Quasi-set theory for a quantum ontology of properties

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Abstract

In previous works, an ontology of properties for quantum mechanics has been proposed, according to which quantum systems are bundles of properties with no principle of individuality. The aim of the present article is to show that, since quasi-set theory is particularly suited for treating aggregates of items that do not belong to the traditional category of individual, it supplies an adequate meta-language to speak of the proposed ontology of properties and its structure.

1 Introduction

After more than a hundred years of the first recognition of quantum phenomena, quantum mechanics continues to be a source of perplexity. In addition to the problem of quantum measurement, which has concentrated most interpretive efforts, the theory presents ontological problems that challenge various basic assumptions of traditional metaphysics. For example, quantum systems cannot be endowed with definite properties in the same way as classical systems –contextuality–, they seem to manifest correlations between properties that cannot be explained in terms of interactions –non-separability–, and they have a statistical behavior that is essentially different than that of collections of classical objects –indistinguishability–.

Each one of these problems has been treated in the literature on the foundations and the philosophy of quantum mechanics, but in general they are considered in isolation. The challenge is, then, to offer a comprehensive view, that is, to delineate an ontological picture that provides a solution to the different challenges in a unified way. This task has been undertaken in previous works (Lombardi and Castagnino 2008, da Costa et al. 2013, da Costa and Lombardi 2014, Lombardi and Dieks 2016), in which an ontology of properties has been proposed: according to this view, in the quantum realm the category of individual is absent and quantum systems are bundles of properties with no principle of individuality. This ontological picture becomes natural in the light of the algebraic approach to quantum mechanics, whose primary element is the algebra of observables, whereas states are represented by functionals on that algebra, which are defined to yield probabilities and expectation values.

Although that ontology of properties proved to be fruitful to tackle the three main ontological challenges of quantum mechanics, the proposal faced an essential difficulty from a formal viewpoint. It is not difficult to see that the algebraic formalism gives an adequate mathematical representation of an ontology of properties. However, it is not easy to find an appropriate meta-language to speak of such an ontology, since traditional logical formalisms, from classical and deviant logic to set theory, are based on both the categories of property and individual. The main aim of this work is to show that, given its specific formal features, quasi-set theory turns out to be particularly suited for treating aggregates of items that do not belong to the ontological category of individual but that of property, and which, for this reason, cannot be formally treated as individual objects. As a consequence, quasi-set theory supplies an adequate meta-language to speak of the proposed ontology of properties and its structure. With this purpose, the paper is organized as follows. In Section 2, the quantum ontology of properties is introduced from a formal-mereological viewpoint, stressing certain relevant links and differences with respect to traditional metaphysics. Section 3 is devoted to recall how such an ontology of properties, deprived from the category of individual, successfully faces the main ontological problems of quantum mechanics: contextuality, non-separability, and indistinguishability. In Section 4, the limitations of traditional logical formalisms to deal with an ontology of properties are discussed. In Section 5, the basics of quasi-set theory are introduced, stressing those formal features that are relevant for its application to the proposed ontology. On the basis of the content of the previous sections, Section 6 shows how quasi-set theory can be used to represent indistinguishable properties and indistinguishable bundles of properties, the ontological correlates of indistinguishable systems. Finally, in Section 7 some general conclusions are drawn.

2 A quantum ontology of properties

It cannot be denied that a formalism does not determine its interpretation. In fact, any non-contradictory formal system has different interpretations. Nevertheless, this does not mean that formalisms are ontologically neutral: different formal systems may suggest different ontologies even if they are equivalent. For example, the theory of natural numbers can be formulated both on the basis of Peano's axioms (Peano 1889) and in terms of Russell's set-theoretic construction (Russell 1903): although mathematically equivalent, the two formulations have different ontological connotations. From Peano's perspective, natural numbers admit a realistic, Platonist interpretation; Russell's formulation, by contrast, is more friendly to a nominalist interpretation, according to which reality is populated by individuals and classes, but not by natural numbers.

As in the above case, different formalisms for standard quantum mechanics, although mathematically equivalent, evoke different ontological pictures. In the Hilbert space formalism, a quantum system is represented by a Hilbert space, whose vectors represent its states; observables are represented by operators acting on the Hilbert space. The mathematical priority of systems and states over observables is easily mirrored by an ontology of individuals, endowed with ontological priority over its properties. By contrast, in the algebraic formalism, a quantum system is represented by an algebra of observables, and states are functionals on that algebra. If this mathematical priority of observables on states is transposed to the ontological domain, the result is an ontology whose primary items are properties, and objects arise as the convergence of those properties. On this basis, an ontology of properties for quantum mechanics has been proposed (Lombardi and Castagnino 2008, da Costa et al. 2013, da Costa and Lombardi 2014, Lombardi and Dieks 2016), described by the algebraic formalism of the theory. Let us develop this idea in more precise terms.

In the algebraic framework, a quantum system is represented by a $\star$-algebra $\mathcal{A}$ of observables $A \in \mathcal{A}$, closed under product, linear combination, and involution. A state of the system is represented by a normed and positive expectation-value functional $\omega : \mathcal{A} \longrightarrow \mathbb{C}$ belonging to the dual algebra $\mathcal{A}'$. A state ω is pure when it cannot be written as a non-trivial convex combination $\omega = \lambda_1\omega_1 + \lambda_2\omega_2$, with $0 < \lambda_1, \lambda_2 < 1$, $\lambda_1 + \lambda_2 = 1$, and $\omega_1, \omega_2 \in \mathcal{A}'$; otherwise it is mixed. If $\mathcal{A}$ is a $C^\star$-algebra, the Gelfand-Naimark-Segal (GNS) construction (Gelfand and Naimark 1943, Segal 1947) shows that $\mathcal{A}$ can be mathematically represented by a set of Hermitian operators $\mathcal{O}$ on a Hilbert space $\mathcal{H}$, and states are represented by normed trace (density) operators ρ on $\mathcal{H}$. When the state represented by the density operador is pure, there is a vector $|\psi\rangle \in \mathcal{H}$ such that $\rho = |\psi\rangle\langle\psi|$ (for different *-algebras, other representations of the algebra have been proved; for instance, a nuclear algebra can be represented by a rigged Hilbert space, see Iguri and Castagnino 1999; for applications of rigged Hilbert spaces to quantum mechanics, see Bohm and Gadella 1989). From now on, we will not distinguish between the abstract algebraic language and the language of the mathematical representation; then, we will say that a quantum system is represented by the algebra $\mathcal{O}$ of observables $O \in \mathcal{O}$, and that the system's states are expectation-value functionals $\rho(O) = \mathrm{tr}(\rho O) = \langle O \rangle_\rho \in \mathbb{R}$, for all $O \in \mathcal{O}$. Moreover, given two component systems represented by the algebras of observables $\mathcal{O}^1$ and $\mathcal{O}^2$, the composite system is represented by $\mathcal{O}^1 \vee \mathcal{O}^2$, that is, the minimal algebra generated by $\mathcal{O}^1$ and $\mathcal{O}^2$.

Let us now describe the ontological counterpart of the algebraic formalism by supplying an interpretation of each physical (mathematical) term:

- The term 'observable' is used in quantum physics to denote certain quantifiable magnitudes of physical relevance, which are mathematically represented by Hermitian operators. Ontologically, they correspond to items belonging to the category of property. We will distinguish between *universal type-properties* (*U-type-properties*) and *instances of universal type-properties* (*I-type-properties*). The ontological counterparts of general physical magnitudes are U-type-properties, and those of observables are I-type-properties. We will symbolize an U-type-property as A^U, and its I-type-properties as $[A]$. An example of U-type-property is energy H^U, which can be instantiated as the energy of a particular system S as $[H]$. Let us stress that, although this talk suggests an ontology of individuals, below we will define the concept of quantum system as an non-individual ontological item.

