp. 41-43].

We shall see that it is a pity Kripke does not give this response more serious consideration. Kripke concludes that, crazy as it seems, his rule-following paradox cannot be solved. He offers a merely "skeptical" solution to it which replaces *truth conditions* for rules such as '+2' with *justification* or *assertability conditions,* as follows:

> All that is needed to legitimize assertions that someone means something is that there be roughly specifiable circumstances under which they are legitimately assertible, and that the game of asserting them under such conditions has a role in our lives. No supposition that 'facts correspond' to those assertions is needed [Kripke 1982, pp. 77-78].

Thus, there is no further fact about what a rule means than what my language-using community happens to do. If we follow this conclusion into mathematics, we must infer that there is no mathematical *truth* over and above what mathematicians currently assent to, and that mathematical *meaning* does not outrun what mathematicians currently understand. This seems so dreadfully conservative that surely it serves as a *reductio ad absurdum* of whatever philosophical arguments led to it.

3. A Peircean Approach to the Rule-Following Problem[9]

The rule-following problem has perplexed philosophers and generated a vast literature attempting to 'solve the problem'. From the perspective of Peirce's philosophy, I believe the problem derives initially from conflating what are in fact two separate questions:

1) What makes me able to project rules into new cases? (*How do I go on*?)

2) What makes my projection into new cases right? (*How do I go on correctly?*)

The fact that the rule-following problem is introduced using a mathematical example arguably obscures this distinction very unhelpfully, as the relationship between merely applying concepts and determining their correct application is especially close there. But speaking generally, why *should* a person's grasp of a concept deliver infallible insight into the truth or otherwise of all sentences containing it? Presenting a Peircean solution to the rule-following problem will give us a different – and I believe richer – understanding of the nature of mathematics and its characteristic necessity than our picture of "rules as rails".

To approach the problem from the Peircean direction, we need to take a brief look at his three philosophical categories, or *'modes of being'*. These offer a direct challenge to Quine's view, now dominant in analytic philosophy, that there is just one mode of being, existence, and *"to be is to be the value of a bound variable…"* (in our best scientific theory). Why does Peirce claim that his fundamental categories are three? We shall begin by asking: *What is the opposite of the particular?* This is usually thought to be "the general", or universal – thus reality is thought to be composed of particular entities instantiating universal entities, and a certain hylomorphism has descended through Western metaphysics since Aristotle. Yet if we identify the particular with the Actual (discrete entities that exist and engage in efficient causal interaction, which Peirce calls 'Secondness' because existential interaction requires two entities), Peirce notes that it is properly counterposed by both the Possible and the Necessary. The Possible, Peirce calls Firstness because it represents the radically new – what can appear on its own and is indescribable in any other terms, and is in that sense monadic. A key example of Firstness is *feeling* in the sense of irreducible *qualia* such as the taste of pineapple. Meanwhile Peirce

[9] This section is adapted from section 2 in [Legg, 2003].

gives the name Thirdness to what exemplifies a kind of 'meaning necessity' whereby reality is mediated by concepts that render it intelligible, such that if we understand the concept 'white', we know that certain other things *must* also receive the label. This is triadic because *mediation* requires two entities to be 'bridged' by a third. Such three-way relationships occur in both physical and mental 'worlds', and thus signs (which have both a material and a logical side) form part of a broad evolutionary system which develops according to final causation. All of this raises the intriguing idea that to treat generality as univocal has been a profound mistake of Western philosophy since Plato. Such, then, are Peirce's categories, and they appear – together and irreducible to one another – in every area of thought (Zalamea identifies them in mathematics itself in an intriguing manner as 'eidal', 'quidditral' and 'archeal' respectively [Zalamea 2012b, p. 174]).

Kripke suggests that we have real mathematical rules if the justification that the rule provides for going on in a certain way can be shown to be somehow reducible to particular existent entities (which Kripke calls "facts"). This can be seen as understanding realism in terms of real Secondness only. Peirce on the other hand argues that we have real Thirdness precisely if general concepts are not reducible to particular things:

> None of the scholastic logics fails to explain that *sol* is a general term; because although there happens to be but one sun yet the term *sol aptum natum est dici de multis*. But that is most inadequately expressed…the idea of a general involves the idea of possible variations which no multitude of existent things could exhaust [CP 5.103].

