The role of representational conventions in assessing the empirical significance of symmetries

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Abstract

This paper explicates the direct empirical significance (DES) of symmetries in gauge theory, with comparisons to classical mechanics. Given a physical system composed of subsystems, such significance is to be awarded to physical differences of the composite system that arise from symmetries acting solely on its subsystems. So my overarching main question is: can DES be associated to the local gauge symmetries, acting solely on subsystems?

In local gauge theories, any quantity with physical significance must be a gauge-invariant quantity. To attack the question of DES from this gauge-invariant angle, we require a split of the state into its physical and its representational content: a split that is relative to a representational convention, or a gauge-fixing. Using this method, we propose a rigorous definition of DES, valid for any state. This definition fills the gaps in influential previous construals of DES, (Greaves & Wallace, 2014; Wallace, 2019a,b,c). In particular, Wallace's need to specialize to 'generic' states is explained and dispensed with.

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1 Introduction

1.1 The debate

Symmetries of the whole Universe are widely regarded as not being directly observable: that is, as having no direct empirical significance. At the same time, it is widely accepted that some of these symmetries, such as velocity boosts in classical or relativistic mechanics (Galilean or Lorentz boosts), are observable when applied solely to subsystems. Thus Galileo's famous thought-experiment about the ship—that a process involving some set of relevant physical quantities in the cabin below decks proceeds in exactly the same way, whether or not the ship is moving uniformly relative to the shore—is used to show that subsystem boosts have a direct, albeit strictly relational, empirical significance. For while the inertial state of motion of the ship is undetectable by experimenters confined to the cabin, the entire system, composed of ship and sea, registers the difference between two such motions, namely in the relative velocity between ship and sea.[1]

Thus the broad notion of 'direct empirical significance' of a symmetry amounts to the existence of transformations of the universe possessing the following two properties (articulated in this way by (Brading & Brown, 2004), following (Kosso, 2000)):

(i): *Global Variance*— the transformation applied to the Universe in one state should lead to an empirically different state; and yet

(ii): *Subsystem Invariance*—the transformation should be a symmetry of the subsystem in question (e.g. Galileo's ship), i.e. involve no change in quantities solely about the subsystem.

I will take the concept of 'directly empirically significant subsystem symmetries' (DES henceforth) to imply observability of those symmetries; but I will prefer the use of the label DES as opposed to 'observable' since I take it to connote an action of a symmetry on a subsystem.

Whether the concept of DES extends to local gauge theories is less settled.[2] Local gauge symmetries are normally taken to encode descriptive redundancy, which suggests local gauge theories cannot illustrate the concept of DES. For surely, a "freedom to redescribe" could not be observable. This argument was developed in detail by Brading and Brown (Brading & Brown, 2004). They take themselves—I think rightly, in this respect—to be articulating the traditional or orthodox answer. In the case of gauge theory, this answer differs from 't Hooft (1980, p. 110)'s claim, that applying distinct rigid phase shifts in the two arms of a beam-splitter experiment would alter the interference pattern. Brading and Brown point out that distinct phase shifts would produce a non-continuous gauge transformation: assuming the two subsytems are contiguous, there would be a mismatch of the phase rotation at the interface, which should not be an allowed operation, on mathematical and physical grounds.

Building on (Healey, 2009) and (Brading & Brown, 2004), Greaves and Wallace resist the orthodoxy by articulating DES for a local gauge theory differently (Greaves & Wallace, 2014). They point out that, since gauge transformations are

[1]There is always some amount of approximation in these notions: we ignore the ripples produced by the ship, assume the sailors don't carry a GPS or look outside for that matter, etc.

[2]DES has been discussed in (Brading & Brown, 2004; Chasova, 2019; Friederich, 2014, 2017; Gomes, 2019, 2021a; Greaves & Wallace, 2014; Healey, 2009; Kosso, 2000; Ladyman, 2015; S. Ramirez & Teh, 2019; S. M. Ramirez, 2019; Teh, 2016; Wallace, 2019a,b). None of these completely encapsulate my own views, but there are large, though of course varying, overlaps of agreement with each.

not physical transformations, we should not demand that they remain continuous at the interface between contiguous subsystems. Rather, we should demand only a continuous transition between the *states* representing such subsystems. Gomes (2021a) argues that this amendment is correct, but it does not go far enough: what matters is a continuous transition *between the physical states of the two subsystems*. Here I will argue that this last demand is more accurate, and it recovers Greaves and Wallace's focus on states for a large number of cases—*just as long as we fix and keep track of representational conventions for the states*, as I will explain in due course.

And indeed, Greaves and Wallace's treatment of DES for local gauge theories bears many similarities to ours here. They focus on subsystems as given by regions and they identify transformations possessing properties (i) and (ii) by first formulating the putative effects of such transformations on the gauge fields in these regions. A more refined treatment that takes into consideration extensions of the symmetries to the measuring apparatus (or subsystems) was developed in (Wallace, 2019c) and applied to particle mechanics in (Wallace, 2019a) and field theory in (Wallace, 2019b). To settle the question of whether local gauge symmetry can be said to have relational 'empirical significance' in the sense of Galileo's ship—viz. in the sense that certain subsystem symmetries can be used to effect a relational difference between a subsystem (e.g. the ship system) and an environment (e.g. the shore)—I will in this paper follow fairly similar routes as Wallace. The two main differences are that:

(1): I will be explicit about the need for, and use of, representational conventions. This first demand is in line with Gomes (2021a): it is a consequence of the focus on the physical, as opposed to the representational, content of the states—a focus that is necessary in order to assess physical significance.

(2): My treatment of the boundary of subsystems—in particular the relation between non-asymptotic and asymptotic boundaries—is different. I believe we should first understand how gauge symmetry behaves in the non-asymptotic case and then translate that understanding to the asymptotic case; whereas Wallace (2019b,c) goes in the other direction.

So naturally, my conclusions will differ somewhat from the previous literature.

That is the debate this paper aims to resolve. I will show that there is a general, coherent formalization of DES which yields (a) the aforementioned Galilean symmetries in the ship-scenario (and its relativistic analogue), but which (b) yields no non-trivial realizer of the concept in the case of *local* gauge symmetries. DES appears only in certain circumstances, and then only related to global—which we will here call 'rigid', as introduced in (Gomes, 2021a)— gauge transformations.[3] Since the case of local gauge theories is the contentious one, it will be my focus in this paper.

1.2 Roadmap

In Section 2, I will lay out some of the conceptual background assumptions, required for the rest of the paper. This Section will consider definitions of subsystems, what notion of isolation is required, and how we construe, mathematically and conceptually, the symmetries that should act intrinsically on the subsystem in comparison

[3]Those transformations depending on a finite number of parameters; i.e. not definable point by point, are the ones called rigid, or global. Their status as regards DES is (much) less disputed.

with those that act on the universe as a whole.

In Section 3, I will give all the general ingredients for DES. In particular I will show why the use of representational conventions is necessary for articulating DES. In this Section I will provide the general structure of gauge symmetry and representational conventions that I will need; I will justify the unobservability of quantities that are variant under symmetries of the entire universe; I will discuss the composition of subsystems using representational conventions; and then finally I will show how to extract empirical significance for the entire universe from subsystem symmetries.

In Section 4, I summarize the treatment of gauge theories of fields. I give more detail about the obstructions to defining regional gauge-invariant dynamical structure, and discuss in brief two ways to overcome this obstruction: edge modes and a careful use of representational conventions that are not taken a priori as anchored to the boundary. I then summarise the details of DES in this context, but leave details of particular examples to the numerous appendices. In Section 5 I conclude.

In Appendix A I show that no non-trivial realizer of DES exists for electromagnetism in a simply-connected Universe in the vacuum sector of the theory (i.e. no matter field). It exists in the sector in which there is matter field but not in the interface between the regions. In Appendix C, I will treat the same sectors using holonomy variables, and find the same conclusions. In Appendix B, I look at what I will call the 'externalist case'. In Appendix D, as a consistency check, I apply the same criteria for DES to theory of particles; and obtain both the Gallilean symmetries of Galileo's ship thought experiment, and the uniformly accelearated solutions representing the Einstein's elevator type of scenario.

2 Background assumptions

In the received view, gauge theory accords physical reality only to certain quantities: those that are invariant under a class of transformations labeled 'gauge'. These transformations are usually construed as mere redescriptions of the same physical state of affairs. While this construal of gauge theory is business as usual, it might seem at first sight inimical to gauge symmetris having DES.

However, according to Section 1.1's condition (ii): *Subsystem Invariance*, empirical significance must involve subsystems, and subsystems introduce a crucial novelty: we leave "God's vantage-point" for a more regional one. That is: we assume our access is restricted to quantities solely about the subsystem—as illustrated by the sailors being cooped up inside the cabin of Galileo's ship. In this context, subsystems bring in what amount to epistemic considerations additional to ontological ones.

But 'subsystem' is a vague concept. In this Section, we will constrain that concept so that it is suitable for our investigations of DES. Section 2.1 gives a first gloss on the idea of a subsystem as reflecting important kinematical features of the larger system of which it is a part. Section 2.2 describes the relationship between symmetries and subsystems that are defined by boundaries. Section 2.3 describes the necessity and use of representational conventions in assessing DES.

2.1 Kinematical subsystem recursivity

In general, observations are modelled as being made from outside the subsystem being studied. Therefore, the importance of subsystems to understanding the observability of symmetries is relatively uncontroversial. As Wallace (2019a, p. 4) points out:

> Observations are physical processes, but they are not normally modelled explicitly within the system being studied, but are considered as external interventions. [... Then a] dynamical symmetry has implications for observability of physical quantities [...] when the symmetry can be extended so as to apply also to the dynamics of those interventions. [...]

I think this is right, and it leads to a natural understanding of Section 1.1's requirements (i) and (ii): *Global Variance* and *Subsystem Invariance*. Indeed, when focusing on subsystems of the Universe, Wallace says (p. 4, ibid): "it becomes relatively simple to understand modal questions in more directly empirical terms: is a situation where the symmetry transformation is applied to this system, but not to other systems, the same as or different from the original situation?". The answer to the question is that the situation should be the same when that subsystem symmetry is part of a global symmetry, and different when it is not.

Thus the empirical significance of a symmetry hinges on how a symmetry, when applied to a subsystem, extends to a larger system of which it is a part; the complementary subsystem is then interpreted as representing the 'environment', or a 'measuring apparatus'. The main question surrounding empirical significane then is about how global symmetries relate to subsystem symmetries. And here I will consider only those theories which satisfy *subsystem recursivity*, i.e. theories that (p. 5-9, ibid)

> have the remarkable and underappreciated feature of being able to reinterpret subsystems of their models, when dynamically isolated, as other models of the same theory. [... in these cases] any model can be interpreted [...as a] dynamically isolated subsystem under certain idealizations about its environment and where, if we want to remove those idealizations, we can embed the model in a model of a larger system within the same theory—and where that larger system in turn is interpretable in the first instance as a subsystem of a still-larger system.

As Wallace argues, 'dynamical isolation' is a term of art in physics, but we will not need to be more precise about this, except that we need to assume that isolation entails a weak form of dynamical autonomy.

Unpacking Wallace's definition of dynamical autonomy, I take it to mean that the dynamical equations governing the motion for the subsystem, up to the level of approximation required by the situation at hand, does not depend on the details of the rest of the system, except insofar as the rest of the system defines *initial* boundary conditions for the subsystem.

But here I am only interested in the behavior of symmetries of the laws, at both subsystem and global levels. And since there is a sense in which symmetries can be seen as 'laws on laws', or 'metalaws' (see Lange (2007)), my requirement about dynamical isolation and subsystem recursivity can be weaked in two senses.

First, I will only be interested in whether the subsystem enjoys the 'same type' of symmetries as the larger system in which it is embedded. This will be labeled *downward consistency*. This weaker requirement allows evolving boundary conditions, if they are symmetry-invariant.

Second, an isolation condition may only hold for a certain interval of time, $I \subset \mathbb{R}$. But I do not want to focus here on the loss of autonomy over time, and so I will only require some small $I \neq 0$. Thus, differently from Wallace, I will focus on the relation between system and subsystem symmetries *for initial states*; assuming only that some $|I| = \delta > 0$ exists in which downward consistency is satisfied.

In the case of gauge theories, the arguments of this paper will require only such kinematical considerations. Due to the locality of the interactions, the weakened form of kinematical isolation—that is only required to obey downward consistency—can always be satisfied by any subsystem defined through a partition of space, as we will see in Section 4.

In the particle theory case, the assumption that the subsystem dynamics inherits the symmetries of the larger universe requires stronger isolation conditions, but these can be encapsulated in our embedding of the subsystem into the larger universe, as done in Section D.

I believe such a kinematical understanding of subsystem recursivity about symmetries can accommodate our intuitions about, and the familiar examples of, direct empirical significance. Consider, for simplicity, a Galileo's ship scenario with the shore (not the sea) taken as the environment, in which the subsystem at $t = 0$ is inertial and at a finite distance d from the shore. Now, for a fixed time interval I, the boosts must be pared down to a scale given by d/I. But we are not concerned with these 'practical matters' when describing the subsystem symmetries; we use certain idealizations, e.g. that the shore is infinitely far away, so that $d \to \infty$. Here I prefer a different idealization, in which I take I to be small. Thus the kinematical understanding of subsystem recursivity avoids some of the fuzziness of dynamical isolation, and yet has the resources to articulate a fruitful construal of DES.

With Wallace, I will take subsystems to be represented as elements X of a collection Ξ, so $X \in \Xi$. The collection Ξ is partially ordered by inclusion, and bounded by a minimal and a maximal element—representing the empty set and the entire universe, respectively. And we define a state space for each X, Φ_X, such that the state spaces respect the partial ordering. Namely, for $X \subset Y$, we define ι_{XY} as the inclusion map (or embedding), $\iota_{XY} : X \to Y$, and, schematically, r_{YX} as the restriction map: $r_{YX} : Y \to Y_{|X}$, with $\iota_{XY} \circ r_{YX} = \mathsf{Id}_X$. The idea is that the restriction on the subsystems gets 'pulled-back' to a restriction on the state spaces, which we can here schematically denote:

$$\iota^*_{XY} : \Phi_Y \to \Phi_X. \tag{2.1}$$

We denote it thus since, in cases of interest in field theory, this "restriction map" is really a type of pull-back.[4] And also with Wallace, we assume "upwards consistency": given $X \subset Y$, for a given $\varphi_X \in \Phi_X$, there exists a $y \in \Phi_Y$ such that $r_{XY}(\phi_Y) = \varphi_X$. This means that any subsystem state is compatible with some global state.

Differently from Wallace, I will further demand that the restriction r_{XY} of (2.1) co-varies with the symmetries of Y, if there are any. Namely, if g_Y is any symmetry of Φ_Y, i.e. a certain type of automorphism $g_Y : \Phi_Y \to \Phi_Y$ (cf. footnote 8), then the

[4]For cases of interest our state space will be the space of sections of some vector bundle (cf. footnote 2.1). If M is the base space of a vector bundle E, and $\iota : N \to M$ is an embedding map, then $\iota^* E$ defines a vector bundle over N by pull-back (i.e. the fiber over $x \in N$ is the fiber over $\iota(x) \in M$).

composition is also a symmetry of the subsystem. In other words, I will demand that subsystems satisfy *downward consistency:* For g_Y a symmetry of Φ_Y and any φ_Y that restricts to a $\varphi_X \in \Phi_X$ of the subsystem, i.e. $\varphi_X = \iota^*_{XY}\varphi_Y$,

$$\iota^*_{XY}(g_Y(\varphi_Y)) \text{ and } \varphi_X \text{ are symmetry-related in } \Phi_X, \tag{2.2}$$

where Φ_X is understood to have its own dynamics, possibly with time-varying boundary conditions. We can equivalently rewrite (2.2) as:

$$\iota^*_{XY} g_Y = g_X \iota^*_{XY}, \quad \text{for some } g_X \text{ symmetry of } \Phi_X. \tag{2.3}$$

This is a watered-down version of subsystem-recursivity: all we require from our definitions of subsystem is that the symmetries are recursive in this way.

Downward consistency demands that the embedding should be symmetry-invariant from the perspective of the entire universe. But gauge theories are local field theories with no action at a distance, and thus already have in-built a weak notion of dynamical isolation of disconnected subsystems. Thus, in their case, I need only demand that subsystems that are demarcated by boundaries (as we will define them in Section 2.2.2) satisfy (2.2). This is a necessary and suffient condition for my weaker notion of subsystem recursivity. And indeed, if it is satisfied, due to the locality of interactions in gauge theory, the regional dynamics can be reinterpreted as the dynamics of other models of the same theory (even if we allow general evolution of boundary conditions). In other words, if (2.2) is satisfied, the *local* equations governing a subsystem that is demarcated by a boundary are identical to those governing a larger bounded system of which it is a part; and, to the extent that boundary conditions differ, that difference does not pare down the symmetry group of the subsystem equations of motion.[5] Thus downward consistency provides a consistent—though weaker, in the sense that subsystems do not be idealized as infinitely far-apart—notion of subsystem-recursivity. And just to give an example, it is this weaker notion that would allow us to model the interior and the exterior of black holes as subsystem and environment: a case of great interest.

In the particle theory case, things are subtler, since forces act a distance.

2.2 Symmetries and boundaries

In what follows I will mostly concentrate on the case of field theories; the considerations for particle systems will differ and will be left for Appendix D. In Section 2.2.1, I will classify symmetries into two general sorts, in a way that is useful for the study of subsystems. In Section 2.2.2 I similarly assess two notions of boundaries, that are naturally paired with the two notions of symmetry. And in Section 2.2.3, I discuss the interplay between these notions of boundary, subsystems and symmetry.

2.2.1 Two notions of symmetry

First, it is helpful to distinguish two types of symmetries:[6]

[5] For instance, in these subsystems, one can always find a representational convention in which the evolution equations are hyperbolic. There are certain complications with elliptic initial value problems, which are to a certain extent non-local. But these complications are under control, as will be discussed in Section 4.2.

[6] There are many precursors to this distinction. (Haro & Butterfield, 2021, Section 5.1)'s idea of stipulated vs accidental symmetries for instance. These roughly correspond to fundamental and dynamical, but are seen as mutually exclusive since the label refers to their origin only. Or Dasgupta (2016)'s formal and ontic symmetries. Formal definitions "define [the notion of symmetry] in purely formal, set-theoretic terms", p. 861; while an

•*'Fundamental symmetries'*: The symmetry is given in purely formal terms. A symmetry group is *defined* as being gauge. So invariance under transformations of the states constrains the laws to respect the symmetries. In this case, there is a symmetry principle, simpliciter, in play; dynamics comes only after.

•*'Dynamical symmetries'*: we define the symmetry transformations as those that leave relevant structures—the state space and the Lagrangian or the Hamiltonian of the theory— invariant.[7] In this case, the symmetries are subservient to, i.e. entirely determined by, the particular features of the state space (e.g. phase space) and of the action functional. Broadly, the laws determine the symmetries; here there is no 'symmetry-principle', simpliciter, in play.[8]

So, given a fundamental symmetry Lie group G, acting on some fields over a space or spacetime manifold, M with value space F—here taken to be a vector space—the fundamental symmetry will be deemed to act uniformly over M. Thus I want to highlight an asymmetry: fundamental symmetries are judged to be dynamical, for the appropriate dynamical structures. But dynamical symmetries are not necessarily fundamental: for example, dynamical symmetries of a field theory may be different on the bulk and boundary of a manifold, and in this case they should not count as fundamental. Indeed, one of the central points I want to argue for here is that for field theories there are two notions of boundary; and for one of them only the dynamical type of symmetries is a natural notion.

Another central point that I want to argue for is that theoretical parsimony and consistency between system and subsystem push us to formulations of subsystems in which the two types of symmetry match. Thus, clearly, downward consistency will have consequences for how we understand DES. But first, we need to develop our ideas about symmetries and boundaries.

2.2.2 Two notions of boundary

Both the Lagrangian and the Hamiltonian formulations of field-theory refer to the fields over the entire universe; we are at first not given any subsystems that DES can latch on to. Subsystems must be somehow "conjured into being", and there

ontic "definition of symmetry [...] requires a symmetry to preserve the laws and preserve certain privileged physical features" p. 862. I would add that, with regards to scientific practice, a hard and fast distinction would over-simplify depiction of how science historically homes in on suitable Lagrangians and associated symmetries. In practice, symmetries that are eventually classified as 'fundamental' can first appear dynamically through the invariance of a Lagrangian but are then elevated to fundamental status and serve as a guiding principle.

[7]This is a simplification: under this general definition we run the risk of allowing models which we would intuitively take to depict physically distinct situations as nonetheless symmetry-related. (Belot, 2013) gives an exposition of the obstacles to a general definition. My definition is closest to what (Wallace, 2019c, p. 3) dubs the 'representational strategy', which "instead builds the representational equivalence of symmetry-related models into the definition [of symmetry], usually by requiring that symmetries are automorphisms of the appropriate mathematical space of models (hence preserve all structure, and thus all representation-apt features, of a model)". As discussed in (Gomes, 2021b, Section 3.3), this definition is still not ideal, since it is slightly circular: structure can be defined implicitly by the symmetry-relation, whatever that is. More generally, I endorse the account of dynamical symmetry in (Gomes, 2021b, Section 1.2). For our purposes in this paper, the vaguer definition above suffices.

[8]In (S. Ramirez & Teh, 2019, p. 8), this distinction has a different label: (A) and (B), with (A) corresponding to 'Fundamental' and (B) (roughly) to 'Dynamical'. They describe the latter as "a more refined [...] notion according to which an (A)-type gauge symmetry is further required to encapsulate redundancy for a particular subservient system, whose states can only be defined after fixing specific boundary conditions". They make a slightly different categorization of (B): it is a subset of (A) that is required to obey boundary conditions.

are two general ways of doing this in field theory: one by introducing a boundary internal to the entire universe; and the other by introducing a boundary 'external' to it. In other words, we can introduce subsystems either by boundaries that define an inside and an outside—the boundaries have two sides—or by ones that define only an inside; boundaries that are one-sided, so to speak.

Thus an

External boundary: imposes a boundary on the whole universe; and an

Internal boundary: imposes boundaries within the bulk of the universe.

In the former case we have a bounded manifold representing the entire universe, and in the latter case we get complementary subsystems demarcated by internal divisions of the entire universe, which, as a whole, is assumed boundary-less.