- Since a physical observable is a quantifiable magnitude, it has possible values, which are mathematically represented by the eigenvalues of the corresponding Hermitian operator. Their ontological counterpart are the *possible case-properties* (*P-case-properties*) of the corresponding I-type-property. Given an I-type-property $[A]$ of a U-type-property A^U, we will symbolize its P-case-properties by a_i. Following with the above example, we can talk of the P-case-properties ω_i (the energy spectrum) of the energy $[H]$ of the particular system S, where $[H]$ is an I-type-property of the U-type-property energy H^U.

- In physics it is implicitly assumed that each observable cannot have more that one value at a time. The effective value acquired by an observable has no direct mathematical representation: there is not formal way to distinguish it from the remaining values. But, ontologically, it is relevant to stress that, given an I-type-property $[A]$ of a U-type-property A^U, *no more than one* of its P-case-properties $[a_i]$ becomes actual; we will symbolize that *actual case-property* (*A-case property*) as $[\mathbf{a}_k]$. In the above example, $[\omega_k]$ is the effective value of the energy $[H]$ of the particular system S.

- Since in the algebraic formalism a quantum system S is mathematically represented by an algebra of observables, its ontological counterpart is a *bundle* $\mathcal{B} = \{[A], [B], [C], \ldots\}$ of the I-type-properties $[A], [B], [C], \ldots$ of the corresponding U-type-properties $A^U, B^U, C^U, \ldots$.

- The physical concept of state is mathematically represented by a expectation-value functional over the space of observables. Precisely due to its probabilistic nature, the state of a system S encodes the *ontological propensities to actualization* for all the P-case-properties of all the I-type-properties belonging to the bundle $\mathcal{B}$, ontological counterpart of S.

Although the above presentation describes the structure of the ontology proposed as the reference of standard quantum mechanics, before going on some philosophical remarks are in order. According to a certain philosophical tradition, properties are *universals*, that is, "one-in-many": a universal is one (e.g. redness), but it is instantiated in a multiplicity of cases (red in this case, red in that case). The so-called "problem of universals" has permeated the history of philosophy from its beginnings, with questions about how universals exist, how they relate with their instances, among many others (see MacLeod and Rubenstein 2006). This is not the place to discuss these matters; the only point to stress here is that the instances of a universal property are many but absolutely indistinguishable: they are only numerically different. In the metaphysics of 20th century a new approach appeared: properties as *tropes*. Tropes are *particular* properties, like the particular shape, weight, and texture of an individual object (see Maurin 2018). Although absolutely similar, tropes are neither absolutely indistinguishable nor only numerically different, because they can be individuated and distinguished by their space-time position (redness here and now), or by the individual to which they apply (red of this individual balloon). Since the elemental items of the quantum ontology should be adequate to supply the foundations to quantum indistinguishability, an ontology of tropes would face the same difficulties as an ontology of individuals, since in both cases they are distinguishable items. For this reason, in our proposal the elemental category of the quantum ontology is that of universal property and its instances.

Another distinction regarding properties, usually not sufficiently taken into account in the philosophy of physics literature, is the traditional difference between *determinables* and *determinates*, which are properties that stand in a distinctive specification relation, let us call it 'determination' (see Wilson 2017). For example, color is a determinable having red, blue, and other specific shades of color as determinates; shape is a determinable having rectangular, oval, and other specific (including many irregular) shapes as determinates; mass is a determinable having specific mass values as determinates. The determination relation appears to differ from other specification relations. In contrast with the genus-species and conjunct-conjunction relations, where the more specific property can be understood as a conjunction of the less specific property and some independent property or properties, a determinate is not naturally treated in conjunctive terms (red is not a conjunctive property having color and some other property or properties as conjuncts, as man can be conceived as the conjunction between animal and rational). And in contrast with the disjunction-disjunct relation, where disjuncts may be dissimilar and compatible (as with 'red or round'), determinates of a determinable are both similar and incompatible (red and blue are similar in both being colors, but nothing can be simultaneously and uniformly both red and blue). Here we have used the terms 'type-properties' and 'case-properties' for determinables and determinates, respectively, just to emphasize that they belong to the ontological category of property.

Although in the present proposal quantum systems are ontologically characterized as bundles of properties, it is important to emphasize the peculiarity of this view. In its traditional version, the *bundle theory* is a theory about individual objects, according to which objects are composed of items of a different ontological category (namely, properties). In other words, the bundle theory is designed to account for individuals objects without appealing to a substratum on which properties inhere (see, e.g., O'Leary-Hawthorne 1995, French 2019). With this purpose, some properties must be selected to play the role of the principle of individuality that supplies synchronic and diachronic identity. Our bundle view, by contrast, completely dispenses with the ontological category of individual: bundles of properties do not behave as individuals at all; they belong to a different ontological category. On this basis, when two bundle-systems combine, the composite system is also a bundle. And since bundles are not individuals, there is no principle of individuality that preserves their identity in the composition: in the composite system the identity of the components is not retained, precisely because they are not individuals.

Finally, it is necessary to say some words about the concept of possibility, whose nature has been one of the most controversial issues in the history of philosophy. Two general ways of conceiving possibility can be distinguished (see Menzel 2018). According to *actualism*, everything that exists, when analyzed in depth, is actual: the discourse about possibility can be reduced to a language that only refers to what actually exists; as a consequence, the predicate 'actual' is redundant. For instance, according to Bertrand Russell (1919) 'possible' means 'sometimes', whereas 'necessary' means 'always'. For *possibilism*, by contrast, possibility is an ontologically irreducible feature of reality: possible items do not need to become actual to be real. In Aristotelian terms, being can be said in different ways: as possible being or as actual being. Given the essential probabilistic nature of quantum phenomena, in the present proposal possibility is conceived in non-actualist terms. The fact that a type-property has possible case-properties, and that the state gives the measure of the corresponding possibilities, has nothing to do with a limitation of our knowledge about an underlying determinate state of affairs. Probabilities measure possibilities conceived as *propensities to actualization*, which are ontologically irreducible to the extent that the theory is irreducibly indeterministic.

3 Facing the ontological challenges

As it is well-known, metaphysics is underdetermined by physics; in particular, quantum mechanics is compatible with different ontological pictures, one containing individuals and another without them (see, e.g., van Fraassen 1985, 1991, French 1989). Thus, arguing for a certain physical ontology over others requires showing how fruitful it is in offering reasonable solutions to interpretive problems. In our case, we will advocate for on ontology of properties by underlining the advantages of this picture to deal with the ontological challenges of quantum mechanics: contextuality, non-locality, and indistinguishability.

3.1 Contextuality

As it is well known, the Kochen-Specker theorem (Kochen and Specker 1967) proves the impossibility of ascribing precise values to all the observables of a quantum system simultaneously, while preserving the functional relations between commuting observables: which observables can be ascribed precise values must be a "contextual" fact. In a traditional ontology of individuals with their properties, quantum contextuality breaches the *principle of omnimode determination*, a basic assumption in modern philosophy (see Lombardi and Dieks 2016). According to this principle, in every individual all determinables are determinate: for instance, if the determinable "shape" applies to an object, it necessarily has some determinate shape, say "round", independently of whether we know which determinate shape is. This holds even from the perspective of the traditional bundle theory, according to which an individual is a bundle of all the determinate properties corresponding to the determinable properties: for example, a particular billiard ball is the convergence of a definite value of position, say here, a definite shape, say round, a definite color, say white, etc. According to quantum contextuality, by contrast, not all the determinables of the system are determinate: for instance, if position is determinate, momentum is not. This challenges the concept of individual, both in the substratum-plus-properties and in the bundle-of-properties view.

