Mathematical Analogies in Physics: the Curious Case of Gauge Symmetries*

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Abstract Gauge symmetries provide one of the most puzzling examples of the applicability of mathematics in physics. The presented work focuses on the role of analogical reasoning in the gauge argument, motivated by Mark Steiner's claim that the application of the gauge principle relies on a Pythagorean analogy whose success undermines naturalist philosophy. In this paper, we present two different views concerning the analogy between gravity, electromagnetism, and nuclear interactions, each providing a different philosophical response to the problem of the applicability of mathematics in the natural sciences. The first is based on an account of Weyl's original work, which first gave rise to the gauge principle. Drawing on his later philosophical writings, we develop an idelaist reading of the mathematical analogies in the gauge argument. On this view, mathematical analogies serve to ensure a conceptual harmony in our scientific account of nature. We further discuss the construction of Yang and Mills's gauge theory in light of this idealist reading. The second account presents a naturalist alternative, formulated in terms of John Norton's account of a material analogy, according to which the analogy succeeds in virtue of a physical similarity between the different interactions. This account is based on the methodological equivalence principle, a simple conceptual extension of the gauge principle that allows us to understand the relation between coordinate transformations and gravity as a manifestation of the same method. The physical similarity between the different cases is based on attributing the success of this method to the dependence of the coupling on relational physical quantities. We conclude by reflecting on the advantages and limits of the idealist, naturalist, and anthropocentric

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Pythagorean views, as three alternative ways to understand the puzzling relation between mathematics and physics.

1 Introduction

The relationship between theoretical physics and mathematics underwent a dramatic change during the 20th century. The mathematical machinery employed by physicists became on the one hand more abstract and detached from the observed phenomena, and on the other hand more strongly intertwined with scientific thought about the world. The language of mathematics, it seemed to many, became the only language through which some central aspects of physical reality can be appropriately described. A common view regards this process as a step forward in the formulation of a scientific worldview that is indifferent to the human perspective. Mark Steiner's *Applicability* (Steiner, 1998) forms a major attack on this naturalist view. In contrast to the Galilean-Newtonian revolution, identified with a 'revolt against anthropocentrism' (p. 56), Steiner argues that the mathematical methodology of modern physics is based on an implicit anthropocentric belief.

Steiner's starting point is the sense of wonder expressed by many physicists with respect to the successful applicability of mathematics in physics. He presents a detailed and original analysis of the kinds of mathematical reasoning that lay at the heart of some of the most significant discoveries in modern physics. Steiner concludes that from a naturalist point of view, physicists are right when they regard the effectiveness of mathematics as a miracle; the process of formulating the laws governing our most successful theories has often been based on purely mathematical considerations—considerations that do not seem to have any physical justification.

What makes the applicability of mathematics in physics particularly remarkable according to Steiner is the role that mathematics plays in scientific discovery. Here, physicists often appeal to mathematics to guess the laws that apply to an unknown and not directly observable domain.[2] Steiner provides a detailed analysis of various examples of such reasoning, and suggests that they are best understood as an appeal to analogy. He argues that the analogies underwriting much of twentieth century physics were explicitly Pythagorean in nature—i.e. analogies that cannot be expressed in any language other than mathematics. On Steiner's view, the problem for the naturalist is that this form of reasoning is inherently anthropocentric. Mathematics is an endeavour guided by human values, defined by an appeal to simplicity, beauty, and convenience, which are all anthropocentric notions. It is this anthropocentric grounding that makes the success of such considerations seem miraculous; the world, according to Steiner (1998, p. 176), appears to be "user friendly".

Steiner's argument is based on a distinction that he draws between mathematical reasoning, which concerns relations between abstract concepts, and physical reasoning, which concerns relations between objects in the world. In an appeal to

[2] Steiner's analysis focuses on reasoning that is non-deductive, i.e. one that is expressed in mathematical language, but does not rely solely on logical inference.

Pythagorean analogies, Steiner argues, physicists apply mathematical reasoning to discover (i.e. successfully guess) facts about the world—a move for which there can be no naturalist justification.

In evaluating Steiner's argument, the question is not only whether his unique form of anthropocentric Pythagoreanism is compelling, but whether the examples taken to support this claim resist an alternative reading. Broadly speaking, there are two possible strategies for presenting an alternative reading of the applicability of mathematics in physics. One is to deny the anthropocentric view of mathematics, and regard it (or at least the mathematics used in physics) as something that has been shaped by the standards of natural science itself. The other is to accept the anthropocentric view of mathematics, and hold the same view of physics, at least to a certain extent. The mathematical description of nature, according to this strategy, does not reflect the way nature is, but rather our way of understanding it. These strategies can be identified with two different schools of philosophical thought: the first would generally aim to deny (or at least to minimize) the significance of *a priori* principles. The second would regard such principles as fundamental to human thought (mathematics and science included). We shall correspondingly refer to the two strategies as naturalist and idealist, respectively.

In this paper, we will look to develop each of these alternatives responses. In particular, we will do so within the context of Steiner's strongest example of Pythagorean reasoning in physics: a famous symmetry principle known as the gauge argument. We will present two different readings of the gauge argument, contrasting them with each other and with Steiner's account. The first follows the idealist strategy, the second follows the naturalist one.

In the development of modern physics, the gauge argument became the primary tool for formulating the laws of fundamental physical interactions. The idea became a central methodological principle in physics following the work of Yang and Mills (1954) who suggested a theory of nuclear interaction by drawing an explicit analogy to the structure of the electromagnetic interaction. Yang and Mills's gauge argument is based on the requirement that every global symmetry of the dynamical law should also hold as a local (i.e. spacetime dependent) symmetry.[3] The long familiar local gauge invariance of electromagnetism, commonly understood as a reflection of a redundancy in the mathematical description of each physically possible state, became the template for all of the fundamental interactions in the standard model of particle physics. However, the gauge argument actually originated much earlier in Weyl's 1918 attempt to provide a unified understanding of electromagnetism and gravity (as described by Einstein's general theory of relativity). The history and methodology of the gauge argument is thus rooted in an analogy between the mathematical descriptions of gravity, electromagnetism, and the nuclear interactions. In his [1989], Steiner singled out the formal nature of the analogical reasoning that gave rise to Yang-Mills theory as an anomaly requiring philosophical attention, and later in the *Applicability* [1998] he portrayed this reasoning as the quintessential Pythagorean analogy.

[3] Here, note that a symmetry of a dynamical law is a transformation of the values of the variables under which the form of the equation expressing the law remains invariant.

Steiner's analysis of the gauge argument anticipated a series of philosophical debates concerning the significance and meaning of gauge symmetries. The central issue in these debates is the invocation of considerations that appear to be devoid of physical content, as a gauge symmetry is commonly understood to be a mere reflection of a mathematical redundancy in our representation of nature. However, the gauge argument has served as a basis for the formulation of novel and successful physical theories, and the generation of new knowledge. This tension between the descriptive redundancy at the core of the gauge argument and its foundational significance presents us with a deep philosophical puzzle concerning the nature of modern gauge theories. While this puzzle mirrors Steiner's challenge concerning the applicability of mathematics in the gauge argument, little attention has been given to the connection between these issues. One of the aims of this paper is to try to draw these two strands of philosophical discussion together.

The two approaches presented in this paper form two alternative ways of understanding the gauge argument as a form of analogical reasoning. The first provides an idealist response to Steiner's argument, based on Weyl's original idealist justification of the gauge methodology. According to this approach, the mathematical analogies underwriting gauge theory serve to extend the constitutive principles of a theoretical framework for physical inquiry. In drawing a mathematical analogy, we do not highlight an existing similarity, but rather construct it. The second account is a naturalist alternative that portrays the analogical aspects of the different interactions as a manifestation of a shared physical property. This property is rooted in an ontology based on relational quantities (in agreement with the recent portrayal of gauge symmetries by Rovelli, 2014). On this view, the dominance of analogical reasoning does present, *prima facie* at least, a particular challenge to the naturalist; a challenge that the naturalist can best confront by adopting a piecemeal approach, reflecting on the methodology of a given successful analogy in an attempt to show that its success can be grounded in physical properties of the world.

In what follows, we will begin by presenting Steiner's general problem (Section 2) and the gauge argument, together with the philosophical problems it invokes (Section 3). We then turn to present the two alternatives. The idealist understanding of gauge (Section 4) and a naturalist one (Section 5). We will then provide a general discussion of the limits of each of the presented approaches and the extent to which they resolve Steiner's problem.

2 Mathematical Analogies in Physics

Analogical reasoning has always been an essential part of physics, and has undoubtedly contributed to the formulation of a number of physical theories throughout the history of science. Joseph Priestley (1775), for example, appealed to an analogy between gravitation and electricity to formulate an inverse square law for the electrostatic force. William Thomson drew an analogy between the flow of heat and electromagnetism, and used it to derive a set of novel dynamical equations governing electromagnetic phenomena (see Hon and Goldstein (2020) chapter 3). These examples already present cases of analogical inference from one physical domain to another based on a conjectured similarity in their physical structure. This similarity was taken to underwrite the postulation of a shared formal (i.e. mathematical) representation.

However, with Maxwell the method of reasoning through mathematical analogy was greatly extended to treat not only physical but also imagined systems. Famously, Maxwell appealed to an analogy with an imaginary incompressible fluid to provide a mathematical account of Faraday's lines of force. He referred to this analogy between an imaginary and a physical system as a "mathematical analogy", contrasting it with Thomson's "physical analogy". Of course, in both cases one typically appeals to an analogy to draw a mathematical inference. The key difference is that in the case of a physical analogy it is the observable similarity between the physical domains that supports the inference, and in the case of a mathematical analogy it is a conjectured similarity between a fictional and a physical domain.