Yet Kripke's sceptic (and many analytic philosophers) will ask – what *metaphysics* lies behind such an understanding? Surely something 'profligate', non-naturalistic? Well perhaps there is no metaphysics to 'ground' our ability to project general concepts, because no metaphysics is required. We would not perform experiments in physics to try to observe and study the derivative (*dy/dx*). This would be a hopeless mistake, because the concept belongs to mathematics, and physics presupposes mathematics. So *why, then, should we search in metaphysics for normativity?*

Peirce's *Hierarchy of the Sciences* – which arranges disciplines in a partial ordering such that they draw principles from sciences above them in the hierarchy and data from those below – is sometimes viewed as a quaint relic of 19th century thinking. But it is actually a big help in detangling the intimidating knot Kripke tied with his presentation of the rule-following problem. *Fig. 2* shows the hierarchy's top levels, in the 1903 version described by Beverley Kent as "perennial" [Kent 1987].

1. MATHEMATICS
2. PHILOSOPHY
   - 2.1 PHENOMENOLOGY
   - 2.2 NORMATIVE SCIENCES
     - 2.2.1 AESTHETICS
     - 2.2.2 ETHICS
     - 2.2.3 LOGIC
   - 2.3 METAPHYSICS

Figure 2
Peirce's 1903 hierarchy of the sciences (top 3 levels)

Let us turn back to the rule-following problem in the light of this arrangement.

3.1 The Rule-Following Problem in Phenomenology.

We see in *Fig. 2* that for Peirce phenomenology precedes logic. Its tools are merely individual observation, as long as it is candid. This approach might seem epistemologically somewhat lax, but when pursuing phenomenology we do not yet have access to a concept of truth. (We do not yet even have propositions.) Phenomenology's methods are to derive fundamental concepts and explore relations of reducibility between them. As 'first' in the three branches of philosophy, phenomenology manifests Firstness, hence the singularity (disconnectedness) of the concepts it explores.

This science is where we explicate rule-following *projectibility* which, as noted, corresponds to how we 'go on', and here all we have to do is notice how things appear to us. We then note that meanings *are* projectible, undeniably – this is simply part of what it is to mean something. Thus Peirce's phenomenological treatment of this issue shows that at least part of the rule-following problem was lurking where Kripke least expected. There *is* a "special feeling" which guides us in rule-following. Concepts *do* each have a special irreducible quality that allows us to reidentify them in applications beyond the finite set of instances with which we have been acquainted. The *Firstness of Thirdness* is a Peircean way to describe this quality. In certain places in Wittgenstein's discussion of rule-following, he seems aware of our having access to such phenomenological resources:

> One does not feel that one has always got to wait upon the nod (the whisper) of the rule…we are not on tenterhooks about what it will tell us next [Wittgenstein 1953, §223].

3.2 The Rule-Following Problem in Logic.

But eventually we have to move beyond merely noting that we 'feel like' extending rules in a particular way, to considering to what extent we are *correct* to do so, which leads us to the science of *logic*. Here Peirce defines the concept of truth in terms of the community of inquiry and the method of science (even in mathematics, which for Peirce consists in 'experimenting on diagrams', as we shall see). After establishing this epistemological framework, we can begin to construct specific propositions (such as 2+2=4) and test them, while of course always keeping a fallibilist attitude.

As noted, it is Kripke's running together of phenomenology and logic that makes the rule-following problem seem hopelessly unsolveable. Only phenomenology can teach us that the 'third' mode of being is irreducible to the other two. But only logic can show us how to determine real from unreal thirdness. No single science can do both of those things. In short, Peirce would likely disagree with Kripke's claim that there is no "straight" solution to the rule-following problem. Mathematical truth is *objective,* although mathematical objects do not *exist* – an alternative that is missed in much contemporary analytic philosophy.[10]

4. Challenge to Basic Empiricism I: Knowledge of Necessity as Knowledge of Structure[11]

Yet the sensitivity to the Firstness of Thirdness that we[12] have is a truly remarkable capacity. Making space for the Firstness of Thirdness would require mainstream philosophy to reconceive perception, and, following that, epistemology, in profound ways.

[10] Most notably in [Benacerraf 1973] and the significant amount of work it has inspired.
[11] This section is adapted from [Legg 2011].
[12] Who is "we" here? Arguably – *creatures who live in the space of reasons.*

For example, we need to escape a certain *basic empiricism*, which is greatly taken for granted, and may be summarised something like this:

> **Basic Empiricism:** Genuine (i.e. synthetic) knowledge consists in nothing but a record of existing objects interacted with causally by the senses.