In the first, external way, the entire universe is taken as a type of subsystem: the 'environment' label can be loosely attached to the boundary itself. In the second, internal way, if one has only two such subsystems, we can label them 'subsystem and environment'.

The external boundary In the first, external way, downward consistency, (2.2), is satisfied vacuously. That is, since the entire universe is considered as a subsystem, no consistency conditions with the symmetries of a larger system can arise. In other words, downward consistency is a condition about the boundary as seen from both the inside and the outside; therefore it does not substantively apply to a one-sided boundary.

Thus, in specifiying the state space and dynamics to which the dynamical symmetries are subservient, there is no obstacle to imposing a fixed representation of the states at the boundary. For instance, one could say: "the configuration space with which I am dealing possesses only one representative of the gauge potential at the boundary". As we will discuss in the next section, one could not have the same type of restriction for an internal boundary without flouting downward consistency.

Of course, if boundary states are pared down, or restricted, they offer an anchor to the representational conventions of the rest of the system. Namely, if the state itself is fixed at the boundary, gauge transformations there are also constrained to preserve that state. The boundary state itself would be gauge-invariant and thus, in the familiar interpretation of gauge theories, accorded physical status.

On the fundamental view, the main issue with pared down states at the boundary is that they will impose boundary conditions that would not be gauge-invariant (or even covariant). Thus we would have to allow a subsystem-quantity that is in this view gauge-*variant*—a quantity such as the boundary value of the gauge potential—to acquire physical significance in the dynamical view. That is, these realizations of the externalist notion of subsystems ascribe gauge-*invariance* under the dynamical view to a quantity that would be viewed, under the fundamental view, as gauge-*variant*.[9]

On the other hand, according to the dynamical view of symmetries, there is no conflict: while external boundaries may curtail or pare down the full set of gauge transformations and gauge representatives, they do not *break* gauge-invariance (nor do they flout downward consistency). For there were no gauge transformations acting on the boundary to begin with.

The external boundary is familiar in the treatment of spatial infinity for field theories (cf. (Gourgoulhon, 2007, Ch. 7)). Spatially asymptotic boundaries are

[9]In (S. Ramirez & Teh, 2019), the externalist account of DES is labeled 'Type II'. As far as I am aware, they give the only other consistent description of DES in these circumstances.

usually construed as boundaries of the entire universe, and the representations of the states can be asymptotically pared down, so as to have a different behavior at those boundaries (Belot (2018) gives a philosophical treatment of this idea).

At this point, I should make a disclaimer. Although I will analyze the externalist notion of subsystem within the dynamical view of symmetries (in Section 3.3), I do not believe this description is as physically relevant as the internalist notion of subsystem. Of course, no one is forbidden from specifying a system where gauge symmetries act differently at the boundary by fiat—as they can in the externalist's notion of subsystem—but the status of such boundaries is not very clear. It is hard to see how such boundaries have ontological significance: even asymptotic boundaries are but a convenient idealisation, and are normally interpreted as describing the way in which a system embeds into a larger system. (And if the notion is epistemic, it should still allow for the possibility that the universe extends beyond the boundary.)

The internal boundary Let us now suppose we would like to introduce subsystems in field theory in the internal way, by embedding a given system into a larger system. Suppose moreover that the entire universe has no boundary, and that the dynamical symmetries of the whole universe are also fundamental: they are given by a symmetry group that acts on some value space, pointwise on spacetime. Following (2.2), if a subsystem is to be demarcated by a boundary of space or spacetime, I will require the boundary conditions to have physical significance. That is, according to the theory as applied to the entire universe, the boundary conditions must be gauge-invariant, or leave the representative of the state *unfixed* there (i.e. subject to gauge transformations). In contrast to the externalist view, in the internalist view, one is not given any boundary-anchor for the representational conventions.

For local field theories such as Yang-Mills gauge theories and general relativity, a subsystem whose boundaries do not break the symmetries of the larger system respects downward consistency. And thus the subsystem inherits the local (fundamental) symmetries of the global system of which it is to be a part.[10] As mentioned in Section 2.1, this is in fact the only feature of subsystem-recursivity that we will require in this paper: that the subsystem enjoys the same type of symmetries as the larger system in which it is embedded.

In the internal boundary case, assuming downward consistency, if the universe as a whole is boundary-less, there is not much to say about the choice of state space: it depends only on the value space of the fields and on the underlying topological properties of spacetime. A gauge-invariance condition of isolation may furthermore be implemented through appropriate sectors of the theory: for example, by saying that the boundary is free of matter.

2.2.3 Symmetries and internal boundaries

But there are subtleties in reconciling the fundamental symmetries of a subsystem defined by an internal boundary with its dynamical symmetries. In particular, there are subtleties about the symmetry-invariance of a bounded subsystem's own

[10]In the case of general relativity, downward consistency would require us to demarcate subsystems using diffeomorphism-invariant conditions; such as Komar-Bergmann scalars (Bergmann & Komar, 1960). This is easy to do asymptotically, and indeed this is one of the great advantages of the treatment of asymptotic infinity through Penrose compactification, see e.g. (Ashtekar A., 1981) and (Ashtekar, 1987, p. 52). There are also many characterizations of black holes that are diffeomorphism-invariant in this way (see e.g. (Hayward, 2013, Chs. 5, 8 and 9)). I will have more to say about this in Section 5.

dynamical structures, such as its intrinsic Hamiltonian, symplectic structure, and variational principles in general. Until recently, subsystems that were so defined were *not* supplied with gauge-invariant boundary conditions. The reason for this was the existence of an obstacle towards a gauge-invariant formulation of subsystems: gauge theories manifest a type of non-locality. Thus the global, physical phase space (or the corresponding global physical Hilbert space) is not factorizable into the physical phase spaces over regions (see footnote 15 for references and more remarks on this issue).

That means that the standard manner of specifying the field dynamics of a subsystem would not be fully gauge invariant if we viewed symmetries as fundamental. The usual response is to pare down gauge symmetries at the boundary. In this way, the boundary conditions and the boundary contributions to the dynamics remain symmetry-invariant, but only in the pared down dynamical view (see e.g. Regge & Teitelboim (1974) for the first paper to enforce this approach explicitly, and, e.g. (Harlow & Wu, 2019, Section 2) and (Geiller & Jai-akson, 2020, Section 2) for more modern treatments). I will return to this issue in greater detail in Section 4.2.

That standard approach treats the lack of fundamental invariance of the subsystem similarly to the one of external boundaries, usually idealized to be infinitely far away, or asymptotic. We saw in Section 2.2.2 that if the whole universe is bounded—there are external boundaries—there is exceptional behavior of symmetries at the asymptotic boundary. So the standard approach takes this to be reflected on subsystems demarcated by internal boundaries.[11]

But if the subsystem symmetries are not fundamental, there is a clear conflict with downward consistency, reflecting the incompatibility between an inside and an outside perspective of the internal boundary. For local field theories, how does the environment, i.e. the entire universe, 'see' the symmetries of the subsystem? Even for a bounded universe and on a dynamical view, the symmetries of the theory that act far away from the asymptotic boundary are unconstrained: they are not pared down. So how should observers from the environment construe a definition of subsystem—a sector of the theory, in Wallace's nomenclature—that does not support the full action of the dynamical symmetries? The standard treatment of bounded subsystems in gauge theory breaks downward consistency, given in Equation (2.2): the action of the universal symmetry on the subsystem is *not* a subsystem symmetry.

Recently, this pared down treatment of internal boundaries of subsystems has been called into question (cf. Carrozza & Hoehn (2021); Donnelly & Freidel (2016); Geiller (2017); Geiller & Jai-akson (2020); Gomes (2019); Gomes & Riello (2021); S. Ramirez & Teh (2019); Riello (2021b)). New geometrical structures, for instance, 'edge-modes', have been devised to maintain the gauge invariance of the internal boundary under the symmetries of the entire universe.[12] Though far from trivial, I will assume that the subsystems in gauge theory that are defined by regional restrictions have fully gauge-invariant dynamical structures. I will briefly return to

[11] Wallace (2019b, p. 11) endorses this view of subsystems, because he takes subsystems as sufficiently isolated so as to warrant an asymptotic-like treatment. So, for instance, the treatment would find no extension to spatially closed manifolds.

[12] In the symplectic case, a resolution requires extensions of the original phase space, to include facts about the representational convention and relational facts about the embedding of subsystem into system Carrozza & Hoehn (2021); Gomes (2019); Rovelli (2014). In worked out examples, cf. (Gomes & Riello, 2021, Section 6), one can however show that the composition of subsystems (which we will tackle on Section 3.2.2) depends only on the symmetry invariant content of each region, and does not depend on any extra, symmetry-variant quantities on the interface of the subsystems.

this topic, in Section 4 for gauge theories and in Appendix D for particle theories.

There are two main upshots of this Section. The first is to suggest a different treatment of asymptotic boundaries, that maintains invariance under symmetries of boundary states. Though this was long ago achieved for null asymptotic infinity through Penrose compactification (Ashtekar A., 1981) and (Ashtekar, 1987, p. 52) (see footnote 10), it has also been developed in the case of Yang-Mills theory for spatial slices in (Riello, 2020), where the spatial subsystem is extended asymptotically.

This resolution is at the crux of my disagreement with Wallace, who (Wallace, 2019b, p. 11) endorses a pared-down version of symmetries on internal subsystems. That is because he takes subsystems as sufficiently isolated to warrant an asymptotic-like treatment, and for external boundaries, there is really no conflict with downward consistency. But recent developments in gauge theory—which will be further discussed in Section 4.2—have shown that we can have finitely bounded subsystems in which e.g. the field-strength $F_{\mu\nu}$ is non-vanishing everywhere, and which still enjoy the same set of fundamental symmetries for their intrinsic dynamics.[13] And although Wallace does not encompass this possibility under his notion of subsystem recursivity, these recent developments in gauge theory show that there is a good notion of subsystem recursivity for subsystems—namely, downward consistency—that does not mimic the asymptotic ideal of perfect isolation. Conversely, there are asymptotic treatments that do not require an anchor state at the boundary, paring down symmetries. I thus conclude that a treatment of internal subsystems in gauge theories that respects the downward consistency of symmetries is conceptually and technically justified.

The second, more practical upshot of this Section, is that, from here on, in the internal boundary case, I will assume a fundamental notion of symmetries, acting intrinsically on the subsystems as well as on the entire universe. In particular, this implies that conventions about the representation of the state are not anchored at the boundary.

2.3 Representational conventions and DES

The great obstacle in assessing the observability of subsystem gauge symmetries (DES) is that physical facts and representational facts come to us highly entangled. This is of course, a common theme. It occurs in the logical positivists' aim of presenting physical theories with a once-and-for-all division of fact and convention; and it was the center of a dispute between Carnap and Quine. I reject this once-and-for-all distinction, both in gauge theory and in the broader philosophical context (for familiar reasons, that I take to be best articulated by Putnam (1975)). But I judge that we can nonetheless assess matters of physical fact. The trick is to anchor these facts to an analogue of a Carnapian framework, that we call a *representational convention*. Each representational convention will have a unique representation of the physical facts. And as long as we stick with a single convention—whatever that is—we can compare and count different physical possibilities unambiguously. Like any good anchor, it will only serve its function if it doesn't move about.

Of course, in the highly regimented domain of mathematical physics we have much better control of the interchangeability of frameworks than we do in the

[13] And this is true even if these subsystems require time-varying boundary conditions! See e.g. Carrozza & Hoehn (2021); Geiller & Jai-akson (2020); S. Ramirez & Teh (2019).

purely philosophical debate. Here we can explicitly articulate which quantities will be independent of the representational convention—the *gauge-invariant* quantities. The existence of these invariant quantities may suggest formalisms that explicitly eliminate the need for conventions—e.g. the holonomy formalism, discussed in Appendix C.[14]

But such formalisms inevitably carry several explanatory and pragmatic deficits (see e.g. (Gomes, 2021c, Section 4.2)). More importantly, these formalisms are inadequate to deal with subsystems of the Universe, in the following sense: the set of invariant quantities of the whole universe does not equal the union of the sets of invariant quantities of partition of the universe into a set of mutually exclusive, jointly exhaustive subsystems. Gauge theories involve a type of holism, or non-separability (cf. Gomes (2019, 2021a); Gomes & Riello (2021), and references therein).

This is often noted even in the classical domain, where it is expressed by the Gauss constraint. For this constraint implies that by simultaneously measuring the electric field flux on all of a large surface surrounding a charge distribution, and integrating, we can ascertain the total amount of charge inside the sphere *at the given instant*. In its quantum version, the non-locality implies the total Hilbert space of possible states is not factorizable.[15] If we seek to employ in our theories only invariant quantities of the subsystems, we may miss important physical facts about the whole universe. In other words, there is a possible gap between regional and universal gauge-invariant quantities.

So we can limit the domain in which the use of representational conventions is necessary as follows. Suppose first that there is *no* concrete, unambiguous, choice-free representation of the physical state-of-affairs. Even, then, in the study of a single physical possibility—describing features of a given solution of the equations of motion for example—a representational convention may be left as implicit. Nothing physically important turns on which representational convention was used, though some conventions may be more convenient than others. On the other hand, if we are to compare different physical possibilities, we must ensure the comparison is made under a fixed representational convention. We will return to this point in Section 3.1.2, once we have introduced some notation.

In sum, even if it is not always inevitable, the use of representational conventions in gauge theory is extremely useful. Moreover, it is not only useful but *necessary* when dealing with subsystems and counting possibilities; as we must to assess DES. In this assessment, we need to keep careful track of which convention we use to anchor our representations; and we must keep track at both the level of the subsystems and of the entire universe. For on both levels we will have to compare alternative possibilities, and this comparison is only meaningful if made under a fixed convention.

Thus, by carefully employing representational conventions at both the subsys-

[14] This discussion is familiar from the debate between the 'eliminativist' and the 'sophistication' approach to symmetries; see e.g. Dewar (2017); Gomes (2021b); Martens & Read (2020).

[15] This type of holism, or non-locality is a well-known issue for theories with elliptic initial value problems: e.g. Yang-Mills theory and general relativity. For a reference that explores this in the context of the holonomy formalism, in the spirit of Appendix C, see Buividovich & Polikarpov (2008). For an attempt to find symmetry-invariant variables in gauge theory, see Berghofer et al. (2021), and for general relativity see e.g. Donnelly & Giddings (2016); for more recent use of this non-factorizability in the black hole information paradox, see Jacobson & Nguyen (2019). For a discussion of the relation between the factorizability of Hilbert spaces and the augmentation of the phase space with 'edge-modes', see Geiller & Jai-akson (2020); S. Ramirez & Teh (2019).

tem and universal level, we will completely characterize the gap between the subsystem and universal symmetry-invariant quantities. And it is this gap that enables a well-defined type of DES for local gauge theories, which we will label as 'relational'. In line with the topic of holism (i.e. the question "does the state of the parts determine the state of the whole?"), the question of relational DES turns on whether or not the (union of) gauge-invariant quantities of the subsystems determine those of the entire system.

This was the basis of the broad argument advanced and explored in (Gomes, 2021a). Here we develop in more detail the use of representational conventions in different settings and with a greater focus on properties of boundaries.[16]

Thus the question of DES in gauge theory will require us to first investigate what constitutes a representational convention in the presence of boundaries. For it is often assumed that, in a division of the universe into subsystem and environment, the latter already comes equipped with its own representational convention. Thus it is assumed that the very existence of an environment serves to anchor representation at the 'edges' of our subsystem. This position is defended in detail in Belot (2018) for asymptotic boundaries, and also in Wallace (2019b). But although endowing environment with a ready-made convention is very often useful, it is in tension with downward consistency, of Equation (2.2), as I discussed in Section 2.2.3.

3 General structure of DES

I will here present the main ingredients for the analysis of DES (and I will I apply these ingredients to field theory and particle mechanics in detail, in the appendices). In Section 3.1, I introduce basic notation about the action of symmetry groups on state spaces globally, or for the entire Universe. There I also discuss representational conventions and the unobservability thesis: that symmetries of the entire universe are not observable. In Section 3.2 I analyse the internalist notion of subsystem and derive DES in this general scenario. In Section 3.3 I provide a criticism of a previous derivation of DES, which takes the environment to come equipped with its own representational convention.

3.1 Preliminaries about symmetry

I start in Section 3.1.1 by describing a group action on a general state space and defining the space of physical states. In Section 3.1.2, I discuss representational conventions, in the generality. Along with (Wallace, 2019c, p. 18), I argue for the importance of a fixed representational convention when assessing differences in states due to the action of a symmetry. In Section 3.1.3, I use the representaional convention to, following Wallace (2019c), demonstrate the unobservability thesis:

[16] In Gomes (2021a) it was found that in the presence of matter, a non-trivial relational DES exists for local gauge theories, but only under conditions allowing conserved regional charges, and for *rigid* symmetries. In the Abelian theory, this would require that the charged matter fields are present solely within each region, and would thereby include scenarios like 't Hooft's beam splitter experiment; see ('t Hooft, 1980, p.110) and (Brading & Brown, 2004, p. 651). In the non-Abelian case, relational DES also requires the regional or boundary field to be highly homogeneous (see (Gomes & Riello, 2021)). For a more complete discussion of circumstances which *would* allow a non-trivial realization of DES in local gauge theories, see (Gomes, 2021a). I will return to these issues in Section A.5.

that the symmetry-invariant degrees of freedom are completely autonomous from the symmetry-variant features of the states.

3.1.1 Group action

Take a system X, with an associated state space Φ_X, on which a group of symmetries, $\mathcal{G}_X$, acts. Omitting the subscript X, we have, for $g \in \mathcal{G}$ and $\varphi \in \Phi$, a map

$$\begin{aligned} \mu : \mathcal{G} \times \Phi &\rightarrow \Phi \\ (g, \varphi) &\mapsto \mu(g, \varphi) =: \varphi^g. \end{aligned} \tag{3.1}$$

The symmetry group partitions the state space into equivalence classes, $\sim$, where $\varphi \sim \varphi'$ iff for some g, $\varphi' = \varphi^g$. We denote the equivalence classes under this relation by square brackets $[\varphi]$ and the orbit of φ under $\mathcal{G}$ by $\mathcal{O}_\varphi := \{\varphi^g, g \in \mathcal{G}\}$.

For the purposes of this paper, we could assume that the state space is phase space; but I will further assume that the symmetries acting on phase space are inherited by symmetries acting on the configuration space of the system under consideration. So, I will take Φ to be configuration space, with the cotangent bundle $T^*\Phi$ its associated phase space, and the symmetry μ from (3.1) then induces an action (for which I use the same label) $\mu : \mathcal{G} \times T^*\Phi \rightarrow T^*\Phi$ that preserves the symplectic structure and leaves invariant the Hamiltonian of the system, which is a function $H : T^*\Phi \rightarrow \mathbb{R}$ that determines the dynamical laws (cf. footnote 7). For this paper, these assumptions suffice: I will not need to provide details of the dynamics (through a specification of the Hamiltonian of the system or otherwise).

Presaging the conclusions of this Section, we call $[\varphi]$ the *physical state*, and $\varphi' \in \mathcal{O}_\varphi$ is its *representative* (when there is no need to emphasise that φ involves a choice of representative, we call it just 'the state' for short). We call the collection of equivalence classes, $[\Phi] := \{[\phi], \phi \in \Phi\}$, *the physical state space.* As written, this is an abstract space, i.e. defined implicitly by an equivalence relation.

Eliminativism about symmetries is a position that seeks an intrinsic parametrization of $[\Phi]$. But such parametrizations are hard to come by, or have serious deficits. In their absence, we opt for a representational convention, that uniquely picks out members of each orbit.

3.1.2 Representational convention, aka gauge-fixing

Suppose we choose one representative per gauge-orbit for each $[\varphi]$. That is, an injective map $\sigma : [\Phi] \rightarrow \Phi$ that takes each equivalence class to a member of the respective orbit. Then, *armed with such a choice of representative for each orbit*, a generic state φ could be written uniquely as the doublet $([\varphi], g)_\sigma$, i.e. $\varphi = \sigma([\varphi])^g$. That is, we identify Φ with $[\Phi] \times \mathcal{G}$ via:

$$\text{For all } \varphi,\ \exists!\, ([\varphi], g) \in [\Phi] \times \mathcal{G} \text{ such that } \varphi = \sigma([\varphi])^g =: ([\varphi], g)_\sigma. \tag{3.2}$$

This representation is guaranteed to satisfy:

$$\varphi^{g'} = \sigma([\varphi])^{g'g} = ([\varphi], g'g)_\sigma. \tag{3.3}$$

To be able to use such doublets in assessing dynamical statements about the action of symmetries, we must moreover assume that the map $\sigma : [\Phi] \rightarrow \Phi$ respects the required mathematical structures of Φ (cf. footnote 8), e.g. smoothness or differentiability. In more formal language, (3.2) provides a structure-preserving map (e.g. a diffeomorphism) from $[\Phi] \times \mathcal{G}$ to Φ.[17]

It is convenient to have a separate label for the state that is in the image of $[\Phi] \times \mathrm{Id}$, where Id is the identity of $\mathcal{G}$:

$$\varphi_\sigma := \sigma([\varphi]), \tag{3.4}$$

so $\sigma : [\Phi] \to \Phi$; acting as $[\varphi] \mapsto \varphi_\sigma$, is a diffeomorphism onto its image. Then any state $\varphi' \in \mathcal{O}_\varphi$, including those not in the section, can be written as in (3.2): $\varphi' = \varphi_\sigma^g = ([\phi], g)_\sigma$, for some $g \in \mathcal{G}$.

Now, as I mentioned, the space $[\Phi]$ is abstract, or only defined implicitly. Therefore it is convenient, if not necessary, to have a definition of the image of σ that only traffics in Φ. That is achieved by *a projection operator* from Φ to the image of $\sigma([\Phi])$:

$$\begin{aligned} h_\sigma : \Phi &\to \Phi \\ \varphi &\mapsto h_\sigma(\varphi) = \sigma([\varphi]), \end{aligned} \tag{3.5}$$

which must of course be invariant, i.e. such that $h_\sigma(\varphi^g) = h_\sigma(\varphi)$. In practice, we only have a concrete or direct implementation of the projection operators, not of σ.