In our ontological language, contextuality is expressed by saying that, given a bundle, not all of its I-type-properties actualize, that is, acquire an (actual) A-case-property among all their (possible) P-case-properties. In other words, the Kochen-Specker theorem introduces a limitation regarding A-case-properties, more precisely, regarding on which P-case-properties of a bundle can enter actuality. But this restriction does not affect our concept of quantum system, because it is defined as a bundle of type-properties (determinables) and not of actual case-properties (determinates) as in the traditional bundle theory. In turn, since the ontology is populated only by properties and bundles of properties, and not by individuals, the principle of omnimode determination of individuals does not apply: the proposed ontology of properties is immune to the challenge represented by the Kochen-Specker theorem.

3.2 Non-separability

Unlike the classical world, the quantum domain admits surprising correlations between the properties of distant non-interacting systems, such as those of the infamous Einstein-Podolsky-Rosen experiment. From the viewpoint of the state of the composite system, those correlations appear when the state is entangled. Taken at face value, EPR-correlations strongly suggest non-locality, that is, non-local influences between distant systems, i.e., systems between which no light signal can travel. However, since this idea seems to be incompatible with special relativity, the exact nature of those quantum correlations is a matter of ongoing controversy. For instance, according to collapse interpretations, EPR-correlations imply a certain action at a distance which, nevertheless, does not allow sending information at a superluminal velocity; in the case of Bohmian mechanics, it is the quantum field that has the necessary non-local features to induce EPR-correlations (see Berkovitz 2016). From a different perspective, those correlations are consequences of the holistic nature of quantum systems, understanding *holism* as the opposite of *separability*. Separability implies that, if a physical object is constructed by assembling its physical parts, then its physical properties are completely determined by the properties of the parts and their relationships. Holism, by contrast, is the feature of some physical objects that are not composed of physical parts but are indivisible wholes; so, EPR-correlations are correlations between properties of a single holistic object (see Healey 2016).

Despite the disagreements about this particular feature of quantum mechanics, in general the arguments are based on the assumption that quantum systems are individual objects, and subsystems are also individuals. As a consequence, the problem is to explain how the properties of those individual subsystems are instantaneously correlated in spite of the fact that they are not in interaction and they might be located in different spatial positions. Or, from a holistic perspective, the challenge is to account for the fact that the individual system is a whole that cannot be decomposed into individual parts. However, from an ontology of properties, the problem appears in a new light.

The ontological category of individual involves some "principle of individuality" that, independently of its specific form, identifies a particular individual as different from others and as the same over time (see discussion in French and Krause 2006). As a consequence, when two individual systems interact as to yield a composite system in an entangled state, they retain their identity as individual parts of the new whole. For this reason EPR-correlations are conceived as the correlations linking the properties of those individual subsystems, and when they are distant in space and do not interact, such correlations become puzzling. By contrast, in our ontology of properties quantum systems are not individuals but non-individual bundles of properties: there is no principle of individuality that allows them to preserve their individuality when merged into a new bundle-system. Therefore, the issue of interpreting EPR-correlations acquires a new formulation from the very beginning. The problem does not longer consist in accounting for correlations between the properties of non-interacting and possibly distant objects. Since the composite bundle is a single whole, non analyzable in component bundles, the EPR-correlations are correlations between properties of a single item and, thus, the mystery of the original formulation vanishes. In a certain sense, this view implies a kind of holism; but since in this case the holistic item is not an individual, the fact that it lacks individual parts turns out to be natural.

3.3 Indistinguishability

Many discussions about the ontological commitments of quantum mechanics point to the serious challenge posed by the indistinguishability of the so-called "identical particles" to the ontological category of individual. The usual story begins by considering the distribution of two particles, 1 and 2, over two states $|a\rangle$ and $|b\rangle$. Tthe question is: How many combinations –complexions– are possible for the composite system? The classical answer is given by the Maxwell-Boltzmann statistics, according to which there are four possible complexions: the principle of individuality, no matter which one, makes particle 1 in $|a\rangle$ and particle 2 in $|b\rangle$ a different complexion from particle 1 in $|b\rangle$ and particle 2 in $|a\rangle$. By contrast, in quantum statistics (Bose-Einstein and Fermi-Dirac), a permutation of the particles does not lead to a different complexion since particles are indistinguishable.

Although the theory has formal resources to deal with quantum statistics, from a conceptual viewpoint the problem is to explain why a permutation of the particles does not lead to different complexions in the quantum case. This has triggered the huge amount of literature about individuals, non-individuals, quasi-individuals, identity, non-identity, labels, strong distinguishability, weak indistinguishability, the Principle of the Identity of Indiscernibles, and many others distinctions directed to cope with the nature of "indistinguishable" particles (see, e.g. French 2019 and the extensive list of references therein). In the light of these multiple debates, at present a kind of underdetermination of the metaphysics by the physics is generally accepted: quantum mechanics is compatible with two distinct metaphysical "packages," one in which quantum systems are regarded as individuals and one in which they are not. However, it is interesting to stress that this conclusion is drawn by considering exclusively the problem of indistinguishability, forgetting the problems of contextuality and non-locality, which also challenge the category of individual from completely different reasons. On the other hand, these discussions are almost always framed in terms of "particles" and in the Hilbert-space formalism. Our ontological view, based on the algebraic formalism, offers a completely different approach to the problem of indistinguishability.

In an ontology of properties, indistinguishability is primarily a relation that holds between two instances of a same universal type-property when they have the same case-properties: *two I-type-properties* $[A]$ and $[A']$ are *indistinguishable* when they are I-type-properties of the same U-type-property A^U and they have the same P-case-properties, $[a_i] = [a'_i]$. From this primary meaning, indistinguishability acquires a derived meaning when applied to bundles: *two bundle-systems are indistinguishable* when their respective I-type-properties are indistinguishable. Both indistinguishable I-type-properties and indistinguishable bundle-systems are only numerically different. Nevertheless, this does not imply that the Principle of Identity of Indiscernibles is false for them: whereas the principle refers to the identity of indiscernible individuals, in this case indistinguihability is a relation between items belonging to the ontological category of property. In other words, the Principle of Identity of Indiscernibles does not apply to the non-individual quantum systems of a quantum ontology of properties.

When indistinguishable bundles combine, it is natural to expect that the I-type-properties belonging to the composite bundle do not distinguish between those component bundles. Said in simple terms, when two indistinguishable bundles merge into a single whole, which component bundle is taken first and which second does not matter at all. Mathematically, this requires that the observables, representing the I-type-properties belonging to the composite system, are symmetric with respect to the permutation of the component bundles. In this way, the so-called Indistinguishable Principle is satisfied: all quantum states differing only by a permutation of indistinguishable particles are observationally indistinguishable, that is, they lead to the same expectation values for any observable of the system. But the principle is satisfied not because the state is symmetrized or anti-symmetrized, but because *only symmetric obervables are allowed* (see Messiah and Greenberg 1964). In summary, in the proposed quantum ontology of properties, the Indistinguishable Principle does not need to be considered an ad hoc postulate of the theory, but turns out to be a consequence of the ontologically motivated symmetry of the observables of systems composed of indistinguishable bundles.

4 How to talk about an ontology of properties?

Up to this point, we have used three different languages, mathematical, physical, and ontological, in order to establish the correspondence among those three domains in the description of the quantum world. This correspondence is summarized in Figure 1.