By contrast, in a recent paper on Peircean modal epistemology, I argued:

> Structural articulation is the source of all knowledge of necessity. Necessary reasoning is in essence just a recognition that a certain structure has the particular structure that it in fact has [Legg 2012, p. 1].

There I noted a curious feature possessed by the words "in fact" in this formulation. This is not a *metaphysical* but an *epistemological* "in fact". It is "…merely meant to invoke rhetorically a certain moment of recognition", where "what is recognised has no content over and above what is represented by the structures that are already present, but nevertheless constitutes some kind of new insight" [Legg 2012, p. 1]. But how is it possible to gain a new insight without seeing any new object? We here turn to Peirce's insights into *hypostatic abstraction*, as a move in logic.

5. Challenge to Basic Empiricism II: Hypostatic Abstraction[13]

Peirce defines hypostatic abstraction as a fact considered as a substantive. A simple example is the passage from "This is green" to "There is greenness in this thing" [Short 1988, 1997]. As a logical move, hypostatic abstraction was famously derided by early moderns as the epitome of pseudo-explanation, following a passage in Molière's play *Malade Imaginaire* which satirises scholastic philosophy. In this passage a candidate for a degree in medicine is asked during his examination why opium makes people sleepy. He replies that it's because the opium has a *dormitive virtue*, to loud acclaim from his examiners, and laughter from the audience.

[13] This section is adapted from [Legg 1999].

This diagnosis of comic uselessness is accepted largely without question by 20th century analytic philosophers. Thus Quine wrote, "The evil of the idea idea is that its use, like the appeal in Molière to a *virtus dormitiva*, engenders an illusion of having explained something" [Quine 1953, p. 48]. Yet Peirce had another view:

> You remember the old satire which represents one of the old school of medical men...who asked why opium puts people to sleep answers very sapiently "because it has a dormitive virtue."…It is a poignant satire, because everybody is supposed to know well enough that this transformation from a *concrete predicate* to an abstract noun in an oblique case, is a mere transformation of language that leaves the thought absolutely untouched.
> I knew this as well as everybody else until I had arrived at that point in my analysis of the reasoning of mathematics where I found that this despised juggle of abstraction is an essential part of almost every really helpful step in mathematics [NEM 4.160] (cited in [Zeman 1982]).

Peirce refers to hypostatic abstraction as "a necessary inference whose conclusion refers to a subject not referred to by the premiss" [CP 4.463]. Such a thing seems extraordinary, logically! Yet let us examine our example as an argument:

(**Premise**) Sleepiness accompanies opium-taking across a wide variety of possible circumstances.

(**Conclusion**) There is something about opium which has a soporific effect.

If the premise really is true, the conclusion arguably follows. Yet the conclusion does mention an object not mentioned in the premise ("something about opium…"). This object is extremely vague, but is *no less new for all that*. This is possible because the hypostatic abstraction is an *ens rationis*, which simply means that its being consists in something else's being true.

How does hypostatic abstraction play an important role in mathematics? Consider the following "picture proof" (*Fig. 3*):

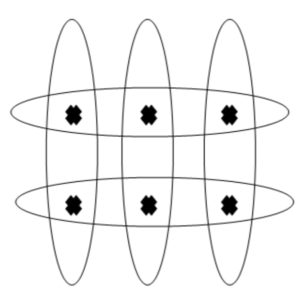

Figure 3
Proof that 3x2=2x3

We may see this diagram as a proof of a number of truths in arithmetic, such as 3x2=6, and 2x3=6. But the diagram might also cause one to notice a certain feature of all multiplication relationships – that they may be reversed as above. One might then give that feature a name: *commutativity*. This kind of thinking is a lodestone of creativity in mathematics.

Note what just happened. A new mathematical insight was gained ("There is such a thing as commutativity") without the mind being causally influenced by any new objects. Here rationalism outruns empiricism. As McDowell would put it: our knowing minds exhibit *spontaneity* as well as *receptivity* [McDowell 1994]. Or, as Zalamea puts it:

> Peirce's architectonics postulates a "dialectics" between indeterminations and determination, opposing processes of progressive determination … to the constant appearance of elements of indetermination and chance ("tychism") that periodically *free* the signs from their sedimentary semantic load [Zalamea 2012a, pp. 62-63].