Here the projection is, essentially, an interpretation of the idea of a *gauge-fixing*, which we will develop in the case of field theories in Section 4.1.[18]

And it is important to note that the decomposition of a given state φ into a doublet, consisting of an equivalence class and a group element is not unique, which is why we have keep the subscript σ on the doublet, indicating this choice (cf. footnote 18 for the analogous construction and notation in (Wallace, 2019c)). That should be clear from the fact that, if one is to change the choice of representative σ, the same state can be represented by different doublets, or, conversely, different states can be represented by the same doublet. That is, we can have

$$([\varphi], g)_\sigma \neq ([\varphi], g)_{\sigma'}, \quad \text{or} \quad ([\varphi], g')_\sigma = ([\varphi], g)_{\sigma'}, \tag{3.6}$$

for $g \neq g', \sigma \neq \sigma'$. It is important to remember this when comparing states at a common boundary, where group elements can match without a matching of the doublet, or vice-versa. In other words, given just the state, φ, we cannot discern any symmetry transformation that has been applied to it. But armed with a choice of representative as in (3.2), we can do exactly that. Thus, as a general principle, any physical significance that we attribute to group elements, or functions of group elements, must make reference to such a choice.

Equally important is the fact emphasized in Section 2.3: that we require a representational convention when combining subsystems. Wallace (2019c, p. 18) highlights this same point:

[17] In the case of non-Abelian field theories, such a global representation of the state space does not exist due to the Gribov obstruction (see Gribov (1978); Singer (1978)). We will have more to say about this when we introduce the notion of a gauge-fixing in the case of Yang-Mills theory, in Section 4.1.

[18] Our notation is slightly different than Wallace (2019c, p. 9)'s, who denotes these doublets as (O, g) (in our notation $([\varphi], g)$), and labels the choice of representative (or gauge-fixing) as φ_O (our φ_σ). We prefer the latter notation, since it makes it clear that there is a choice to be made. As with coordinate systems, the interesting quantities will be invariant under these choices; nonetheless, we need to keep them fixed. This requirement will become nuanced when we are comparing different subsystems, with each other and with the joint system.

given configurations $(q; q')$ of the systems separately, we have not been given enough information to describe their joint configuration: that requires, in addition, a representational convention as to how points in the two configuration spaces are to be compared. Such a convention is inevitably required whenever we combine subsystems into a joint system. (In practice, the convention is often given by a choice of coordinate systems, and/or of reference frames, in the two subsystems.) Prior to stipulating any such convention, there is no sense in which (q, q') specifies a different configuration from $(R(g)q, q')$, since q and $R(g)q$ are representationally equivalent.[19] Given a choice of representational convention [i.e. σ], though, it is clear that applying the symmetry transformation to one system gives rise to a different total configuration (and that this is true independent of what the actual representational convention is). So: symmetry-related configurations can be understood as representing different possible configurations *if we hold fixed the choice of representational convention.* [my italics]

The requirement of a fixed representational convention is paramount for DES, since it discloses whether a symmetry transformation has been applied to a given state. But it is easy to see that one cannot just leave all these choices implicit when composing subsystems. For instance, the representational convention of the universe may not, when restricted, respect the representational convention of its subsystems. To give a simple example, in the non-relativistic particle case: if the convention employs the center of mass, there will be a conflict between the center of mass of the subsystems and of the whole. A similar issue appears in gauge theory.

3.1.3 Unobservability and other theses about symmetry

The central idea of dynamical symmetries (cf. footnote 8) can now be put as follows: given some notion of dynamical evolution, U, then $\varphi(t)$ satisfies the evolution equation, $U(t')\varphi(s) = \varphi(t' + s)$, if and only if $g(t)\varphi(t)$ also satisfies it. Once we assume a well-defined gauge-fixing exists, we can translate the central idea of dynamical symmetries from a statement about the dynamics of φ to one about the dynamics of $([\varphi], g)_\sigma$. Then, from (3.3) it is easy to show (see, for example, (Wallace, 2019c, p. 10)) that for a dynamical symmetry the future evolution of φ_σ depends only on the present value of φ_σ, with no additional dependence on g. Since the map $[\Phi] \times \mathrm{Id} \to \mathrm{Im}(\sigma)$ is a diffeomorphism, we get to translate these statements into ones about the equivalence classes: the future evolution of $[\varphi]$ depends only on the present value of $[\varphi]$—which is how it is is stated by (Wallace, 2019c, p. 10) (where this last step of translation from $\mathrm{Im}(\sigma)$ to $[\Phi]$ is omitted).

The natural interpretation is that there is "a self-contained dynamics for the invariant degrees of freedom of the system that is quite independent of the $\mathcal{G}$-variant features" (Wallace, 2019c, p. 10). If one moreover assumes that "the system under investigation is rich enough to model its own dynamics, and that the system is measuring itself rather than being observed from outside," this demonstrates *the unobservability thesis*: given a family of models of a global system which are related by a symmetry transformation, it is impossible to determine empirically which model in fact represents the system.

In a similar spirit, Wallace (2019c, p. 7-8) provides four theses about symmetries in general, and I completely endorse his demonstrations regarding these. In

[19]In our notation, introduced below in Section 3.2.1: $q \equiv \varphi_+, q' \equiv \varphi_-, R(g)q \equiv \varphi_+^{g+}$

particular, I will further jointly assume that: "given a family of models of a theory which are related by a [dynamical] symmetry transformation, insofar as one model successfully represents a system, so do all the others"; and that "two states of affairs related by the action of a symmetry transformation are really the same state of affairs, differently described."

3.2 DES and gluing

In section 1.1, we defined DES as transformations of the Universe possessing the following two properties:

(i): (*Global Variance*) the transformation should lead to an empirically different scenario, and

(ii): (*Subsystem Invariance*) the transformation should be a symmetry of the subsystem in question.

We also saw that physical quantities in gauge theories are characterized as gauge-invariant quantities, and that this obtains for both the subsystems and for the entire system.

Therefore, to earn the labels 'direct' and 'empirical', DES must be construed as referring solely to universal and subsystem gauge-invariant concepts.

Here, properties (i) and (ii) will be taken to apply to a Universe composed of a subsystem and an environment (as two subsystems). Following the internalist's symmetric treatment of subsystem and environment, (ii) will be taken to apply to all subsystems, i.e. to subsystem and environment.[20]

In Section 3.2.1 I will develop the definition of the internalist subsystem begun in Section 2.2.3. In Section 3.2.2 I describe how DES emerges from the gluing of physical states, using representational conventions.

3.2.1 Internalist subsystems

In our discussion of subsystems it is important to note that, in the internalist case, none of the symmetries here are, in the words of (Wallace, 2019c, p.12), 'subsystem-specific'. That means that the symmetries of a subsystem are extendible to the symmetries of other subsystems of the same universe. So given two subsystems, Φ_1, Φ_2, and $\mathcal{G}_1$, there is an action $\mathcal{G}$ defined on $\Phi_1 \times \Phi_2$ that extends elements of $\mathcal{G}_1$. This is in line with what I labeled downward consistency in Section 2.2.2.

[20] Couldn't we allow for a transformation that also changes the physical state of the environment? Invoking a physically significant change in the environment leads to the concern (voiced in, for example, (Friederich, 2014, p. 544)) that the empirical significance intended for the subsystem gauge symmetry is in fact completely due to the change in the environment state, thus leaving no room for the gauge symmetry to do any work. For example, in the Galileo's ship thought experiment, a transformation that leaves the ship and its relation to the shore as they are but changes a grain of sand on the other side of the Universe satisfies (i) and (ii). This is an observable change perhaps, but it has little to do with symmetries. Greaves & Wallace (2014, p. 68, 86 and 87) react to this concern by requiring a further condition: there should be a 'principled connection' between the putative change in the environment and the gauge symmetry. But they say nothing further about what such a connection might be. On the other hand, if (ii) applies symmetrically to all subsystems, empirical significance is encoded solely in the relations between them. In (S. Ramirez & Teh, 2019), a 'principled connection' is taken to exist when the changes in the subsystem are taken to be generated by a charge on the boundary. Here, we will be able to make some sense of such 'principled connections' in the externalist case, where the environment is just the boundary (see section B). In later work, Wallace (2019c) also restricts attention to extendible symmetries, and thereby to a relational understanding of the observability thesis (which is tantamount to the question of DES). Cf. the following footnote 22.

The extension may be unique or not. Wallace calls symmetries with unique extensions 'subsystem-global'; I call them *rigid*. In the rigid case, for each g_1, we get a unique $g_1 \times g_2 = g$. But instead of taking the (extendible) alternative to these—what he calls *subsystem-local*—to be ones in which the extension g_2 is given by an independent action of the same symmetry group, he defines them as one in which, for any action of the symmetry in one subsystem, a composition of that symmetry with the identity on the second subsystem is still a symmetry of the universe. Namely, for him, a subsystem-local symmetry is one for which, for every $g_1 \in \mathcal{G}_1$, $g_1 \times \mathrm{Id} = g$ is a symmetry of $\Phi_1 \times \Phi_2$.

In the case of field theory, the malleable symmetries on two subsystems that lie on *completely disjoint subsets* of spacetime (i.e. ones whose closure are nowhere intersecting) are independent, and thus conform to this definition. But when two subsystems are contiguous, this definition of subsystem-local symmetries is in clear tension with my assumption of downward consistency, as defined in Section 2.2.2. Indeed, I understand Wallace's construal of subsystem-local symmetries to be imposing an unnecessary further restriction on the behavior of symmetries at the common boundary of the subsystems; and the tension with downward consistency will be carried forward to a tension between the representational conventions for each subsystem. This will become clear, below, in Section 3.2, when we learn how to compose physical states belonging to the subsystems.

Adopting the internalist perspective, we do not require such a restriction. We must carve up the system into two (mutually exclusive, jointly exhaustive) subsystems whose state spaces we label Φ_+ and Φ_-, or $\Phi_\pm$, for short (a mnemonic notation to think of the subsystems as complements of one another, and intersecting only at a common boundary, e.g. 0). When these subsystems are made to correspond to regions, we will name the regions $R_\pm$. These are taken as subsets of the spatial manifold M, i.e. such that $R_+ \cup R_- = M$, and I will moreover assume that the intersection of the closure of the regions is a *boundary manifold*, S, i.e.

$$\overline{R}_+ \cap \overline{R}_- = S. \tag{3.7}$$

As discussed in Sections 2.1 and 2.2.3, under the assumption of downward consistency (2.2), the universal symmetries bequeath symmetries, through the split, to the subsystems, by mere restriction. Thus we write $\mathcal{G}_\pm$; and similarly, we extend the use of the equivalence class notation and of the square brackets: $\varphi_\pm \sim_\pm \varphi'_\pm$ iff $\varphi'_\pm = \varphi_\pm^{g_\pm}$ for some $g_\pm$, in which case $\varphi'_\pm \in [\varphi_\pm]$. Note that *no* extra conditions on the gauge transformations at the boundary are imposed, thus in particular these symmetries are not required to be subsystem-local in the sense of Wallace (2019c). Subsystem symmetries are just the symmetries obtained in each subsystem through the restriction of the symmetries of the larger system; this is the assumption of downward consistency of Section 2.2.2.

3.2.2 DES in terms of the physical states

We can now translate:

• *Global Variance:* $[\varphi] \neq [\varphi']$: the two physical states of the Universe are distinct according to the $\sim$ relation.

• *Subsystem Invariance:* $[\varphi_\pm] = [\varphi'_\pm]$: regionally the states are physically indistinguishable according to the $\sim_\pm$ relation; that, is for each $(\pm)$ subsystem, the primed and unprimed states are symmetry-related according to their internal models. Two subsystem physical states $[\varphi_\pm] \in [\Phi_\pm] := \Phi_\pm / \sim_\pm$ are composable iff they jointly descend from a global state, $[\varphi]$.

Note that only the 'physically significant', i.e. gauge-invariant, content of the subsystem and Universe states is relevant in the characterization of DES. The physical difference between $[\varphi]$ and $[\varphi']$ clearly must lie in the different possibilities for composing the two regional states, $[\varphi_\pm]$. The transformation must leave the subsystems physical content alone, but change their relation. This is possible because there are different domains for the equivalence relations—subsystem or universe—and therefore a Universal empirically significant difference may arise from a transformation that doesn't change the subsystem states, but *does* change their relation.[21] This idea is further explored in Gomes (2021a), under the lable of *holism*. Let us see how it plays out in more detail.

By introducing some, yet-to-be-defined, composition of physical states, $\boxplus$, and writing

$$[\varphi_+] \boxplus [\varphi_-] =: [\varphi] \neq [\varphi'] := [\varphi'_+] \boxplus' [\varphi'_-] = [\varphi_+] \boxplus' [\varphi_-] \tag{3.8}$$

we indicate more clearly that the very concept of DES needs to be gauge-invariant, i.e. physical. Note also that the *subsystem states* are intrinsically identical between the $[\varphi]$ and the $[\varphi']$ Universes, i.e. between the left and right hand sides of (3.8). Therefore the difference between the two sides of the equality must lie in the relation between the subsystems; this is signalled by (3.8)'s use of $\boxplus$ as well as $\boxplus'$. Thus here DES appears when there is a type of *holism*: when the subsystem physical states $[\varphi_\pm]$ do not suffice to determine the physical state of the joint system.[22]

Of course, as mentioned Section 3.1, equivalence classes are abstract and implicit, and notoriously resistant to explicit mathematical manipulation. In particular, we cannot articulate a notion of composition using only equivalence classes (see e.g. (Dougherty, 2017; Gomes, 2019; Nguyen et al., 2018)). To analyze (3.8) explicitly, we must refer back to local representatives, i.e. to representational conventions, as argued in Sections 2.3 and 3.1.2.

For local, smooth representatives in field theory, there is a straightforward definition of composition, as smooth composition, or gluing. More especifically, in the field-theories we will study here, we are given a Lie group G and a gauge transformation is a map from the spatial manifold M to G, i.e. $\mathcal{G} := C^\infty(M, G)$. It is a group on its own right, whose structure is inherited pointwise from the composition properties of G. M is also the manifold on which the global states of Φ are represented, usually as maps $\varphi : M \to V$, where V is some value space of the field.[23]

[21] Here I disagree with (Friederich, 2017), who seems to exclude the possibility of DES almost by assumption. He demands that no difference in relations should be present. E.g. on page 155: "The present article explores the idea that two subsystem ontic variables designate one and the same physical subsystem state only if the states designated by them are empirically indistinguishable both from within the subsystem and from the point of view of arbitrary external observers." and then again (p. 157): "In other words, s and s' designate the same physical state if they are empirically equivalent both from within the subsystem and from the perspectives of arbitrary external observers."

[22] In later work, Wallace (2019c, p.13-14) has a similar characterization: focusing on the extent to which the orbits of the subsystems determine the orbit of the joint system, and attributes failures of this determination to relational information.

[23] More accurately, we would have a fiber bundle over M, given by a manifold E with a well-defined, surjective operator $\pi : E \to M$, such that $\pi^{-1}(x) = V_x$ are the isomorphic fibers of E, i.e. $V_x \simeq V_y \simeq V$, for all $x, y \in M$. A field as we are defining it above would then be a section: $\varphi : M \to E$ such that $\pi \circ \varphi = \mathrm{Id}_M$. This is useful to construe gauge transformations as certain automorphisms of the bundle (e.g. spacetime dependent changes of bases for V_x that are representations of G). On the other hand, writing the fields just as maps, as I have

Suppose the regional state spaces are given by $\Phi_\pm$ and the regional gauge transformations are given by $\mathcal{G}_\pm$.[24] Then we can write the conditions on the composition operation, $\boxplus$, for physical, i.e. symmetry-invariant states as follows:

In field theory the two physical states are composable iff there exist states in each orbit, $\varphi_\pm \in \mathcal{O}_{\varphi_\pm}$, such that the value of φ_+ and all its derivatives at the boundary S match those of φ_-. We call such a notion of composition *gluing*. (see Appendix D for subsystem composition in the case of particles).

Given $[\varphi_\pm]$, and representational conventions $\sigma_\pm$, the condition of composition can thus be translated into the following gluing condition: there exist gauge transformations, $g_\pm \in \mathcal{G}_\pm$, such that:

$$\sigma_+([\varphi_+])^{g_+} =_{|S} \sigma_-([\varphi_-])^{g_-}, \tag{3.9}$$

where the subscript $_{|S}$, restricting the equality to S, is understood as also matching derivatives.[25]

I will use the notation $\oplus$ with the meaning of 'composition of *representatives*'; I do not restrict $\oplus$ to mean 'direct sum'. So, if (3.9) is satisfied, we translate the physical compositions of (3.8), i.e. $[\varphi_+] \boxplus [\varphi_-]$, into:

$$\sigma_+([\varphi_+])^{g_+} \oplus \sigma_-([\varphi_-])^{g_-} \tag{3.10}$$

In the field theory case, we can usually understand $\oplus$ just as addition in some vector space of smooth functions.

Two important things to note: though we are not specifying the choice of convention, we label each choice and do not leave implicit the fact that one is being made. Note also that we cannot eliminate either of $g_\pm$ in (3.9) and (3.10), since they act on different spaces and are therefore not subject to the same representational convention.

For point-particle systems, as we will see in section D, $\oplus$ requires an embedding of the subsystems into a common Euclidean space, and then it signifies vector addition.

Again, the way to make the two conditions for DES precise and clear is by using fixed representatives, for $\Phi_\pm$ and also Φ. Namely, *Subsystem Invariance* just means that we have just two Subsystem Invariance classes, $[\varphi_\pm]$, that are composable. Global Variance means then that there exist $g_\pm$ and $g'_\pm$ such that, given the same representational convention for the global state, σ, the glued states will differ. Simplifying the notation by writing $\sigma([\varphi]) =: \varphi_\sigma$ as in (3.4), the condition for DES is simply:

$$\varphi_{\sigma_+}^{g_+} + \varphi_{\sigma_-}^{g_-} =: \varphi_\sigma \neq \varphi'_\sigma := \varphi_{\sigma_+}^{g'_+} + \varphi_{\sigma_-}^{g'_-} \tag{3.11}$$

This is the most important equation for the matter of DES. This rendering of the physical significance of symmetries employs fixed representational conventions; it is this convention that allows us to unambiguously compare different possibilities, as is required in our construal of DES (cf. Sections 2.3 and 3.1.2).

done above, requires many other assumptions, e.g. about the topology of M and E. Nonetheless, I judge these issues to be unimportant to this paper, and will thus proceed with the simplified presentation above.

[24] Once downward consistency is respected, given regions $R_\pm$ and the restriction maps $r_\pm : M \to M|_{R_\pm}$, or alternatively the embedding maps $\iota_\pm : R_\pm \to M$, we would write $\mathcal{G}_\pm := \iota_\pm^* \mathcal{G} = \mathcal{G} \circ \iota_\pm$ and $\Phi_\pm := \iota_\pm^* \Phi$.

[25] For the standard notion of continuity, i.e. when all we require is the value of f at the boundary, and not also of its derivatives, we employ no bar, i.e: $f =_S f'$ iff $f(x) = f'(x)\ \forall x \in S$.

3.3 An incomplete derivation of DES

This paper started with the question: it is widely ackowledged that rigid symmetries in particle mechanics can have a (relational) DES when applied to subsystems; do local gauge theories also realize the concept of DES?

Although our answers differ, the treatment of this question by Greaves & Wallace (2014); Wallace (2019b,c) bears many similarities to ours here: they focus on subsystems as given by regions; they think of subsystems as given by a splitting of the universe; they identify transformations possessing properties (i) and (ii) in Section s 1.1 and 3.2.1 by first formulating the putative effects of such transformations on the gauge fields in these regions; and they construe DES essentially as a relational property.[26]

But *unlike* our results, they claim that there is relational DES transformations in 1-1 correspondence with the following quotient:

$$\mathcal{G}^{\rm GW}_{\rm DES}(\varphi) \simeq \mathcal{G}_+^{\varphi_-}/\mathcal{G}_+^{\rm Id}, \tag{3.12}$$

where $\mathcal{G}_+^{\varphi_-}$ are the elements of $\mathcal{G}_+$ which are 'in the φ_-–sector', that is, that can be composed with φ_- (cf. (Wallace, 2019b, p.10-11)). These transformations need preserve (only) the state φ_+ *at the boundary* of the region—which we call the *boundary*-stabilizer group for φ_+, as in (A.1)—and $\mathcal{G}_+^{\rm Id}$ are the gauge transformations of the region which are the identity at the boundary (and thus preserve all states at the boundary); the latter symmetries make up what he calls 'subsystem-local' symmetries (see Section 3.2.1).

I start in Section 3.3.1 by presenting a sketch of the standard derivation, and then I criticize this derivation in Section 3.3.2.

3.3.1 The derivation

Assuming the subsystem physical states are composable, given two global states

$$\varphi := \varphi_+ \oplus \varphi_- \quad \text{and} \quad \varphi' := \varphi'_+ \oplus \varphi'_-,$$

the condition *Subsystem Invariance* translates into:

$$\varphi' := \varphi'_+ \oplus \varphi'_- = \varphi_+^{g_+} \oplus \varphi_-^{g_-} \quad \text{for some pair of elements} \quad g_\pm \in \mathcal{G}_\pm \tag{3.13}$$

Now, *Global Variance* demands that, for DES to be realized, there can be *no* g such that $\varphi' = \varphi^g$. That is, *Global Variance* implies:

$$\text{there is no universal } g \text{ such that} \quad g_{|R_+} = g_+, \quad g_{|R_-} = g_-, \tag{3.14}$$

for otherwise $\varphi' = \varphi^g \sim \varphi$ and the two states are entirely physically equivalent.

The result (3.12) requires an assumption: when looking for the realizers of the conditions *Global Variance* and *Subsystem Invariance* (cf. Section 3.2.1), one

[26] Although Greaves and Wallace allow for the larger, non-strictly relational quotient group, of all subsystem symmetries quotiented by the interior ones, at the theoretical level they do not investigate this larger (infinite dimensional) group, whose physical meaning—if any—is unclear (Greaves & Wallace, 2014, p.86,87). They only mention Einstein's elevator and the Faraday cage as examples of this extension, but have little to say about what are the principled connections that render these, and only these (?) as bona-fide examples of DES, while disallowing examples where only an irrelevant change in the environment accompanies the subsystem symmetry; e.g. a change of a grain of sand on the beach should not be associated to a DES of Galileo's ship. See their footnote 17, p.74. The later papers Wallace (2019a,b,c) are less ambiguous in this respect, and are more aligned with our relational view here, cf. e.g. (Wallace, 2019c, p. 13-14).

may keep one of the regional subsystems—labeled 'the environment'—not only physically fixed (both are physically 'fixed' according to *Subsystem Invariance*), but also representionally fixed. In other words, there is an assumption that we have a fixed representative φ_- of the physical environment state $[\varphi_-]$ which we can employ as a reference to externally assess the capacity of regional gauge transformations g_+ to produce empirically distinguishable differences. (But no mention of an explicit use of a representational convention is made.) This restricts the states $\varphi_+ \in \Phi_+$ to be, in the language of Wallace (2019b, p. 10), in the *φ_--sector of the theory* (and in the nomenclature of Section 2.1, that parallels that of Wallace, called Φ_{φ_-}). It is thus usually taken for granted that we can assume the environment is in this implicitly given representation and restrict attention to g_- being the identity transformation.