It is clear that we can describe the ontological items listed in the third column by means of the mathematical items of the first column. But, how can we talk, in a direct way, about the items listed in the third column? How can we rigorously treat the items populating an ontology deprived from individuals? The ontology of individuals and properties is captured by the structure of natural languages, in which proper names or definite descriptions are assigned to individuals. The fact that an individual possesses properties is linguistically expressed by predicates applied to a subject. As Peter Strawson states in his classical book *Individuals*, an individual is "[a]nything whatever can be introduced into discussion by means of a singular, definitely identifying substantival expression" (1959: 137). This structure is also that of the usual systems of logic, which make use of constants and variables that are subject to predication and, thus, denote classical individuals. For instance, in first order logic, the sentence 'Pa' says that the property corresponding to the predicate 'P' applies to the individual denoted by the individual constant 'a'; likewise, in the expressions '$\forall x Px$' and '$\exists x Px$ ', the range of the variable x is understood to be a domain of individuals. Wittgenstein is clear about this point when he says that "the variable name 'x' is the proper sign of the pseudo-concept object. Wherever the word 'object' ('thing', 'entity', etc.) is rightly used, it is expressed in logical symbolism by the variable name. For example in the proposition 'there are two objects which...' by '$\exists x, y$'." (1921, Proposition 4.1272). As Wittgenstein thus emphasizes, "object" is not a concept that is defined within a logical language, but is rather a category that is presupposed by a language. Categories are not "said" but are "shown" by language. The category of object is shown by the language's structure: it can be read off from the use of constants and variables. The essential role of individual constants and variables is not limited to traditional logic: the vast majority of systems of logic, even extensions of traditional logic and deviant systems, also appeal to them (see Haack 1974, 1978). And the same applies to traditional set theory: '$a \in A$' expresses the fact that the element 'a' belongs to the set of individuals represented by 'A'.

An ontological domain populated exclusively by properties and non-individual bundles of properties cannot be adequately apprehended by any language that includes individual constants and variables. Recently, Steven French (2020) noticed that our view of non-individual bundles is conceptually close to ontic structural realism (Ladyman 1998), according to which physical objects are not elemental items of the ontology, but rather are "reduced to mere 'nodes' of the structure, or 'intersections' of the relevant relations" (French 2006: 173). French and Ladyman also notice the limitations of traditional languages to describe this kind of ontology, due to the "the descriptive inadequacies of modern logic and set theory which retains the classical framework of individual objects represented by variables and which are the subject of predication or membership respectively" (French and Ladyman 2003: 41).

In a quantum ontology as that proposed here, systems do not belong to the category of individual but are bundles of I-type-properties. We have expressed this idea by saying that a system, mathematically represented by an algebra of observables, has its ontological counterpart in a bundle $\mathcal{B} = \{[A], [B], [C], \ldots\}$ of the I-type-properties $[A], [B], [C], \ldots$ of the corresponding U-type-properties $A^U, B^U, C^U, \ldots$. However, it is clear that here the symbol "$\{\ldots\}$" does not represent a set: bundles are not sets, precisely because their "members" are not individuals but items belonging to the ontological category of property. But, if this is the case, how to formalize the relationship between a bundle and its I-type-properties if the bundle is not a set? How to say that a certain I-type-property $[A]$ "belongs" to the bundle $\mathcal{B}$ if they are not linked by the traditional relation of membership? It is precisely for this reason that we need an adequate formal language for our quantum ontology of properties. In the following section, we will search for this language in the quasi-set theory.

5 Elements of the quasi-set theory

Quasi-set theory was first introduced by Décio Krause (1990) (see also Krause 1992) to cope with collections (quasi-sets or *qsets*) of indistinguishable items, which are called 'indiscernible' in the context of the theory. Those items are conceived as "*solo numero*" different, that is, only numerically discernible; as a consequence, they challenge the Principle of Identity of Indiscernibles. The theory makes possible to deal with collections of indiscernible items without the usual tricks of confining them into equivalence classes or non-rigid (deformable) structures: in quasi-set theory one can speak and treat those items without those standard tricks and with logical precision.

Although quasi-set theory was devised to supply a formalism for indiscernible items, it also has a "standard" part, which is equivalent to the ZFA set theory (Zermelo-Fraenkel with the Axiom of Choice plus Urelemente, see Jech 2003: 250ff), in whose context all the standard mathematical formalism can be developed. This means that adopting quasi-set theory is not an obstacle to the precise characterization of the Hilbert space or the algebraic formalisms.

Let us use $\mathfrak{Q}$ to denote a first order theory whose primitive vocabulary contains, besides the vocabulary of standard first order logic without identity (propositional connectives, quantifiers, etc.), the following specific symbols: (i) three unary predicates m, M, Z, (ii) two binary predicates $\in$ and $\equiv$, and (iii) one unary functional symbol qc. Notice that identity is not part of the primitive vocabulary, and that the only terms in the language are variables and expressions of the form $qc(x)$, where x is an individual variable, and not a general term (this restriction prevents, for instance, $qc(qc(x))$ from being a term). The intuitive meaning of the primitive symbols is the following:

(i) $x \equiv y$ (x is *indiscernible* from y)

(ii) $m(x)$ (x is an *m-atom*, denoting an item that can stand in a relationship of indiscernibility with others)

(iii) $M(x)$ (x is an *M-atom*, an atom of ZFA)

(iv) $Z(x)$ (x is a '*set*' –a copy of a ZFA set)

(v) $qc(x)$ (*the quasi-cardina*l of x)

Notice that identity is not part of the primitive vocabulary; the theory has a defined notion of *extensional identity*, symbolized by '$=_E$', which is defined only for M-atoms that belong to the same quasi-sets or quasi-sets having the same elements (where the precise meaning of 'the same' will be given by Definition 1(v)). The underlying logic of $\mathfrak{Q}$ is a *non-reflexive logic*, a logic where the standard theory of identity is limited in some way; in our case, by restricting the application of identity (see French and Krause 2006: Chapter 8, Arenhart 2014).

Some relevant definitions are given, with their corresponding intuitive interpretation:

Definition 1 *(i)* $Q(x) := \neg(m(x) \vee M(x))$
(x is a qset)

(ii) $P(x) := Q(x) \wedge \forall y(y \in x \rightarrow m(y)) \wedge \forall y \forall z(y \in x \wedge z \in x \rightarrow y \equiv z)$
(x is a pure qset, having only indiscernible m-atoms as elements)

(iii) $D(x) := M(x) \vee Z(x)$
(x is a Ding, a 'classical object' in the sense of Zermelo's set theory, namely, either a set or a 'macro Urelement')

(iv) $E(x) := Q(x) \wedge \forall y(y \in x \rightarrow Q(y))$
(x is a qset whose elements are qsets.)

(v) $x =_E y := (Z(x) \wedge Z(y) \wedge \forall z(z \in x \leftrightarrow z \in y)) \vee (M(x) \wedge M(y) \wedge \forall z(x \in z \leftrightarrow y \in z))$
(extensional identity)— we shall write simply $x = y$ *instead of* $x =_E y$ *from now on. Notice that although the expression* $x = y$*, when either* x *or* y *is an* m*-atom, can be written, it does not have any meaning in the theory.*[1] *The notion of identity applies to sets and to* M*-atoms only. Furthermore, just to explain the terminology, sometimes we use relativized quantifiers: for instance,* $\forall_Q x\gamma$ *means* $\forall x(Q(x) \to \gamma)$*, while* $\exists_Q x\gamma$ *means* $\exists x(Q(x) \wedge \gamma)$*; the same for predicates other than* Q*.*

(vi) $x \subseteq y := \forall z(z \in x \to z \in y)$
(x *is a subqset of* y*) (A subtle remark is in order here. Since identity does not hold for* m*-atoms, we may be in trouble in trying to prove that a certain* m*-atom belongs to a certain qset, for it should be identical to some element of it. This fact does not matter for our purposes. The definition says that if* z *belongs to* x *then* z *belongs to* y*. In* $\mathfrak{Q}$*, it suffices to prove (or to assume) that there is an indiscernible from* z *in* x*.*[2]

As we have said, $\mathfrak{Q}$ is compatible with the existence of two kinds of Urelements, m-atoms and M-atoms, and also of collections formed either by atoms or by other collections, the qsets, or by both, atoms and qsets. The theory does not postulate the existence of atoms. Some qsets are specially important: when their transitive closure does not contain m-atoms, they contain only what we call the 'classical objects' of the theory (objects satisfying D); items fulfilling this condition satisfy the predicate Z and match with the sets in ZFA. So, classical mathematics can be built in the context of $\mathfrak{Q}$, in its *classical part*.

