

A PARTITION THEOREM FOR A RANDOMLY SELECTED LARGE POPULATION

(Forthcoming article in *Acta Biotheoretica*, Springer)

Arni S.R. Srinivasa Rao

Laboratory for Theory and Mathematical Modeling,
Division of Infectious Diseases-Department of Medicine
Medical College of Georgia,
and

Department of Mathematics, Augusta University,
1120, 15th Street, AE 1015
Augusta, GA, 30912, USA,
Tel: +1-706-721-5906 (office).
Email: arrao@augusta.edu

ABSTRACT. A theorem on the partitioning of a randomly selected large population into stationary and non-stationary components by using a property of the stationary population identity is stated and proved. The methods of partitioning demonstrated are original and these are helpful in real-world situations where age-wise data is available. Applications of this theorem for practical purposes are summarized at the end.

[2000]2020 AMS Subject Classification: 92D25 *Keywords:* Population partitions, stationary population and non-stationary populations

1. STATIONARY POPULATION

Stationary population assumptions and related mathematical formulations were of interest to Edmond Halley, an astronomer who also published in demography (in 1693, [1]), to influential mathematician Leonard Euler in the eighteenth century [2, 3, 4],

as well as to twentieth-century famous theoretical population biologist Alfred Lotka [20, 21]. A population is said to be stationary if it has a zero growth rate and a constant population age-structure, and if not the population is said to be non-stationary. Lotka associated his theory of stationary populations with the average rate at which a woman in her lifetime will be replaced by a girl (which we call the net reproduction rate). In fact, Lotka [20, 21] logically argued that the rate of natural growth, r , of a population will be zero in (1.1) when the net reproduction rate of the population, R_0 , will be equal to one (the notation R_0 was introduced by Lotka). The relation between r and R_0 is given by

$$(1.1) \quad r = \frac{\ln(R_0)}{L},$$

where L is the length of the generation of a population, which is expressed as,

$$(1.2) \quad L = \frac{1}{R_0} \int_0^{\infty} xa(x)s(x)dx,$$

where $a(x)$ is the age-specific fertility rate of women of age x to give births to girl babies, and $s(x)$ is the survival probability function for women to live up to the age x . Although R_0 is a popular demographic measure, its practical applicability in timely measuring the stationary status of a population is questioned. See, for example, [5].

We define below the stationary population identity (SPI) or the Life Table Identity for a discrete population as given in [17].

Definition 1. Stationary Population identity (SPI) or the Life Table Identity: Let, for each age between 0 and a maximum age present in a discrete stationary population or discrete stationary life table population, X denote the set of elements representing the proportions of individuals present in the population for these ages. Furthermore, let Y denote the set of elements representing the proportions of individuals in the population whose remaining times to live are equal to these ages. We say that the SPI holds if $X = Y$.

Let, for a discrete stationary population with maximum age ω , A denote the set of all integers from 0 to (and including) ω : $A = \{0, a_1, a_2, \dots, \omega\}$. So, A equals the set of discrete ages in a discrete stationary population or discrete stationary life table. The statement that the SPI holds for a discrete stationary population or discrete stationary

life table means that the proportion $f_1(x)$ of individuals with age $x \in A$ in the population is equal to the proportion $f_2(x)$ of individuals in the population who will live for x years. i.e., $f_1(x) = f_2(x) \forall x \in A$.

This identity was theoretically demonstrated by means of partitioning large populations [17]. This is frequently referred to also as the life table identity in mathematical population biology and demography literature. Such identities are found in various published literature, for example, see [6, 7, 8, 9, 10, 11]. There are other related equalities in stationary populations which suggest that the average age in a stationary population is equal to the average expectation of remaining life time (for continuous versions) [18, 19]. Although Lotka and Cox in their respective works have chosen continuous frameworks, in this article the population ages are treated in a discrete framework.

In this article, we state and prove a novel theorem that states a criterion to partition a randomly selected large discrete population into stationary and non-stationary components. The procedure consists of comparing $f_1(x)$ and $f_2(x)$ and see if they are identical. Instead of the life table population, the theorem suggests comparing the fractions of the population for each age of the actual population at a time point with the fractions of the population that have remaining ages in the corresponding life table population constructed at the same time.

In general, we do not often see stationary populations, except in life tables. But a careful investigation of any large population data suggests that a sub-population of a population could obey the properties of a stationary population. Now, through the partition theorem that is proved in this article, it becomes easier to understand the stationary component of large populations instantly, without being dependent on measures like NRR (net reproduction rate). Moreover, the theorem guarantees with certainty if a component of a large population is stationary.

2. PARTITION THEOREM AND PROOF

Theorem 2. Stationary Population Identity (SPI) partitions a randomly selected large population into stationary and non-stationary components (if such components are non-empty).