Then, *if* some physical states already satisfy *Global Variance* and *Subsystem Invariance*, instead of (3.13), the assumption is that we have representatives of the states fulfilling:

$$\varphi = \varphi_+ \oplus \varphi_- \quad \text{and} \quad \varphi' := \varphi_+^{g_+} \oplus \varphi_-. \tag{3.13'}$$

If (3.13') is assumed, and we moreover assume that φ_- has only the trivial stabilizer, meaning there are no $g_- \neq \mathrm{Id}$ such that $\varphi_-^{g_-} = \varphi_-$ (see Section A.1.1), we can similarly rewrite (3.14) as follows:

$$\text{there is no } g \in \mathcal{G} \text{ such that} \quad g_{|R_+} = g_+, \quad g_{|R_-} = \mathrm{Id}. \tag{3.14'}$$

Of course, jointly, the assumptions above would then mean that *Global Variance* requires $g_{+|S} \neq \mathrm{Id}$. By quotienting all the transformations that do have this boundary behavior, namely those that preserve the Φ_{φ_-}-sector of the theory, by those such that $g_{+|S} \neq \mathrm{Id}$, we arrive at (3.12).

3.3.2 The gap in the previous derivation

The assumption that $g_- = \mathrm{Id}$ (or equivalently, that the transformation is *subsystem-local* in the narrow understanding discussed in Section 3.2.1) is consequential for the issue at hand.

The assumption is that we do not need to make reference to the representational convention for the environment; that it can be left implicit. We are just 'given' a φ_-. It, like the asymptotic states, will therefore pare down the symmetries at the boundary. Of course, there is no a priori, or canonically preferred, representational convention for the environment: $g_- = \mathrm{Id}$ is not a representational convention (or a gauge-fixing); it doesn't fix a map from the equivalence classes to the representative states, as in (3.4).

What we are in fact given is an equivalence class, or a physical state (according to the theory as applied to the environment), and we must choose a representational convention for the environment just as we must for the subsystem in question and as we will have to for the universe as a whole. But as one can show, once one makes the representational convention explicit, subtleties arise when comparing the global states, for there is the issue of how these conventions mesh. Let me expand this argument in more detail.

As we agreed in Sections 2.3 and 3.1.2, we must keep fixed a representational convention in order to evaluate the observability of subsystem symmetries. It is true that in many, if not most, circumstances we need not make that convention explicit: it suffices that we acknowledge one exists and is kept fixed; we often talk about a representative of the physical state without discussion of how that representative is defined. However, when investigating subsystems and their relation to the entire universe more than one representational convention is at play, and they may be incongruous, in the following sense. Suppose one fixes the representational convention for subsystems and universe, $\sigma_\pm$ and σ, respectively. Still, the representational convention of the global state may have its restrictions to the subsystems fail to satisfy the regional represenational convention. This is very clear in the point particle case discussed in Section D: we choose subsystem center of mass coordinates, but, upon composition, a new center of mass will emerge, and we will have to 'readjust' both our previous representational conventions.

In the field theoretic case, something similar happens. Using the nomenclature of Section 3.2.2 and the projection operator (3.5), we may have:

$$\iota_\pm^* h_\sigma \neq h_\sigma^\pm. \tag{3.15}$$

Thus, in order to count global possibilities given just the physical state, or, equivalently, the $h_\sigma^\pm$, some adjustment between the two states in their regional representational conventions may be allowed or even required; that is, we should allow a $g_- \neq \mathrm{Id}$ (which we did in (3.11); cf. footnote 43 in Section 4.4.2). We will see this issue emerge explicitly in Sections D and 4.4.2 (see footnote 43 and equation (4.13)).

Indeed, as one can show, for internal boundaries respecting downward consistency, by rejecting the 'God-given' representation of the environment, no relational empirical significance in the vacuum, simply-connected case, can be identified. In this sector of the theory, (Gomes, 2021a, Section 4, Equation 4.2) provides an explicit counter-example to the definition (3.12).[27]

In more recent work, the type of counter-example of footnote 27 is excluded by narrowing the focus of the definition to 'generic' states (Wallace, 2019b, p.9).[28] But this assumption is not used or sufficiently justified in the rest of the paper, and thus its imposition seems to me slightly *ad hoc*. To be more precise, in Wallace (2019b,c), the generic property is mentioned at the same stage that I mentioned it: between (3.14') and (3.13'). The idea is that, if the environment does not have any stabilizer, a gauge transformation that preserves the boundary state and is not the identity will necessarily be continued into a transformation that doesn't preserve the environment state, and is therefore "witnessed" by the environment. But I find this confusing, since part of the initial assumption was precisely that the representation of the environment state *is fixed* as φ_-.

In sum: if all we have access to, according to the theory, are the physical content of the states, then we require a representational convention to represent the physical

[27]The counter-example is as follows: for electromagnetism, for a configuration (or sector) that happens to be in vacuum, any element of $\mathcal{G}_+$ that goes to a constant $c \neq 1$ at the boundary will provide a representative of $\mathcal{G}_+^{\varphi_-}/\mathcal{G}_+^{\mathrm{Id}}$ in (3.12). Moreover, this can occur for any notion 'isolation' of the subsystem. But in fact, for two states, φ' and φ, as in (3.13'), related by such a a transformation, one is always able to explicitly find a global g such that $\varphi^g = \varphi'$, thus foiling *Global Variance*. I should note that Greaves and Wallace do not overtly narrow down their formal prescription for DES to include matter. In particular, their derivation does not mention matter or the lack thereof. The failure of that derivation in sectors in which matter is absent is neither explained nor mathematically expected; there is nothing in their definition that gives any hint to why this should be the case.

[28]A'generic' subset here is not defined as usual: it is defined as the set of states with only the trivial stabilizer (cf. (A.1)). In field theory, were one to use an actual definition of generic subspaces of Φ as dense and open subsets, then there would be no DES. For in the presence or absence of matter, generic states would have matter on their boundaries, and thus would not have any non-trivial boundary stabilizer, and thus $\mathcal{G}_+^{\varphi_-} = \mathcal{G}_+^{\mathrm{Id}}$.

state. Without such a convention, one is liable to be led astray in the internalist case. Employing representational conventions, in Section 4 we will assess DES for any state (even non-generic ones, cf. footnote 27).

4 The gauge theory of fields

Here I will describe the basic setting with which I will treat the local gauge theory of fields, taking as my model vacuum Yang-Mills theory on a simply-connected manifold, M. The stated results should be taken as applying to both Abelian and non-Abelian interactions alike, and the extension to non-simply-connected manifolds and to the inclusion of matter are straightforward but notationally cumbersome; exceptions and differences to these generalizations will be explicitly flagged. Having said this, I will, as a simplification, only *explicitly* treat Abelian gauge fields (like electromagnetism).[29]

In Section 4.1, I develop further the ideas presented in Section 3.1.2, about representational convention, and describe what those ideas have to do with gauge-fixing, with an eye towards the application to Yang-Mills theory. In Section 4.2 I describe in a bit more detail what I will take subsystems to be in Yang-Mills theory and discuss recent developments for gauge-invariant subsystem dynamics. In Section 4.3, I write down the specific field content and its symmetry transformation properties, specializing to the case of electromagnetism, and to subsystems defined by gauge-invariant boundaries. And in Section 4.4, I finally put these constructions to use in finding DES, by unpacking the main equation defining DES, Equation (3.11), in the case of electromagnetism. This viewpoint expresses DES in terms of uniqueness properties of coupled partial differential equations with particular boundary conditions.

4.1 Gauge-fixing: the general ideas

In the type of field theories we will focus on in this paper, the procedure for fixing the representative of the state, or finding a representational convention, as in Section 3.1.2 and Section 3.1.2, is intimately related to a procedure called 'fixing the gauge'. The procedure is necessary to extricate physically significant properties of the state from the unphysical ones, that are not invariant under the symmetries. In other words, by fixing the gauge, no physical property is lost. Thus important physical effects, such as the Aharonov-Bohm effect, quantum anomalies, interference, are all perfectly expressible in a gauge-fixed setting, as I define it here[30]

A gauge fixing provides, in the language of Section 3.1.2, a *a fixed representational convention* with which to compare different states. As we saw in that Section (see Equation (3.5)) gauge-fixing can be seen as a sort of projection on state space, which allows us to judge whether two given representatives, φ, φ', unrelated in principle, are physically the same, i.e. give the same value for *all* gauge-invariant quantities. In other words, two configurations are physically the same if and only

[29] For a more complete treatment, see (Gomes, 2021a; Gomes & Riello, 2021).

[30] Much as in other representtions of gauge-invariant quantities—such as in the holonomy interpretation—fixing the gauge is non-local in the following sense: just as $\int A$ requires the value of of A at several points simultaneously as an input, the projected state $h_\sigma(A)$ requires the value of A throughout the region as in input. This is just a reflection of the non-local aspects of gauge-invariant functions (cf. (Earman, 1987, p. 460), (Healey, 2007, Ch. 4.5), and (Gomes, 2019; Strocchi, 2015)).

if they are *identical* once gauge-fixed. Thus a gauge-fixing resolves problems of physical identity.

In the language of fiber bundles, a gauge-fixing is a choice of *section* of the configuration space, seen as a (possibly infinite-dimensional) principal bundle. A choice of section is essentially an embedded submanifold on the state space Φ that intersects each orbit once. In practice, the gauge-fixing procedure relies on the given representational convention $\sigma([\varphi])$ satisfying some auxiliary condition. That is, we impose further functional equations that the state in the aimed-for representation must satisfy. This is like defining a submanifold indirectly, through the regular value theorem. E.g. defining a co-dimension one surface $\Sigma \subset N$ for some manifold N, as $F^{-1}(c)$, for $c \in \mathbb{R}$, and F a smooth and regular function, i.e. $F : N \to \mathbb{R}$ such that $\mathrm{d}F \neq 0$. Once the surface is defined, σ, as defined in Section 3.1.2, will be the embedding map for one such surface, e.g. $\sigma : [\Phi] \to F^{-1}(0) \subset \Phi$. Once the surface is defined, we can define a projection map, that projects any configuration to this surface, and this projection will be gauge-invariant.[31]

Now I will describe the two conditions expected of a complete gauge-fixing.

In general, we fix the gauge freedom by imposing conditions on the representative gauge potential, i.e. by imposing a local functional equation $F(A) = 0$, for some F which, besides being regular, ideally must satisfy two further conditions:

• *Universality* (or existence): For all $\varphi \in \Phi$, the equation $F(\varphi^g) = 0$ must be solvable by a functional $g_\sigma(\varphi)$. Here, $g_\sigma(\varphi)$ is a gauge transformation required to transform φ to a configuration $\varphi^{g_\sigma(\varphi)}$ which belongs to the gauge-fixing section σ. So g_σ must be such that $F(\varphi^{g_\sigma(\varphi)}) = 0$. That is:

$$g_\sigma : \Phi \to \mathcal{G}, \text{ is such that } F(\varphi^{g_\sigma(\varphi)}) = 0, \text{ for all } \varphi \in \Phi. \tag{4.1}$$

This condition ensures that F doesn't forbid certain states, i.e. that each orbit possesses at least one intersection with the gauge-fixing section.

• *Uniqueness*: If g_σ as above satisfies $F(\varphi^{g_\sigma(\varphi)}) = 0$, then $\varphi^{g_\sigma(\varphi)} = \varphi'^{g_\sigma(\varphi')}$ if and only if $\varphi \sim \varphi'$. That is, the representatives coincide iff they represent the same physical state, $[\varphi]$. That is, a gauge-fixing resolves mattes of physical identity between representative states.[32]

Since g_σ should act as a projection operator on Φ, onto the gauge-fixing surface, it is convenient to explicitly define this projection as in (3.5), but now explicitly including g_σ:

$$h_\sigma : \Phi \to \Phi \tag{4.2}$$

$$\varphi \mapsto h_\sigma(\varphi) := \varphi^{g_\sigma(\varphi)} \tag{4.3}$$

And, as expected, $h_\sigma(\varphi)$ is a gauge-invariant functional, in the sense that $h_\sigma(\varphi^g) = h_\sigma(\varphi)$, i.e. it is invariant under the group action on Φ as its domain. Of course, we can still act on the surface itself, i.e. act with the group on the image of h_σ.[33]

[31] Take $\mathbb{R}^2$, and a choice of a graph, $y(x)$, defined by some function $F(x, y) = 0$. Now we can project any doublet, (x, y) onto $y(x)$, namely, $(x, y) \mapsto (x, y(x))$. The projection is independent of y, and, if we identify translations in the y-directions as 'gauge', the projection is gauge-invariant.

[32] Jumping ahead, in Section A.1.1, I introduce one subtlety in the concept of gauge-fixing, due to *stabilizers*, which plays an important role in the definition of DES. Certain states are not "wrinkly enough", do not have features that are detailed enough, to completely fix the representation. These states have stabilizers. Stabilizers are degeneracies in the representational convention, that foil uniqueness for physical reasons.

[33] It is important to stress that $h : \Phi \to \Phi$ is a projection, as opposed to a reduction, $\mathrm{pr} : \Phi \to [\Phi]$. In (Gomes, 2019, 2021a), the construal of a gauge-fixing as a projection, and not as a quotienting, was argued to be funda-

Assume both the *Universality* and *Uniqueness* conditions hold for some choice of F. Then, as stated in Section 3.1.2, we can describe any element of Φ as h_σ^g, where h belongs to the surface in question, that is, satisifies the condition $F(h_\sigma) = 0$; and $g \in \mathcal{G}$ describes a gauge transformation as applied to the given element of the section.

A gauge-fixing thus yields a one-to-one relation: $[\varphi] \leftrightarrow h_\sigma(\varphi) := \varphi^{g_\sigma(\varphi)}$, which is what is meant when we say that the entire gauge-invariant content of a configuration is contained in its gauge-fixed form. In other words, h_σ is the representational convention, articulated as a projection from a member of an equivalence class to a unique representative of that class.

We will see explicit examples of $h_\sigma(\varphi)$ in the appendix. It is also important to distinguish $h_\sigma(\varphi)$, which is itself a gauge potential, from $g_\sigma(\varphi)$, which is a group transformation taking φ to $h_\sigma(\varphi)$. To unclutter notation, we will remove the subscript σ from all functionals unless explicit reference to σ is needed as a reminder.

In the upcoming Section 4.2, we define the generic subsystems that we will focus on. This Section explains the obstacles towards satisfying downward consistency when dealing with subsystems of a gauge theory.

4.2 The subsystems

Now, I can briefly, and at a pedestrian level, address the issues posed by the non-locality of gauge theories on consistent definitions of subsystems, as mentioned in Sections 2.2.3 and 2.3.

First, I will just schematically introduce the issue, as seen through the Lagrangian formalism. In the Abelian case, we define the field strength $F_{\mu\nu} := \partial_{[\mu} A_{\nu]}$, where square brackets denote anti-symmetrization. The Yang-Mills action in vacuum then is:

$$S(A) := \int_{M\times\mathbb{R}} F_{\mu\nu} F^{\mu\nu}. \tag{4.5}$$

mental for the gluing of regions: for both h and pr are gauge-invariant with respect to gauge-transformations on the common domain, Φ, i.e. $\mathrm{pr}(\varphi^g) = \mathrm{pr}(\varphi)$ as well as $h(\varphi^g) = h(\varphi)$, but only the projection h allows further transformations to be enacted on its range, and therefore allows for a change of representational convention. Gomes (2019, 2021a) distinguishes between two sorts of action of $\mathcal{G}$:
Subsystem-intrinsic gauge transformations: Given $h : \Phi \to \Phi$, a subsystem-intrinsic gauge-transformation acts solely on the domain of h. The projection h is invariant under subsystem-intrinsic gauge transformations: $h(\varphi^g) = h(\varphi)$. The label 'intrinsic' stands in opposition to 'extrinsic'. Such gauge transformations are all that is needed for unique description of the entire Universe. But if we have more than one subsystem and we want to satisfy the gluing condition (3.11), we may need to change the representative of $[\varphi]$—from the outside, as it were.
Subsystem-extrinsic gauge transformations: Given $h : \Phi \to \Phi$, we can define *subsystem-extrinsic* gauge transformations g^{ext}, as those transformations which act on the *range* of h as

$$h(\varphi) \mapsto h(\varphi)^g. \tag{4.4}$$

Of course such a transformed field would no longer satisfy (A.7). In what follows we omit the superscript 'ext'. There are the transformations that are required when we need to change representational conventions, as we must when we glue subsystems.

It is instructive to compare the two possibilities of action of $\mathcal{G}$ to the use of homotopy type theory (HoTT) in gauge theory, as advocated by (Ladyman, 2015). Ladyman says HoTT "both (a) distinguishes states conceived of differently even if they are subsequently identified, and (b) distinguishes the identity map from non-trivial transformations that nonetheless might be regarded as delivering an identical state". Here we have two sorts of transformations: the subsystem-intrinsic one, $\varphi \mapsto \varphi^g$, which does not change $h(\varphi)$—satisfying Ladyman's (b)—, and the subsystem-extrinsic one, that does the work of Ladyman's (a).

On a bounded submanifold, say, $R \times \mathbb{R}$, where $R \subset M$ is a spatial submanifold of M, a variation of the action yields, after integration by parts:

$$\delta S(A) = -\int_{M\times\mathbb{R}} \delta A^\nu(\partial^\mu F_{\mu\nu}) + \oint_{S\times\mathbb{R}} s^\mu F_{\mu\nu}\delta A^\nu, \tag{4.6}$$

where s^μ is the normal to the hypersurface $S \times \mathbb{R}$ in $M \times \mathbb{R}$. Now, for the first term of (4.6) to vanish for arbitrary variations of the gauge potential it suffices that the gauge potential satisfies the vacuum Maxwell equations. But the second term vanishes only if either the electromagnetic field tensor vanishes along the boundary or δA^μ vanishes at the boundary. The first condition is severely limiting; the second is not a gauge-invariant condition.[34]

In the symplectic formalism, we witness a similar obstruction: in brief, denoting the symplectic 2-form by Ω (i.e. a closed, non-degenerate 2-form on phase space), infinitesimal generators of gauge transformations, $\xi \in C^\infty(M, \mathfrak{g})$ are usually characterized by generating phase space vector fields $\xi^\sharp$ in the kernel of the symplectic-form, that is, gauge transformations satisfy:[35]

$$\mathbb{i}_{\xi^\sharp}\Omega \approx 0, \tag{4.7}$$

where $\approx$ means the equality holds after we impose the kinematical constraints, or conservation laws (see (Henneaux & Teitelboim, 1992, Ch. 1) and (Butterfield, 2007; Gomes & Butterfield, 2021) for philosophical introductions). For Yang-Mills theories, with a general, non-Abelian algebra $\mathfrak{g}$, (4.7) always obtains in the absence of boundaries. But in the presence of boundaries, it only obtains if $\xi_{|S} = 0$ or $f = 0$, which, again, are either severely limiting isolation conditions or do not respect downward consistency (Equation (2.2)).[36]

[34]It is important here that these are time-like boundaries; for the spacelike initial and final surfaces, one can implement whatever initial conditions one likes. And the boundary term gives rise to the symplectic potential: $\theta = \int E^i \delta A_i$, which defines the symplectic structure of the theory, $\Omega := \delta\theta$.

[35]Indeed, the null directions of $\mathfrak{i}^*(\omega)$, where $\mathfrak{i}$ represents the embedding of the constraint surface into phase space, are necessary and sufficient to characterise the generators of gauge symmetry. For suppose that what we know is that a certain class of vector fields X_I is such that $\omega(X_I, \bullet) = 0$. Since the exterior derivative d commutes with pullbacks, if ω is closed, $\mathfrak{i}^*\omega =: \tilde\omega$ is also closed. Thus using the Cartan Magic formula relating Lie derivatives, contractions i and the exterior derivative d:

$$L_{X_I}\tilde\omega = (\mathrm{d}i_{X_I} + i_{X_I}\mathrm{d})\tilde\omega = 0;$$

i.e. the first term also vanishes because $\tilde\omega(X_I, \bullet) = 0$. So $\tilde\omega$ itself is invariant along X_I. Moreover, if we take the commutator of X_I, X_J, i.e. $[X_I, X_J] = L_{X_I}X_J$, contract it with $\tilde\omega$, and remember the formula:

$$L_{X_I}(\tilde\omega(X_J, \bullet)) = \tilde\omega(L_{X_I}X_J, \bullet) + (L_{X_I}\tilde\omega)(X_J, \bullet),$$

we obtain that, since both $L_{X_I}(\tilde\omega(X_J, \bullet)) = 0$ and $L_{X_I}\tilde\omega = 0$, it is also the case that $\tilde\omega([X_I, X_J], \bullet) = 0$. Thus, by the Frobenius theorem the kernel of the pullback $\mathfrak{i}^*(\omega)$ forms an integrable distribution which integrates to give the orbits of the symmetry transformation. This means we can define a projection operator $\pi : \Gamma \to \Gamma/G$; and, ultimately the degeneracy of $\mathfrak{i}^*\omega$ allows one to define a *reduced symplectic form*, ϖ, on the space of orbits, given by $\pi^*\varpi = \mathfrak{i}^*\omega$. See (Marsden, 2007, Ch. 1). This will be picked up in footnote 39, below.

[36]In more detail, let $\Omega = \int \mathrm{tr}(\delta A \wedge \delta E)$. Then we obtain:

$$\mathbb{i}_{\xi^\sharp}\Omega = \int \mathrm{tr}(\mathrm{d}\xi \mathbb{d}E + [A, \delta E] + [\delta A, E]) = \int \mathrm{tr}(\xi\, \delta(\mathrm{D}_A E)) + \oint \mathrm{tr}(\xi \delta f),$$

where D_A is the gauge-covariant derivative (cf. footnote 55). We can extract two important pieces of information from this equation: (1) the flow of gauge transformations is Hamiltonian, i.e. such that for each ξ we have a generating function on phase space, H_ξ such that: $\mathbb{i}_{\xi^\sharp}\Omega = \delta H_\xi$ iff $\delta\xi = 0$ and either $\xi_{|S} = 0$ or $\delta f_{|S} = 0$. But, unless $f = 0$, f is not gauge-invariant in the non-Abelian theory, and therefore we cannot fix $\delta f = 0$

In line with these considerations, the resolution pursued in Gomes (2019); Gomes et al. (2019); Gomes & Riello (2017, 2018, 2021); Riello (2021a) is to consider all variations to be performed within the same representational convention. The dependence on the representational convention then appears explicitly in the variational procedure through the projection operator, h_σ, given in (3.5). By taking into account the phase-space dependence of this projection, the dynamical structures of the subsystem become suitably gauge-invariant (cf. (Gomes & Riello, 2021, Section 3)).