The main idea motivating the development of the theory is that some items are non-individuals (roughly speaking, items for which the standard notion of identity does not apply), and which do not satisfy the notion encapsulated in the definition of extensional identity. As explained above, this concept is not defined for m-atoms,which intuitively represent indistinguishable items. Thus, on one hand, those items "do not have identity", that is, it does not make sense to say they are identical or different; on the other hand, the relation of indiscernibility holds for every item of the theory, so m-atoms may be indiscernible without being identical. It is important to notice that a qset of indiscernible m-atoms may have a quasi-cardinal greater than one, say 5, so five items can be conceived although they cannot be discerned from each other in any way. Furthermore, quantified expressions must be interpreted adequately; when a qset C of indistinguishable objects is considered, the universal quantifier means 'all elements of C', while the existential quantifier means 'some element of C'. Thus, universal quantification does not refer to 'each' element of the qset (which would presuppose identity) as the standard interpretation suggest (for more on

[1] This is similar to name $\mathcal{R}$ the collection $\mathcal{R} = \{x : x \notin x\}$ (Russell's set), which can be expressed in the language of ZFC but is not a *set* of this theory, supposed consistent.

[2] For instance, in a Litium atom $1s^2 2s^1$, it suffices to say that there is one electron in the outer shell; it does not matter which one (really, it makes no sense to identify a particular electron there).

this point, see Krause and Arenhart 2017).

Besides the postulates for classical first-order logic without identity (which we shall not list here), quasi-set theory introduces specific postulates for $\mathfrak{Q}$.

($\equiv_1$) $\forall x(x \equiv x)$

($\equiv_2$) $\forall x \forall y(x \equiv y \to y \equiv x)$

($\equiv_3$) $\forall x \forall y \forall z(x \equiv y \wedge y \equiv z \to x \equiv z)$

($=_4$) $\forall x \forall y(x = y \to (\alpha(x) \to \alpha(y)))$ (where $\alpha(x)$ is a formula with free variable x) with the usual restrictions (Mendelson 1997: 95).

The first three postulates say that indiscernibility is an equivalence relation. Due to the fact that this relation does not preserve structure with respect to the relation of membership of qsets, indiscernibility is not a congruence. In turn, since it is not necessarily compatible with other primitive predicates, identity and indiscernibility can be kept as distinct concepts. In fact, if x and y are indiscernible m-atoms and z is a qset, then $x \in z$ does not entail that $y \in z$, and conversely. The fourth postulate says that substitutivity holds only for identical items, that is, for "classical" items. This last remark is fundamental for the understanding of the theory: substitutivity *does not hold in general.* It does not hold for m-atoms since they cannot be identical but only indiscernible; for instance, if $x \in z$ and $x \equiv y$, nothing in the theory enables us to conclude that $y \in z$ (in fact, models where just $x \in z$ holds can be constructed). In other words, substitutivity of indiscernibles does not hold.

Remark

Somebody may say that we are presupposing identity in the metalanguage when we say that variables x and y are different. This is true, but does not collapse the theory. A similar situation occurs, for instance, with some paraconsistent logics (da Costa et al. 2007), which are apt to deal with contradictory sentences: the Principle of Contradiction in the form $\neg(\alpha \wedge \neg\alpha)$ does not hold in general. However, in elaborating such systems, that is, in the metalevel, the principle is used as being true, for nobody would suggest that some expression (finite sequence of symbols of the language), say, is a formula *and* is not a formula (this "constructive" character of scientific theories $-$and of logics$-$ is discussed in Krause and Arenhart 2017: Chapter 3).

Other postulates are:

($\in_1$) $\forall x \forall y(x \in y \to Q(y))$
If something has an element, then it is a qset; in other words, atoms have no elements (in terms of the membership relation).

($\in_2$) $\forall_D x \forall_D y(x \equiv y \to x = y)$
Indiscernible *Dinge* are extensionally identical. This makes $=$ and $\equiv$ coincide for this kind of items.

($\in_3$) $\forall x \forall y[(m(x) \wedge x \equiv y \to m(y)) \wedge (M(x) \wedge x = y \to M(y)) \wedge (Z(x) \wedge x = y \to Z(y))]$

($\in_4$) $\exists x \forall y(\neg y \in x)$
This qset can be proved to be a set (in the sense of obeying the predicate Z), and it is unique, as it follows from the axiom of weak extensionality we shall see below. Thus, it can be denoted, as usual, by '$\emptyset$'.

($\in_5$) $\forall_Q x(\forall y(y \in x \to D(y)) \leftrightarrow Z(x))$
This postulate grants that something is a set (obeys Z) iff its transitive closure does not contain m-atoms. That is, sets in $\mathfrak{Q}$ are those entities obtained in the "classical" part of the theory.

($\in_6$) $\forall x \forall y \exists_Q z(x \in z \wedge y \in z)$

($\in_7$) If $\alpha(x)$ is a formula in which x appears free, then

$$\forall_Q z \exists_Q y \forall x(x \in y \leftrightarrow x \in z \wedge \alpha(x)).$$

This is the *Separation Schema.* We represent the qset y as follows:

$$[x \in z : \alpha(x)], \;\; \textit{or} \;\; [x : \alpha(x)]_z.$$

When this qset is a set, we write, as usual, $\{x \in z : \alpha(x)\}$.

($\in_8$) $\forall_Q x(E(x) \to \exists_Q y(\forall z(z \in y \leftrightarrow \exists w(z \in w \wedge w \in x)))$.

From ($\in_6$), by the Separation Scheme and using $\alpha(w) \leftrightarrow w \equiv x \vee w \equiv y$, a subqset of z is obtained, which can be denoted as

$$[x, y]_z$$

This is the qset of the indiscernible elements from either x or y that belong to z. When $x \equiv y$, this qset reduces to

$$[x]_z$$

which is called the qset of the indiscernibles from x that belong to z. The qset $[x, y]_z$ does not necessarily have only *two* elements (that is, it may happen that $qc([x, y]_z) > 2$), for there may be more than just one indiscernible of x or y in z. Given the qset z and one of its elements, x, the collections $[x]$ and $[x]_z$ stand for *all* the indiscernible from x and for the qset of the indiscernible from x that belong to z, respectively (usually, $[x]$ is too large to be a qset $-$as in general it is a collection of *all* objects so and so, as in standard set theory).

With the postulates of quasi-cardinality (see below), it can be proved that $[x]_z$ has a subqset whose quasi-cardinal equals to 1, written as

$$[[x]]_z$$

which will be called '*strong singleton*' of x (actually, *a* strong singleton of x, since its unicity cannot be granted). The strong singleton of x has just one element, which can be conceived *as if* it were x; however, from the definition it follows that all can be said is that $[[x]]_z$ contains *one object of the "species"* x. In other words, since $qc([[x]]_z) = 1$, there is one item indiscernible from x in this qset. In order to prove that this item is x, identity sould be necessary. But notice again that the theory makes the equality $qc([[x]]_z) = 1$ meaningful because it deals with cardinals, which are "classical objects".