Proof. Let a randomly selected large population be given, and let the maximum age present in the population be denoted by ω . Let $P(t)$ denote the size of the population at time t , and for $x \in A$, where $A = \{0, a_1, a_2, \dots, \omega\}$. For $x \in A$, let $P_x(t)$ denote the

number of individuals with age x in the population at time t . In the current proof, we consider P as a summation over discrete ages. Then

$$(2.1) \quad P(t) = \sum_{x=0}^{\omega} P_x(t).$$

Let $g_1(x, t) = \frac{P_x(t)}{P(t)}$ for all $x \in A$. We define the stationary component of $P(t)$, denoted by $M(t)$, as the size of the sub-population of all (variously aged) individuals for which there exists a value $y \in A$ such that $g_1(y, t) = f_2(y, t)$ holds, where $f_2(y, t)$ denotes the proportion of individuals in the population at time t who will have y remaining years to live. The non-stationary component of $P(t)$, denoted by $N(t)$, is defined as the size of the sub-collection of individuals for which $g_1(z, t) \neq f_2(z, t)$ for any $z \in A$. It will be clear that at any t the equality

$$(2.2) \quad P(t) = M(t) + N(t)$$

holds. Let us choose all individuals at age y in $P(t)$ and consider

$$(2.3) \quad g_1(y, t) = \frac{P_y(t)}{P(t)}.$$

From the life table constructed for the population $P(t)$ (assumably for the single ages of the set A), we can obtain the proportion of the population who have expected remaining years within A , and compare this with the proportion of the population at each age $x \in A$ in $P(t)$. Note that a life table is a mathematical model to synthetically demonstrate the age-specific death rates of a population and to compute remaining average years to live at each age in A . There are several books available to understand details of life table constructions, for example, see [13, 14, 15, 12, 16]. We will compare $g_1(y, t)$ with another proportion $f_2(x, t) \forall x \in A$, where

$$(2.4) \quad f_2(x, t) = \frac{L_x(t)}{L(t)}.$$

Here $L_x(t)$ is the life table population at time t whose remaining years to live is x and $L(t)$ is total life table population at t . First, we match $\frac{P_{y_1}(t)}{P(t)}$ for a given $y_1 \in A$ in (2.3) with $\frac{L_x(t)}{L(t)}$ for each $x \in A$. If $\frac{P_{y_1}(t)}{P(t)}$ is equal to $\frac{L_x(t)}{L(t)}$ for some x , then we call the corresponding life table proportion of the population as $\frac{L_{y_1}(t)}{L(t)}$ (say, $f_2(y_1, t)$). That

is, $\frac{L_{y_1}(t)}{L(t)}$ is the proportion of life table sub-population who have y_1 years to live. This follows,

$$(2.5) \quad g_1(y_1, t) = \frac{P_{y_1}(t)}{P(t)} = \frac{L_{y_1}(t)}{L(t)} = f_2(y_1, t).$$

Suppose $\frac{P_{y_1}(t)}{P(t)}$ does not equal to any of the proportions $\frac{L_x(t)}{L(t)}$ for $x \in [0, \omega)$, then we denote y_1 in (2.5) by z_1 and write this situation as $g_1(z_1, t) \neq f_2(z_1, t)$. That is, for any of the proportions $\frac{L_x(t)}{L(t)}$ for $x \in [0, \omega)$, the remaining years to live is not equal to y_1 .

We will continue matching $\frac{P_{y_2}(t)}{P(t)}$ for some $y_2 \neq y_1$ and $y_2 \in [0, \omega)$ with $\frac{L_x(t)}{L(t)}$ for each $x \in A$ except for $x = y_1$. If there is a value of $\frac{L_x(t)}{L(t)}$ that equals $\frac{P_{y_2}(t)}{P(t)}$, we call the corresponding proportion in the life table population as $\frac{L_{y_2}(t)}{L(t)}$ (say, $f_2(y, t)$). $\frac{L_{y_2}(t)}{L(t)}$ is the proportion of the life table sub-population who have y_2 years to live. That is,

$$(2.6) \quad g_1(y_2, t) = \frac{P_{y_2}(t)}{P(t)} = \frac{L_{y_2}(t)}{L(t)} = f_2(y_2, t).$$

Suppose $\frac{P_{y_2}(t)}{P(t)}$ is not equal to any of the proportions $\frac{L_x(t)}{L(t)}$ for $x \in A$, then we denote y_2 in (2.6) by z_2 (if z_1 already arises in an earlier situation such that $g_1(z_1, t) \neq f_2(z_1, t)$), and we write $g_1(z_2, t) \neq f_2(z_2, t)$. However, if $g_1(y_1, t) = f_2(y_1, t)$ exists but $\frac{P_{y_2}(t)}{P(t)} \neq \frac{L_x(t)}{L(t)} \forall x \in A$, then we denote y_2 as z_1 and write $g_1(z_1, t) \neq f_2(z_1, t)$.