It is also interesting to note that, in a given convention, $h_\sigma(\mathbf{A})$ only captures the content of a principal connection, ω, in directions that lie along the section, or representational convention, σ. The vertical component of ω—which is dynamically inert, since it is determined by gauge covariance—can be seen (in a suitable interpretation of differential forms, cf. Bonora & Cotta-Ramusino (1983)) as the BRST ghosts; see Thierry-Mieg (1980). When we have two regions, we have two sections, or two representational conventions. In the Thierry-Mieg interpretation, an infinitesimal relation between states $h_{\sigma_\pm}(\mathbf{A}_\pm)$ is given by the vertical part of ω; integrating this difference we obtain transition functions. We can think of that transition as our $g_{\sigma_+}(\mathbf{A}_-)$, defined in (4.1), defined at the boundary. So (1): there is an intimate relation between ghosts and the projection operators h_σ; and (2) both mathematical objects are only dispensable in the classical domain with a single, unbounded region. Once we need to take into account multiple physical states—as we must in either the quantum regime or in the presence of boundaries—we need a representational convention. The relationship between ghosts, representational conventions and the gluing of regions is elaborated in Gomes (2019); Gomes & Riello (2017). Thus we speculate: the restoration of invariance of the regional dynamical structures is due to the use of the classical BRST differential, that becomes manifest only upon either the gluing of regions or upon quantization; and that this is the main reason Gomes and Riello's functional connection-form works to restablish gauge-invariance.

The resolution is pursued differently in Donnelly & Freidel (2016) and follow-up papers (see e.g. Geiller & Jai-akson (2020) for a full list), which adds degrees of freedom at the boundary with appropriate gauge-covariance properties so as to cancel out the unwanted terms. The two approaches are related through a suitable interpretation of the new degrees of freedom as our g_σ of (4.1) (see e.g. (Riello, 2021b, Section 5), (Regge & Teitelboim, 1974, Section 5) (Carrozza & Hoehn, 2021, Section 4)).

But let us focus on the Riello and Gomes's resolution through explicit representational convention. In the symplectic formalism, their choice of representational convention symplectically pairs $h_\sigma(A)$ with the radiative content of the electric field. In more detail, the electric field can be split into a radiative and Coulombic component, as discussed in (Gomes & Riello, 2021, Section 6.5). The radiative component corresponds, roughly, to radiation (also in the non-Abelian case), and it does not depend on the contemporaneous distribution of charges nor on the value of f at the boundary; whereas the Coulombic component is entirely determined by these two pieces of information. The crucial mathematical property for the split in phase space is that the Coulombic component is symplectically orthogonal to the $h_\sigma(A)$.[37]

(unless $f = 0$); (2) $\xi^\sharp$ is in the kernel of the symplectic form iff $\xi_{|S} = 0$ or $f = 0$. But using representational conventions, one can find gauge-invariant regional structures labeled by the fluxes at the boundary; see (Riello, 2021b) and footnote 39.

[37]That representational convention is a generalization of Coulomb gauge (see Gomes & Butterfield (2021) for

The radiative/gauge-fixed regional phase space structure is fully gauge-invariant, but it leaves out a part of phase space that pairs up the group-valued g_σ of (4.1) with the electric flux.[38]

When we take into consideration the full phase space, there are superselction sectors: i.e. different symplectic spaces attached to each gauge orbit of the electric flux, and these sectors are dynamically decoupled from each other. Although the full phase space structure of a regional subsystem is therefore indexed by the gauge-invariant class of f at the boundary, this superselection becomes redundant once we have at hand both the charged matter content and the radiative/gauge-fixed symplectic pair of each region (Gomes & Riello, 2021, p. 57):

> Once both regional radiatives are known, even the regional Coulombic components are completely determined—including the electric flux f through S, which is thus no longer an independent degree of freedom once the radiative modes are accessible in both regions. Thus, in this case—when the larger (glued) region M has no boundary—the regional radiative modes encode the totality of the degrees of freedom in the joint system. In particular, the conclusion reached in section 3.4 from a regional viewpoint that f through S must be superselected is a mere artifact of excluding [radiative] observables in the complement of that region. The addition of charged matter does not change this conclusion.

In sum, using a representational convention and the appropriate variational principles, we can use a gauge-invariant characterization of the regional phase space, in accord with downward consistency, given in (2.2). Thus we can consider the configuration space of the theory over any spacetime as built out of the configuration spaces of the theory over subregions of that spacetime. The internalist's splitting of the manifold naturally induces an identification of subsystems and regions, and ensuing identifications of their respective state spaces and symmetries.

4.3 Preliminaries: from Yang-Mills to vacuum electromagnetism

Now we especialize to Yang-Mills theories. We do not need to explicitly exhibit the Lagrangian or the Hamiltonian, since in these cases the fundamental and the dynamical symmetries match, we just let $g \in \mathcal{G}$, where $\mathcal{G} := C^\infty(M, G)$, for a spatial manifold M. In the vacuum Yang-Mills case, φ are identified as the field configurations, A; they are representatives of the equivalence classes, $[A]$. Here $A \sim A'$ iff $A' = A^g$, and, in vacuum,

$$\Phi \equiv \mathcal{A} := \{A \in \Lambda^1(M, \mathfrak{g})\} \tag{4.8}$$

(the space of Lie-algebra-valued smooth one-forms on M). The momentum variables conjugate to A are the electric fields—Lie-algebra-valued vector fields, which we denote by $E \in \mathfrak{X}(M, \mathfrak{g})$.

For the detailed exposition, in Appendix A, we will specialize to A_i as the electromagnetic gauge field. The fundamental, or *charge group*, of this theory is $G = \mathrm{U}(1)$, with an associated Lie-algebra $\mathfrak{g} = \mathbb{R}$. When we include matter in the

a philosophical/conceptual analysis of the relation between Coulomb gauge and the radiative/Coulombic split of the electric field).

[38] This is an alternative characterization of edge-modes: as the conjugate to the electric flux. Gomes & Riello (2021) make the symplectic pairing mathematically rigorous and find that this definition is imprecise: it is the entire Gauss constraint, together with the boundary flux, that becomes conjugate to the g_σ in the entire region.

case of electromagnetism, we will assume it is of the Klein-Gordon type, , e.g. a map $\psi : M \to \mathbb{C}$. With an appropriate choice of units, the gauge transformations are:

$$\begin{cases} A_i & \to (A^g)_i := A_i + i\partial_i \ln g \\ E^i & \to (E^g)^i := E^i \\ \psi & \to \psi^g := g\psi \end{cases} \tag{4.9}$$

for some U(1)-valued function $g(x)$ (i.e. $g(x)$ is smooth complex-valued function satisfying $|g(x)| = 1$). That is, in the vacuum case the φ of the previous section would here be the electromagnetic potential, A, which changes non-trivially under the gauge transformation, whereas its conjugate variable, E^i is invariant under it.

Given the embeddings $\iota_\pm : R_\pm \to M$, in the vacuum case, we get $\Phi_\pm \equiv \mathcal{A}_\pm := \{A_\pm \in \Lambda^1(R_\pm, \mathfrak{g})\}$, where $\Lambda^1(R_\pm, \mathfrak{g})$ are the Lie-algebra-valued (i.e. here $\mathbb{R}$-valued) smooth 1-forms on the spatial submanifolds $R_\pm$. Here we take the surface S to not impose any condition on the states, and thus it is specified gauge-invariant. As per the considerations of Section 4.2, the dynamics of the region then has an invariance group: $\mathcal{G}_\pm := \mathcal{G} \circ \iota_\pm = C^\infty(R_\pm, G)$ so that the the subsystem satisfies downward consistency.[39] We could also include matter in Φ, as long as we implement boundary conditions that are gauge invariant according to (4.9); e.g. if matter is absent from the boundary.

4.4 Finding DES in gauge theories

In Section 4.4.1, I summarize the main ideas involved in gluing or composing physical Yang-Mills states and articulate the matter of DES as a remaining physical variety after gluing. Section 4.4.2 summarizes the main technical achievements of the approach, which are considered in detail in Appendix A, where I describe the procedure explicitly in the representational convention that corresponds to Coulomb gauge.

4.4.1 General considerations

The question of DES as I have construed it here amounts to whether there are *physically distinct* ways that the composition of *physically identical* subsystems states can go. We need to assess the possibilities for satisfying (3.8), finding $[\varphi] = [\varphi_+] \boxplus [\varphi_-] \neq [\varphi'] = [\varphi'_+] \boxplus' [\varphi'_-]$ and non-trivially satisfying DES according to Section 3.2. Our main aim will be to unpack (3.11), which we reproduce here:

$$\varphi^{g_+}_{\sigma_+} + \varphi^{g_-}_{\sigma_-} =: \varphi_\sigma \neq \varphi'_\sigma := \varphi^{g'_+}_{\sigma_+} + \varphi^{g'_-}_{\sigma_-}. \tag{4.10}$$

We can translate (4.10) using the present notation as:

$$h^{g_+}_+ \Theta_+ + h^{g_-}_- \Theta_- =: h \neq h' := h^{g'_+}_+ \Theta_- + h^{g'_-}_- \Theta_-. \tag{4.11}$$

[39] More accurately, we would have to partition the phase spaces $T^*\Phi_\pm = \cup_{[f]} \Phi^{[f]}_\pm$, where $[f]$ is the equivalence class of electric fluxes (in the Abelian theory, since f is gauge-invariant, no squre brackets are necessary). For each value of $[f]$ there is a well-defined gauge invariant regional symplectic form. Indeed, we can abstractly define the projected, or reduced symplectic form: $\Omega^{[f]}_{\text{red}}$, as $\pi^*\Omega^{[f]\pm}_{\text{red}} = \mathfrak{i}^*_{[f]\pm}\Omega^\pm$, where $\mathfrak{i}_{[f]\pm}$ is the embedding of the Gauss constraint surfaces for $[f]$ (as in footnote 35. See (Riello, 2021b, Section 4) for the derivations of these facts and for the relation between the reduced symplectic form and the symplectic form in a representational convention.

The $\Theta_\pm$ in (4.11) are the (Heaviside) characteristic functions of regions $R_\pm$.[40] We are given the physical content of the regional configurations, $[A_\pm]$ as input, and that is enough for our purposes of assessing DES; and while the $h_\pm$ that represent these physical states might not smoothly join, they may still jointly correspond to a physically possible global state. As we saw in Section 3.2, whether two regional gauge-fixed states $h_\pm$ are composable turns on whether there are gauge transformations on each region such that the transformed states—no longer of the form $h_\pm$—smoothly join, or glue. But if they are composable there will be many such transformations. Equation (4.11) selects only those transformations that lead to a global state in the chosen representational convention, which thus allows us to infer physical differences from differences of the represented states.

4.4.2 A summary of gluing the Abelian gauge potential in the Coulomb representational convention

Here we summarize, in a more perdestrian language, the conclusions of Appendix A.

The existence of gauge transformations smoothening out the transition between h_+ and h_- is a necessary and sufficient condition for their compatibility. The condition is that there exist gauge transformations satisfying:[41]

$$(h_+ - h_-)_{|S} = i\text{grad}(\ln g_+ - \ln g_-)_{|S}; \tag{4.12}$$

(in spacetime index-free notation[42]) which is the appropriate rewriting of the gluing condition (3.9).

There could be many such possible "adjustments" of h; there are either none or an infinite amount of $g_\pm$ that will satisfy (4.12) and we need to partition all of these possibilities into physical equivalence classes. For the remaining question—whether the composition of regional states is physically unique—we employ a gauge-fixing of the global state, i.e. we demand that the global state is also given in some representational convention, F, or σ. Indeed, as discussed in Section 3.1.2 (see in particular the quote from Wallace (2019c)), that is the only way we can assess physical differences between alternative global states.

Thus we are given $h_\pm$ that are in the regional representational convention (i.e. satisfy (A.11)) and want to glue them into a state that satisfies the global representational convention, h. That is:

$$h := (h_+ + i\text{grad}(\ln g_+))\Theta_+ + (h_- + i\text{grad}(\ln g_-))\Theta_-, \tag{4.13}$$

must satisfy, for some $g_\pm$, the unbounded gauge-fixing condition (A.3) of Appendix A.1 (so that we uniquely determine the universal physical state).

It is important to emphasize that the use of the gauge-fixed fields has eliminated all local redundancy. For instance, imposing that the regional states in their representational convention should match—$(h_+ - h_-)_{|S} = 0$—would restrict our analysis to only a subset of compatible physical configurations. In general, the universal h's do not themselves restrict to $h_\pm$'s, as we saw in Section 3.3.2 (see

[40] The assumption of states as supported on the regions $R_\pm$ and adjacency of the regions fixes the embedding of the subsystems through the distributions $\Theta_\pm$. Then conditions for gluing become simply smoothness conditions.

[41] Note that we can still change the representational convention itself. In footnote 33, these types of transformations are labeled subsystem-extrinsic. This is how the smoothening gauge transformations need to be interpreted.

[42] Using indices, the equation is: $(h^\mu_+ - h^\mu_-)_{|S} = i\partial^\mu(\ln g_+ - \ln g_-)_{|S}$.

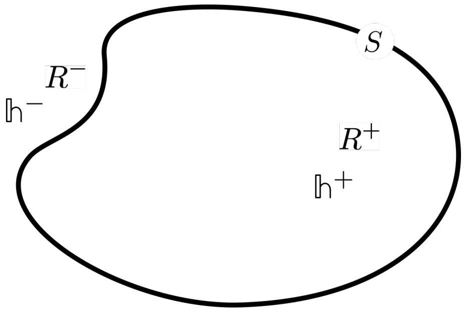

Figure 1: The two subregions of M, i.e. $R_\pm$, with the respective horizontal perturbations $h^\pm$ on each side, along with the separating surface S.

Equation (3.15)).[43] Thus, as stated in Section 3.3, if we are to use representational conventions, we cannot, when satisfying (4.11), assume $g_- = \mathrm{Id}$ irrespective of the conventions used.[44]

Finally, since we have also partitioned the global state space, and identified 1-1 representatives of the physical equivalence classes, whatever information beyond the specification of the subsystem physical states that is required to determine h will reveal a gap between $[\mathcal{A}]$ and the union of $[\mathcal{A}_\pm]$. Thus any remaining underdetermination will have physical significance, fulfilling the notion of DES. In other words, the DES transformations—satisfying criteria Global Variance and Subsystem Invariance for DES, of Section 1.1—*will take the form of symmetries* on the subsystems, arising from the underdetermination of the global state by the gluing conditions that respect the global representational convention.[45]

5 Conclusions

In gauge theories, empirical significance can be obscured by redundancy of representation. Ultimately, that is why the direct empirical significance (DES), or the observability of symmetries continues to be a debated question. Nonetheless, the standard treatment of DES is almost silent about fixing representation,[46] with the exception of (Gomes, 2021a) and Wallace (2019c), where the assumption is partially flagged, as noted in Section 3.1.2, but not fully examined. Here I have paid due attention to this issue.

In Section 5.1, I summarize the findings of this paper. In Section 5.2, I discuss asymptotic idealisations in relation to the externalist view of subsystems.

[43] Namely, $\iota_\pm^* h \neq h_\pm$. The generality of this inequality is a consequence of the non-locality of the gauge-fixing, implicit in the inverse Laplacian. That is, the restrictions of universal h's over M—satisfying (A.3)—to the regions $R_\pm$ are not necessarily themselves of the form of $h_\pm$, i.e. do not necesarily satisfy Neumann boundary conditions.

[44] Moreover, it would be impossible to meet in a manifold that requires many charts (see Section 4.2).

[45] In the nomenclature of footnote 33, these are subsystem-extrinsic symmetries, and as such conform to our intuition about symmetries with DES being applied from a perspective outside the subsystem and being undetectable from within.

[46] As noted in Section 3.3 yielding equation (3.12), if φ_- is not kept fixed, one can always extend g_+. Such extensions have caused some confusion in the literature (see (Friederich, 2014)).

5.1 Summary

The main question I have investigated here is precisely how to establish a choice of representational convention in the context of our search for DES. This approach to DES reveals the inadequacy of the standard construction of Section 3.3 and provides a straightforward alternative. And, while I agree with most aspects of Greaves & Wallace (2014) and Wallace (2019b,c)'s analysis of symmetries, I have recast the topic to focus on gauge-*invariant* information about regions, by explicitly using representational conventions. This approach yielded a more precise formulation of the question of DES.

The upshot is that gauge-fixing disentangles the issue of redundant representation from DES, for *all* types of systems and their subsystems. Thus we do not have to make extra assumptions (e.g. about the lack of bulk stabilizers): our construction is able to discern the existence of DES for any state.

The procedure identifies a type *holism* lies at the core of DES: as articulated at length in Gomes (2021a), there is a difference between $[\Phi]$ and $[\Phi_+] \cup [\Phi_-]$, even when $M = R_+ \cup R_-$. If this is so, we should see the same sort of difference in other formalisms, that do not necessarily use gauge-fixing. We checked that this indeed occurs in the holonomy formalism for electromagnetism in Appendix C.[47]

As another consistency-check, I then applied the gauge-fixed approach to particle mechanics (Section D). Thus, using precisely the same type of constructions as for gauge theories, I recovered the standard DES associated to Galileo's ship. In that context, DES arises from the different ways to embed intrinsically identical subsystems into the universe.[48]

The externalist case requires configuration space to be (non-covariantly) pared down. That is, we limit not the set of physical possibilities, $[\varphi]$, but the set of representatives, φ, at the boundary. These boundary conditions would not be gauge-covariant under a fundamental view of symmetries. But by abandoning the requirement that gauge symmetries act equably on all configurations, the restriction does not break any symmetry. This finding is entirely consistent with the idea that the environment state, whatever it is, provides a representational convention. There is no gluing, and thus no requirement of meshing the global representational convention with the subsystem one, and no requirement that the subsystem symmetries must be compatible from the inside and the outside perspective on the boundary.

For many decades, the pared down asymptotic treatment of symmetries was assumed for the treatment of isolated subsystems. Thus, until recently, attempts to treat the finite, bounded subsystem of gauge theory were scarce (and mostly focused on computations of the entanglement entropy of black holes; the trailblazers were Carlip (1997); Sorkin (1983); Srednicki (1993)). In our language, internal boundaries were not distinguished from external ones. Wallace (2019b, p. 11) endorses the ensuing view of subsystems, because he takes subsystems as sufficiently isolated to warrant an asymptotic-like treatment. One drawback is that such a treatment would find no extension to spatially closed manifolds (as discussed in footnote 11). Another, is that the notion of isolation requires very strong boundary conditions, such as $F_{\mu\nu|S} = 0$. But recent developments in gauge theory have shown that we can have (non-asymptotic) bounded subsystems, in which e.g. $F_{\mu\nu}$ is non-

[47] In this we go beyond Greaves & Wallace (2014); Wallace (2019b,c), which explicitly open an exception to the use of holonomy variables (see e.g. (Greaves & Wallace, 2014, p. 67)).

[48] No such possibilities exist in the field theoretic case, for the simply-connected manifold partitioned into regions; the embedding is determined by the partition.

vanishing everywhere, and which still enjoy the same set of fundamental symmetries for their intrinsic dynamics (even if they require time-varying boundary conditions). So clearly, there are good, weaker notions of subsystem recursivity that do not mimic the asymptotic ideal of perfect isolation. These recent developments were discussed in Section 4.2, and they warrant a treatment of internal subsystems in gauge theories that respects the downward consistency of symmetries.

Classifying DES for gauge theory: Using our treatment based on representational conventions to assess DES, we can classify its occurrence for different types of systems (as computed in Gomes (2021a); Gomes & Riello (2021) and summarized in the Appendices). As discussed in Appendix A.1.1, stabilizers represent certain degeneracies within any given representational convention, and thus are crucial in articulating the results below. When they exist, stabilizers form a rigid (or subsystem-global) group of gauge transformations.

First, assuming trivial topology, and an internal boundary:

(i) in the Abelian case there is no physical variety in the absence of charged matter; or when charged matter is present at the interface between the regions. That is because, to have underdetermination, the regional stabilizers of the gauge potential *cannot* stabilize the state of all of the fields; but to preserve compatibility of the states at the boundary, it *must* stabilize the boundary states (and Klein-Gordon matter fields have only the trivial stabilizer). Thus the sector of the theory in which one has observable symmetries corresponds to regions that have charged matter in the bulk, but not in the interface of the regions. This sector contains the situation depicted by 't Hooft beam splitter experiment (see ('t Hooft, 1980, p.110) and (Brading & Brown, 2004, p. 651)); and likewise, the group of symmetries with DES is a rigid phase shift, given by $U(1)$.

(ii) In the non-Abelian case, we must distinguish a few possibilities. As in the Abelian case, if the regions have the same set of stabilizers, and if a subgroup of stabilizers of the gauge potential act non-trivially on the regional states as a whole (e.g. by acting non-trivially on the matter fields), then there will be a physical variety, corresponding to the subgroup of the (rigid) group of stabilizers. But such a condition is generically forbidden: generic states in non-Abelian Yang-Mills theory have only a trivial stabilizer. Moreover, if the state at the interface of the region has stabilizers—meaning that there are gauge transformations that act as the identity only on the boundary states—then we also get one physical global state per choice of boundary stabilizer. These are what I take to be the *physically relevant* notion of edge modes (see also Carrozza & Hoehn (2021), for a similar argument).

A comparison of these two cases with the familiar Aharonov-Bohm phases in the Abelian theory is also worthwhile. There, M is taken to have a non-trivial topology, and the cohomology class of the gauge potential represents holistic physical information, that can nonetheless be represented at the boundary by suitable transition functions. Here too: there is a discrepancy between the tensor product of the regional physical state spaces and the physical state space of the union of the regions. The discrepancy represents holistic physical information about the total system that is not contained in the individual subsystems. Nonetheless, we can represent this physical information through suitable mathematical operations either at the boundary between the two regions or on each region.