Further postulates are:

($\in_9$) $\forall_Q x \exists_Q y \forall z (z \in y \leftrightarrow z \subseteq x)$,
The *Power qset* of x is written $\mathcal{P}(x)$.[3]

($\in_{10}$) $\exists_Q x (\emptyset \in x \wedge \forall y (y \in x \rightarrow y \cup [y]_x \in x))$,
The *Infinity Axiom.*

($\in_{11}$) $\forall_Q x (E(x) \wedge x \neq \emptyset \rightarrow \exists_Q y (y \in x \wedge y \cap x = \emptyset))$,
The *Axiom of Foundation*, where $x \cap y$ is defined as usual.

Some definitions are necessary to introduce the correlates of certain classical notions:

Definition 2 ***Weak Ordered Pair****. The weak ordered pair $\langle x, y \rangle_z$ is defined as*

$$\langle x, y \rangle_z := [[x]_z, [x, y]_z]_z$$

Definition 3 ***Cartesian Product****. Let z and w be two qsets. the cartesian product $z \times w$ is defined as*

$$z \times w := \langle x, y \rangle_{z \cup w} : x \in z \wedge y \in w$$

Functions and relations can neither be defined as usual because, when m-atoms are involved, a mapping may not distinguish between arguments and values. Therefore, a wider definition for both concepts, which reduce to the standard ones when restricted to classical entities, must be provided.

Definition 4 ***Quasi-relation****. A qset R is a binary quasi-relation between two qsets z and w if its elements are weak ordered pairs of the form $\langle x, y \rangle_{z \cup w}$, with $x \in z$ and $y \in w$.*

Definition 5 ***Quasi-function****. f is a quasi-function between two qsets z and w if and only if f is a quasi-relation between z and w such that for every $x \in z$ there is a $y \in w$ such that if $\langle x, y \rangle_{z \cup w} \in f$, $\langle u, v \rangle_{z \cup w} \in f$, and $x \equiv u$, then $y \equiv v$.*

[3] It is interesting here that we would be in trouble to teach quasi-set theory to children. For instance, take a qset x with cardinal 2 so that its elements (call them y and z) are indiscernible. Now try to write the qset $\mathcal{P}(x)$. It cannot be done in a menaingful way. Actually, the two subsets with quasi-cardinal 1 are indiscernible (by the Weak Extensionality Axiom, see below), so something like $\mathcal{P}(x) = [\emptyset, [y], [z], x]$ has no clear meaning. Nevertheless, as we will see from axiom (qc_7), the quasi-cardinal of $\mathcal{P}(x)$ is 4.

In simple words, a quasi-function maps indiscernible items onto indiscernible items. More specific kinds of functions, that is, injections, surjections and bijections, can be defined with some restrictions, but we will not introduce them here (see French and Krause 2006: Chapter 7).

We already know that in $\mathfrak{Q}$ the standard notion of identity is not defined for some entities (Definition 1(v)). However, identity is essential to define many of the usual set theoretic concepts of standard mathematics, such as well order, the ordinal attributed to a well ordered set, and the cardinal of a collection. Since identity makes no sense for some items in $\mathfrak{Q}$, how can these notions be employed? One alternative is looking for formulations based on methods that do not rely on identity. Another possibility is introducing these concepts as primitive and giving adequate postulates for them.

Concerning the notion of cardinal, there are interesting issues to be acknowledged. First, in $\mathfrak{Q}$ there cannot be well-orders on qsets of indiscernible m-atoms. In fact, a well-order would imply, for example, that there is a least element relative to this well order, a notion which could only be formulated if identity were defined for m-atoms, for this element would be different from any *other* element in the qset. Second, the usual claim that aggregates of quantum entities can have a cardinal but not an ordinal demands a distinction between the notions of ordinal and of cardinal of a qset; this distinction is made in $\mathfrak{Q}$ by the introduction of cardinals as a primitive notion, called quasi-cardinals.[4]

Let us recall the postulates for quasi-cardinals (for details and motivations, see French and Krause 2006, French and Krause 2010). Here $\alpha, \beta, ...$ stand for cardinals (defined as usual in the classical part of $\mathfrak{Q}$, that is, in the theory $\mathfrak{Q}$ when we rule out the m-atoms):

(qc_1) $\forall_Q x \exists_Z y(y = qc(x)) \rightarrow \exists! y(card(y) \wedge y = qc(x) \wedge (Z(x) \rightarrow y = card(x)))$
In words, if the qset x has a quasi-cardinal, then its (unique) quasi-cardinal is a cardinal (defined in the 'classical' part of the theory) and coincides with the cardinal of x *stricto sensu* if x is a set. As recalled above, this axiom does not grant that every qset has a well defined quasi-cardinal.

(qc_2) $\forall_Q x(\exists y(y = qc(x) \rightarrow x \neq \emptyset \rightarrow qc(x) \neq 0))$
Every non-empty qset that has a quasi-cardinal has a non-null quasi-cardinal.

(qc_3) $\forall_Q x(\exists_Z \alpha(\alpha = qc(x)) \rightarrow \forall \beta(\beta \leq \alpha \rightarrow \exists_Q z(z \subseteq x \wedge qc(z) = \beta)))$
If x has quasi-cardinal α, then for any cardinal $\beta \leq \alpha$, there is a subqset of x with that quasi-cardinal.

In the remaining axioms, for simplicity, we shall write $\forall_{Q_{qc}} x$ (or $\exists_{Q_{qc}} x$) for quantifications over qsets x having a quasi-cardinal.

(qc_4) $\forall_{Q_{qc}} x \forall_{Q_{qc}} y(y \subseteq x \rightarrow qc(y) \leq qc(x))$

[4] As shown by Domenech and Holik (2007), we can define quasi-cardinals for finite qsets in $\mathfrak{Q}$, without resulting that the qset will have an associated ordinal in the usual sense.

(qc_5) $\forall_{Q_{qc}} x \forall_{Q_{qc}} y(Fin(x) \wedge x \subset y \to qc(x) < qc(y))$

It can be proven that if both x and y have a quasi-cardinal, then $x \cup y$ has a quasi-cardinal. Then,

(qc_6) $\forall_{Q_{qc}} x \forall_{Q_{qc}} y(\forall w(w \notin x \vee w \notin y) \to qc(x \cup y) = qc(x) + qc(y))$

In the next axiom, $2^{qc(x)}$ denotes (intuitively) the quantity of subquasi-sets of x. Then,

(qc_7) $\forall_{Q_{qc}} x(qc(\mathcal{P}(x)) = 2^{qc(x)})$

This last postulate enables us to think of subqsets of a given qset in the usual sense; for instance, if $qc(x) = 3$, the postulate says that there exists $2^3 = 8$ subqsets, and according to (qc_3), there are subqsets with 0, 1, 2 and 3 elements. Furthermore, as explained above, in $\mathfrak{Q}$ it can be proved that, given any object $a \in z$ (either an m-atom, M-atom or quasi-set), the strong singleton of a, $[[a]]_z$, can be obtained, whose quasi-cardinal is 1. It is worth to insist that saying, within $\mathfrak{Q}$, that a is the only element of $[[a]]_z$ makes no sense, to the extent that identity is necessary to prove such a statement. Anyway, $\mathfrak{Q}$ is consistent with this idea and we may reason *as if* this is really so. So, one can think that, in the context of $\mathfrak{Q}$, one may have *a certain m-atom*, without identifying it, except that it has some characteristics or properties and not others (for instance, it may be discernible from another m-atom b). That indiscernible m-atoms may have different properties can be seen from the fact that $\mathfrak{Q}$ *doesn't prove* the substitutivity of indiscernibles, that is,

$$\mathfrak{Q} \nvdash a \equiv b \to \forall_Q z(a \in z \leftrightarrow b \in z).$$

In order to prove this result, let us consider $[[a]]_z$. Since $qc([[a]]_z) = 1$, a and b cannot belong both to this qset, except if a is identical to b, which cannot be assumed in the case of m-atoms. So, in an extensional context ($\mathfrak{Q}$ is also a kind of extensional theory, although this should be qualified), we can read $a \in z$ as a having a certain "property" (whose "extension" is z). Then, even indiscernible m-atoms may have different properties.[5] Furthermore, if a and b are two indiscernible m-atoms, the strong singletons $[[a]]_z$ and $[[n]]_z$ are also indiscernible (by the Weak Extensionality Axiom, see below), hence there are no differences among them (yet they are not *the same* qset).