But how do we know that this new insight is *true*? Again we need to heed the boundary between the sciences of phenomenology and logic, and where we stand at any given time. The initial hypostatic abstraction is merely something like: "3x2=2x3 exhibits commutativity". This is merely name-giving (again: "a fact viewed as a substantive"). But extending the insight further (e.g. "All multiplication is commutative") will require real research in the mathematical community of inquiry.

The simple picture of mathematical understanding as "rules as rails" is now undone, giving us a small glimpse of the profound creativity which drives the discipline. We can now see that hypostatic abstraction (or, as one might express it, *"thinking of new ways to go on"*) has a strong aesthetic dimension of careful noticing the appearance of what is before one. Peirce would assimilate this to his category of Firstness. Contemporary understandings of *reason* seem to see it as almost coextensive with algorithmic deductive proof (possibly overly influenced by recent research in AI). But an older understanding exists in Philosophy whereby it was precisely rationalist philosophers who invoked such notions as intuition, and clear and distinct perception.

Peirce comments insightfully on this aesthetic dimension to mathematical discovery:

> It has long been a puzzle how it could be that, on the one hand, mathematics is purely deductive in its nature, and draws its conclusions apodictically, while on the other hand, it presents as rich and apparently unending a series of surprising discoveries as any observational science. The truth ... appears to be that all deductive reasoning, even simple syllogism, involves an element of observation; namely, deduction consists in constructing an

> icon or diagram the relations of whose parts shall present a complete analogy with those of the parts of the object of reasoning, of experimenting upon this image in the imagination, and of observing the result so as to discover unnoticed and hidden relations among the parts [Peirce CP 3.363][14].

This necessary element of creative observation points the way to a further profound difference from the rules as rails model. In an important sense, mathematics is *not epistemologically transparent to itself*, but rather full of surprises.[15] Although mathematics has the reputation of being the most rational of all sciences, ironically, from another view ('looking inside our heads') it is one of the most mysterious of all, as Peirce noted in a number of places:

> The action of Nature is a wonder to us; but that of Reason is not usually so…We seem to comprehend Reason. We flatter ourselves that we grasp its very *noumenon*. But it is really as occult as Nature. It is only because its effects are for the most part familiar to us from infancy that they are not surprising [Peirce 2010, p. 51: "On the Logic of Quantity" 1895].

> ... mathematics brings to light results as truly occult and unexpected as those of chemistry; only they are results dependent upon the action of reason in the depths of our own consciousness, instead of being dependent, like those of chemistry, upon the action of Cosmical Reason, or Law [CP 6.594].

Peirce argued that understanding hypostatic abstraction is particularly important for analysing the logic of mathematics because mathematics is particularly prone to cannibalising itself by taking its own predicates as objects of investigation (performing "operations on operations"). So it is that abstraction reaches heights in mathematics undreamed of in the physical sciences. Quine thought that the only question raised by hypostasis in the philosophy of mathematics was whether to be ontologically committed to classes![16] Thus preoccupied by metaphysics, he completely missed the *logical* question raised by hypostatic abstraction – namely what is its unique and vital role in working inference? And moreover, challenging questions such as *why such a form of inference should be so incredibly useful*.[17]

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[14] For a valuable extended discussion of these matters, see [Stjernfelt 2007].

[15] Zalamea has noted that when contemporary mathematicians are asked to predict how their discipline will look in the next few decades, they typically claim to have absolutely no idea [Zalamea 2012b].

[16] E.g. [Quine 1953, pp. 121-127].

[17] This paragraph is taken from [Legg 1999, p. 671].

6. Mathematics as Learning about Reality

It might here be protested that as we have admitted that hypostatic abstractions are *entia rationis*, they are not real. But this would be to miss just what is distinctive about Peirce's realism: his commitment to the reality of Thirdness. When Peirce looks to the pragmaticist maxim for a clarification of the meaning of *real*, he finds that something is real if "all the practical consequences of it are true". He then notes that the very definition of a hypostatic abstraction is of something whose being consists in the truth of another fact. This in fact *guarantees* the reality of the hypostatic abstraction:

> On pragmatistic principles *reality* can mean nothing except the *truth* of statements in which the real thing is asserted. To say that opium has a dormitive virtue means nothing and can have no practical consequences except what are involved in the statement that there is some circumstance connected with opium that explains its putting people to sleep... Indeed, nobody but a metaphysician would dream of denying that opium *really* has a dormitive virtue [NEM 4.162].