Moving on to the external boundary, vacuum case:

(iii) In this case, for either the Abelian or the non-Abelian case, gauge-fixings formulated strictly in terms of the gauge fields satisfy *Uniqueness* and *Universality* (as described at the end of Section 4.1) *only if* each transformation that stabilizes the boundary is continuable to a transformation that stabilizes the universe. Otherwise, gauge-fixings must also be indexed by the choice of boundary stabilizer.

But these indices do not belong to the configuration space $\mathcal{A}$: they should be seen as additional degrees of freedom, which, ultimately, represent the externalist's version of DES. They find a counterpart in the internalist scenario, in which the internal boundary state has stabilizers not shared by the bulk of the regions, as described in (ii) above.[49] That is, in the appropriate regime, each choice of stabilizer intrinsic to the boundary corresponds to a different physical state, matching the findings of (Greaves & Wallace, 2014) (cf. equation (3.12)) and, in the asymptotic case of external boundaries, conforming to the intuition of Belot's 'generalized shifts' (Belot, 2018).

5.2 Externalist boundaries and asymptotics

As a last remark, I admit that the externalist's notion of boundaries is ubiquitous in asymptotic treatments of symmetries. In fact, we model the solar system in this way: the standard spatially asymptotically flat spacetime imposes a particular form of the metric as one approaches the asymptotic boundary; it is not a diffeomorphism-invariant, geometric boundary condition. That is, the treatment of asymptotic symmetries cannot fall under the fundamental approach to symmetries, as discussed in Section 2. In this way, coordinates at the boundary acquire some physical meaning. And in this way, all the coordinates compatible with some given condition on the field acquire physical meaning: this variety is represented by the non-trivial boundary stabilizers, and they are what Wallace (2019b) describes as observable symmetries; or what Greaves & Wallace (2014) describe as symmetries with DES.

Note, moreover, that the non-trivial case of the asymptotic treatment arises precisely when there is a gap between the boundary and the bulk stabilizers. For instance, only the completely flat state (i.e. Minkowski space) extends to the bulk all the boundary stabilizers of a generic asymptotically flat spacetime. But as we find in the externalist case, we would only obtain DES if the bulk did not share the boundary stabilizers; e.g. if there is matter in the bulk or if the metric only asymptotes to the Minkowski metric. In Wallace (2019c), this intuition is preserved by a restriction of focus to cases without bulk stabilizers. But this restriction, as it stands there, is *ad hoc*, while here our formalism includes all cases.

The interesting examples are the ones for which different stabilizers intrinsic to the boundary correspond to different physical states. We can find such cases in our formalism, and they *agree* with (Greaves & Wallace, 2014) and with the more general characterization of 'physical symmetries' as corresponding to those of the quotient group $\mathcal{G}_S/\mathcal{G}_{\text{Id}}$ (cf. (Giulini, 1995)).

Therefore I grant that even if, ideally, boundary conditions should be gauge-covariant so that the dynamical treatment of symmetries coincides with the funda-

[49]The only way I know to make this construction kosher in the external boundary case, is that explored in (S. Ramirez & Teh, 2019), which endows these boundary 'gauge' degrees of freedom with their own dynamics, following the work of (Donnelly & Freidel, 2016). But one should not confuse these degrees of freedom with the generalized (non-Abelian) electric flux and its conjugate. These latter quantities should not be interpreted as new degrees of freedom in the same way as edge-modes, and they do not contribute to the issue of DES (cf. (Gomes & Riello, 2021, Section 6) and Section 4.2).

mental treatment of symmetries, the externalist approach—which does not abide by that ideal—may work well for some purposes.

But, to obtain a solid conceptual footing, the externalist's notion os subsystem requires further conceptual analysis (see (Belot, 2018) which partially lays the groundwork for such an analysis). Thus I believe we should not leave the underlying assumptions about asymptotic symmetries unexamined simply because they are useful; lest we acquiesce to what amounts to a 'shut up and calculate' mentality in the treatment of gauge and asymptotics.

A more conceptually grounded approach is also a more conservative one: it goes from small systems to big ones. We should first properly understand gauge systems in finite regions and then move to the asymptotic regime by progressive enlargement, keeping careful track of how objects and relations maintain or lose their properties in the (singular) limit.[50]

This should be possible: the internalist case imposes only gauge-covariant boundary conditions, and thereby unifies the fundamental and the dynamical treatment of symmetries. Not only that: it recovers qualitatively similar observable symmetries as we found in the externalist case.

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APPENDIX

A Coulomb gauge

In Section A.4 I sketch a solution to this mathematical problem. In Section A.5 I glimpse how this solution extends to other cases that were left out of this paper in the interest of simplicity: namely, I briefly discuss the necessary alterations and caveats incurred by the addition of matter, non-trivial topology, and non-Abelian gauge groups.

A.1 Coulomb gauge for the closed universe

Now we will illustrate the previous definitions explicitly, by employing an explicit gauge-fixing functional F. I will describe how this works when the manifold is closed but without boundary. Formally, this is simpler than the bounded case, which we will leave to Appendix A.3. Nonetheless, the simpler case already suffices to illustrate many of the intricacies of gauge-fixing. This Section is more technically involved. Its main purpose is to illustrate: (i) how a representational convention

[50]This is how Riello relates the subsystem divisions used here to the asymptotic regime, in the case of Yang-Mills theory (Riello, 2020). By doing so, he finds a singular limit for the asymptotic charges, recovering precisely the results of (Ashtekar A., 1981), who also treats boundary conditions in a diffeomorphism-invariant manner in general relativity, through Penrose compactification.

can fail to fix all representational redundancy, due to a lack of 'wrinkles' of the represented state; and (ii) how a representational convention operates like a projection on configuration space, and is in that sense gauge-invariant.

I will start in Section A.1.1 by posing the intricacies of fixing the gauge in the presence of stabilizers of the gauge potential. And in Section A.2, I will lay out the details of the Coulomb gauge (and the corresponding projection operator, etc) in the bounded case. In Section A.3, I do the same for the bounded case.

A.1.1 How to fix representational conventions: the problem of stabilizers

Physically, fixing representational conventions uses features of the state to nail down representational redundancy of that state.

However, for certain states there may be group elements that have no grip on representation. In other words, there may be φ, and certain $\tilde{g}$ for which $\varphi^{\tilde{g}} = \varphi$. Viz. there are certain group elements that act trivially on certain states. In these cases, the orbits formed by the action of the group, $\mathcal{O}_\varphi$, will also be, in certain respects, singular.[51] Thus we define the *stabilizer*:

$$\text{Stab}(\varphi) := \{\tilde{g} \in \mathcal{G} \mid \varphi^{\tilde{g}} = \varphi\}. \tag{A.1}$$

It is easy to see that $\text{Stab}(\varphi^g) = g^{-1}\text{Stab}(\varphi)g$,[52] and thus the conjugacy class of the stabilizer group is a property of the entire orbit (i.e. it is not dependent on the representative of the physical state).

In field theory, the group $\mathcal{G}$ of local (or malleable) gauge transformations is infinite-dimensional, since it is a space of maps from a smooth manifold M to some value space. In practice, the features of the states used to fix the representation belong to the gauge potential, A, and not to the matter fields or the electric field.

That is because, either configurations of the matter fields or of the electric fields, transforming via (4.9), generically will possess stabilizers.[53] For instance, configurations in which the matter fields vanish *anywhere*, will have stabilizers; if $\psi(x) = 0$, then $\psi^g(x) = 0$: the group cannot change representation on those points because it has no grip there. Thus, for example, if the matter fields vanish on an open set, a gauge transformation that is only non-zero on that open set will stablize the state.[54]

But to fix representational conventions we would like to choose those fields that generically have no stabilizer, i.e. that on an open and dense set of Φ have no stabilizer. This criterion selects the gauge potential as the field used to fix representational conventions. With that choice, a meagre set of states of the fields *will* possess stabilizers, but the group of stabilizers will be rigid, or finite-dimensional,

[51] The quotient $[\Phi]$ in these cases form 'stratified manifolds', which are essentially a concatenation of bounded manifolds of different dimensions, with the manifolds of smaller dimension being obtained by the states with more symmetry and being the boundaries of the manifolds of higher dimension. See Fischer (1970); Kondracki & Rogulski (1983); Mitter & Viallet (1981).

[52] A quick proof: for $\varphi' := \varphi^g$, we take $g' = g^{-1}\tilde{g}g$, where $\tilde{g} \in \text{Stab}(\varphi)$. Then $\varphi'^{g'} = \varphi^{\tilde{g}}g = \varphi^g = \varphi'$.

[53] This is also true for the electric field in the non-Abelian case, transforming under conjugation by the group element.

[54] Of course, we could be interested in sectors of the theory in which the matter fields do not vanish anywhere; configurations in which they form a plenum. In these sectors, it is legitimate to use matter fields to fix representational conventions. And indeed, in these cases, as we will comment on in Section A.5, there is no DES. That is, the gauge theory is separable. This is in line with (Wallace, 2014) (see Gomes (2021a)). But I disagree with Wallace (2014) that such a plenum is generic, in light of quantum de-localization. Specifying configuration or phase space is prior to quantization; and de-localization is a higher level consequence of quantization that has no bearing there.

so that the values of a stabilizer on an open region determines that stabilizer everywhere.

The importance of these stabilizers for DES is that they are left unfixed by representational conventions, even if that convention fixes all local redundancy, i.e. completely fixes the representation of A. For instance, within fixed representational conventions for A, a disparity between stabilizers of two fixed subsystem states may give rise to DES as follows. Suppose two subsystem stabilizers $\tilde{g}_\pm$ do *not* conjoin to form a global stabilizer of the joint region, $\tilde{g}$. Since we assume the representational convention has fixed all local redundancy, it will not allow a representational change that smoothens out the difference between the stabilizers. However, even if stabilizers do not change the representation of A, they can change the representation of the matter fields. In this case, incompatible stabilizers in each region will give rise to a different global state, which, within a fixed representational conventions, implies a physical difference, that is, *Global Variance* will be satisfied.

In Gomes (2021a, p. 87), it is argued that stabilizers are the only natural notion of global symmetries in a gauge theory, since they can, independently of any representational convention, pick out rigid subgroups from the local groups of gauge transformations. And indeed, the difference between stabilizers is an entirely gauge-invariant quantity. Thus finding observability criteria in terms of functionals of these stabilizers is consistent with gauge-invariance and a welcome development.

Here we will select the gauge potential as the field that orients the representational conventions. That is, representational conventions will be chosen by specifying particular forms for the gauge potential.

I will only consider gauge-fixings that completely fix the representation of A, but note that rising to this challenge does not require the solution $g(A)$ of (4.1) to be unique. For, even if $F(A) = 0$ underdetermines $g(A)$, as long as this underdetermination is only up to a stabilizer of A, as in (A.1), the gauge-fixed representative of $[A]$ will not be underdetermined. In other words, suppose that $g(A)$ and $g'(A)$ both satisfy (4.1), as long as they differ by stabilizers of their argument, say $g'(A) = \tilde{g}(A)g(A)$, we will still obtain:

$$h(A) = A^{g(A)} = (A^{\tilde{g}(A)})^{g(A)} = A^{g'(A)} \tag{A.2}$$

and so the difference does not show up at the level of the projection operator.

The presence of non-trivial stabilizers implies that features of $[A]$ do not possess enough variety—"wrinkliness"—to completely fix the *gauge transformations* that carry an $A \in [A]$ to an $h(A)$. In other words, a representational convention can fail to fix all representational redundancy, due to a lack of 'wrinkles' of the represented state. Nonetheless, the represented state cannot register any difference due to this remaining degeneracy, precisely because the state is not 'wrinkly' enough for that remaining redundancy to get a grip on. In other words, if the only "slack" left in the determination of $g_\sigma(A) \in \mathcal{G}$ is due to stabilizers, it is idle: there is no effect on the resulting gauge-fixed $h(A)$. That is because that slack has a trivial action on the configuration.

For non-Abelian groups, $\mathcal{A}$ is generically stabilizer-free and so stabilizer groups are generically trivial, i.e. just the identity. Nonetheless, particular physical states, such as the physical state of "no A field", i.e. $A \sim 0$, allow stabilizers, and thus do not allow gauge transformations to be uniquely fixed. In the Abelian case, all configurations share the same stabilizer, viz. the group of constant gauge transformations (cf. the discussion in Section A).

Indeed, for the purposes of this paper, this is the most important distinction between Abelian and non-Abelian theories. Namely: stabilizers are the same for all Abelian field configurations—they are the constant transformations—and, on the other hand, are trivial for generic non-Abelian field configurations.

A.2 Details of Coulomb gauge

Let us start by introducing a standard gauge-fixing for the entire manifold: Coulomb gauge.[55] Following the nomenclature of Section 4.1 for the gauge-fixing section σ, we define:

$$F(A) := \mathrm{div}(A) = 0. \tag{A.3}$$

As we will now see, it is easy to see that this gauge-fixing satisfies *Universality* and *Uniqueness*. First, *Universality*: given a general A, not necessarily belonging to the given gauge-fixing section, i.e. A which can be such that $h(A) \neq A$, we must ensure that there exists a gauge transformation that takes A to that section. Second, *Uniqueness*: we must ensure that whatever slack remains in the determination of this transformation cannot be "detected" by any A.

As to the first demand, equation (4.9) yields:

$$\mathrm{div}(A^g) = \mathrm{div}(A) + i\nabla^2(\ln g) = 0 \tag{A.4}$$

$$\therefore\; g(A) = \exp(i\nabla^{-2}(\mathrm{div}(A))) \tag{A.5}$$

where ∇^{-2} are the Green's functions associated to ∇^2.[56] For all A, we can find a solution $g(A)$ to $\sigma(A^{g(A)}) = 0$ and thus a projection, $h(A)$. Therefore the gauge-fixing is *Universal*.

Consistently, the 1-form $h_i(A) := A_i^{g(A)}$, defined through (A.5) does satisfy (A.4). That is, it is easy to verify, since $\mathrm{div}(\mathrm{grad}) = \nabla^2$, that

$$h(A) := A + i\,\mathrm{grad}(i\nabla^{-2}(\mathrm{div}(A))), \tag{A.6}$$

satisfies

$$\mathrm{div}(h(A)) = 0. \tag{A.7}$$

Moreover, the projection h is invariant under gauge transformations: $\forall g \in \mathcal{G}, h(A^g) = h(A)$:

$$\begin{aligned} h(A^g) &= A + i\,\mathrm{grad}(\ln g) + i\,\mathrm{grad}((i\nabla^{-2}(\mathrm{div}(A + i\mathrm{grad}(\ln g))) & \text{(A.8)} \\ &= A + i\,\mathrm{grad}(i\nabla^{-2}(\mathrm{div}(A))) & \text{(A.9)} \\ &= h(A) & \text{(A.10)} \end{aligned}$$

[55] Our specific findings do not depend on the particular gauge-fixing, as long as it adheres to the definitions in Section 4.1; these definitions imply that some non-locality, or integration over the spatial region, will be involved in finding the particular gauge-fixed representation. And it will also necessarily satisfy '*Uniqueness*', as explained at the end of Section 4.1. All the constructions presented here have an exact analogue in the non-Abelian case. Essentially, the analogue replaces ∂_i by the gauge-covariant $\mathrm{D}_i = \partial_i + [A, \cdot]$, and constant gauge transformations are replaced by the more general concept of stabilizer (A.1). See (Gomes, 2021a; Gomes et al., 2019; Gomes & Riello, 2021).

[56] Roughly, on the space of functions, for $f \in C^\infty(M)$ we define the Green's function as an operator inverse to the Laplacian, i.e. $\int_M \nabla^{-2}_{xy}(\nabla^2 f)(x) = f(y)$. For a closed manifold, the operator exists and is unique on the space of non-constant functions, see e.g. (Gilbarg & Trudinger, 2001). When the metric is sufficiently homogeneous, Green's functions can be obtained in explicit form.

Moreover, if $h(A) = h(A')$, then, putting all the gradients to one side of the equation, we obtain that $A - A' = i\text{grad}f$,[57] and so $h(A) = h(A')$ iff $A \sim A'$. In this way, $h(A)$ captures the full gauge-invariant content of A.

Lastly, notice something we glossed over when we found (A.5): F doesn't determine $g(A)$ uniquely. However, the underdetermination is solely due to stabilizers. Here in the Abelian case, $g(A)$ and $g'(A)$ are solutions to (A.5), if and only if $g'(A) = g(A) + c$, where c is a constant.[58] Nonetheless, since here stabilizers have a trivial action on the gauge potential (from (4.9), since $\partial \ln c = 0$), the gauge-fixing still satisfies *Uniqueness*; i.e. $A^{g(A)} = A^{g'(A)}$.

It is important to note that fixing the representative requires finding something like $g(A)$, and this is always a non-local process. That is, since A is related to g through a derivative, going the other way—tying g to A—always requires integration.[59] This is manifested in g_σ, and passed on to h_σ, by the presence of the Green's function.

A.3 Coulomb gauge-fixing for the bounded case

Now we want to generalize (A.3) to the bounded case, i.e. for regions $R_\pm$ bounded by S, as in the introduction to Section 4.

Again, as in the discussion towards the end of Section 4.1, we want to fix the gauge using functions that exploit the 'wrinkliness' of the states.

Since in the bulk of the manifold we have a scalar second-order differential equation for g, viz. equation (A.4), we want to impose scalar conditions for g on the boundary: either Dirichlet or Neumann boundary conditions, which are, respectively, of zeroth and first order in derivatives. However, as per the definition of the gauge-fixing section σ, these conditions should descend to g from $F(A)$; we do not simply impose Neumann or Dirichlet boundary conditions on g. Apart from being conceptually uncouth and falling outside our previous definition of gauge-fixing sections, such imposed conditions on the boundary values of g would not be gauge-covariant, a rather unwelcome side-effect. That is, the gauge-fixing projection would depend on the initial, entirely arbitrary choice of A; we would thus have $g_{\sigma, A_{|R}}$, where S is the boundary.[60] In this case, if I were to choose one A' to start with, and you chose another, A'', and we use the same boundary conditions *on* g, we would find distinct gauge-fixed fields for the same A.

Dirichlet or Neumann (scalar) boundary conditions imply that we must fix one, and only one, scalar degree of freedom of A at the boundary (A has $\dim(M)$-many such degrees of freedom, of course). Since A can only constrain the gradients of g through their relation in equation (4.9), this rules out Dirichlet conditions. And since only the spacetime direction normal to the boundary is singled out by the introduction of a boundary, we can only naturally introduce Neumman boundary conditions by fixing the normal component A_n of the gauge potential (see Section 3.3 for more on this). There are mathematically and physically more upright ways

[57] For $f = \nabla^{-2}(\text{div}(A)) - \nabla^{-2}(\text{div}(A'))$.

[58] For both to be solutions of (A.4), we must have $\nabla^2(g(A) - g'(A)) = 0$. But it is easy to show that $\nabla^2 f = 0$ iff f is constant: $0 = \int f\nabla^2 f = \int |\text{grad}f|^2 \Leftrightarrow \text{grad}f = 0 \Leftrightarrow f =$const (where we used integration by parts in the second equality).

[59] A simple example: radial, or axial gauge, $A_r = 0$. This is not a complete gauge-fixing, but we still find $g(x, r) = \int_0^r dr' A(r', x)$, where r are radial coordinates, and x are the remaining coordinates.

[60] Unless there is a fixed choice for the boundary A, as in the externalist scenario. Then, of course, we may omit this dependence.

to introduce these boundary conditions (cf. Gomes & Butterfield (2021); Gomes & Riello (2021)), but this simple argument here suffices.

Lastly, if we do not want to introduce further arbitrary parameters in our gauge-fixing, the simplest choice for $F(A_\pm)$ as given by:

$$F(A_\pm) \equiv \begin{cases} \mathrm{div}(A^\pm) & = 0 \\ A_n^\pm & = 0. \end{cases} \tag{A.11}$$

Given arbitrary regional configurations $A_\pm$, we solve the following second-order set of differential equations with field-dependent, covariant boundary conditions:

$$\nabla^2(\ln g_\pm) \;=\; i\mathrm{div}(A_\pm) \tag{A.12}$$
$$\partial_n(\ln g_\pm) \;=\; \pm i A_n \tag{A.13}$$

where the $\pm$ signs on the right hand side of (A.13) come from the opposite directions of the normal at S. The solution is as in (A.5), namely,

$$g_\pm(A_\pm) = \exp(\pm i \nabla^{-2}_{\mathrm{Neu}(\pm A_n)}(\mathrm{div}(A_\pm))) \tag{A.14}$$

with the difference that now the Green's functions (inverse Laplacian), $\nabla^{-2}_{\mathrm{Neu}(A_n)}$ are defined for the field-dependent, Neumann boundary conditions (A.13); therefore $\partial_n \ln g_\pm(A_\pm) = \pm i A_n$ holds automatically.

In precise analogy to (A.6), we obtain

$$h_\pm(A_\pm) := A_\pm + i\,\mathrm{grad}(i\nabla^{-2}_{\mathrm{Neu}(\pm A_n)}(\mathrm{div}(A_\pm))), \tag{A.15}$$

The projected field $h_\pm$ also satisfies (A.11). That is, $h(A_\pm)_n = 0$, even if $A_n^\pm \neq 0$. In other words, even though we have not restricted the set of $A_\pm$'s, independently of its behavior at the boundary, any $A_\pm$ can be brought to satisfy equations (A.11) through a gauge transformation—also generically non-trivial at the boundary. The reason is simple: the system of equations (A.12) and (A.13) always has solutions (existence).[61] This shows that we are respecting the '*internalist*'s mantra for the internal boundary: no truncation of gauge transformations or of configuration variables is needed at this internal boundary. The projection works just fine without such truncations.

Moreover, as expected,: $h(A_\pm) = h(A'_\pm)$ iff $A' = A^{g_\pm}$ for some $g_\pm \in \mathcal{G}_\pm$ (the proof is a little more complicated than the unbounded case, due to the field-dependent boundary conditions, but it proceeds in much the same way as (A.8)-(A.10)).