According to Hon and Goldstein (2012, 2020) Maxwell's methodology of mathematical analogy was one of the first instances of the practice of scientific modelling. Steiner (1998, p. 78) also uses the term 'model' in his account of Maxwell's reasoning. However, he describes the development of Maxwell's thought as a process in which "the model gradually lost its appeal", such that only the mathematics remained. Steiner concludes that "Maxwell's reasoning was Pythagorean. By this I mean that once he had a mathematical structure which described many different phenomena of electricity and magnetism, the mathematical structure itself, rather than anything underlying it, defined the analogy between the different phenomena." This new Pythagorean strategy is thus based on drawing analogies between fictional mathematical concepts associated with different physical domains.

This strategy, according to Steiner, became increasingly significant in theoretical physics with the advent of quantum theory. Physicists were confronted with the prospect of describing an imperceptible quantum reality, one that appeared to obey laws that were drastically different from those of classical theories. As a result, Steiner argues, physicists began to reason by mathematical analogy, i.e. to look for laws that were *mathematically* similar to the classical ones they were trying to replace.

In the context of quantum theory, Steiner suggests that the analogies that guided the development of the theory were neither 'physical' nor simply 'mathematical', they were *Pythagorean*—i.e. not expressible outside the language of mathematics. In contrast to Priestley's and Thomson's analogies, here there was nothing but the conjectured mathematical similarity to justify the analogy. It is mathematics itself that "provided the framework for guessing the laws of the atomic world" (1998, p. 4). The construction of Schrödinger's equation, the basic law of quantum mechanics, is one prominent example of this strategy, as it was based on an analogy between a complex function (conjectured to describe material particles) and the mathematical properties of waves.

The problem, as Steiner sees it, is that this form of reasoning is inherently anthropocentric. The naturalist has no reason to think that purely mathematical reasoning could serve as the basis for identifying physical similarities. Yet, this Pythagorean strategy has been incredibly successful. The central thesis of the *Applicability* is that naturalism is inconsistent with this appeal to Pythagorean analogies.

In addition, Steiner also notes that the Pythagorean strategy further blurs the border between the physical and the mathematical in a deeper sense. Physicists often assume that a mathematical fact corresponds to an actual physical possibility. For example, it is a mathematical fact that (provided certain conditions) the product space of a vector space of dimension two and a vector space of dimension three is a vector space that is isomorphic to the sum space of a vector space of dimension two and a vector space of dimension four. From this mathematical fact, according to Steiner's analysis, physicists conclude that there is a possible physical process in which two particles, a pion (whose isospin space is three-dimensional) and a nucleon (whose isospin space is two-dimensional), are transformed into a quantum superposition of a Lambda baryon (whose isospin space is four-dimensional) and a nucleon.

The reasoning here, according to Steiner (p. 90), is an appeal to the "Pythagorean hypothesis", according to which mathematical equivalence *is* physical equivalence. The relation here is not between two physical systems, but rather directly between mathematical and physical facts. Thus, while the main focus of *Applicability* is on Pythagorean methods in theoretical physics, Steiner also outlines a metaphysical form of Pythagoreanism, stating that at least some of the ultimate natural kinds of science are those of mathematics.

3 What is Gauge?

The gauge theory of Yang and Mills (1954) provides the methodological basis for the construction of the standard model of particle physics, spelling out the idea that "symmetry dictates interactions" (Yang, 1980). The principle, presented in parallel as a methodological principle also by Utiyama (1956), states that we should promote every global symmetry to a local symmetry. Namely, when we are faced with a dynamical law whose form remains unaltered by a transformation of the variables that is uniform over space and time, we should require (by altering the law) invariance under the non-uniform (spacetime dependent) counterpart of the transformation. The requirement for invariance with respect to the local symmetries is said to force the introduction of fields and constrain their modes of interaction. Analogical reasoning played a significant role at each stage of the development of the gauge principle: first, by Weyl, through an analogy between gravity and electromagnetism and later, by Yang and Mills, through an analogy between electromagnetism and the nuclear interaction.

Steiner holds that the development of Yang-Mills Theory was based on an appeal to a Pythagorean analogy, and that Weyl's earlier versions of the gauge principle were a step in the development of this form of mathematical reasoning. In this section, we briefly describe the evolution of the concept of gauge symmetry, and then the relevant philosophical discussion on their significance.

3.1 A Brief History of the Gauge Principle

The concept of a 'gauge invariance' was first introduced by Hermann Weyl (1918). The paper starts with a discussion of the geometrical foundations of general relativity. Weyl complains that the theory contains a "residual element of rigid geometry" (p. 25), the metric, $g_{\mu\nu}$, which allows for the magnitude of two vectors to be compared at two distant points. In other words, upon translation (from one point in spacetime to another) a vector may change its direction but not its length. However, this seemed to contradict the 'local' nature of the theory. To Weyl, the local choice of measurement standard, or 'gauge', should be just as arbitrary as the choice of a coordinate system. This arbitrary choice could be described by the multiplication of all lengths by some spacetime dependent scale factor, $\lambda(x)$. To account for this freedom in the choice of gauge, Weyl argued that the theory of general relativity ought to have a double invariance. First, it should be invariant under the usual arbitrary smooth coordinate transformations. Second, it should be invariant under an arbitrary scale transformation of the metric $g_{\mu\nu}(x) \rightarrow \lambda g_{\mu\nu}(x)$, where $\lambda(x) > 0$.

However, this conformal structure must be supplemented to describe the continuous change in magnitudes due to parallel transport. In this purely infinitesimal geometry, Weyl notes that the metrical connection over spacetime will depend not only on the quadratic form $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, but also on an additional linear form $dl = A_\mu dx^\mu$. It is this 'length connection' A_μ that allows for length comparisons between two infinitesimally close points. A change of gauge would then correspond to a change in the representation of this connection. Surprisingly, Weyl suggested that given the formal similarity between the curvature tensor that one can derive from this 'length connection' and the structure of the electromagnetic field, we could interpret the length connection, A_μ, as the electromagnetic potential. The result was striking, as from a purely geometrical perspective, gravity and electromagnetism could now be seen to "spring from the same source" (p. 25). Indeed, Weyl showed that "*just as* [...] *the four conservation laws of matter* (of the energy momentum tensor [i.e. $\nabla_\mu T^{\mu\nu} = 0$]) *are connected with the invariance of the Action with respect to coordinate transformations*, expressed through four independent functions, the electromagnetic conservation law [i.e. $\nabla_\mu j^\mu = 0$] is connected with the new scale-invariance, expressed through a fifth arbitrary function" (p. 32, emphasis in original). Thus, through imposing a form of gauge invariance, Weyl was able to derive the equations governing the electromagnetic field and unify gravity and electromagnetism.

Despite this profound result, Weyl's 1918 gauge theory was taken to be empirically inadequate, due to the non-integrability of the scale factor, and the resulting path-dependence. As Einstein was quick to point out, this seemed to contradict the observed fixed spacing of atomic spectral lines. Weyl disagreed, but with the advent of quantum theory, he had already begun to reconsider his attempts to ground a unified field theory on a purely geometrical foundation. However, with the development of quantum theory, Weyl's notion of a 'gauge invariance' was found to reemerge in a surprising new setting—i.e. the complex amplitude of the wave-funtion (London, 1927).

In his 1929, Weyl took up this suggestion and set out to formulate a gauge theory that embraced gravity, electromagnetism, and matter in the new context of the developing quantum theory. Here, he (p. 121-122) notes that the "Dirac field-equations for ψ together with the Maxwell equations for the four potentials ϕ_μ of the electromagnetic field have an invariance property which is formally similar to the one which I called gauge-invariance in my 1918 theory of gravitation and electromagnetism". The key point is that "ψ now plays the role that Einstein's ds played before."—i.e. it is only determined up to a 'gauge-factor' $e^{i\lambda}$. Weyl notes that if this gauge factor was 'localized', as in his 1918 theory, i.e. $e^{i\lambda} \to e^{i\lambda(x)}$, then the invariance requirement can be maintained by introducing iA_μ, an imaginary counterpart of the 1918 connection form.

Through the 'localization' of the symmetry, one is required to introduce a new connection, a so-called 'gauge field'. Once more, given the formal similarity between the curvature tensor that one derives from this connection and the structure of the electromagnetic field, Weyl suggested that we could again associate the gauge field, A_μ, with the electromagnetic potential. But in this case, he was able to demonstrate that the problematic path-dependence that plagued his original theory is no longer a concern due to the fact that the phase factor lacks direct physical significance.

By the end of the 1920s, Weyl was able to clarify the sense in which the electromagnetic field and its interaction with charged particles can be derived from a form of gauge invariance. From this result, he formulated what is now known as the 'gauge principle' (e.g. see Weyl, 1928/1931, p. 100). Weyl's 'gauge principle' was further developed by Pauli (1980/1933,1941) in his influential review articles, and was formally divorced from its origins in general relativity. It was in this form that the gauge argument was widely disseminated in the physics community.