The *Weak Extensionality Axiom* generalizes the usual Extensionality Axiom. Intuitively, it grants that two qsets with the same quantity of the same kinds of elements are indiscernible. For that, two new definitions are needed: the notion of similarity between qsets, denoted by *Sim*, and the notion of Q-similarity, denoted by *Q-sim*. Intuitively speaking, similar qsets have elements of the same kind, and Q-similar qsets have elements of the same kind, and in the same quantity:

[5] When quasi-set theory is applied to elemental particles, this allows us to say that the two electrons in an Helium atom in its fundamental state have different values of spin in a given direction.

Definition 6 *(i)* $Sim(x, y) := \forall z \forall w(z \in x \wedge w \in y \rightarrow z \equiv y)$;

(ii) $Q\text{-}sim(x, y) := Sim(x, y) \wedge qc(x) = qc(y)$.

The **Weak Extensionality Axiom** reads as follows:
$(\equiv_{12})$ $\forall_Q x \forall_Q y((\forall z(z \in x/\equiv \rightarrow \exists t(t \in y/\equiv \wedge\ Q\text{-}sim(z, t)))) \wedge \forall t(t \in y/\equiv \rightarrow \exists z(z \in x/\equiv \wedge\ Q\text{-}sim(t, z))) \rightarrow x \equiv y)$

Intuitively speaking, qsets that have "the same quantity" of elements (given by their quasi-cardinals) of the same kind are indiscernible.

The following theorem proves the invariance under permutations in $\mathfrak{Q}$. Its demonstration requires assuming that $y \subseteq x$ entails $qc(x - y) = qc(x) - qc(y)$; let us call this result Theorem $(\star)$ (for the proof, see French and Krause 2006: Chapter 7). So, the theorem goes as follows:

Theorem 7 ***Invariance under permutations***. *Let x be a finite qset such that $\neg(x = [z]_t)$ for some t, and let z be an m-atom such that $z \in x$. If $w \in t$, $w \equiv z$, and $w \notin x$, then there exists $[[w]]_t$ such that*

$$(x - [[z]]_t) \cup [[w]]_t \equiv x$$

Proof: Case 1: the only element of $[[z]]_t$ does not belong to x. Then $x - [[z]]_t = x$. Let w be so that its only element belongs to x (for instance, it may be z). Then $(x - [[z]]_t) \cup [[w]]_t = x$, hence the theorem. Case 2: the only element of $[[z]]_t$ belongs to x. Then $qc(x - [[z]]_t) = qc(x) - 1$ by the mentioned Theorem $(\star)$. Let $[[w]]_t$ be such that its only element is w itself, so $(x - [[z]]_t) \cup [[w]]_t = \emptyset$. Hence, by Postulate (qc_7), $qc(x - [[z]]_t) = qc(x)$. Thus, by the Weak Extensionality Axiom, the theorem follows (for more details, see French and Krause 2006: Chapter 7).

This theorem shows that, when two indiscernible elements z and w, with $z \in x$ and $w \notin x$, expressed by their strong-singletons $[[z]]_t$ and $[[w]]_t$, are "permuted", the resulting qset x remains indiscernible from the original one. The hypothesis that $\neg(x = [z]_t)$ grants that there are indiscernible from z in t which do not belong to x. In this context, the *Axiom of Choice* reads as follows:

$$\begin{aligned} \forall_Q x(E(x) \wedge \forall y \forall z(y &\in x \wedge z \in x \rightarrow y \cap z = \emptyset \wedge \neg(y = \emptyset)) \rightarrow \\ \exists_Q u \forall y \forall v(y &\in x \wedge v \in y \rightarrow \exists_Q w(w \subseteq [v]_y \wedge qc(w) = 1 \wedge w \cap y \equiv w \cap u))). \end{aligned}$$

6 A quasi-set representation for an ontology of properties

Although quasi-set theory is a formal theory, it was devised by its author with an intended interpretation: m-atoms were conceived as the correlates of elemental particles, which ontologically are *nonindividuals* (see Arenhart, Bueno, and Krause 2019), so that the formal relation of indiscernibility represented the physical relation of indistinguishability between elemental particles of the same kind. Nevertheless, precisely due to its formal nature, quasi-set theory can be detached from that intended interpretation and can be applied to items belonging to any ontological category, whenever the logical "behavior" of those items agrees with the structure of the theory. In fact, here quasi-set theory will be applied to a universe of properties, in particular, of I-type-properties, which under certain circumstances can be considered as legitimately indistinguishable due to their own ontological nature.

Let us recall that, in the proposed ontology of properties, indistinguishability is not a relation between particles or other individual systems, precisely because there are no individuals in the ontology. Indistinguishability is primarily a relation that holds between I-type-properties, when they are instances of a same U-type-property and they have the same P-case-properties. Quantum systems, since bundles of I-type-properties, are only derivatively indistinguishable when their elements are indistinguishable. Then, let us begin by characterize those I-type-properties and bundles whose indistinguishability has to be formally characterized.

In quantum mechanics, space and time are Galilean: time is homogeneous, space is homogeneous and isotropic, and their properties are represented by the group of the Galilean transformations. The Galilean group is defined by the commutation relations between its generators. In absence of external fields, these generators represent the basic magnitudes of the theory: the energy, the three momentum components, the three angular momentum components, and the three boost components. Mathematically, the Galilean group is a Lie group, whose Casimir operators (operators that commute with all the generators of the group and are thus invariant under all the transformations of the group) are the operators corresponding to the observables mass M, internal energy W, and squared spin S. In turn, the Galilean group has irreducible representations, in which the Casimir operators are multiples of the identity: $M = mI$, $W = wI$, and $S = s(s+1)I$. As a consequence, each irreducible representation is labeled by a triplet (m, w, s).

In physics it is assumed that each irreducible representation of the Galilean group represents a kind of *elemental particle*, characterized by its mass m, internal energy w and its spin s. Following previous works (da Costa et al. 2013), the ontological correlate of the physical concept of elemental particle will be called '*atomic bundle*': A bundle is *atomic* if (i) there is no more than one I-type-properties of each U-type-properties in it, (ii) the I-type-properties $[M]$, $[W]$, and $[S]$ of the U-type-properties mass M^U, internal energy W^U, and squared spin S^U, respectively, always belong to it, and (iii) those I-type-properties have each a single P-case-property (they are mathematically represented by operators that are multiples of the identity). Therefore, an atomic bundle is defined by a collection of I-type-properties as follows: $\mathcal{B} = \{[M], [W], [S], [A], [B], [C], \ldots\}$. As already pointed out, $\mathcal{B}$ is not a set because its members are not individuals; as a consequence, it will be precisely defined by means of the tools of quasi-set theory.