We need to get past the "tyranny of existence" in philosophy. Basic empiricism claims there is no knowledge except that given by *prior causal interaction with existent objects*. This model has been pervasive through so-called "modern" philosophy. It easily generalises to a demand that no belief be accepted except on sufficient prior warrant – nowadays frequently referred to as evidentialism. Peirce suggests a whole new model of knowledge, based on what we might call future-directed justification, which draws on the Biblical maxim, "By their fruits shall ye know them". Rather than attempting to determine prior reasons to hold a belief, this model looks for *fruitful abductions*.

And now we come to a key point: *A hypostatic abstraction is a doorway to abduction*. Consider our derivation of the concept of commutativity. Once imagined, this new concept becomes an organizing pivot for a host of further explanations in mathematics. Peirce notes that abduction is a form of thought that characteristically argues *from part to whole of a system*. He also notes that this is the way mathematical thought grows:

> ...a mathematician often finds what a chess player might call a gambit to his advantage; exchanging a smaller problem that involves exceptions for a larger one free from them. Thus, rather than suppose that parallel lines, unlike all other pairs of straight lines in a plane, never meet, he supposes that they intersect at infinity. Rather than suppose that some equations have roots while others have not, he supplements real quantity by the infinitely greater realm of imaginary quantity... [Peirce 2010, p. 31: "Minute Logic" 1902].

The systematic character of mathematical thinking is also unaccounted for by the "rules as rails" model. Zalamea expresses this radical turnaround in epistemology towards future-directed justification beautifully in his bold and creative rereading of Murphey's famously dismissive "castle in the air" metaphor, which Murphey used to deprecate Peirce's use of the mathematical concept of continuity as the lynchpin of his philosophy:

> ...explanation is only really needed and appropriately lodged when it goes beyond particulars and then fuses into the general (the continuum)...Thus, the logic of abduction becomes in fact one of the basic supports of Peirce's pragmaticist architectonics and general synechism. Abduction serves as a regulatory system for the Real, for that plastic weaving (third) formed by facts (seconds) and hypotheses (firsts) where hypotheses are subject to complexity tests until they continuously fuse with facts … we hope to have been able to show that Peirce's "castle" – very *real*, but not reducible to existence – is far from just flying in the air [Zalamea 2012a, pp. 101-103].

Conclusion

It is tempting to say that there are moments in mathematics where the talented mathematician "goes *off* the rails", and sees a *different, yet equally necessary* way to go on in a given mathematical diagram. Such moments have been assumed to be 'where genius is touched by Heaven' (so to say), and thus ineffable (following the Romantic tradition). We have seen in this paper how the "rules as rails" model suggests that mathematical results are predetermined by the very definition of mathematical terms, and that insofar as this leaves no room for creativity in mathematics, the model is false. Yet where Romanticism counterposes against the idea of rigidly determined rules, which we might call the Secondness of Thirdness, a kind of pure poesis which we might call Absolute Firstness, Peirce is much more rigorous and ambitious than this. He plans to open the creative moment to logical modelling – a modelling which constructs none of his philosophical categories as opposites but rather brings all three together in structured insight.

To put the same point another way, we might go beyond merely rejecting rules as the sole model for mathematical epistemology, to criticise the whole idea of a 'mathematical rule' as incoherent insofar as it draws on a simple binarism that when one contemplates the true richness of mathematical reality cannot be sustained. Zalamea discusses how philosophers' unthinking reliance on classical logic and set theory has led philosophers of mathematics to an "illusion of precision" [Zalamea 2012b, p. 87] – that any object *x* either has property *P* or it does not, and correspondingly, the application of the rule which defines *P* to *x* is either right or wrong. Philosophers have even imagined that such clear boundaries are symptomatic of exact thought. Zalamea suggests that nothing could be further from the truth, and shows exactly how the naïve assumption is exploded by the work of contemporary mathematicians, in that such binary structures form only one small corner of a much larger realm of possible structures which it is the mathematician's task to explore (e.g. [Zalamea 2012b, p. 101, p. 177n]).

Once this is understood, we can uncover a wealth of concepts developed recently in mathematics that might be used to lead philosophers' discussions of ontology beyond classical logic and set theory. We might view ontology for instance as "a sophisticated sheaf of methods and constructions for systematic explorations of the transitory" [Zalamea 2012b, p. 90]. Then, as we witness new deep and enigmatic concepts, such as 'gluings', 'fibers', 'coverings', 'decantation of reality', emerging as Firstnesses at the phenomenological edge of mathematics, we might compare them with the metaphors birthed by our previous assumptions – such as 'atomism', 'dualism', 'foundation', [Zalamea 2012b, p. 343]. We might then decide for ourselves which concepts we might marshal in the discipline of philosophy to further explore our own real Thirdness.