As in the previous, unbounded case, the only ambiguity in solutions is due to stabilizers. Again, in the Abelian case, this means $g_\pm(A_\pm)$ and $g'_\pm(A_\pm)$ are solutions to (A.12) and (A.13) iff $g'_\pm(A_\pm) = g_\pm(A_\pm) + c$; as is easy to check.[62]

And again, for the same reasons, this ambiguity will have no effect on the representative. In other words, the associated projected potentials, $h(A_\pm) := A_\pm^{g(A_\pm)}$ and

[61] There is also the added benefit, in the dynamical 3+1 setting of Yang-Mills gauge theories, that such gauge-fixings correspond to *Helmholtz decompositions* separating the Coulombic from the radiative degrees of freedom of the region. Radiative degrees of freedom are those that are intrinsic to a region; they do not depend on further incoming information at the boundary. See (Gomes & Riello, 2021, Sec. 3) for more on this point.

[62] As in footnote 58, to be solutions to (A.4), we must have $\nabla^2(g - g') = 0$ and $\partial_n(g - g') = 0$. Again, calling $f = (g - g')$, we have $0 = \int f\nabla^2 f = \int |\mathrm{grad} f|^2 + 2\oint f\partial_n f \Leftrightarrow \mathrm{grad} f = 0$, and since $\partial_n f = 0$, $f =$const. In the non-Abelian case, we can have stabilizers of the boundary that are not shared by the bulk: each such stabilizer will likewise contribute to a degeneracy in the gluing.

$h'(A_\pm) := A_\pm^{g'(A_\pm)}$, are identical. Thus σ satisfies both *Universality* and *Uniqueness* and provides a bona-fide *gauge-fixing.*

So, not only is the configuration space $\mathcal{A}$ defined by the spaces $\mathcal{A}_\pm$, but each such space has its own principal fiber bundle structure.[63]

A.4 A sketch of the solution

After this stage setting, we sketch the solution to our original problem: in the type of systems we have focussed on—vacuum, simply connected Universe—do regional physical states uniquely determine the entire physical state?

Essentially, to find $g_\pm$ as above, we obtain, from (A.7), i.e. from $\text{div}(h) = 0$ (and $\text{div}(h_\pm) = 0$) that $\nabla^2(g_\pm) = 0$; and the action of the divergence operator on the Heaviside functions on (4.13) (and the Neumann conditions $h_n^\pm = 0$) enforces a continuity equation for $g_\pm$ in terms of $h_\pm$ (the gluing condition, (4.12)). This gives us enough information to fix the appropriate boundary conditions for the solutions $g_\pm$ (see (Gomes & Riello, 2021, Sec.4, p.30-33)).

When all the chips have fallen, one can prove existence and almost uniqueness for the $g_\pm$ of (4.13). Unsurprisingly, the only degeneracy left is again made up of regional stabilizers; they form the only well-defined rigid (or global) subgroup of the local gauge symmetries.

To finish an assessement of DES, we now need to give more information about sectors of the theory. In the case studied here—the Abelian case, in the absence of charged matter fields—any degeneracy in the stabilizers is idle: it is not felt by the gauge fields. This result is irrespective of the boundary conditions on the fields E and A, as long as these conditions are posed gauge-invariantly (i.e. respect downward consistency, as described in Section 2.2.2). Moreover, since in the present case E is gauge-invariant, there are no further considerations that impinge on the gluing of subsystems ($E_\pm$ is just required to match at S).

In other words, we find *unique* $g_\pm$ *strictly as functionals of* the values of $h_\pm$ pulled-back to the boundary, $i^*h_\pm =: h_\pm^S$, where $i : S \to M$ (no derivatives of $h_\pm$ at the boundary are necessary, cf. footnote 25) and of regional stabilizers $c_\pm$:[64]

$$g_\pm = g_\pm(h_\pm^S, c_\pm) \quad \text{with} \quad g_\pm(h_\pm^S, c_\pm) = g_\pm(h_\pm^S, 0) + c_\pm. \tag{A.16}$$

Thus the difference between two solutions is entirely due to stabilizers.[65]

As before, since we are in vacuum, stabilizers—for electromagnetism, constant gauge transformations—do not affect the gauge potential. That is, some internal

[63] What we have just shown for each space is essentially equivalent to the existence of a local *slice*, which is the mathematical jargon for a gauge-fixing section (local on field-space) on infinite-dimensional configuration spaces. The existence of a local slice is *the* characterizing feature for (the closest analogues of a) principal fiber bundle structure in this context. See, e.g. (Kondracki & Rogulski, 1983; Mitter & Viallet, 1981; Wilkins, 1989).

[64] Also note that we are using S as a superscript to denote the intrinsic—pulled-back—quantity, that is different from the S subscript that denotes mere restriction of the base point of vector quantities (cf. footnote 25).

[65] For illustration purposes, I display the solution here:

$$\ln g_\pm = \zeta_{(\pm)}^{\pm\Pi} \quad \text{with} \quad \Pi = \left(\mathcal{R}_+^{-1} + \mathcal{R}_-^{-1}\right)^{-1} \left((\nabla_S^2)^{-1}\text{div}_S(h_+ - h_-)_S\right),$$

where the subscript S denotes operators and quantities intrinsic (i.e. pulled-back) to the interface surface S; $\zeta_{(\pm)}^u$ is a harmonic function on (respectively) $R_\pm$ with Neumann boundary condition $\partial_n \zeta_{(\pm)}^u = u$, and $\mathcal{R}$ is the Dirichlet-to-Neumann operator. For the meaning of these operators, and also the analogous solution for the general non-Abelian Yang-Mills gauge theories, see (Gomes & Riello, 2021, Sec. 4), and (Gomes, 2021a, Appendix D).

directions are not fixed by gluing, but they also do not change the vacuum states, as we saw in (A.2). Thus the underdetermination of $g_\pm$ *cannot* be converted into a *physical* variety (Gomes, 2021a). Therefore, given $h_\pm$, there is a unique h which can be obtained from their union. In this particular case, we are left *without* DES for local gauge theory.

A.5 Matter, non-Abelian, and non simply-connected M: the observability of symmetries in other theories and other sectors, glimpsed

In contrast to the vacuum sector studied in the previous Section, in the presence of matter, both for Abelian and non-Abelian, the stabilizer redundancy *can* lead to real physical difference. It can do this because it may act non-trivially on the matter and electric fields. That is, as we saw in (A.16), our gluing procedure left a redundancy, corresponding to certain rigid—more commonly known as global—symmetries acting on each region.

Now we must more carefully consider what kind of boundary conditions defining our subsystems would allow DES. In the non-Abelian case, gauge symmetries act on the electric field, and so (4.9) is no longer valid. The gluing condition (3.9), acquires two more sets of equations beyond (the non-Abelian analogue of) (4.12).[66]

And as before, as required by downward consistency, sectors should be defined so that they cannot discern between gauge-related boundary values of these fields either, as discussed in Section 2.2.3 (but I will not discuss the non-Abelian case at length).

In the Abelian case, the representation of a Klein-Gordon charged scalar is nowhere stabilized by a non-trivial action of the gauge transformations, as can be seen from (4.9). So, in the simple Abelian case of $U(1)$ symmetry, if the initial state has matter fields on S, no mismatch of stabilizers can maintain the composition of the states in their representational convention (cf. footnote 66). But if there are no matter fields on S, *prima facie* we would have an initial variety corresponding to the action of $\mathrm{U}(1) \times \mathrm{U}(1)$. This would have dynamical significance for as long as matter did not wander into the boundary S. And the sector such that S has no matter fields and which gives some spatial partition of the manifold still respects downward consistency, since it is a gauge invariant specification. In other words, were we to write down an action for the subsystem, the boundary contribution from S would be gauge-invariant, and, for an interval I for which matter does not cross S, we would have observable rigid symmetries corresponding to a physical variety of joint states.

To find out precisely what the physical variety here is, we also need to reinstate the action of the global stabilizer: the global representational convention was still left ambiguous up to a global stabilizer, as we saw in Section A.1. And, as per the *unobserability thesis* of Section 3.1.3, a global symmetry is unobservable (i.e. not empirically significant).

[66]Reinstating σ for the choice of convention—that is a functional of the gauge potential alone, we define $h^\psi_\pm := g^\sigma_\pm(A)\psi_\pm$ and $h^E_\pm := \mathrm{Ad}_{g^\sigma_\pm(A)}E_\pm$. We then have: in the Abelian case, the states $\psi_{\pm|S}$ the externally applied gauge transformations $g_\pm$ would have to satisfy $(g_+h^\psi_+ - g_-h^\psi_-)_{|S} = 0$ and, in the non-Abelian case, $(\mathrm{Ad}_{g_+}h^E_+ - \mathrm{Ad}_{g_-}h^E_-)_{|S} = 0$. If $g_\pm$ are defined up to some degeneracy $\tilde{g}_\pm$ (e.g. the stabilizers of $A_\pm$). Composing all the group transformations (and since both the $\mathcal{G}_\pm$ overlap on S) it is easy to see that this will only occur if both of the following conditions are satisfied: $(\mathrm{Ad}_{\tilde{g}_-^{-1}\tilde{g}_+}E_+ - E_+)_{|S} = 0$ and $(\tilde{g}_-^{-1}\tilde{g}_+\psi_+ - \psi_+)_{|S} = 0$. And so the combination of stabilizers must preserve the boundary value of the other fields.

From (A.16), any c_+ in R_+ has a unique–e.g. subsystem-global, cf. §3.2.1—extension to c_+ acting on R_-. Thus applying a global $-c_+$ symmetry, we find that for any choice of $c_\pm$, the regional states can always be seen as transformed by:

$$g_+(h^S_\pm, 0) \quad \text{and } g_-(h^S_\pm, c_- - c_+), \tag{A.17}$$

for a given choice of $\bar{c} = c_+ - c_-$. Thus we obtain a remaining U(1) variety of observationally distinct global states. This is precisely what is expected from e.g. 't Hooft's beam splitter thought-experiment (cf. ('t Hooft, 1980, p.110) and and (Brading & Brown, 2004, p. 651)).

We can phrase this result in Wallace (2019c, p. 13)'s notation (cf. Equation 11): since the rigid symmetry $\varphi_+ \mapsto \varphi_+^{c_+}$ is subsystem-global, and thus has a unique extension to Φ_- which does not alter either the representational convention or the gluing, we can write this global action as:

$$([\varphi_+], g_+; [\varphi_-], g_-)_\sigma \mapsto ([\varphi_+], c^+ g_+; [\varphi_-], c^+ g_-)_\sigma \tag{A.18}$$

where we have reinstated the subscript-σ notation of Section 3.1.2 (used to designate the use of representational conventions to link the equivalence classes to the states; see (3.2)). Then, as remarked by (Wallace, 2019c, p. 13): "Importantly, since the symmetry acts simultaneously on the two systems, the symmetry-invariant information about the combined system is not exhausted by O and O' but also includes the relational quantity $g'^{-1}g$."[67]

In the non-Abelian case, we can only articulate the analogue of (4.13) perturbatively, for reasons mainly to do with the Gribov problem (Gribov, 1978) (the Gribov problem says there is no gauge-fixing section that covers the entire configuration space, cf footnote 17). But again we could find the same type of variety of global physical states, or same variety of observable symmetries if we have shared regional stabilizers, i.e such that the stabilizer of one region can be uniquely extended to act on the other region. Thus, for instance, for $G = SU(N)$, a configuration that is in the orbit of $A = 0$, has $SU(N)$ stabilizers of the gauge potential: infinitesimally, the constant generators of the Lie-algebra. The existence of observable symmetries would then depend on the sector of the theory we are in. According to footnote 66, we would only have DES for those boundary conditions that were also stabilized by the constant generators. For instance, if τ^I is an element of the Lie-algebra basis $\mathfrak{g}$, we would require, at the boundary: $[\tau^I, E_+]_{|S} = 0$. Note that such conditions are gauge-invariant, since both the electric field and the stabilizers transform in the adjoint representation and thus they respect downward consistency. We would only get a full set of $SU(N)$ symmetries with DES if the sector was defined with vanishing electric field at the boundary.[68]

Thus the question of DES is not dependent on the detail of the boundary contributions to the dynamics: it depends only on the compatibility between the boundary values of the fields and the stabilizers of A. Again, for the time interval in which these conditions hold—namely, such that A maintains the stabilizers in time, and the boundary values of E and ψ are also stabilized throughout evolution, in the

[67] In our notation: $O \equiv [\varphi_+], O' \equiv [\varphi'], g' \equiv g_-, g \equiv g_+$.

[68] Moreover, in the non-Abelian case, it is possible to have a stabilizer of the boundary that is not shared by the bulk of the region. In that case, we will also have non-uniqueness of the composition (see also footnote 62). We will come back to this last point in Section B. Of course, as mentioned in Section 4.3, below (4.9), stabilizers are trivial for generic non-Abelian field configurations, in both bulk and boundary.

sense above—we will have the corresponding observable symmetries.

In case M is not simply-connected, there is more freedom in how one embeds, or puts together, the regions. This topological redundancy produces physical variety even in the absence of matter. Such a variety will be equivalent to Aharonov-Bohm phases (cf. (Gomes, 2021a; Gomes & Riello, 2021)).[69]

B Using gauge-fixings for the externalist's subsystem

Let us now see in more detail how our analysis through gauge-fixing, when applied to the dynamical view of symmetries in the externalist's notion of subsystem recovers the results of (Greaves & Wallace, 2014; Wallace, 2019b).

First, it should be clear that there is, *prima facie*, a tension between a fundamental approach to symmetries (as discussed in Section 2) and assigning a fixed boundary value to the states. It is in fact, not hard to show that only the dynamical approach works in these cases, and we will do so below. At least, that is, if the externalist is saddled with providing a specification of the state at the boundary as in (B.1)—an assumption that I am making.[70]

Thus suppose that instead of the covariant boundary conditions used in the internalist boundary case, (A.11), we implement $A_{|S} = \lambda$ for some fixed boundary 1-form, λ. That is (omitting the subscript σ on g_σ):

$$F(A^g) \equiv \begin{cases} \mathrm{div}(A^g) & = 0 \\ {A^g}_{|S} & = \lambda \end{cases} \tag{B.1}$$

To require $A^g_{|S} = \lambda$ as a boundary condition, we must appropriately pare down configuration space, so that only $A_{|S} \equiv \lambda$ are allowed, i.e. $\mathcal{A}' = \{A_i \in \Lambda^1(M, \mathfrak{g}) \ , \ A^I_{i\,|S} \equiv \lambda^I_i\}$.[71] This is the space where the projection $h : \mathcal{A}' \to \mathcal{A}'$ will be taken to operate. Here λ is functioning as the fixed environment state, and this boundary condition is analogous to fixing the representation of the environment in equation (3.13')—one of the dubious suppositions at stake in Section 3.3.

The reason we must pare down configuration space is the same reason that we cannot take a fundamental view of symmetry with the boundary conditions of (B.1). The obstruction is that the boundary-value problem (B.1) is over-determined for g_σ if $\mathcal{A}$ and $\mathcal{G}$ are not constrained at the boundary (where I reinsteated the subscript, for clarity). Namely, knowing the normal component of A at the boundary suffices for a complete solution, since it determines a boundary-value problem for g_σ in terms of $\mathrm{div}(A)$ and A_n, but the boundary state also gives two more boundary conditions (given by the other components of A). In more detail, given *any* A, the g_σ must satisfy (B.2) with $\partial_n \ln(g_\sigma) = A_n - \lambda_n$. This fixes g_σ. But the remaining gradients of $\ln(g_\sigma)$ will not in general coincide with the remaining components of $A_n - \lambda_n$. Even if we pare down the space $\mathcal{A}$ where the gauge-fixing projection is operating to $\mathcal{A}'$, such that $A_{|S} \equiv \lambda$, we now have a Neumann boundary problem for g_σ, but the remaining gradients of g_σ at the boundary are also constrained to vanish.

[69] Such topological variety is more akin to the standard Galileo ship case, as we will see in Section D.2.

[70] Were we able to provide a gauge-invariant specification of A at the boundary, it wouldn't help fix the gauge: it would then be underdetermined.

[71] Also recall the notation $|_S$ denotes equality of all derivatives at the boundary: cf. (3.9) and footnote 25.

Thus, instead of (A.13), solving these equations for $g(A)$ in $F(A^g) = 0$, for consistency we must simultaneously pare down the configuration space $\mathcal{A}$ and require the boundary condition $\partial_i(\ln g)_S \equiv 0$. More generally, the same argument would apply in the non-Abelian case, where preserving the boundary condition implies that the gauge transformations must be boundary-stabilizers, called $\mathcal{G}_S(A)$ in Section 3.3.

We can then choose one of these stabilizers, as a non-covariant—there is no need for covariance, since A is fixed at the boundary—Dirichlet boundary condition for the gauge transformations. Namely, $g_{|S} = \tilde{g}_{|S} =: \kappa$, for some arbitrary boundary-stabilizing $\tilde{g} \in \mathcal{G}_S$. So we have, in analogy to (A.12) and (A.13), the system:

$$\nabla^2(\ln g) = i\mathrm{div}(A) \tag{B.2}$$
$$g_{|S} = \kappa \tag{B.3}$$

Differente choices of κ can be thought of as related by the action of the stabilizer group at the boundary (even if the action on A there is trivial). Such changes correspond what Belot calls 'generalized shifts' (Belot, 2018): these are 'transformations' that don't change the fixed state at the boundary.

We can only claim this choice satisfies *'Uniqueness'*, thereby yielding a bona-fide gauge-fixing as seen in Section 4.1 if different choices of κ produce the same $h(A)$. Otherswise, the surface in (the pared down) $\mathcal{A}'$ defined by (B.1) may depend on the choice of κ (which does not appear in the defining equation, (B.1)). The above conditions demand g stabilizes the boundary state, but that is it; each choice $g_{|S} = \kappa$ can in principle yield a different gauge-fixed A.

That is, for $\kappa \neq \kappa'$, we may have substantially different solutions. Augmenting the notation to include κ as a subscript, and understanding σ as implicit, we may have $g_\kappa(A) \neq g_{\kappa'}(A)$, and perhaps even such that their difference is not due to stabilizers, and therefore $h_\kappa(A) \neq h_{\kappa'}(A)$.

There are three possibilities: (i) A's boundary state has only the trivial stabilizer; or (ii) every κ can be extended to a universal stabilizer; or (iii) some boundary stabilizers are not so extendible. Let us examine these in turn.

Suppose first (i), that A has only the trivial boundary stabilizer: then there is no DES, for $\kappa =$Id (the same conclusion holds from (3.12)). This matches Wallace (2019c)'s conclusion about what he defines as subsystem-local symmetries, since these obligatorily go to the identity at the boundary.

Now suppose that A has some stabilizers intrinsic to the boundary. The system (B.2) and (B.3) has a different unique solution $g_\kappa(A)$ for each κ. If we are in possibility (ii) and these solutions were related by a universal stabilizer, the difference between $g_\kappa(A)$ and $g_{\kappa'}(A)$ would not affect h. So in vacuum, the difference would be immaterial; and if there is matter in the bulk, the difference would again be physically relevant. This case describes electromagnetism, since there $g_\kappa(A) = g_{\kappa'}(A) + (\kappa - \kappa')$ as in (A.16).

But if we are in possibility (iii) and they *are not* related by a universal stabilizer, that is, if the boundary stabilizer does not extend throughout the bulk, different κ *will* produce different physical states even in vacuum. Since in this case A is assumed not to have a universal stabilizer, the gauge-fixed, or projected states h_κ (cf. (A.6)), will differ, depending on the boundary value of the gauge-group: $h_\kappa(A) \neq h_{\kappa'}(A)$. In this case, , each A corresponds to a collection of $h_\kappa(A)$'s, parametrized by a choice of stabilizer intrinsic to the boundary, κ. In vacuum, this can only occur in the non-Abelian case. And although the equations would no longer be (B.2) and (B.3), the general manipulations still apply.[72]

Since in possibility (iii) the group of boundary-intrinsic κ that are not extendible is isomorphic to the quotient (3.12), it is then true that we have leftover physically inequivalent configurations in that same amount, even in vacuum. They can be taken to possess 'non-relational DES' if you will, because these inequivalent possibilities are related solely by 'gauge transformations' of the boundary conditions: κ and κ' *would be* symmetry-related under a fundamental view after all, and these transformations do not change the state of A at the boundary.[73] More importantly, such degeneracy has no representation as the action of a rigid group on the bulk of the region, as it does when the stabilizer of the boundary is shared by a bulk infused with charges. The only plausible view on κ is that it represents degrees of freedom intrinsc to the boundary.

We can now summarize our findings: in either the externalist or the internalist scenario, in vaccuum and in the simply-connected case, we find that stabilizers intrinsic to the boundary that do not correspond to either regional or universal stabilizers give observable boundary-intrinsic symmetries; and the physical difference between these has no immediate realization through the action of a symmetry group in the bulk of the region; but this scenario can only occur in the non-Abelian theory. If both kinds of stabilizers—bulk and boundary—match-up (trivially or not), neither the internalist nor the externalist obtains DES in vacuum. Moreover, in this case, the internalist and the externalist also agree about DES in the presence of charged matter within the region(s) (cf. footnote 69 and Section C.2): they exist only when bulk charges are present and the stabilizer is non-trivial.

C Comparison with the holonomy formalism

The holonomy interpretation of electromagnetism takes as its basic elements assignments of unit complex numbers to loops in spacetime. A loop is the image of a smooth embedding of the oriented circle, $\gamma : S^1 \to \Sigma$; the image is therefore a closed, oriented, non-intersecting curve. One can form a basis of gauge-invariant quantities for the holonomies (cf. (Barrett, 1991) and (Healey, 2007, Ch.4.4) and references therein),[74]

$$hol(\gamma) := \exp\,(i\int_\gamma A). \tag{C.1}$$

C.1 The basic formalism

Let us look at this in more detail. By exponentiation (path-ordered in the non-Abelian case), we can assign a complex number (matrix element in the non-Abelian case) $hol(C)$ to the oriented embedding of the unit interval: $C : [0,1] \mapsto M$. This

[72] We cannot proceed in precise analogy to footnotes 58 and 62 here. Writing g as the (path)exponential of an infinitesimal ξ for simplification, we have $\mathrm{D}^2(\xi_\kappa(A) - \xi'_\kappa(A)) = 0$, in the non-Abelian analogue. But now the integration by parts trick of footnote 62 no longer works, because we are using Dirichlet, not Neumann conditions.

[73] In the treatment of 'non-relational' DES of (S. Ramirez & Teh, 2019), κ's are treated in a *dynamical* fashion. See also (Mathieu, 2020) for the categorical geometrical treatment. See also Donnelly & Freidel (2016).