Similarly, for a randomly selected age $y_i \in A$, the proportion $\frac{P_{y_i}(t)}{P(t)}$ (i.e. $g_1(y_i, t)$) is matched with the proportions $\frac{L_x(t)}{L(t)} \forall x \in A$. If there is a value of $\frac{P_{y_i}(t)}{P(t)}$ that matches with $\frac{L_x(t)}{L(t)}$ then we denote it by $f_2(y_i, t)$, otherwise we denote it by $f_2(z_i, t)$ (if the previous z value in the order of unmatched proportions was denoted as z_{i-1} for $i = 2, 3, \dots$). Through this procedure we will match for all the values of $y_i \in A$, and decide whether or not $g_1(y_i, t)$ is equal to the $f_2(y_i, t)$. The criteria is: of for any age y_i in a randomly selected large population (actual population) the, if the value of the proportion $\frac{P_{y_i}(t)}{P(t)}$ is equal to any of the life table sub-population proportions $\frac{L_x(t)}{L(t)} \forall x \in A$ whose remaining years to live is exactly y_i , then $g_1(y_i, t) = f_2(y_i, t)$.

Existence of two sets $\{y_{k_j}\}$ and $\{z_{k_j}\}$:

In general, at each age $x \in A$ we can check whether

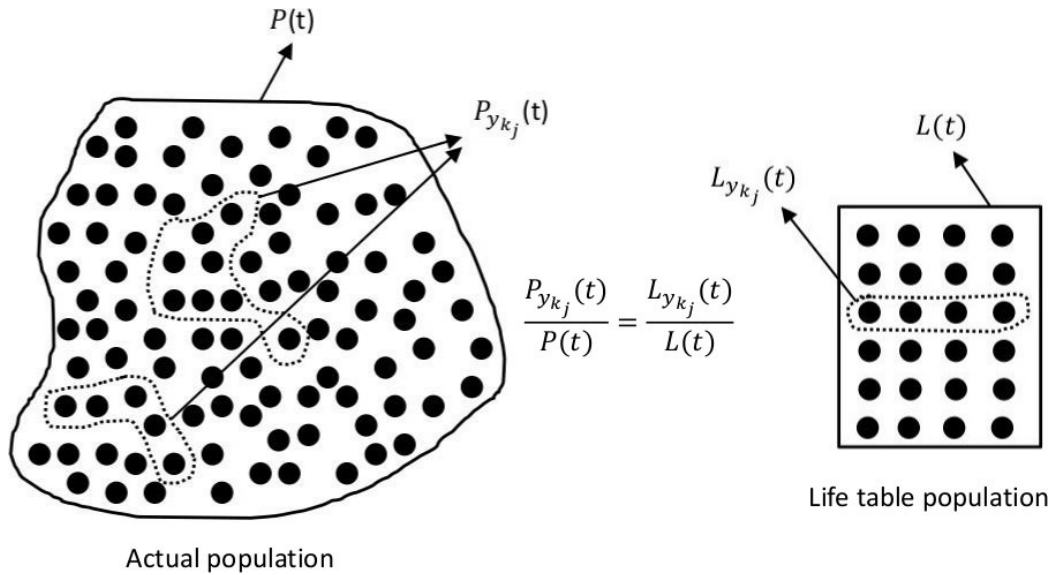


FIGURE 2.1. Proportions of sub-population in an actual population and in a life table. $P_{y_{k_j}}(t)$ denotes the size of sub-population whose individuals are of age y_{k_j} at time t , and $P(t)$ is the total actual population at time t . $L_{y_{k_j}}(t)$ is the life table population at time t of these individuals whose remaining years to live at time t is equals y_{k_j} and $L(t)$ is the total life table population at t . Here time is not to be measured on a continuous scale.