REFERENCES.

[Balaguer 1998] M. Balaguer, *Platonism and Anti-Platonism in Mathematics*, Oxford: Oxford University Press, 1998.

[Benacerraf 1973] P. Benacerraf, "Mathematical Truth", *Journal of Philosophy* **70** (1973), pp. 661-679.

[Havenel 2008] J. Havenel, "Peirce's Clarifications of Continuity", *Transactions of the Charles S. Peirce Society* **44(1)** (2008), pp. 86-133.

[Kent 1987] B. Kent, *Charles S. Peirce: Logic and the Classification of the Sciences*, Montreal: McGill-Queens University Press, 1987.

[Kripke 1982] S. Kripke, *Wittgenstein on Rules and Private Language*, Cambridge MA: Harvard University Press, 1982.

[Legg & Franklin 2015] C. Legg and J. Franklin, "Perceiving Necessity", *Pacific Philosophical Quarterly* (2015, online).

[Legg 1999] C. Legg, "Extension, Intension and Dormitive Virtue", *Transactions of the Charles S. Peirce Society* **35(4)** (1999), pp. 654-77.

[Legg 2003] C. Legg, "This is Simply What I Do", *Philosophy and Phenomenological Research* **46(1)** (2003), pp. 58-80.

[Legg 2012] C. Legg, "The Hardness of the Iconic Must: Can Peirce's Existential Graphs Assist Modal Epistemology?", *Philosophia Mathematica* **20(1)** (2012), pp. 1-24.

[Legg 2014] C. Legg, "Necessary Connections Between Distinct Existences: A Peircean Challenge to Humeanism", *Cognitio* **15(1)** (2014), pp. 89-112.

[Maddalena & Zalamea 2009] G. Maddalena and F. Zalamea, "A new analytic/synthetic/horotic paradigm: from mathematical gesture to synthetic/horotic reasoning", preprint, *Kurt Gödel Research Prize Fellowships*, 2009.

[McDowell 1994] J. McDowell, *Mind and World*, Cambridge: Harvard University Press, 1994.

[Morgan 1979] C. Morgan, "Modality, Analogy, and Ideal Experiments according to C.S. Peirce", *Synthese* **41** (1979), pp. 65-83.

[Peirce 2010] C. S. Peirce, *Philosophy of Mathematics* (ed. Moore), Bloomington: Indiana University Press, 2010.

[Quine 1953] W.V.O. Quine, *From a Logical Point of View*, Cambridge MA: Harvard University Press, 1953.

[Short 1988] T.L. Short, "Hypostatic Abstraction in Physical Science", *Grazer Philosophische Studien* **32** (1988), pp. 51-68.

[Short 1997] T.L. Short, "Hypostatic Abstraction in Self-Consciousness", in *The Rule of Reason* (eds. Brunning and Forster), Toronto: University of Toronto Press, 1997, pp. 289-308.

[Stjernfelt 2007] F. Stjernfelt, *Diagrammatology: An Investigation on the Borderlines of Phenomenology, Ontology, and Semiotics*, Dordrecht: Springer, 2007.

[Wittgenstein 1953] L. Wittgenstein, *Philosophical Investigations* (trs. G.E.M. Anscombe), Oxford: Basil Blackwell, 1953.

[Wright 1989] C. Wright, "Critical Notice of *Wittgenstein on Meaning* by Colin McGinn", *Mind* **98(390)** (1989), pp. 289-305.

[Wright 2007] C. Wright, "Rule-Following without Reasons: Wittgenstein's Quietism and the Constitutive Question", *Ratio* **20(4)** (2007), pp. 481-502.

[Zalamea 2003] F. Zalamea, "Peirce's Logic of Continuity: Existential Graphs and Non-Cantorian continuum." *The Review of Modern Logic* **9.1(2)** (2003), pp. 115-162.

[Zalamea 2012a] F. Zalamea, *Peirce's Logic of Continuity: A Conceptual and Mathematical Approach*, Boston: Docent Press, 2012.

[Zalamea 2012b] F. Zalamea, *Synthetic Philosophy of Contemporary Mathematics*, New York: Sequence Press, 2012.

[Zeman 1982] J. J. Zeman, "Peirce on Abstraction", *The Monist* **65(2)** (1982), pp. 211-229.