The foundation of modern gauge theories was laid down in a famous paper by Yang and Mills (1954), who sought to describe the strong nuclear interaction between nucleons (protons and neutrons). Here, they further developed Heisenberg's suggestion that, given their roughly equivalent mass, the proton and neutron may actually be two states of the same particle, differing only in their 'isotopic spin' state.[4] At the time, isotopic spin appeared to be a conserved quantity, and Yang and Mills (1954, p. 192) noted that the conservation of isotopic spin could be related to the invariance of all interactions under global isotopic spin rotation.

However, they point out that this global condition "is not consistent with the localized field concept that underlies the usual physical theories [...] " (p. 192). The violation of this locality desideratum motivates them to promote the global symmetry to a localized symmetry, aiming to "explore the possibility of requiring all interactions to be invariant under independent rotations of the isotopic spin at all spacetime points, so that the relative orientation of the isotopic spin at two spacetime points becomes a physically meaningless quantity". The analogical reasoning is explicit in the way they motivate this suggestion:

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[4] Isotopic spin (or isospin) is an abstract space in which the directions correspond to different kinds of particles (protons and neutrons in the original theory). Rotations in this abstract space are defined by analogy to the rotation of the spin property of particles.

> We wish to point out that an entirely similar situation arises with respect to the ordinary gauge invariance of a charged field which is described by a complex wave function ψ. A change of gauge means a change of phase factor $\psi \to \psi'$, $\psi' = (\exp i\alpha)\psi$, a change that is devoid of any physical consequences. Since ψ may depend on x, y, z, and t, the relative phase factor of ψ at two different space-time points is therefore completely arbitrary. In other words, the arbitrariness in choosing the phase factor is local in character.
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> We define isotopic gauge as an arbitrary way of choosing the orientation of the isotopic spin axes at all spacetime points, in analogy with the electromagnetic gauge which represents an arbitrary way of choosing the complex phase factor of a charged field at all space-time points.

The desired local isospin gauge invariance is then achieved, once again in analogy to electromagnetism, by the introduction of a new gauge field B_μ. Like in the electromagnetic case, the new field functions in the equations as a geometrical connection, i.e. it can define parallel transport of isospin directions along paths in spacetime. The requirement for gauge invariance, Yang and Mills claim, uniquely defines how this field interacts with itself and any field carrying isotopic spin.

While the proposal itself did not succeed as a theory of the strong nuclear interaction, physicists in the 60s and 70s continued to use Yang and Mills's method in the construction of new theories of the nuclear interactions. A central motivation was the conjectured renormalizability of theories of this type (see, for example, Kibble, 2015; Veltman, 1997).[5] These theories turned out to be particularly successful, both empirically, and in providing unified description of elementary particles and the electromagnetic and nuclear interactions among them.

Despite the historical roots of the gauge argument in Einstein's general theory of relativity, it is still a matter of debate whether and how gravity can be understood in a gauge-theoretical framework.[6] These questions are closely intertwined with fundamental questions on space and time as well as with ongoing attempts to formulate a unified field theory encompassing all known interactions.

3.2 Is Gauge a Pythagorean Analogy?

Steiner cites the history of gauge theory as the one of the clearest examples of Pythagorean reasoning in all of physics. Starting from Weyl's gauge argument, Steiner provides a brief account of the considerations of local conservation of charge in Weyl (1929) that provide the basic motivation for requiring local phase invariance. This invariance is a localized $U(1)$ symmetry from which the 4-vector potential, and

[5] Renormalization techniques are mathematical methods introduced already in the context of quantum electrodynamics more than a decade before Yang and Mills's paper in order to deal with infinities that arise and hinder the calculations in quantum field theories, and became a crucial part of the quantum field theoretic program. See Cao (1999).

[6] Gauge theories of gravity (see Hehl, 2017) aim to go beyond general relativity. The extent of the mathematical analogy between electromagnetism and Newtonian gravity has also received recent attention (Dewar, 2018; Teh, 2018).

it's law of transformation, are derived using his gauge argument.[7] "The baffling conclusion is: the move from global to local invariance in quantum mechanics is equivalent to the existence of the classical electromagnetic field as described by Maxwell" ((Steiner, 1998), p. 172).

In the construction of Yang-Mill theory, the analogy is based on the generalization of the procedure from the $U(1)$ group of electromagnetism to other groups. Yang and Mills considered the global continuous $SU(2)$ symmetry of the isospin transformation of a proton into a neutron and vice versa.[8] Using a localization procedure similar to Weyl's, Yang and Mills were able to write down the analogs of Maxwell's equations for the new gauge field, which they identified with the nuclear force.

Framed in this manner, Steiner argues that the Pythagorean nature of the analogy is far more salient in the Yang-Mills case. One reason has to do with the fact that the Yang-Mills field, in contrast to the electromagnetic field, has no classical parallel. "[T]he electromagnetic field was a well-established empirical phenomenon, detectable on the macroscopic level, prior to its so-called "quantization," the "classical" gauge fields of the Yang-Mills program were hardly real objects." In fact, they "existed only to be quantized away, for they cannot be detected at all as classical objects" (1998, p. 174-175). Second, he notes that the "symmetries in question are abstract symmetries (as opposed, for example, to spacetime symmetries). The validity of the projection from the success of one instance of this rule to another is heavily dependent upon the idea that we must categorize nature through the categories of mathematics.

The empirically successful theories that constitute the standard model of particle physics, Steiner contends, were therefore constructed on the basis of Pythagorean reasoning, that implicitly assumes the conformity of nature to mathematical categories that originate in anthropocentric values.

3.3 Responses to Steiner's Challenge

One possible line of response to Steiner is to argue that the aesthetic notions that guide the development of mathematics and its application in theoretical physics are not derived from human values, but rather from experience. Steven Weinberg (1994) famously expressed this view:

> A physicist who says that a theory is beautiful does not mean quite the same thing that would be meant in saying that a particular painting or a piece of music or poetry is beautiful. It is not merely a personal expression of aesthetic pleasure; it is much closer to what a horse trainer means when he looks at a racehorse and says that it is a beautiful horse. The horse trainer is of course expressing a personal opinion, but it is an opinion about an objective fact: that, on the basis of judgments that the trainer could not easily put into words, this is the kind of horse that wins races. (p. 133)

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[7] Roughly speaking, the unitary group $U(1)$ is the group of all possible phase transformations.

[8] The special unitary group $SU(2)$ is the group whose members describe all possible rotations in isospin space.

> The racehorse trainer has been at the track for many years—has experienced many horses winning or losing—and has come to associate, without being able to express it explicitly, certain visual cues with the expectation of a winning horse. (p. 158)

Can such reasoning explain the analogy employed in the case of the gauge argument? Steiner's answer is that it cannot. Even if one puts the theoretical physicist in line with the horse-breeder, their analogical reasoning is different. Fast horses are easily understood as natural kinds. In contrast, it does not make sense to trust a Pythagorean analogy unless one presupposes that the natural kinds of science are those of mathematics, a presupposition that is irreconcilable with naturalism in Steiner's view. Steiner further emphasizes that the mathematics used in many cases (e.g. the fibre-bundle representation of gauge theories) was developed independently by mathematicians based on considerations that had nothing to do with physics.[9]

A different strategy is to deny that the argument is in fact a Pythagorean analogy. Bartha (2010) (Section 6.3) adopted this non-Pythagorean strategy with respect to some of Steiner's examples, including Maxwell's derivation of the displacement current and the construction of Schrödinger's equation. However, the question of where to draw the line between physics and mathematics in the case of gauge symmetries, as we shall see below, is a much tougher nut to crack.

3.4 Is Gauge Mere Convention?

The challenge presented by the gauge principle to naturalist philosophy involves more than the difficulty of accounting for the historical development of physics. The heart of the matter is the concept of a gauge symmetry itself. This had become clear as the philosophical interest in the gauge principle began to take off (Teller, 1997; Brown, 1999; Teller, 2000; Lyre, 2000; Redhead, 2002; Martin, 2003; Ben-Menahem, 2012).

To get a sense of the problem, we can return to Yang and Mills's 1954 description of a gauge freedom in terms of a local convention:

> "The difference between a neutron and a proton is then a purely arbitrary process. As usually conceived, however, this arbitrariness is subject to the following limitation: once one chooses *what to call* a proton, what a neutron, at one space-time point, one is then not free to make any choices at other space-time points. [...] In the present paper we wish to explore the possibility of requiring all interactions to be invariant under independent rotations of the isotopic spin at all spacetime points, so that the relative orientation of the isotopic spin at two space-time points becomes *a physically meaningless quantity*" (p.192, emphasis added).

This passive view of a gauge transformation (as a change in the mathematical description of a physical system, as opposed to a change in the system itself) became part of the standard view among physicists. It was also given a philosophical formulation by Auyang (1995), in her book that presents quantum field theory in a Kantian categorical framework. Many of the philosophical concerns regarding the

[9] However, Bangu (2006) points out major contributions of anti-anthropocentric values to the development of mathematics, arguing against Steiner's anthropocentric view of mathematics.

gauge principle can be traced to Teller's [1997] review of Auyang's book, which identified this point as requiring further investigation:

> The gauge argument in quantum field theories is often presented in terms of linguistic conventions. Since only relative phase matters and absolute phase is arbitrary, it makes no difference which phase we call 'zero'. Thus a local phase transformation is seen as no more than a local change in the conventions of what we call what. But then this fact about mere conventions has dramatic repercussions in seeming to force the introduction of an otherwise neglected physical field! (p. 516)

This puzzle about the relation between descriptive mathematical conventions and novel physical phenomena, as was soon pointed out (Brown, 1999; Norton, 2003), seems to echo the debates concerning the role of general covariance ((i.e. the requirement for invariance under arbitrary coordinate transformations) in general relativity. Einstein (1916) has described this principle using the terminology of conventional choice of representation of the actual phenomena in his famous point coincidence argument:

> "there is nothing for it but to regard all imaginable systems of co-ordinates, on principle, as equally suitable for the description of nature. [...]
>
> The introduction of a system of reference serves no other purpose than to facilitate the description of the totality of such coincidences."