In order to simplify the presentation, a further step is in order. According to the spectral decomposition theorem, any operator $A \in \mathcal{O}$ representing an observable can be expressed as

$$A = \sum_i a_i \Pi_i^A \in \mathcal{O}$$

where $a_i \neq a_j$ f or i$\neq j$ are the eigenvalues of A, and the Π_i^A are the eigenprojectors of A. Therefore, if an observable represented by the operator A corresponds to the I-type-property $[A]$ of a bundle-system, without loss of generality it can be replaced by the observables represented by the eigenprojectors Π_i^A of A. Each Π_k^A, which has only two eigenvalues, represents an observable correlated to an I-type-property $[\Pi_k^A]$ with two possible P-case-properties: 1, corresponding to the P-case-property a_k of the I-type-property $[A]$, and 0 corresponding to any of the other P-case-properties a_i, with $i \neq k$, of the I-type-property $[A]$. As a consequence, without loss of generality the atomic bundle-system can be expressed as:

$$\mathcal{B} = \{[M], [W], [S], [\Pi_1], [\Pi_2], [\Pi_3], \ldots\}$$

with $[\Pi_1], [\Pi_2], [\Pi_3], \ldots \in \mathcal{T}$, where $\mathcal{T}$ is the collection of all the I-type-properties mathematically represented by the projectors projecting on the closed subspaces of the Hilbert space corresponding to the system. On this basis, now indistinguishability can be transferred from I-type-properties to atomic bundles, in order to supply the ontological correlate of indistinguishability as a relation between elemental particles, in the way in which it is conceived in the physical language. In the context of this ontology of properties, *two atomic bundles are indistinguishable* when their respective I-type-properties are indistinguishable.

As already emphasized, although this ontological picture has a clear metaphysical meaning, the problem is how to formally deal with bundles that cannot be characterized as sets due to the fact that their members are not individuals but properties, which are items without identity. It is at this point that quasi-set theory comes to our aid. The first step is to consider that, for each I-type-property $[\Pi]$, there is a qset $U([\Pi])$, which can be assumed to have a denumerable infinite quasi-cardinal, and whose members are indiscernible from each other (since they are I-type-properties of the same U-type-property and have the same P-case-properties). The qsets $U([M])$, $U([W])$, and $U([S])$ can be characterized in an analogous way. The second step is to apply the Axiom of Choice, in order to build a new qset with only one element of each one of the above qsets. With this purpose, let us define the following proposition:

$$\begin{aligned} R(x) := {} & (qc(x \cap U([M])) = 1) \wedge (qc(x \cap U([W])) = 1) \wedge (qc(x \cap U([S])) = 1) \\ & \wedge (\forall \Pi (qc(x \cap U([\Pi])) = 1)) \end{aligned}$$

where $qc(x)$ represents the quasi-cardinal of x. In natural language, $R(x)$ can be expressed as follows: 'the quasi-cardinal of the intersection of x and $U([M])$ is one, the quasi-cardinal of the intersection of x and $U([W])$ is one, the quasi-cardinal of the intersection of x and $U([S])$ is one, and for all $[\Pi] \in \mathcal{T}$, the quasi-cardinal of the intersection of x and $U([\Pi])$ is one'. Intuitively, the x that satisfies proposition $R(x)$ is a qset that picks up one I-type-property of each qset $U([M])$, $U([W])$, $U([S])$, and $U([\Pi])$ for all $[\Pi] \in \mathcal{T}$, no matter which one since the members of each qset are indiscernible. Therefore, the qset x is an adequate formal representation of the atomic bundle $\mathcal{B}$.

Let us now define the qset Q as the union of $U([M])$, $U([W])$, $U([S])$, and $U([\Pi])$ for all $[\Pi] \in \mathcal{T}$, and let be $\mathcal{P}(Q)$ the power qset of Q, which will also have a non-denumerable infinite quasi-cardinal. On this basis, the following qset can be defined:

$$\mathcal{Q}(\mathcal{B}) := \{x \in \mathcal{P}(Q) \mid R(x)\}$$

The elements of $\mathcal{Q}(\mathcal{B})$ are qsets that are indiscernible from x, so any of its members can represent the bundle $\mathcal{B}$. It is important to introduce a remark about the existence of any element of $\mathcal{Q}(\mathcal{B})$: How can the fact that $\mathcal{Q}(\mathcal{B})$ is not empty be guaranteed? In order to prove the existence of at least one element of $\mathcal{Q}(\mathcal{B})$, the Axiom of Choice can be used: it grants the existence of the desired qset. On this basis, it can be guaranteed that all the members of $\mathcal{Q}(\mathcal{B})$ are indiscernible from each other: if $x \in \mathcal{Q}(\mathcal{B})$ and $y \in \mathcal{Q}(\mathcal{B})$, then $x \equiv y$ since their corresponding members are indiscernible.

Up to this point, given a certain atomic bundle $\mathcal{B}$ (in physical terms, a certain elemental particle) the qset $\mathcal{Q}(\mathcal{B})$ supplies the formal tool to talk about all the atomic bundles (elemental particles) that are indistinguishable from $\mathcal{B}$ and indistinguishable from each other. Nevertheless, physics requires being able to talk about a finite and definite number of indistinguishable atomic bundles (a finite and definite number of indistinguishable particles, in the physical language). Let us consider that such a number is N: any collection of N indistinguishable bundles of type $\mathcal{B}$, with N finite, can be represented by a qset $\mathcal{N}(\mathcal{B}) \subset \mathcal{Q}(\mathcal{B})$ with quasi-cardinal equal to N and whose members are indiscernible from each other. Of course, the question about which precise members of $\mathcal{Q}(\mathcal{B})$ are used to compose $\mathcal{N}(\mathcal{B})$ makes no sense, since all the elements belonging to $\mathcal{Q}(\mathcal{B})$ are indiscernible. Nevertheless, $\mathcal{N}(\mathcal{B})$ makes possible to treat in a precise formal way a collection of N indistinguishable items (N indistinguishable particles in the physical language) that are only numerically different, that is, that are N despite their indistinguishability.

7 Conclusions

Since the middle of the 20th century, the study of the formal properties of the mathematical structure of quantum mechanics produced many relevant results, unknown by the founding fathers of the theory. Those results greatly improved the understanding of the deep challenges that any interpretation must face. But only in the last decades interpreters of quantum mechanics have begun to focus their interest decidedly on ontological issues. This work is aligned with this trend: the main goal is to design and ontological picture that supplies a unified solution to the ontological problems posed by the peculiarities of quantum mechanics.

In the ontology of properties proposed here, deprived form the category of individual, the elemental items are universal properties and their instances. In this context. the ontological correlate of the physical notion of particle is the concept of bundle: a collection of property-instances lacking any principle of individuality. Therefore, the ontological challenges derived from conceiving quantum systems by means of the traditional category of individual simply dissolve in an only-property ontology.

On the other hand, quasi-set theory was originally devised to offer a formalism framework capable to deal with indistinguishable particles, conceived not as individuals but as "quasi-individuals" (French and Krause 2006). However, as emphasized above, since completely formal, quasi-set theory can be employed to deal with any kind of indistinguishable items. In the proposed quantum ontology, indistinguishability is primarily a relation between certain instances of the same universal property, and the indistinguishability between the so-called "elemental particles" turns out to be ontologically the indistinguishability between atomic bundles, which is inherited from the indistinguishability of the bundles' components. As shown in the present work, quasi-set theory is an appropriate formalism for the treatment of indistinguishable property-items. In particular, collections of indistinguishable property-instances can be formally characterized as qsets, and bundles can be formed by applying the the Axiom of Choice to those qsets: a bundle is a qset built with only one element of each one of them. Moreover, qsets of bundle-qsets, with definite quasi-cardinality, can also be defined in order to treat collections of indistinguishable bundles with a definite number of members in spite of the fact that those members can neither be counted nor labeled. In summary, a quantum ontology of properties finds in quasi-set theory an appropriate meta-language to speak of properties and collections of properties in a formally precise way.

8 References

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