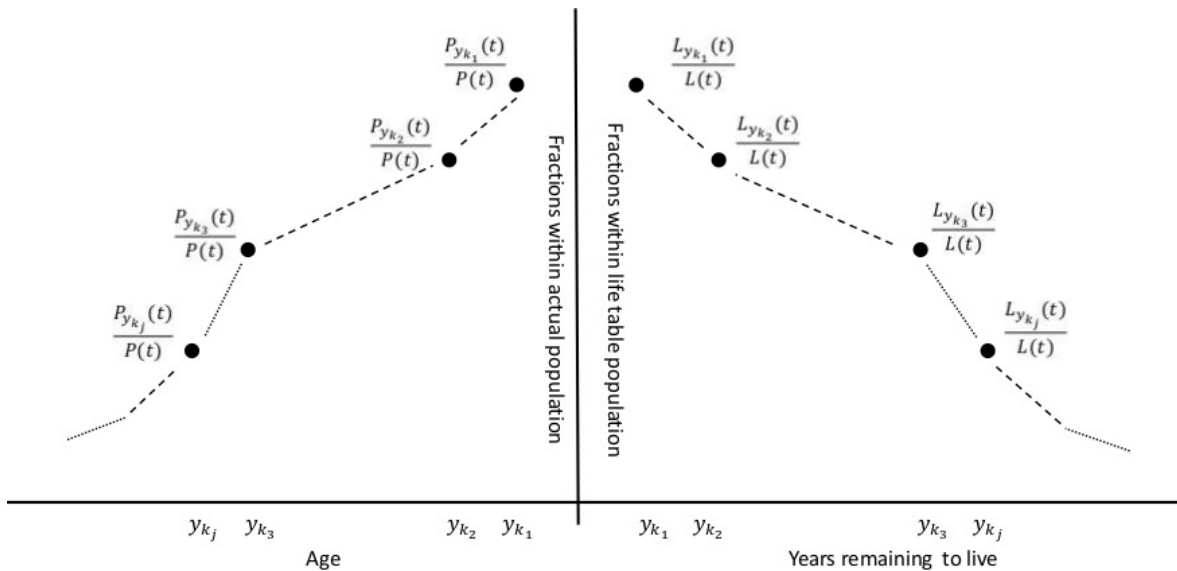


FIGURE 2.2. Stationary component of a randomly selected large population. Proportional sub-populations at ages y_{k_1}, y_{k_2}, \dots in the actual total population are identical with the proportional life table sub-populations whose individuals have remaining years y_{k_1}, y_{k_2}, \dots to live.

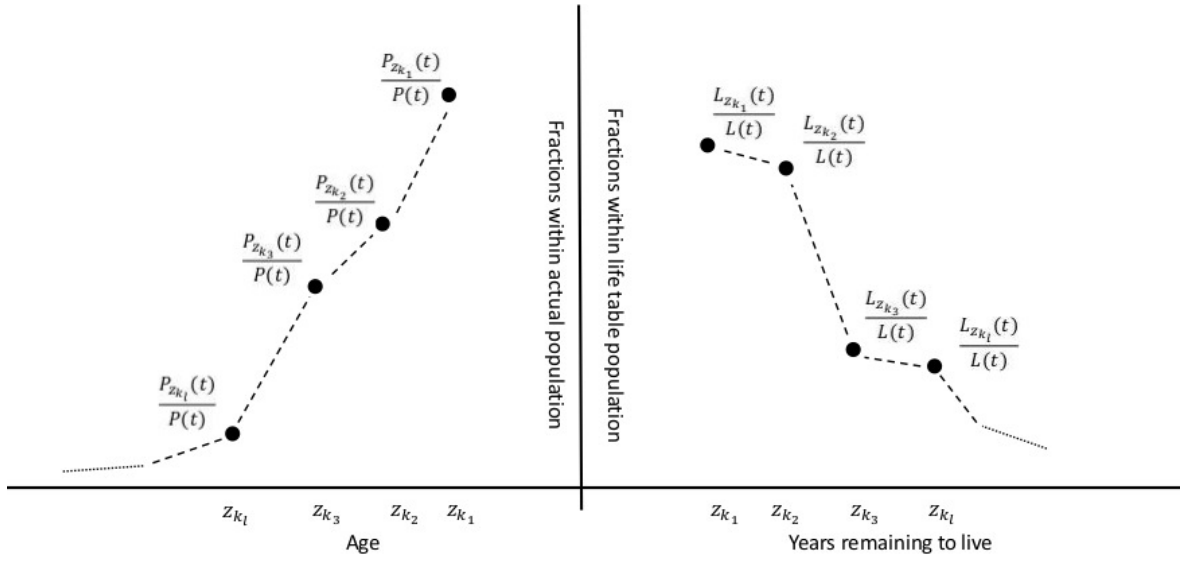


FIGURE 2.3. Non-stationary component of a randomly selected large population. Proportional sub-populations at ages z_{k_1}, z_{k_2}, \dots in the actual total population are not identical with any of the life table proportional sub-populations whose individuals have remaining years to live equal to z_{k_1}, z_{k_2}, \dots

$$(2.7) \quad \frac{P_x(t)}{P(t)} = \frac{L_x(t)}{L(t)}$$

holds, or

$$(2.8) \quad \frac{P_x(t)}{P(t)} \neq \frac{L_x(t)}{L(t)}$$

holds. See Figure 2.1 for application of (2.7), Figure (2.2) for the stationary component, and Figure (2.3) for the non-stationary component.

Suppose we start this procedure from age 0. At $x = 0$, either (2.7) holds or (2.8) holds, but not both. If (2.7) holds, then let us denote y_{k_1} for 0, or, if (2.8) holds, then let us denote z_{k_1} for 0. Hence, at age 0 one of the y_{k_1} or z_{k_1} exists, and

$$(2.9) \quad \{y_{k_1}\} \cup \{z_{k_1}\} = \{0\}$$

Now, let us consider age $a_1 > 0$ for $a_1 \in A$. At age $x = a_1$, either (2.7) holds or (2.8) holds, but not both. If (2.7) holds at $x = a_1$, and y_{k_1} exists, then let us denote y_{k_2} for a_1 . If (2.7) holds at $x = a_1$ and z_{k_1} exists, then let us denote y_{k_1} for a_1 . If (2.8) holds

at $x = a_1$ and y_{k_1} exist, then let us denote z_{k_1} for a_1 . If (2.8) is true at $x = a_1$ and z_{k_1} exists, then let us denote z_{k_2} for a_1 . From the arguments constructed so far, we have shown the existence of one of the following sets:

$$(2.10) \quad \{\{y_{k_1}, y_{k_2}\}, \{z_{k_1}, y_{k_1}\}, \{y_{k_1}, z_{k_1}\}, \{z_{k_1}, z_{k_2}\}\}$$