The possibility to obtain physical knowledge through the principle of general covariance was immediately disputed and remains controversial (Norton, 1993; Pooley, 2010), mainly due to the question of the physical content of the constraint on the mathematical formulation.

It therefore seems that we are facing here not only a miraculous analogy, but rather an analogy between two miracles: that of general covariance and that of gauge invariance. Novel laws of interaction seem to pop out, in both cases, out of a merely formal constraint on the mathematical representation of nature.

Steiner's discussion of symmetries highlights a similar problem, describing symmetry arguments in particle physics as "doubly Pythagorean" (p. 84). Given that symmetries are understood in terms of similarity relations between structures, Steiner notes that not only is this notion of a similarity purely mathematical, but the structures from which the analogies set off are irreducible to any apparent physical properties. Internal symmetries in particular are merely defined as groups of automorphisms of the mathematical structure[10]; "The groups invoked in the theory of elementary particles today", he stresses, "express symmetries only in this question-begging sense [i.e. the sense of mathematical automorphism]; they do not express empirical or geometrical symmetries. The analogies here were therefore, in fact, Pythagorean." (p. 94).

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[10] Automorphism is the mathematical concept that is also used to describe the replacement of one convention with another. See for example Redhead (2002).

4 Gauge: an Idealist Reading

In light of the manner in which the gauge argument tends to blur the distinction between mathematical and physical reasoning, it may be natural to first consider a Kantian response to Steiner's Pythagoreanism, one which highlights, at the outset, the mathematical constitution of scientific thought. Steiner (1998, p. 9) considered such a response. He notes that on the

> Kantian account of mathematical discovery: the world is the way it is, in part because of our contribution to our own experiences. Mathematics is the lens through which we view the Universe, meaning the phenomenal, or experienced, Universe (about things in themselves we know nothing). This is also a valid attempt to explain away [this appeal to Pythagorean analogies], but it will have to come to grips with the nature of contemporary science, which deals with objects beyond the realm of spatiotemporal experience.

Here, we will briefly consider not only whether it is possible to defend a Kantian perspective on the analogies underwriting the gauge argument, but also the extent to which such an account actually served as an early justification for Weyl's gauge theory.

4.1 A Little Historical Context

In the early twentieth century, philosophers of science in the Kantian tradition were faced with the problem of articulating a new epistemology of science in light of recent developments in both the foundations of mathematics and physics. Hilbert's formalist program in the foundations of mathematics had made it clear that the Kantian epistemological program was in need of a novel reformulation, as Kant's account of the synthetic nature of mathematical thought was no longer tenable. Kant held that mathematical concepts are founded upon an act of construction in intuition (the faculty through which they are exhibited). On Kant's account, it is only by way of our 'pure' intuitions of time and space, that arithmetic and geometry can be formally constructed. However, both the logicist (e.g. Fregean) and formalist (e.g. Hilbertian) programs in the foundations of mathematics had made it clear that mathematical concepts need not be exhibited in intuition to be definable.

Furthermore, Einstein's work on the development of the theories of special and general relativity seemed to present a profound challenge to the Kantian account of the constitutive framework of physical enquiry. Kant held that a fixed Euclidean spatial and temporal structure (along with a set of fundamental conceptual categories), define the constitutive framework of scientific thought. He argued that this framework serves as a necessary presupposition of scientific cognition, in the sense that it makes the scientific understanding of nature possible. However, Kant's thought was couched in the Newtonian worldview, which was no longer taken to provide a viable foundation for scientific thought (at least from the perspective of early twentieth century physics). The development of the theories of special and general relativity had shown that the privileged *a priori* status that Kant assigned to Euclidean spatial geometry was no longer justified. In addition, given the interweaving of geometrical and dynamical structure in the theory of general relativity, it was no longer clear if one could even defend a traditional Kantian account of the constitutive geometrical framework of scientific thought. Yet, the question remained whether the Kantian epistemological program could be suitably reinterpreted in light of these developments in the foundations of mathematics and physics.[11]

By the 1920s, these difficulties gave rise to two strands of philosophical thought that sought to develop a novel understanding of the epistemology of modern mathematical physics. These were the 'early' logical positivist and neo-Kantian traditions.[12] In a certain sense, both of these traditions looked to reinterpret the Kantian program in light of the recent developments in the foundations of mathematics and physics—highlighting both the formalist tradition in the philosophy of mathematics and the epistemological lessons gained from the general theory of relativity (for more, see Friedman (1999); and Ryckman (2005)). In this pursuit, they were forced to abandon the traditional Kantian account of the synthetic *a priori*.[13] But without the anchor of Kant's account of the necessary and universal *a priori* forms of intuition, the worry seemed to be that the Kantian epistemological program may be set adrift in the sea of relativism. Thus, the debate hinged on how to best understand the formal systems of physical enquiry in this new light.

Weyl's philosophy of science offered a deep and thoughtful analysis of the nature of modern scientific thought, and directly engaged with foundational issues in both the logical positivist and neo-Kantian traditions (e.g. see Weyl (1949)). Weyl, better than anyone else, understood the fundamental implications of the interweaving of geometry and dynamics in the dynamical spacetime structure of general relativity, and the blurring of mathematics and physics that it seemed to entail. This was particularly important to Weyl in light of the pioneering work of Klein's Erlanger program on the group-theoretic foundations of geometry. Weyl's broadly idealist philosophy was motivated by the desire to better understand the rich interplay between mathematics and physics, in the context of group theory, and on this basis, to present a unified mathematical-physical philosophy of nature.

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[11] Here, and in the paragraph above, we would like to acknowledge that the presented characterization of Kant's thought is neither universally accepted nor unambiguous. The aim is to merely give a sense of some of the central concerns that guided the relevant philosophical discussion in the early twentieth century.

[12] The former associated with Schlick, Reichenbach, and Carnap, and the latter, most notably, with Cassirer.

[13] For example, Cassirer (1910/1923) offered a regulative reading of the relativised constitutive principles of physical enquiry. Reichenbach [1920/1965] offered a similar account of the coordinative principles of scientific thought. However, in later years, this was reinterpreted in line with Schlick's particular reading of Poincare's conventionalism (see Friedman (1999); Ben-Menahem (2006)).

4.2 Weyl's Philosophy of Science and the Gauge Argument

In 1927 (with a major revision in 1949), Weyl reflected on the foundations of gauge theory in a text entitled *Philosophy of Mathematics and Natural Science*. Here, he puts forth a broadly Kantian account of scientific enquiry.[14] Concerning the applicability of mathematics in physics, the fundamental problem as Weyl (1949, p. 123) saw it was that "the mere positing of the external world does not really explain what it was meant to explain, the question of the reality of the world mingles inseparably with the question of the reason for its lawful mathematical harmony."[15] In line with traditional neo-Kantian critiques of the applicability of mathematics in science (e.g. Cassirer (1910/1923)), Weyl (1949, p. 113) held that the 'reality' depicted in modern physics is "ultimately a symbolic construction". However, he (1949, p. 116) points out that within the "natural sciences the conflicting philosophies of idealism and realism signify principles of method which do not contradict each other". On the one hand, we

> construct through science an objective world in which all perceptions are founded on objective facts in "reality". Here natural science proceeds realistically. [...] On the other hand science concedes to idealism that its objective reality is not given but to be constructed, and that it cannot be constructed absolutely but only in relation to an arbitrarily assumed coordinate system and in mere symbols [i.e. to a given symbolic representation]. (Weyl, 1949, p. 117).

Weyl then turns to a discussion of the nature of this theoretical construction.

Reflecting on the conformal geometry underlying his gauge theory of 1918, Weyl was able to draw an important lesson concerning what he took to be the nature of the Kantian program in modern physics. Following Kant, he held the symbolic construction of nature can only be defined on the "field of the *a priori* existing possibilities" (Weyl, 1949, p. 131). However, Weyl notes that the '*a priori* field of possibilities' is far more general than previously thought. In a general dynamical spacetime theory, the metrical structure is not given *a priori* as it was for Kant. The material content determines the local metrical structure of any given spacetime region, *a postriori*. It is only the Pythagorean-Riemannian form of the metric which is given *a priori*. Thus, the Kantian line between the *a priori* and *a posteriori* has shifted, as only the form of the local metrical structure is predetermined (Weyl, 1949, p. 134-137).

In the transition to Weyl's 1929 gauge theory, the physical setting may have changed, but certain key aspects of his methodological approach remained the same. By the mid-1920s, Weyl had given up the hope of a geometrical unification of the fundamental forces of nature, in line with what one might call a more traditional form of neo-Kantian idealism. He had already seen that such a unification would

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[14] However, it is important to note that Weyl's philosophy was not purely idealistic (Scholz, 2004), but it seems much of his philosophical thought was motivated by both the Kantian and Husserlian traditions (e.g. see Ryckman (2005)).