[74] Of course, any discussion of matter charges and normalization of action functionals would require e and $\hbar$ to appear. However, I am not treating matter, so these questions of choice of unit do not become paramount. As before, if needed, I set my units to $e = \hbar = 1$; as is the standard choice in quantum chromodynamics (or as in the so-called Hartree convention for atomic units).

makes it easier to see how composition works: if the endpoint of C_1 coincides with the starting point of C_2, we define the composition $C_1 \circ C_2$ as, again, a map from $[0, 1]$ into M, which takes $[0, 1/2]$ to traverse C_1 and $[1/2, 1]$ to traverse C_2. The inverse C^{-1} traces out the same curve with the opposite orientation, and therefore $C \circ C^{-1} = C(0)$.[75] Following this composition law, it is easy to see from (C.1) that

$$hol(C_1 \circ C_2) = hol(C_1)hol(C_2), \tag{C.2}$$

with the right hand side understood as complex multiplication in the Abelian case, and as composition of linear transformations, or multiplication of matrices, in the non-Abelian case.

For both Abelian and non-Abelian groups, given the above notion of composition, holonomies are conceived of as smooth homomorphisms from the space of loops into a suitable Lie group. One obtains a representation of these abstractly defined holonomies as holonomies of a connection on a principal fiber bundle with that Lie group as structure group; the collection of such holonomies carries the same amount of information as the gauge-field A. However, only for an Abelian theory can we cash this relation out in terms of gauge-invariant functionals. That is, while (C.1) is gauge-invariant, the non-Abelian counterpart (with a path-ordered exponential), is not.[76]

C.2 DES and separability

As both Healey (Healey, 2007, Ch. 4.4) and Belot ((Belot, 2003, Sec.12) and (Belot, 1998, Sec.3)) have pointed out, even classical electromagnetism, in the holonomy interpretation, evinces a form of non-locality, which one might otherwise have thought was a hallmark of non-classical physics. But is it still the case that the state of a region supervenes on assignments of intrinsic properties to patches of the region (where the patches may be taken to be arbitrarily small)? This is essentially the question of *separability* of the theory (see (Healey, 2007, Ch.2.4), (Belot, 2003, Sec.12), (Belot, 1998, Sec.3), and (Myrvold, 2010)).

Clearly, the question of DES asked in the present paper is intimately related to the one of separability. For DES, in many of its incarnations, e.g. (Brading & Brown, 2004; Friederich, 2014; Greaves & Wallace, 2014; Teh, 2016), is conditional on the existence of universal gauge-invariant quantities that are *not* specified by the regional gauge-invariant content. But we are not interested here in cases of "topological holism", as related to the Aharonov-Bohm effect. We are asking whether a vacuum, simply-connected universe still displays non-separability. For this topic, we can directly follow Myrvold's definition (Myrvold, 2010, p.427) (which builds on

[75] It is rather intuitive that we don't want to consider curves that trace the same path back and forth, i.e. *thin* curves. Therefore we define a closed curve as *thin* if it is possible to shrink it down to a point while remaining within its image. Quotienting the space of curves by those that are thin, we obtain the space of *hoops*, and this is the actual space considered in the treatment of holonomies. I will not call attention to this finer point, since it follows from a rather intuitive understanding of the composition of curves.

[76] For non-Abelian theories the gauge-invariant counterparts of (C.1) are Wilson loops, see e.g. (Barrett, 1991), $W(\gamma) := \mathrm{Tr}\,\mathcal{P}\exp\left(i\int_\gamma A\right)$, where one must take the trace of the (path-ordered) exponential of the gauge-potential. It is true that all the gauge-invariant content of the theory can be reconstructed from Wilson loops. But, importantly for our purposes, it is no longer true that there is a homomorphism from the composition of loops to the composition of Wilson loops. That is, it is no longer true that the counterpart (C.2) holds, $W(\gamma_1 \circ \gamma_2) \neq W(\gamma_1)W(\gamma_2)$. This is due solely to the presence of the trace. The general composition constraints—named after Mandelstam—come from generalizations of the Jacobi identity for Lie algebras, and depend on N for SU(N)-theories; e.g. for $N = 2$, they apply to three paths and are: $W(\gamma_1)W(\gamma_2)W(\gamma_3) - \frac{1}{2}(W(\gamma_1\gamma_2)W(\gamma_3) + W(\gamma_2\gamma_3)W(\gamma_1) + W(\gamma_1\gamma_3)W(\gamma_2)) + \frac{1}{4}(W(\gamma_1\gamma_2\gamma_3) + W(\gamma_1\gamma_3\gamma_2) = 0$.

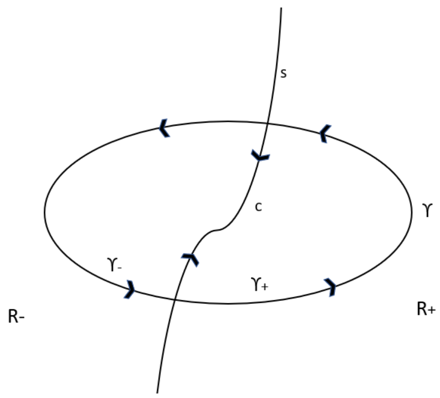

Figure 2: Two subregions, i.e. $R_\pm$, with the separating surface S. A larger loop γ crosses both regions. But, since γ_1 and γ_2 traverse S along C in opposite directions, $\gamma = \gamma_1 \circ \gamma_2$.

Healey's notion of Weak Separability (Healey, 2007, p. 46) and on Belot's notion of Synchronic Locality (Belot, 1998, p. 540)):

• *Patchy Separability for Simply-Connected Regions.* For any simply-connected spacetime region R, there are arbitrarily fine open coverings $\mathcal{N} = \{R_i\}$ of R such that the state of R supervenes on an assignment of qualitative intrinsic physical properties to elements of $\mathcal{N}$.

If *Patchy Separability for Simply-Connected Regions* holds, there will be no room for DES. And indeed, in vacuum, it is easy to show that it *does* hold. In Figure 2, we see a loop γ not contained in either R_+ or R_-. However, we can decompose it as $\gamma = \gamma_+ \circ \gamma_-$, where each regional loop $\gamma_\pm$ does not enter the complementary region ($R_\mp$, respectively). Following (C.2), it is then true that, since holonomies form a basis of gauge-invariant quantities, the universal gauge-invariant content of the theory supervenes on the regional gauge-invariant content of the theory.

It is also easy to see how *Patchy Separability for Simply-Connected Regions* fails when charges are present within the regions but absent from the boundary S (see in particular (Gomes & Riello, 2021, Sec. 4.3.2), and footnote 70 in (Gomes, 2021a)). For, in the presence of charges, we can form *gauge-invariant* functions from a non-closed curve C' that crosses S and has one positive and one negative charge, $\psi_\pm(x_\pm)$, at each end of C', at $x_\pm \in R_\pm$. That is, the following quantity is a gauge-invariant function:

$$Q(C', \psi_\pm) = \psi_-(x_-) hol(C') \psi_+(x_+)$$

for $C'(0) = x_-, C'(1) = x_+$. It is easy to check from the transformation property $\psi \mapsto g\psi$, that Q is gauge-invariant. Moreover, we cannot break this invariant up into the two regions, since we have assumed no charges lie at the boundary. This is just a translation of the results mentioned at the end of section A.4 into the holonomy formalism.

Unfortunately, this holonomy-based analysis cannot be reproduced for non-Abelian theories (see footnote 76); and it does not apply to an externalist's notion of boundaries; and it cannot be translated to the point particle language. Since we will have to analyse point-particles and the externalist's notion of boundaries, and since we want our formalism to apply also to the non-Abelian case, a treatment with holonomies—even if good for illustration—will not do.

D Point-particle systems

To compare the local gauge theory discussed above to the case that originally motivated the notion of DES—Galileo's ship—we introduce representational conventions to the study of point particles in Euclidean space.[77]

Adopting subsystem recursivity, as discussed in Section 2.1, and in particular, downward consistency, we assume the sectors of the theory are defined in symmetry-invariant way (with respect to the global symmetries). In particular, this implies that the subsystem inherits the full group of symmetries of the universe. Of course, these limited assumptions will only allow us to discuss observable symmetries for the initial state. How these symmetries extend in time will depend on the details of how we embed the subsystem in the rest of the universe, and if they are to extend at all one requires this embedding to respect some condition of dynamical isolation.

In sum, we would like here to gauge-fix the Galileian symmetries, for two subsystems, replacing ship and shore respectively. After some prescription for composing the system, we would still like to evaluate whether different compositions are physically distinguishable or not, and therefore we must again choose a representational convention for the global state.

In Section D.1 I introduce natural representational conventions in the particle case, and in Section D.2 I use these conventions to find the standard notion of DES for Galilean symmetries.

D.1 Gauge-fixing

For particle systems, it is straightforward to fix translations by the center of mass and rotations by diagonalizing the moment of inertia tensor around the center of mass. It is again true that these choices of gauge-fixing/representational conventions may not satisfy '*uniqueness*'. In the case of translations, this can happen for infinite, homogeneous mass distributions; there just is no unique center of mass to speak about. For rotations, the lack of uniqueness will obtain when the configuration has some rotational symmetry along an axis. We will only consider a finite number of pointlike mass particles, leaving only the degeneracy of rotations as relevant.

To be more explicit, the total system is given by N particles of mass m_α, $\alpha \in I = \{1, \cdots N\}$, with position vectors $\mathbf{q}_\alpha$ in some arbitrary inertial frame of $\mathbb{R}^3$, constituting the configuration space $\mathcal{Q} = \mathbb{R}^{3N}$; with conjugate momentum variables $\mathbf{p}^\alpha$. The subsystems here are defined by selecting two subsets of these particles, $I_\pm \subset I$, so that $I_+ \cap I_- = \emptyset$ and $I_+ \cup I_- = I$; that is, they are mutually exclusive and jointly exhaustive. The subsets define sectors that satisfy downward consistency, i.e. are symmetry-invariant, and are the analogues of $R_\pm$, whereas the relevant configuration space, $\mathcal{Q}$, is analogous to $\mathcal{A}$, and $\mathcal{Q}_\pm$ to $\mathcal{A}_\pm$. Thus we assume that the same global symmetries that act on the entire universe act on these subsystem configuration spaces.

The translations act as $T : \mathbf{q}_\alpha \mapsto \mathbf{q}_\alpha + \mathbf{t}$, for a given vector $\mathbf{t}$. The rotations act as $R : \mathbf{q}_\alpha \mapsto \mathbf{R}\mathbf{q}_\alpha$, where $\mathbf{R} \in SO(3)$, acting in coordinates as $R : q^i_\alpha \mapsto R^i_j q^j_\alpha$. A Galilei boost is just the translation in momentum space (i.e. an infinitesimal translation of the type $\mathbf{t}(t) := \mathbf{v}t$, where t is time), acting as $B : \mathbf{q}_\alpha \mapsto \mathbf{q}_\alpha$ and $B : \mathbf{p}^\alpha \mapsto \mathbf{p}^\alpha + \mathbf{v}$.

[77] This discussion echoes (Rovelli, 2014), which considers precisely the question of matching physical information about point-particle subsystems. The thought-experiment is made more explicit in the context we are exploring here in (Gomes, 2019, Sec 2). For an enlightening discussion of the topic, see also (Teh, 2016).

The action of the group on configuration space is a semi-direct product of the two groups $\mathcal{G} = SO(3) \ltimes \mathbb{R}^3$, with group action $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$ denoted by '$\cdot$', i.e. $(g, g') \mapsto g \cdot g'$. We will focus on this action, and denote $g_\pm = (R_\pm, t_\pm) \in \mathcal{G}_\pm$, with $\circ$ the action of rotations and translations on the configurations, e.g. $\circ : (g, \mathbf{q}) \mapsto g \circ \mathbf{q}$.

For each (sub)system, $J = I, I_+$ or I_-, we first fix center of mass coordinates through the gauge-fixing $F_t(q) = 0$, as:

$$F_t(q) = \sum_{\alpha \in J} m_\alpha \mathbf{q}_\alpha + \mathbf{t} = 0 \tag{D.1}$$

and so define $\mathbf{t}_\sigma(q) = \sum_{\alpha \in J} m_\alpha \mathbf{q}_\alpha$. Fixing the rotations is slightly more complicated. We first define the translationally fixed positions through the translationally fixed coordinates, as $\overline{\mathbf{q}}_\alpha := \mathbf{q}_\alpha + \mathbf{t}_\sigma(q)$. Now we can define the moment of inertia tensor as $\mathbf{L}$ with components:

$$L^{ij} := \sum_{\alpha \in J} m_\alpha (\|\overline{\mathbf{q}}_\alpha\|^2 \delta^{ij} - \overline{q}^i_\alpha \overline{q}^j_\alpha)$$

L^{ij} is a real symmetric matrix. A real symmetric matrix has an almost unique eigendecomposition into the product of a rotation matrix and a diagonal matrix. We therefore fix rotations through $F_R(q) = 0$, as:

$$F_R(q) = \mathbf{R}^T \mathbf{L} \mathbf{R} - \mathbf{\Lambda} = 0, \tag{D.2}$$

where $\Lambda = \mathrm{diag}(\Lambda_1, \Lambda_2, \Lambda_3)$ is a diagonal matrix whose non-zero elements are called the principal moments of inertia. When all principal moments of inertia are distinct, the principal axes through the center of mass are uniquely specified. If two principal moments are the same, there is no unique choice for the two corresponding principal axes. If all three principal moments are the same, the moment of inertia is the same about any axis. These constitute the possible degeneracies in the determination of $\mathbf{\Lambda}$. And so we find the configuration-dependent rotation matrix $\mathbf{R}_\sigma(q)$. As with the translation element $t_\sigma(q)$, this matrix depends on the positions of all the particles, $\{\mathbf{q}_\alpha\}$, a dependence we denote simply by (q).

We have thus completely fixed the coordinate system for the particles, and therefore a complete representational convention of the configurations is given by the n-tuples:

$$\mathbf{h}(q)_\alpha = \mathbf{R}_\sigma(q)(\mathbf{q}_\alpha + \mathbf{t}_\sigma(q)) = g_\sigma(q) \circ \mathbf{q}_\alpha \tag{D.3}$$

in perfect analogy with our definition of $h(A)$ in (A.6); e.g.

$$\mathbf{h}(g \circ q)_\alpha = \mathbf{h}(q)_\alpha \tag{D.4}$$

where $g_\sigma(q) = (\mathbf{R}_\sigma(q), \mathbf{t}_\sigma(q))$ is the necessary translation and rotation to bring the configurations to the frame chosen by σ, i.e. so that $F_t(h) = F_R(h) = 0$. This configuration-dependent group element obeys:

$$g_\sigma(g' \circ q) = g_\sigma(q) \cdot g'^{-1} \tag{D.5}$$

which is what guarantees (D.4). Again we can see $\mathbf{h} : \mathcal{Q} \to \mathcal{Q}$ as a projection from configuration space to configuration space, and such that the image of $\mathbf{h}$ is the gauge-fixing surface and this image is invariant under gauge transformations on the domain.[78]

[78]But, as before, in equation (4.4), we can also apply subsystem-extrinsic gauge transformations to the range of $\mathbf{h}$; cf. footnote 33.

D.2 Finding DES

Again, the idea is:—assume each subsystem employs these representational conventions. Then we ask: in how many physically distinct ways can we compose given physical states of the subsystems?

In the beginning of Section 4.4 we saw that a universal gauge-fixed field, h, did not necessarily restrict to the corresponding regional gauge-fixed fields, $h_\pm$, because of non-locality—this is why subsystem-extrinsic gauge transformations were required (cf. footnote 43). Again it is true that, given a universal configuration in the preferred coordinates, $\mathbf{h}_\alpha(q)$, restriction to subsystems—to the analogue of $R_\pm$—will not be in their center of mass and diagonalized moment of inertia. Therefore, again: in order to relate an h to the $h_\pm$, we must allow some adjustments (or subsystem-extrinsic transformations) so that we find an expresion of the glued states in the global representational convention (omitting the particle indices):

$$\mathbf{h} = (g_+ \circ \mathbf{h}_+) \oplus (g_- \circ \mathbf{h}_-). \tag{D.6}$$

And in particular, we cannot assume one of the subsystems will remain in the center of mass coordinates (so that $g_- =$Id).

But the main difference to the previous, field theoretic case is the following: there we nailed down the composition $\oplus$ in terms of the embeddings of the manifolds: it amounted to smooth composition along a shared boundary. For fields, the splitting of the universe into adjacent regions nails down the embedding of the regions supporting the subsystems into the larger spacetime manifold. Consider: were the two regions $R_\pm$ not adjacent, and had their placement been left free, we would have had a further freedom of composition given by the possible embeddings of one submanifold with respect to the other.[79] Of course, the two regional states then would not have determined the global state, and so adjency is implied by completeness. In the field theory example, by stipulating that the two regional subsystems descended from a splitting of the universe and were to jointly determine the global state, we topologically fixed the embeddings of the regions.[80]

In contrast, here in the particle case an analogue of the gluing condition, (4.12), is missing. So even if we hold that the two subsystems should jointly describe the state of the universe, we have the extra step of stipulating how to embed the subsystems. It is *this* freedom that gives rise to Galileo's ship and Einstein elevator realizations of DES. For it is still possible to respect downward consistency, and have the subsystem symmetries be Gallilean, by embedding that system into the universe with a force that acts equally on all its components (see e.g. (Saunders, 2013) for a thorough analysis of the constant but non-zero force, and its relation to Newton's Corollary VI). Of course, this will only be possible for certain embeddings: those that satisfy downward consistency and, for an arbitrary time-dependent acceleration, it may well be near impossible to find an environment for which the embedding satisfies the equations of motion of the universe.

Thus, instead of finding explicit $g_\pm$ in (D.6), we divide the process into two parts: we first arbitrarily embed the subsystems into the same Euclidean space, and then we find a transformation that brings the newly defined composite system

[79] And indeed the topological ambiguity related to the Aharonov-Bohm effect is the effect of such an added freedom, since then adjancy still leaves some features of the embedding undetermined.

[80] Indeed, for non-simply connected manifolds, adjacency does *not* fix the topological embedding uniquely, giving rise to a DES for gauge systems associated to the Aharonov-Bohm effect; see (Gomes & Riello, 2021, Sec.4.5).

to its gauge-fixed frame. At the end, we want to find out what information is required to determine $\mathbf{h}$ beyond that provided by the subsystem-intrinsic physical information, given by $\mathbf{h}_\pm$.

Here $\oplus$ firstly designates an embedding of the two frames into the larger universe. We embed them by defining a new frame, which is related to the ones used in $\mathcal{Q}_\pm$ by arbitrary transformations $g^{\text{emb}}_\pm \in SO(3) \ltimes \mathbb{R}^3(t)$, where $\mathbb{R}^3(t)$ denotes translations that have an arbitrary time-dependence. We thus obtain a universal configuration,

$$\mathbf{q}_\alpha = \begin{cases} g^{\text{emb}}_+ \circ \mathbf{h}^+_\alpha & \text{for} \qquad \alpha \in I_+ \\ g^{\text{emb}}_- \circ \mathbf{h}^-_\alpha & \text{for} \qquad \alpha \in I_- \end{cases} ; \tag{D.7}$$

with the understanding that α runs through the appropriate domains for $I_\pm$, we can replace those indices by $\pm$. The positions of the particles are now all seen to inhabit the same Euclidean 3-space, and $\oplus$ becomes simple vector addition.

Of course, this $\mathbf{q}_\alpha$ is not yet in the form of $\mathbf{h}_\alpha$; that is, it is not in a universal center of mass and eigenframe of the moment of inertia coordinate system. As above, a gauge-fixing yields $g(q_\alpha)$, and therefore, by linearity (omitting particle indices):

$$\mathbf{h} := g(q) \circ \mathbf{q} = (g(q) \circ (g^{\text{emb}}_+ \circ \mathbf{h}_+)) + (g(q) \circ (g^{\text{emb}}_- \circ \mathbf{h}_-)). \tag{D.8}$$

But we can put (D.8) in a slightly more concise form. Since here the symmetries act universally (i.e. they are subsystem-global, cf. Section 3.2.1) and we know the covariance property (D.5) holds (this is what guarantees (D.4)), there is no loss of generality if we replace (D.7) by:

$$\mathbf{q}'_\alpha = \begin{cases} g^{\text{emb}'}_+ \circ \mathbf{h}^+_\alpha & \text{for} \qquad \alpha \in I_+ \\ \mathbf{h}^-_\alpha & \text{for} \qquad \alpha \in I_- \end{cases} \tag{D.9}$$

where $g^{\text{emb}'}_+ := (g^{\text{emb}}_-)^{-1} \cdot g^{\text{emb}}_+$ (we can compose them since they all act on the same Euclidean space). Thus, finally, we can write our solution (again omitting the index α) as:

$$\mathbf{h} = (g(q') \circ (g^{\text{emb}'}_+ \circ \mathbf{h}_+)) + (g(q') \circ \mathbf{h}_-), \tag{D.10}$$

where '+' is now simply vector addition in the center of mass frame.

We can write $g(q') = g(\mathbf{h}_\pm, g^{\text{emb}'}_+)$. Therefore the solution is uniquely defined, in terms of $g^{\text{emb}'}_+$ and $\mathbf{h}_\pm$. Now $g^{\text{emb}'}_+$ is universally gauge invariant: it is a quotient of two rigid symmetries, as we obtained in Section A.5; we can no longer get rid of it by a universal change of coordinates. But $g^{\text{emb}'}_+$ is *not* solely determined by $\mathbf{h}_\pm$. This is in contrast to what we found for the field theory, in equation (A.16), where, for a simply-connected, vacuum universe, up to stabilizers, the transformations were uniquely determined by $h_\pm$. Here in the particle case, there is no way to associate $g^{\text{emb}'}_+$—the information required beyond $\mathbf{h}_\pm$—with stabilizers of the configurations.

The physical variety, i.e. the variety of ways to compose physical states of subsystems, is therefore given by $g^{\text{emb}'}_+$: namely, by *how we embed one of the subsytems with respect to the other*. Everything else is uniquely determined by $\mathbf{h}_\pm$.[81]

Again, dynamical considerations would come into play once we take into account the time interval in which the subsystems remain (approximately) isolated, and, more importantly, in determining the type of environment for which $g^{\text{emb}'}_+$ is a dynamically allowed embedding. This needs to analysed in a case by case basis.

[81]Namely, for two ships, these $\mathbf{h}_\pm$ would be the description of all the particles of each ship with respect to its own gauge-fixed coordinates (center of mass and diagonal moment of inertia).

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