The union of all the elements of the set (2.10) is

$$(2.11) \quad \{\{y_{k_1}, y_{k_2}\} \cup \{z_{k_1}, y_{k_1}\} \cup \{y_{k_1}, z_{k_1}\} \cup \{z_{k_1}, z_{k_2}\}\} = \{0, a_1\}.$$

Now, let us consider age a_2 for $a_2 > a_1 > 0$ and $a_2 \in A$. At age $x = a_2$, either (2.7) holds or (2.8) holds, but not both.

Suppose (2.7) holds at $x = a_2$. For $x = 0$ and $x = a_1$, there is a possibility of occurrence of one of the four sets of (2.10). For $x = a_2$ exactly one of the following four possibilities then will occur:

- (i) (2.7) holds at $x = a_2$ and $\{y_{k_1}, y_{k_2}\}$ exists; then let us denote y_{k_3} for a_2 .
- (ii) (2.7) holds at $x = a_2$ and $\{z_{k_1}, y_{k_1}\}$ exists; then let us denote y_{k_2} for a_2 .
- (iii) (2.7) holds at $x = a_2$ and $\{y_{k_1}, z_{k_1}\}$ exists; then let us denote y_{k_2} for a_2 .
- (iv) (2.7) holds at $x = a_2$ and $\{z_{k_1}, z_{k_2}\}$ exists; then let us denote y_{k_1} for a_2 .

Suppose (2.8) holds at $x = a_2$. For $x = 0$ and $x = a_1$, there is a possibility of occurrence of one of the four sets of (2.10). Also here one of the following possibilities will occur:

- (v) (2.8) holds at $x = a_2$ and $\{y_{k_1}, y_{k_2}\}$ exists; then let us denote z_{k_1} for a_2 .
- (vi) (2.8) holds at $x = a_2$ and $\{z_{k_1}, y_{k_1}\}$ exists; then let us denote z_{k_2} for a_2 .
- (vii) (2.8) holds at $x = a_2$ and $\{y_{k_1}, z_{k_1}\}$ exists; then let us denote z_{k_2} for a_2 .
- (viii) (2.8) holds at $x = a_2$ and $\{z_{k_1}, z_{k_2}\}$ exists; then let us denote z_{k_3} for a_2 .

Through (i) to (viii) we have shown the existence of one of the following sets:

$$(2.12) \quad \left\{ \begin{array}{l} \{y_{k_1}, y_{k_2}, y_{k_3}\}, \{z_{k_1}, y_{k_1}, y_{k_2}\}, \{y_{k_1}, z_{k_1}, y_{k_2}\}, \{z_{k_1}, z_{k_2}, y_{k_1}\} \\ \{y_{k_1}, y_{k_2}, z_{k_1}\}, \{z_{k_1}, y_{k_1}, z_{k_2}\}, \{y_{k_1}, z_{k_1}, z_{k_2}\}, \{z_{k_1}, z_{k_2}, z_{k_3}\} \end{array} \right\}$$

The union of 2^3 number of sets is

$$(2.13) \quad \{0, a_1, a_2\}.$$

Continuing in a similar manner with $x = a_3$ we construct the following 2^4 number of sets:

$$(2.14) \quad \left\{ \begin{array}{l} \{y_{k_1}, y_{k_2}, y_{k_3}, y_{k_4}\}, \{z_{k_1}, y_{k_1}, y_{k_2}, y_{k_3}\}, \{y_{k_1}, z_{k_1}, y_{k_2}, y_{k_3}\}, \{z_{k_1}, z_{k_2}, y_{k_1}, y_{k_2}\} \\ \{y_{k_1}, y_{k_2}, z_{k_1}, y_{k_3}\}, \{z_{k_1}, y_{k_1}, z_{k_2}, y_{k_2}\}, \{y_{k_1}, z_{k_1}, z_{k_2}, y_{k_2}\}, \{z_{k_1}, z_{k_2}, z_{k_3}, y_{k_1}\} \\ \{y_{k_1}, y_{k_2}, y_{k_3}, z_{k_1}\}, \{z_{k_1}, y_{k_1}, y_{k_2}, z_{k_2}\}, \{y_{k_1}, z_{k_1}, y_{k_2}, z_{k_2}\}, \{z_{k_1}, z_{k_2}, y_{k_1}, z_{k_3}\} \\ \{y_{k_1}, y_{k_2}, z_{k_1}, z_{k_2}\}, \{z_{k_1}, y_{k_1}, z_{k_2}, z_{k_3}\}, \{y_{k_1}, z_{k_1}, z_{k_2}, z_{k_3}\}, \{z_{k_1}, z_{k_2}, z_{k_3}, z_{k_4}\} \end{array} \right\}$$