[15] In this case, the positing of an external world was meant to explain the objective nature of scientific knowledge.

not be possible in the context of the new quantum theory of matter. However, when reflecting on this transition, Weyl maintained the methodological commitment to a broadly Kantian account of the constitutive framework of scientific enquiry. Weyl still sought to ground the theoretical construction of reality on a '*a priori* field of possibilities", but this field was now defined in terms of a more general symmetry structure underlying physical theory. In the process, one could argue that the constitutive framework had shifted once again. This time to an even more abstract level, based on a group-theoretic account of objectivity.

In Weyl's appeal to a group-theoretic account of objectivity, we can discern the outlines of what we might call a Weylian form of neo-Kantian idealism. Reflecting on the structure of mathematics and physics, Weyl held that the relevant notion of objectivity is the same in both cases. A relation can only be determined to be objective if it remains invariant under a given set of transformations. However, drawing on Klein's Erlanger Program, Weyl (1949, p. 73) notes that to define invariance one presupposes a group operation, as "only once a group is given [do] we know what like-ness or similarity means". It is in this sense that Weyl held that there was a close harmony between the notions of mathematical and physical symmetry. In mathematics, any group can be taken to define the symmetries which characterize the objective features of an associated structure. Thus, the domain of possible groups demarcates the domain of possible structures, up to an isomorphism. On Weyl's view, the same is true in physics. In physics, one characterizes the objective symmetries of a physical structure through the group operations that define its invariance structure. In fact, Weyl held that physical symmetries are formally a subset of a broader class of mathematical symmetries. In this sense, the generalization, or rather extension, of the mathematical symmetries of a physical theory would correspond to a potential extension of its physical symmetries. The extension would serve to ground a new 'field of *a priori* existing possibilities'. Indeed, this is how one might read Weyl's (1952) famous dictum that all *a priori* statements have their origin in symmetry.

However, to simply point out that a symmetry structure serves in a certain sense as a precondition of scientific objectivity is not to say much at all if any symmetry group could serve as an equally valid basis for the theoretical construction of 'reality'. Certainly, in mathematics, one is free to choose whatever group one wishes. But in physics, the freedom one has in specifying the relevant symmetry structure underlying physical theory is greatly constrained. Weyl stressed that our physical theories must be confronted with nature as a whole (1949, p. 61). This confrontation with nature 'as a whole', concerns the harmony, or "concordance" (p. 62), that a given theoretical construction brings to our understanding of nature, and not all extensions of the basic group structures of modern physics will lead to novel physical insight (e.g. see Scholz (2018)). Thus, on Weyl's view, physics is in a continual process of development, searching the domain of structural extensions for those that lead to a more encompassing picture of 'reality'. Throughout this process, one finds fundamental group structures that persist, and these structures serve to guide scientific thought.

Certainly, more work remains to clarify the nature of the constitutive framework of this Weylian view, but one can already make out its broader methodological implications. If we can only understand nature through a given symmetry structure, then it is no miracle that we find such structures throughout modern physical theory. In addition, if the development of science entails, at least in part, a generalization of the underlying local group structures of a physical theory, then it would not come as a surprise that this trend continued with Yang-Mills theory through to the development of the standard model of particle physics.

4.3 Weylian Idealism, Analogies, and the Gauge Argument

On this reading of Weyl's idealist philosophy, the appeal to analogy serves to extend the domain of the symmetries underwriting the constitutive framework of scientific thought. This is by showing that the extension, or generalization, of the mathematical framework of a physical theory could correspond to a potential extension of its physical domain of application. In a certain sense, one could argue that Weyl's gauge methodology was based on a principle of analogy—it is only by way of analogy that Weyl could ensure that the right constructive concepts evolve for the account of nature. The 'gauge principle' merely expresses an essential feature of this methodology, as the appeal to formal analogies served to secure a methodological continuity. Indeed, this could be seen as the reason why Weyl sought in 1929 to connect his later work on gauge theory with its early foundation in general relativity. The analogies in the development of gauge theory may have been more about guiding the symbolic construction of reality than about specific physical analogies between gravity and electromagnetism. The analogies underwriting the gauge principle serve to constrain the mathematical formalism for the symbolic construction of nature, by showing that diverse phenomena can be understood in terms of the same structure—thus, *bringing them into harmony with one another*.

Weyl's philosophy of science served to both motivate the development of the gauge argument and justify its application in the context of a broader 'mathematical methodology' of physical inquiry. Here we can discern an interesting idealist response to Steiner's anthropocentric Pythagoreanism. In the historical development of science, following Weyl, we might argue that we obtain a better understanding when we can show that diverse phenomena can be characterized through the same basic structural relations. In drawing an analogy between different domains, we do not highlight a similarity, but *construct it*.

In picking up the gauge argument, whereby the fundamental interactions in nature are *derived* from the postulated gauge invariance, we could argue that Yang and Mills (1954) implicitly assumed the norms of the methodology that supported it. In fact, writing in 1989, Mills reflects on the methodological principles underwriting the gauge argument. He notes that what underlies the gauge argument is the growing realization "of the importance of symmetry to our basic understanding of the universe, to the point where it is now felt that it is the underlying symmetry of physical laws that *drives* the system—that determines the structure of the laws and the number and character of elementary particles" (p. 494).

This methodology was grounded on an appeal to Noether's theorem(s), which Mills paraphrases as "*for every symmetry of nature there is a corresponding conservation law and for every conservation law there is a symmetry*" (Mills, 1989, p. 494). In the context of a discussion of conservation laws, he notes that it is "quite possible that Noether's theorem is the more fundamental fact—that the physical theories that we devise to describe the universe about us have the structure they do *because* of this fundamental relationship between symmetries and conservation laws." In this case, Noether's theorem becomes a principle through which theories are constructed, "like the principles of equivalence in special and general relativity; we should say than that classical physical laws take the Lagrangian form and quantum theory takes its characteristic Hamiltonian form as a consequence of Noether's principle" (p. 494). Mills elevates, by way of analogy, a mathematical theorem to a physical principle!

Following Steiner, this reasoning could be understood as an explicit avowal of Pythagoreanism. On the Pythagorean view, the symmetry-dictates-interaction paradigm is simply reified into a principle of nature. From a naturalist perspective, it thus became miraculous. Steiner may be right on this point. However, the real problem may be that within gauge theory, the line between the physical and the mathematical has become inextricably blurred. Thus, the most natural justification of modern appeals to the gauge argument may lie in the idealist tradition that first motivated its development.

The idealist's disagreement with Steiner primarily concerns the nature of scientific thought. Steiner holds that such thought relates directly to the natural world. In this case, the anthropocentrism implicit in the Pythagorean analogies of modern science tends, given their miraculous success, to indicate that the world must be in some sense 'user-friendly'. The idealist finds no miracle in this, as the 'natural world' is only given according to the constitutive framework of scientific thought. Thus, it is by definition 'user-friendly'. However, both would agree that the 'Pythagorean analogies' one finds in modern science pose a distinct challenge to Naturalism.

5 Gauge: a Naturalist Account

The term naturalism relates to a range of epistemic and ontological approaches that give central place to scientific standards and methods of inquiry and to scientific knowledge. Steiner contrasts naturalism with anthropocentrism. The aspect of naturalism that is the focus of Steiner's critic in the *Applicability* is the belief that the human perspective should not be a privileged one. While it may appear clear to some that scientific standards allow no place for the anthropocentric values, the anti-anthropocentric aspect of naturalism is seldom an explicit part of presented self-proclaimed naturalist views. One such view which does make it explicit is *subject naturalism* advocated by Huw Price (2011), presented as "the philosophical viewpoint that begins with the realization that we humans (our thought and talk included) are surely part of the natural world" (p. 5). Steiner's argument seems to imply that this view is undermined by the observation that human values are the driving force that gives rise to major scientific discoveries. Steiner problem may be stated as the observation that while the content of our scientific theories is in harmony with this form of naturalism, scientific methods of discovery are not. This conflict between the discovery and content of theories leads Steiner to doubt whether “it is profitable to construct a philosophy based both on what scientists say and what they do” (p. 10).

This section does not intend to provide a general answer to these questions, but rather to focus on the example of the gauge methodology and outline a naturalist way of understanding it. It is a naturalist approach in two respects. First, it sets off from the invariance requirement, understood as a naturalist desideratum that concerns the indifference of the represented physical phenomena to its theoretical mode of representation (it is therefore a stronger version of naturalism than Price’s aforementioned version, which aims to present a naturalist philosophy disengaged from representationalist intentions). Second, this approach attributes the success of certain symmetry considerations to a contingent property of the natural world. In contrast to the approach presented in Section 4 as well as to Steiner’s approach, here the reconstruction of the methodology is only the starting point. The focus thus shifts from the method (together with the values and ideas that motivated it) to the conjectured physical basis for its success.

Let us begin with the notion of invariance that is sometimes used interchangeably with that of symmetry. An invariance requirement amounts to the demand that we formulate laws in a manner such that their form remains intact under certain transformations. Some invariance requirements, including those which are at the basis of the gauge argument, can be regarded, at their core, as a naturalist desiderata. The scientific aim to formulate laws of nature in an invariant form is an expression of the belief that nature does not care about our particular choice of representation. Einstein expressed such a view when he described the reasoning that led him to the general theory of relativity (Einstein, 1919, 1954a). Similar reasoning is sometimes used in contemporary expositions of gauge theories. Nakahara (2003), for example, emphasizes the similarity between gauge symmetry and general covariance when he presents the gauge principle as stating that “physics should not depend on how we describe it”.