The union of the 2^4 sets in (2.14) is

$$\{0, a_1, a_2, a_3\}.$$

Note that the elements of each of the sets in (2.14) are drawn from the set

$$\{y_{k_1}, y_{k_2}, y_{k_3}, y_{k_4}, z_{k_1}, z_{k_2}, z_{k_3}, z_{k_4}\}$$

which is of size $2 \cdot 4 = 8$.

In a similar way, by induction, the number of possible sets at age $x = a_j$ for $a_j > a_{j-1} > \dots > a_1 > a_0$ would be 2^{j+1} . Each of these 2^{j+1} sets will be of size $j + 1$, and the $j + 1$ elements for each set are drawn from the following unique combinations of set of $2j + 2$ elements:

$$(2.15) \quad \{y_{k_1}, y_{k_2}, \dots, y_{k_{j+1}}, z_{k_1}, z_{k_2}, \dots, z_{k_{j+1}}\}.$$

The union of the 2^{j+1} sets would be

$$(2.16) \quad \{0, a_1, a_2, \dots, a_j\}.$$

When $a_j = \omega$ in (2.16), this union becomes the set A . The elements of A are drawn from

$$\{y_{k_1}, y_{k_2}, \dots, \omega, z_{k_1}, z_{k_2}, \dots, \omega\}.$$

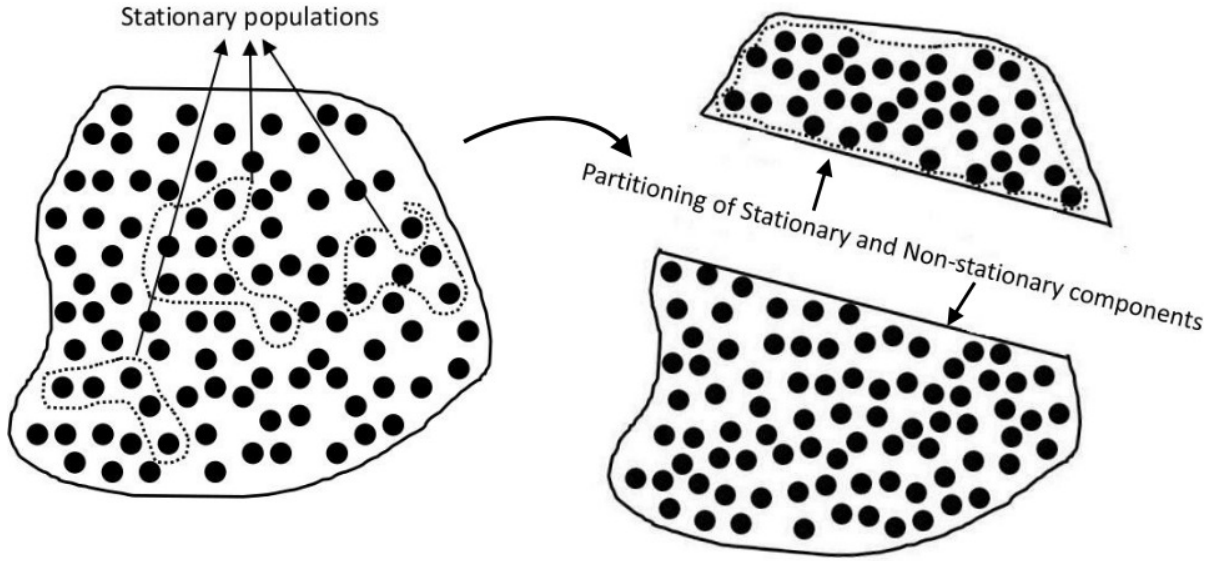


FIGURE 2.4. Partitions into stationary and non-stationary components of a randomly selected large population. All the populations within the dotted lines on the left hand side form the stationary component and others make the non-stationary component of the total population, as shown in the right side of the Figure. Note that partitioning is not about geographical partitioning.

Here,

$$(2.17) \quad \bigcup_{p=1}^j \{y_{k_p}\} \cup \{z_{k_p}\} = A,$$

and the corresponding sub-populations' totals for the individuals who are all at ages $\{y_{k_j}\}$ and $\{z_{k_l}\}$ are $P_{y_{k_j}}(t)$ and $P_{z_{k_l}}(t)$, respectively. See Figure (2.4) for partitioned components. Due to (2.17), we can write,

$$(2.18) \quad \sum_{j=1}^{\omega} P_{y_{k_j}}(t) + \sum_{l=1}^{\omega} P_{z_{k_l}}(t) = P(t),$$

where $\sum_{j=1}^{\omega} P_{y_{k_j}}(t)$ is $M(t)$ formed by satisfying the SPI at the ages $\{y_{k_j}\}$, and $\sum_{l=1}^{\omega} P_{z_{k_l}}(t)$ is $N(t)$ formed by not satisfying the SPI at the ages $\{z_{k_l}\}$. Hence, the proof. \square

Example 3. Suppose for an arbitrary age $a_m \in A$, we have

$$\frac{P_{a_m}(t)}{P(t)} = \frac{L_{a_m}(t)}{L(t)},$$

then the population at a_m belongs to the stationary component, and for an arbitrary age $a_n \in A$, suppose

$$\frac{P_{a_n}(t)}{P(t)} \neq \frac{L_{a_n}(t)}{L(t)},$$

then the population at the age a_n belongs to the non-stationary component.

Remark 4. If (2.7) is satisfied for all the ages in A then the entire randomly selected population is stationary. If (2.7) is not satisfied for all the ages in A then the entire randomly selected population is non-stationary.

Remark 5. Since we are calculating the proportions of sub-populations at each age of the actual population, we might come across values of these proportions calculated at two or more ages that could be identical. All the ages with identical proportions would fall within the same component of the population, i.e. stationary or non-stationary by the construction explained in the proof of the Theorem 2.

Remark 6. The partition theorem can be extended to more than two partitions if there are multiple decrement life tables available for a randomly selected large population.

3. CONCLUSIONS

The main theorem stated and proved is the first such observation in the literature. Moreover, we have not come across in the literature where the life table identity was being used to relate to actual populations and applied to decide on stationary and non-stationary components of a large randomly selected actual population. The partitioning procedure described in this work can be used to decide what proportion of the population is stationary and what proportion is not by considering the world population as a whole as one unit, individual countries, continents and groups of countries, etc. We can also apply this procedure to test the stationary and non-stationary status of all sub-regions of a large country.

The partition theorem stated and proved is not an improvement of any previously shown results in stationary populations. The statement of the partition theorem in the article is original and the method demonstrates whether a component of a large population is stationary or non-stationary did not exist previously. Cases of a given large population or subpopulation showing oscillatory behaviour of transitioning from a

stationary to a non-stationary state or vice versa were earlier given in [17]. The theory and partition methods described could be used in insect demography [22, 23, 24], plant sciences [25, 26], or other research areas where age- and aging-based stratification is done, for example [27, 28, 29].

With many countries in the world approaching or at replacement levels but often with widely varying if not unstable age distributions, the partitioning theory outlined here has the potential to provide new metrics on age structure in particular, and on overall population dynamics in general. Metrics to measure population stability status between two populations were proposed in [30]. Both of these can be used for both between-country comparisons as well as for projections into the future at both regional, national and global levels.

Acknowledgments. J.R. Carey (U.C. Davis) provided appreciation and encouragement when the author posed the statement of the partition theorem for populations for the first time in 2018. This inspired the author to finish the proof. S. Tuljapurkar (Stanford University), J.R. Carey (UC Davis), two anonymous reviewers, Editor in Chief F.J.A. Jacobs provided very helpful and constructive comments that helped in revising the article. I am greatly thankful to all. ASRS Rao has no funding support to disclose that is related to this project.

REFERENCES

- [1] Halley, E.: An estimate of the degrees of the mortality of mankind, drawn from curious tables of the births and funerals at the city of Breslaw; with an attempt to ascertain the price of annuities upon lives. *Phil. Trans. Roy. Soc. London* 17, 596f1610 (1693)
- [2] Euler, L.: Recherches gfnirales sur la mortalitfi et la multiplication du genre humain. *Hist. Acad. R. Sci. B.-Lett. Berl.* 16, 144f164 (1760/1767). eulerarchive.maa.org
- [3] Euler, L.: Sur la multiplication du genre humain. In: Leonhardi *Euleri Opera omnia*, Ser. I, vol. 7, 545f552. Teubner, Leipzig (1923)
- [4] Euler, L.: Introductio in analysin infinitorum, Tomus primus. Bousquet, Lausanne (1748). *Leonhardi Euleri Opera omnia*, Ser. I, vol. 8, Teubner, Leipzig (1922)
- [5] Rao, A.S.R.S. (2021). Is NRR Time-Sensitive in Measuring Population Replacement Level? PAA Affairs, <https://www.populationassociation.org/blogs/emily-merchant1/2021/01/26/is-nrr-time-sensitive>
- [6] Muller H .G., Wang J - L, Carey J.R., Caswell - Chen E.P., Chen C., Papadopoulos N., Yao F. (2004) Demographic window to aging in the wild: Constructing life tables and estimating survival functions from marked individuals of unknown age. *Aging Cell* 3 , 125 - 131.