This view, however, raises more questions than it answers. It is obviously dubious, particularly from a naturalist point of view, to trust such a philosophical desideratum as a guide to new knowledge. Moreover, it is unclear to what degree this kind of reasoning played an actual role in the historical development of gauge theories. Nevertheless, this view forms a central motivation for a recent conceptual reconstruction of the gauge argument (Hetzroni, 2021). The principles employed in this account are plainly analogous to Einstein’s reasoning in presenting general relativity (Hetzroni, 2020). In this approach, the invariance requirement is supported by a ‘relationist’ view of gauge, according to which the fundamental physical quantities are relations between different entities. The aim is naturalist: to explain the success of the formalism by appealing to contingent physical properties (in this case, relational quantities). Here, however, we shall aim at a sharper separation between the suggested method and its interpretation. We begin below by presenting the methodology, as a reconstruction of the gauge argument, that sets off from the invariance requirement but includes an additional element, the methodological equivalence principle. We leave the discussion of the naturalist interpretation to Section 5.3, in which we outline a reconstruction of the analogy between gravity, electromagnetism and Yang-Mills theory as a *material analogy*. Section 5.4 discusses the extent to which this account can be considered a response to Steiner's problem.

5.1 The Methodological Equivalence Principle

A general lesson from the debates described in Section 3.4 over the role of general covariance in general relativity and the gauge argument in particle physics is that it is not possible to understand the conceptual relation between symmetries and interactions based on the invariance requirement by itself. The methodological equivalence principle aims to capture the additional element in the methodology.

The fundamental observation is that while invariance is a formal constraint, it is pursued in many different cases according to a specific prescription that rests on the non-invariance of an existing theory. The non-invariance of a theory, by definition, involves a preferred class of representations in which the dynamical laws take a simple form, and additional or modified terms that appear under a transformation (for example, inertial frames in special relativity are preferred ways of assigning coordinates to spacetime points, such that the laws of motion take a particularly simple form). The prescription that allows physicists to guess the laws of interaction is based on the exact form of these additional terms. More formally, the principle states:

> *The Methodological Equivalence Principle*: Given a non-invariant dynamical law, in the sense that that it obtains a simple form in some preferred representations while in arbitrary representations it includes modified/additional expressions, construct an invariant law by replacing the modified/additional expressions with new dynamical fields, whose set of possible local values is identical to that of the modified/additional expressions, and obey the same transformation rules.

This principle aims to convert Einstein's equivalence principle into a general method suitable for modern field theories, in which the equivalence is not an empirical fact known in advance, but is rather conjectured.

For example, in quantum mechanics the Schrödinger equation describing the temporal evolution of the wavefunction of a free quantum particle $i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\sum_i \frac{\partial^2\psi}{\partial x_i^2}$ changes its form under a local change of the phase convention: $|x\rangle \rightarrow e^{i\Lambda(x)}|x\rangle$ into the form $i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\sum_i \left(\frac{\partial}{\partial x_i} - \omega_i\right)^2 \psi$ (with $\omega_i = \frac{\partial\Lambda}{\partial x_i}$). This change does not seem to reflect anything physical; it is obtained by pure mathematics from the change of local convention in which the basis states are re-defined. The Schrödinger equation corresponding to the situation in which the particle is subjected to classical (static) magnetic influence can be obtained using the methodological equivalence principle by formally replacing ω_i with the components of a gauge field identified with the magnetic vector potential. Similarly, the Dirac equation

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0 \tag{1}$$

is not invariant under change of local phase convention in spacetime: $\psi \to e^{i\Lambda(x)}\psi$ (here x is a 4-coordinate in spacetime), as it's form changes into

$$i\gamma^\mu \left(\partial_\mu + i\partial_\mu \Lambda\right) \psi - m\psi = 0. \tag{2}$$

The methodological equivalence principle urges us to replace the term that breaks the invariance $\partial_\mu \Lambda$ with a physical field A_μ (accompanied by a coupling constant e corresponding to the charge of the particle).[16] This change would indeed lead to the form of the Dirac equation with electromagnetic coupling[17]

$$i\gamma^\mu \left(\partial_\mu + ieA_\mu\right) \psi - m\psi = 0. \tag{3}$$

In the case of several spinor fields with equal mass parameter, the introduction of the Yang-Mills field using the requirement for invariance under local change in isospin convention is completely analogous.

The same method can be used as the basis of gravitational coupling prescriptions in a wide range of physical situations using the requirement for general covariance (see Hetzroni and Read (2021) for details and a discussion on the relation of the principle to standard formulations of Einstein's equivalence principle). Setting off from a free particle described according to the special theory of relativity, the trajectory of the particle $x^\mu(\tau)$ satisfies the equation $\frac{d^2x^\mu}{d\tau^2} = 0$ (expressing the relativistic Newtonian law of motion with τ the proper time). The equation is not generally covariant due to the transformation of the derivative under coordinate transformation. In terms of arbitrary coordinates $\xi^\mu(x^\nu)$ the equation for $\xi^\mu(\tau)$ takes the form:

$$\frac{d^2\xi^\mu}{d\tau^2} + \gamma^\mu_{\alpha\beta}\frac{d\xi^\alpha}{d\tau}\frac{d\xi^\beta}{d\tau} = 0, \tag{4}$$

with $\gamma^\mu_{\alpha\beta}$ determined by the transformation according to

$$\gamma^\mu_{\alpha\beta} = \frac{\partial \xi^\mu}{\partial x^\nu}\frac{\partial^2 x^\nu}{\partial \xi^\alpha \partial \xi^\beta} \tag{5}$$

and obeys the usual transformation law for Christoffel symbols. The methodological equivalence principle motivates and prescribes the formulation of an invariant law by postulating the existence of an interaction, such that the local form of the equation of motion is:

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[16] Note that according to the methodological equivalence principle the possible local values (at a point) of the new term eA_μ are the possible local values of the term $\partial_\mu \Lambda$, yet, the space of possible functions eA_μ is bigger than that of $\partial_\mu \Lambda$, as the latter is constrained to be irrotational vector fields.

[17] The introduction of the Maxwellian term $-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ into the Lagrangian would require additional principles. Here we limit our attention to the interaction terms in the equations of motion.

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0, \tag{6}$$

The Christoffel symbol in this equation is understood to represent the influence of a gravitational-like interaction. It can be interpreted geometrically as a symmetric affine connection over a linearly connected space. The pseudo-Riemannian geometry of the general theory of relativity in which (6) is the familiar geodesic equation is a special case (the case of Weyl geometry with scale symmetry of the metric is another special case). A metric-like tensor $g^{\mu\nu}$ and the corresponding (Riemannian) Levi-Civita connection can be obtained by applying the same method to the relativistic point-particle Lagrangian $L = \eta_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = \eta_{\alpha\beta}u^\alpha u^\beta$ and deriving the new equations of motion.[18]

The methodological equivalence principle therefore extends the physical scope as well as the empirical content of a theory by introducing new fields. The initial non-invariance of the dynamics under a transformation understood as a change in the mathematical convention not only motivates this change, but also specifies its form. Gravity, electromagnetism and Yang-Mills theories are fully analogous in this respect.

5.2 A Formalist Analogy?

The presentation of the gauge argument in terms of the methodological equivalence principle emphasizes the role of mathematical conventions in the argument. It does not by itself resolve any of the epistemological worries about gauge; actually it seems to make things worse. The different mathematical descriptions of the motion of a free particle in flat spacetime appear to be mysteriously connected to the contingent way in which a particular physical interaction, gravitation, would alter the trajectory of a particle. The different possible phase conventions through which an interaction-free quantum field can be described seem to give us a broad hint about the way this field would interact with an electromagnetic field. Different local isospin conventions, through which non-interacting fermion fields can be described, appear to correspond, for some reason, to different possible settings of an additional Yang-Mills field that is associated with a new interaction.

All this should sound deeply troubling. In all those cases, the purely mathematical relations between different conventions associated with a given system reveal some contingent facts about the possible interaction of the system with an external field, through reasoning that lead to successful and accurate predictions.

Recall that invariance has been presented above as a naturalist desideratum. Indeed, it makes sense to ask "what has nature to do with our coordinate systems\phase conventions\isospin conventions? If it is necessary for the purpose of

[18] It is quite a general result, which also holds in field theories, that the results of applying the methodological equivalence principle for arbitrary coordinate transformations at the level of the Lagrangian coincide with those obtained by the coupling prescription of general relativity (Hetzroni and Read, 2021).

describing nature, to write our equations in a particular coordinate system\phase convention\isospin convention arbitrarily introduced by us, then its choice ought to be subject to no restriction".[19] But, as we have seen, by adhering to this principle the naturalist might be shooting herself in the foot: the statement expresses a formal and *a priori* requirement that seems to involve the form of the theory rather than it's physical content.

The introduction of the interaction, however, involves more than manipulating the different conventions, as has been pointed out (Brown, 1999; Martin, 2003); it requires a non-trivial generalization in order to turn a flat connection into a curved one, and other considerations in order to construct the interaction and the dynamics of the free field. The mathematical conventions don't know *everything* about the previously unknown interaction, yet the method presented in the previous section can easily leave one with the feeling that they know too much!

While it is impossible to know what Steiner's opinion on this presentation of the gauge argument would have been, we can point out that his philosophical analysis can provide a deeper perspective on this issue, suggesting that the methodological equivalence principle is, by itself, a mathematical analogy. In addition to the 'horizontal' analogies between gravity and electromagnetism, or between electromagnetism and the nuclear interactions, this is a 'vertical' analogy. The mathematical analogy here bridges the system that is described by the interaction-free theory and the greater, extended, system that is described by the theory of interaction. More precisely, the analogy conjectures a mathematical similarity between the initially known mathematical representations of the original system, and possible configurations of the gauge field that governs the introduced interaction.

Steiner described the gauge argument as a Pythagorean analogy, i.e. an analogy between physically distinct interactions that have nothing in common except for the conjectured similarity in the mathematical form of their descriptions. The vertical analogy seems even more troubling in the way it crosses the border between physics and mathematics. Steiner, in a different context, described such analogies as *formalist analogies*:

> "In some remarkable instances, mathematical notation (rather than structures) provided the analogies used in physical discovery. [...] So the analogy was to the form of an equation, not to its mathematical meaning. This is a special case of Pythagorean analogies which I will call 'formalist' analogies." (p. 4)
>
> "I single out formalist analogies because, from the 'naturalist' standpoint, formalist analogies are (or should be) particularly repugnant." (p.54)

According to the approach defended in *Applicability*, the mathematical language is shaped by anthropocentric values such as convenience and beauty. Accordingly, the practice of using the different mathematical representations of a physical system to guess its law of interaction with other systems makes no sense from the naturalist's point of view. Therefore, understanding the gauge argument in terms of the methodological equivalence principle turns out to highlight an additional way in which the

[19] Paraphrasing Einstein (1919), p. 230.

empirical success of the gauge argument indicates the user-friendly nature of the world, undermining naturalist philosophy.

5.3 Relational Quantities and Material Analogies

We began this section by arguing that the naturalist is well motivated to pursue theories whose laws are invariant under change of representation. Yet, the way this invariance is pursued is through a formalist analogy whose recurring success appears miraculous by the naturalist own standards. The situation, however, is not hopeless. A mathematical analogy can make sense from a naturalist point of view if the formal similarity on which it is based is seen as associated with an actual physical similarity. Note that the question regarding the exact nature of the physical similarity arises only for those analogies that have already been shown to lead to empirically successful theories. The aim to learn from the success of a given guess may involve considerations that are different from the actual lines of reasoning used by physicists who initially made the guess.

Norton [2020, chp. 4] presented a minimalist "material" account of analogy which centres on the factual basis of the analogy. In this section, we will appeal to Norton's account to defend a naturalist understanding of the gauge argument (gravitational coupling in general relativity included) through the lens of the methodological equivalence principle.

According to Norton's analysis, a material analogy is based on a *fact of analogy*, i.e. "a factual state of affairs that arises when two systems' properties are similar, with the exact mode of correspondence expressed as part of the fact." Norton stresses that "[t]here is no general template to which the fact must conform". In particular, it could either be an empirical fact, subjected to direct observation, or a fact that is merely conjectured. In many cases, this fact is a property shared by the two systems. A second essential element is *analogical inference* "warranted by a fact of analogy".

In our case we are looking for a physical property that is common to the gravitational, electromagnetic and nuclear interactions, that can justify the analogical inferences that give rise to the laws of interaction through the gauge principle.

Rovelli (2014) is concerned with a closely related question: why is the world described so well by gauge theories? Rovelli notes that many physical systems can be described either using gauge independent variables (e.g. electromagnetic fields) or gauge dependent variables (such as the electromagnetic potentials). However, the description of the interaction of the system with other systems (e.g. coupling of electromagnetism to matter fields) involves gauge *dependent* variables of both systems, that together form new gauge independent quantities. For example, the gauge dependent local phase gradient of a fermion quantum field couples to the electromagnetic potential to form the covariant derivative operator that describes the coupling of the particle to electromagnetism. Without the coupling, it might seem like the gauge independent variables of the two systems provide a full description, but the presence of an interaction reveals that this is not the case. The new gauge independent variable formed using gauge dependent variables of the two systems is interpreted as a quantity that represents a relation between the two systems, on which the interaction depends. Rovelli therefore suggests that gauge invariance is an indication of an ontological property: the relational nature of fundamental physical quantities, that "do not refer to properties of a single entity. They refer to relational properties between entities".

Hetzroni (2021, 2020) has more recently showed that this observation can be used to understand the working of the gauge argument. According to this account, the invariance requirement reflects the relationist desideratum that "the laws of physics should depend on the relations between fundamental physical objects, and not on the relations of the physical objects to a fictitious mathematical frame of reference" [2020, p. 130].

In order to use this desideratum to present gauge as a material analogy, let us first note that the formal similarity is already manifest in the interaction-free theories. The laws of motion in special relativity and the Dirac equation of the free field both define a class of preferred mathematical representations: Lorentz frames in the former case, and the Schrödinger representation (in which the momentum operator is proportional to the derivative operator) in the latter. In those preferred representations, the equations of motion take a unique simple form. The choice of mathematical convention does make a difference, in contrast to the invariance requirement.

We know from experience that it is possible to find certain domains in spacetime in which special relativity is empirically adequate. The motions of material bodies (and the dynamics of matter fields) in such a domain take a unique form in certain coordinate systems connected to each other by Lorentz transformations. General relativity, however, tells us that these preferred coordinate systems are actually a local manifestation of the gravitational field. In other words, the equations of motion of special relativity that appear to describe the motion of bodies with respect to Lorentz frames, actually describe the motion of bodies with respect to the local gravitational field.

The analogical reasoning in this case conjectures that, in the case of the quantum spinor field, the preferred Schrödinger representation in a given region is a manifestation of an additional field, that, similarly, is not taken into account in the original theory. The fact of analogy in this case is thus the existence of the additional field.

Next thing to ask is how does the fact of analogy warrant the analogical inference. Recall that according to the relationist view of gauge, the aim of the gauge heuristics is to construct a variable that represents the relation between the original system and the introduced gauge field. Again, the crucial point here is that the preferred representation in the interaction-free theory is a manifestation of a special case of an interaction: the possible values of $\gamma^{\mu}_{\alpha\beta}$ in Eq. (4) are no more than different representations of the uniform field. The construction of Eq. (6) from Eq. (4) can be seen as consisting of two distinct elements. The first, which concerns the notation and its interpretation, is interpreting the coefficients $\Gamma^{\mu}_{\alpha\beta}$ as a field that acts in the settings described by the original coordinates x^{μ}, rather than a manifestation of a coordinate transformation. The postulate that can warrant this step is that the range of different representations of a given uniform field correspond to the range of possible uniform fields that can act at a point. In other words, the assumption is that the mathematical representation of the space of physical possibilities is an adequate one. The second element is the generalization from a uniform field to a non-uniform field, i.e. to a general $\Gamma^{\mu}_{\alpha\beta}$ that does not obey the constraint of Eq. (5). It is this element that makes the introduction of the new field physically meaningful, achieving more than formal invariance. The uniform field settings, in which special relativity is applicable, is therefore conceived here as a subspace of a more general space of possibilities. The local relational variables that describe the settings of the gravitational field with respect to the material content are generalized from those that describe the special uniform field. Again, the assumption that can warrant this generalization is that the mathematical structure corresponds to the space of physical possibilities, this time the space in question is this of the new gravitational theory.

In order for the analogical inference to the electromagnetic case to be warranted by the conjectured existence of the gauge field, similar conditions should hold. Primarily, the free Dirac equation has to be understood as describing not merely the limit in which the strength of the interaction is zero, but rather a special case of an interaction. The analogy does not have to be simple and straightforward; in this case we are obviously not talking about uniform electric or magnetic field. There is indeed a famous situation that demonstrates that equation (1) is valid in a certain domain that is influenced by the electromagnetic interaction: the Aharonov-Bohm effect. In this case, the electromagnetic interaction changes the phase relation between two domains, even though each of them separately is described by the free Dirac equation (1). Thus, the transition from Eq. (2) to the theory of interaction described by (3) can be similarly understood in the same basis. Eq. (3) with the constraint $\partial_\mu A_\nu - \partial_\nu A_\mu = 0$ describes a gauge field whose possible local states correspond to the different representation of the situation described by the interaction-free theory. Relaxing this condition generalizes the space of local possible relations between the fermion field and the gauge field. Similarly, in Yang-Mills theory the preferred local isospin convention motivates the conjecture of the existence of the Yang-Mills field. The relation between the preferred convention to other, general local isospin conventions, corresponds to possible relations between the state of the fermion field and the new field B_μ.

5.4 A Response to Steiner Challenge?

The idea that gauge symmetries express the relational nature of physical quantities is a new idea. If this idea is correct, it means that physicists who developed gauge theories somehow managed to suggest successful theories, but for the wrong reasons. Surely, this would not be the first instance of physicists formulating successful theories based on considerations or concepts that are later abandoned or replaced by other justifications. Indeed, this might be the situation in the case of Yang and Mills's theory. According to 't Hooft (2005, pp 1-2), in contrast to Yang and Mills's locality desideratum (see Section 3.1), it is actually "easy enough to write down equations for perfectly localized fields that show only global, continuous symmetries such as isospin". 't Hooft further states that "As often happens with obviously false statements, they were eagerly embraced by some enthusiastic followers; yet this cannot be the real reason why the theory became as important as it is today. [...] And so it happened that, by asking a rather ill-posed question, Yang and Mills made a momentous discovery: electro-magnetism and gravity are not the only force laws one can write down that have a local symmetry".

However, even if one accepts the claim that the reason for the success of the gauge methodology is the significance of relational quantities, to what extent would it constitute an answer to Steiner's challenge? Even if the naturalist would find *ex post facto* a physical similarity at the basis of each and every mathematical analogy that contributed to our successful theories, would it make the world appear less "user friendly"? After all, Steiner's claim for anthropocentricity is based upon the very possibility of discovering the laws through mathematical analogies.