- [7] Carey, JR; Müller, HG; JL Wang, JL; Papadopoulos, NT; Diamantidis, A; Koulousis, NA (2012). Graphical and demographic synopsis of the captive cohort method for estimating population age structure in the wild. *Experimental Gerontology* 47 (10), 787-791.
- [8] Rao A.S.R.S. and Carey J.R. (2015). Carey's Equality and a theorem on Stationary Population, *Journal of Mathematical Biology*, 71: 583-594.
- [9] Brouard, N. 1986. *Structure et dynamique des populations La pyramide des années à vivre*, aspects nationaux et exemples régionaux.4: 157-168
- [10] Carey, J.R., Silverman, S., Rao, A.S.R.S. (2018). The life table population identity: Discovery, formulations, proofs, extensions and applications, pp: 155-185, *Handbook of Statistics: Integrated Population Biology and Modelling Part A*, volume 39 (Eds: Arni S.R. Srinivasa Rao and C.R. Rao), Elsevier.
- [11] Swanson, D.A., Tedrow, L.M. On Mathematical Equalities and Inequalities in the Life Table: Something Old and Something New. *Can. Stud. Popul.* (2021). <https://doi.org/10.1007/s42650-021-00044-0>
- [12] Carey J.R. and Roach D (2019). *Biodemography. An Introduction to Concepts and Methods*, Princeton University Press, Princeton.
- [13] Wachter, K.W. (2014). *Essential Demographic Methods*, Harvard University Press (1/e).
- [14] Misra, B.D. (1998) An Introduction to the Study of Population, by South Asian Publishers, New Delhi, 1995.
- [15] Preston S, Heuveline P, Guillot M (2000). *Demography: Measuring and Modeling Population Processes* 1st Edition Wiley-Blackwell.
- [16] Lahiri S. (2018). Survival Probabilities From 5-Year Cumulative Life Table Survival Ratios ($T_x + 5/T_x$): Some Innovative Methodological Investigations, pp: 481-542, *Handbook of Statistics: Integrated Population Biology and Modelling Part A*, volume 39 (Eds: Arni S.R. Srinivasa Rao and C.R. Rao), Elsevier/North-Holland, Amsterdam.
- [17] Rao A.S.R.S. and Carey J.R. (2019). On three properties of stationary populations and knotting with non-stationary populations, *Bulletin of Mathematical Biology*, 81, 10: 4233-4250 (10.1007/s11538-019-00652-7)
- [18] Kim, Y. J. and Aron, J. L. (1989). On the equality of average age and average expectation of remaining life in a stationary population. *SIAM Review* 31(1): 110-113.
- [19] Cox, D. R. (1962). *Renewal Theory*. London: Methuen and Co.
- [20] Lotka A. J. (1925). *Elements of Physical Biology*, Baltimore, Williams and Wilkins Company.
- [21] Lotka, A.J. (1939). On an integral equation in population analysis. *Annals of Mathematical Statistics*, 10:144-161.
- [22] S Unnikrishnan, R Gadagkar (2020). The effect of age on non-reproductive division of labour in the tropical primitively eusocial wasp, *Ropalidia cyathiformis Int J Dev Biol* 64, 277-283.
- [23] H Chi, JW Fu, MS You (2019). -Age-stage, two-sex life table and its application in population ecology and integrated pest management, *Acta Entomologica Sinica*, 62 (2), 255-262.
- [24] MS Tasnin, BJ Kay, T Peek, K Merkel, AR Clarke (2021). Age-related changes in the reproductive potential of the Queensland fruit fly. *Journal of Insect Physiology* 131, 104245

- [25] H Mohammadi-Tabar, R Talaei-Hassanloui, H Allahyari et al. (2015). Study on the life table parameters of diamondback moth, *Plutella xylostella* (Lep., Plutellidae) on three plant hosts under laboratory conditions, *Iranian Journal of Plant Protection Science* 46 (2), 277-284.
- [26] Chalmandrier, L., Hartig, F., Laughlin, D.C. et al. Linking functional traits and demography to model species-rich communities. *Nat Commun* 12, 2724 (2021). <https://doi.org/10.1038/s41467-021-22630-1>
- [27] P Mondal, H Nagendra, R DeFries (2016). Preface-Addressing Issues of Climate Change Impacts, Adaptation and Vulnerability on the Ground: Challenges and Opportunities, *Current Science* 110 (7), 1193-1194
- [28] Ning, S., Zhang, W., Sun, Y. et al. (2017). Development of insect life tables: comparison of two demographic methods of *Delia antiqua* (Diptera: Anthomyiidae) on different hosts. *Sci Rep* 7, 4821 (2017). <https://doi.org/10.1038/s41598-017-05041-5>
- [29] Srivastava, U., Singh, K.K., Pandey, A. et al. Experiments in modeling recent Indian fertility pattern. *Sci Rep* 11, 6592 (2021). <https://doi.org/10.1038/s41598-021-85959-z>
- [30] Rao, A.S.R.S (2014). Population Stability and Momentum, *Notices of the American Mathematical Society*, 62, 9: 1062-1065.