The account presented above, however, provides another perspective on the applicability of mathematics in physics, by emphasizing the methodological significance of the conjectured relation of representation between certain mathematical concepts and physical objects. Namely, the physicists who used the gauge argument to make guesses were not completely misguided: they were correct in regarding the states connected by gauge transformations as representing the same state of affairs. This amounts to rejecting the anthropocentric view of mathematics (at least when it comes to the mathematics relevant in the physical context), and regard the development of the mathematical representation of the world in physics as an outcome of a gradual process of scientific inquiry through trial and error. Indeed, the development of physics provides numerous examples of mathematical analogies that lead to failed theories. In our context it seems relevant to note the gravitational theory of Barbour and Bertotti (1977), which was constructed using an invariance requirement manifestly analogous to the one at the basis of the gauge argument, but led to empirically inadequate predictions. Particle physics itself may (controversially) provide another example, in the recent failures to detect Supersymmetry (whose analogy with the case of gauge was explicitly noted, for example, in Yang, 1980). Such cases highlight the significance of retrospective understanding of the applicability of mathematics. The physicist who is trying to guess a previously unknown law is facing a completely different challenge from the one who is trying to understand why a particular guess led to an empirically successful theory while other ones have failed.

Thus, in the course of the scientific inquiry, we update on the one hand the way nature is represented in our theories according to laws that proved to be empirically adequate. On the other hand, in order to extend the laws and try to replace them with more general ones, scientists regard certain mathematical concepts as a representation of reality. Applying invariance requirements is a manifestation of this reasoning, reflecting Wigner's [1967, p. 962] desideratum that "two different descriptions of the same situation should develop, in the course of time, into two descriptions which also describe the same physical situation". Which distinct mathematical representations should be regraded as a description of one reality is of course a matter of one's ontological picture. The naturalist's ontological picture is shaped by the progress of science and reflects the empirical knowledge available.

6 Taking Stock

Steiner's anthropocentric Pythagoreanism non-trivially combines two different assertions: that the human perspective on the universe is a privileged one, and that physics can be reduced to mathematics (or at least that some of its natural kinds are those of mathematics).[20] The glue that holds these two elements together is Steiner's anthropocentric view concerning the applicability of mathematics. On this view, mathematics is a human pursuit, it is a domain demarcated by the anthropocentric values that guide its development. Thus, there seems to be no naturalist justification for the apparent reliance on purely mathematical considerations as a guide to novel scientific discovery.

The crux of Steiner's argument concerns his account of what constitutes mathematical, as opposed to physical, reasoning. It is a question of where the line between physics and mathematics should be drawn. On this view, mathematics is a human pursuit, it is a domain demarcated by the anthropocentric values that guide its development. Thus, there seems to be no naturalist justification for the apparent reliance on purely mathematical considerations as a guide to novel scientific discovery. In this paper, we have presented two different approaches to this question.

In accounting for the success of the mathematical analogies underwriting the gauge argument, the idealist and naturalist face different challenges. For the idealist, the concern is to respond to Steiner's appeal to a form of Pythagoreanism. The strategy adopted in Section 4 is to highlight the constitutive role that mathematics plays in scientific cognition. It was suggested that the analogies underwriting the application of the gauge principle serve to extend the constitutive framework, through which nature is understood, to a new domain. This framework entails that nature is grasped in terms of the group structures that define the objective features of 'reality'. They serve as a necessary presupposition for the 'gauge-theoretic' worldview. In this sense, one appeals to analogies to construct a theoretical harmony in our account of nature. However, in adopting this idealist account, one takes a much stronger view than Steiner concerning the role of mathematics in physics. The line between mathematics and physics is not merely blurred, it is no longer existent. Thus, modern theoretical physics is seen to be anthropocentric, in a similar sense as mathematics.

In contrast, for the naturalist, the most pressing worry is to avoid any form of anthropocentrism. The strategy adopted in Section 5 is to portray the mathematical description of the world in our physical theories as the outcome of a process of scientific inquiry. The invariance requirement that initiates the gauge argument is a major example of the way naturalist values (in the anti-anthropocentric sense) play a role in this process.

[20] For more on Steiner's Pythagoreanism see Ben-Menahem (2021).

The lingering concern for the naturalists is whether, in trying to avoid anthropocentrism, they are forced to reduce their account of reality to the mathematical concepts of our theories, and thereby adopt a strong form of Pythagoreanism. In order to argue that the formal similarity underlying Pythagorean analogies should be associated with a physical similarity, the naturalist needs to draw a line between physics and mathematics. Without this line, it remains unclear what content, if any, the claim for a physical similarity has beyond the mathematical similarity.

The heart of the naturalist account of gauge presented here is the conjectured existence of a physical object that was not part of the interaction free theory. This is the basis for the physical similarity between the different applications of the argument. However, the motivation for this step is grounded in the formalism (the existence of a preferred class of representations), and accordingly, at the end of the day the account yields a mathematical description ("gauge field") of the properties required to define this preferred class of representations.

Is the naturalist account, therefore, no more than a form of Pythagoreanism? To try to answer this question, let us note that while a gauge field is described mathematically and has a similar form to familiar mathematical concepts (a connection), it is also distinguished from such concepts due to its dynamical properties. The difference between the fixed mathematical connection in equations (2), (4) and the fields in equations (3), (6) demonstrates the difference. The state of the field is contingent and dynamically influenced by physical properties such as mass and charge of matter.[21] These properties of the interaction are robust, and do not depend on a realist, literal reading of the theory. It is enough to acknowledge that the gauge field is a mathematical description of some aspect of the modal structure that makes the empirical success of the theory possible. This role can not be fully understood in terms of the mathematical properties of a connection. They are captured instead in physical notions such as causality, action-reaction principle, scattering processes etc., all of which are arguably essential non-mathematical parts of a physical theory. It is not possible to account for the practice through which predictions are extracted from the formalism without appealing to such non-formal notions. Whether these aspects suffice to dispel the worries of Pythagoreanism depends, among other things, on one's definition of Pythagoreanism. We shall leave this question open-ended.

Another worry raised by Steiner is that many theoretical physicists seem to suffer from "intellectual schizophrenia" (p. 73). They adopt a naturalist belief while at the same time adopting "anthropocentric methods of discovering physical laws". Here again the naturalist and the idealist would disagree. From the naturalist perspective presented here, Steiner's problem is a genuine concern in the context of the gauge methodology. It urges the naturalist on the one hand to emphasize the significance of non-anthropocentric (or even anti-anthropocentric) values to the scientific practice,

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[21] Norton (1995) notes that Pauli as well as Weyl raised similar points in 1921 with respect to the contingent and dynamical nature of the metric coefficients in general relativity. Regarding Yang and Mills's field, 't Hooft (2005) (p. 3) similarly notes: 'One could regard it as a mere mathematical artifact; today we would call such a field a 'background field'. They [Yang and Mills] emphasize that this would be physically unacceptable. If these fields exist at all, they must be endowed with dynamical properties'.

and on the other hand to to look for a material explanation for the success of the argument as an alternative to Steiner's anthropocentric explanation. For the neo-Kantian idealist, this is not a concern, given that it is the constitutive framework of scientific thought that serves to guide the broader application of mathematics in the formulation of novel physical theories.

However, the problem is that this idealist account seems to entail that the empirical success and failure of modern physics is, at least in part, based on a form of self-deception. Nietzsche (1873/2010) expressed this view, in a discussion of natural law from a Kantian perspective:

> what is a law of nature as such for us, anyway? [...] all we really know is what we ourselves bring to them—time, space, hence relations of succession and number. Everything wondrous that we marvel at in the laws of nature, that demands explanation and could lead us to distrust idealism, however, lies precisely and exclusively in the mathematical rigor and inviolability of the representations of time and space. This, though, we produce in ourselves and out of ourselves with the same necessity with which the spider spins its web; if we are constrained to conceive all things only under these forms, then it is no wonder that we do in fact conceive of all things in just these forms, for they must all bear in themselves the laws of number, and number is precisely what is most astonishing in things. The lawlike uniformity that so impresses in the orbits of the stars and in chemical processes ultimately coincides with those properties we ourselves bring to things, so that it is we who are impressing ourselves. (p. 39-40)

To the Weylian idealist, the context has changed, but not much else. The 'miracle' of gauge theory is now simply contained within the mathematical representations of the local group structures that underwrite our understanding of the abstract symmetries of modern physics. But on a 'Weylian' reading, both the 'geometrical' constitutive framework of classical physics and the more 'abstract' framework of modern physics can be seen to 'spring from the same source', to paraphrase Weyl (1918) and it is in this source that we can look to ground the 'miraculous' success of gauge theory. From a naturalist point of view, this kind of account may be regarded, at the most, as a description of mental, social or intellectual processes through which certain mathematical concepts found their way into our theories. It cannot be regarded as explaining the empirical success or failure of those theories (that in many cases came decades after the formulation of the theories).

On this backdrop, one may wonder to what extent we should regard the idealist and the naturalist approaches as providing competing answers to the same set of questions. Rather, they may merely be focused on different aspects of scientific practice and thought. While these questions will not be settled here, it does seem clear at this point that any meaningful discussion of them in the context of modern physics has to faithfully describe the mathematical forms of reasoning employed in theory construction. Steiner's rich concepts of mathematical, formal and Pythagorean analogies provide a general yet insightful framework for such a discussion.

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