Reinterpreting the universe-multiverse debate in light of inter-model inconsistency in set theory

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Abstract

In this paper I apply the concept of *inter-Model Inconsistency in Set Theory* (MIST), introduced by Antos, to select positions in the current universe-multiverse debate in philosophy of set theory: I reinterpret H. Woodin's Ultimate L, J. D. Hamkins' multiverse, S.-D. Friedman's hyperuniverse and the algebraic multiverse as normative strategies to deal with the situation of *de facto* inconsistency toleration in set theory as described by MIST. In particular, my aim is to situate these positions on the spectrum from inconsistency avoidance to inconsistency toleration. By doing so, I connect a debate in philosophy of set theory with a debate in philosophy of science about the role of inconsistencies in the natural sciences. While there are important differences, like the lack of threatening explosive inferences, I show how specific philosophical positions in the philosophy of set theory can be interpreted as reactions to a state of inconsistency similar to analogous reactions studied in the philosophy of science literature. My hope is that this transfer operation from philosophy of science to mathematics sheds a new light on the current discussion in philosophy of set theory; and that it can help to bring philosophy of mathematics and philosophy of science closer together.

1 Introduction

For some decades philosophers have alleged the existence of important inconsistent scientific theories (Feyerabend 1975; Rescher 2000; Lakatos 1970). Examples discussed include Bohr's model of the hydrogen atom, Newtonian cosmology, classical electrodynamics and the early calculus. The issue of philosophical interest is that these theories were nonetheless successfully reasoned with, meaning that these cases of inconsistency have not been a threat to scientific rationality: the inferences were not explosive. While these examples have been called into question as representing cases of scientific theories involving inconsistent beliefs (Vickers 2013; Muller 2007; Davey 2014), the study of reactions to inconsistencies in science continues to be a topic of interest in recent philosophy of science (Martínez-Ordaz and Estrada-González 2017; Bueno 2017; Martínez-Ordaz 2020).

Beside inconsistency avoidance or consistency preservation, in the recent literature a further reaction has been studied, namely, that of inconsistency-toleration mechanisms (Martínez-Ordaz and Estrada-González 2017), where the strategies used to make such a toleration possible range from moderate or radical scientific pluralism (Chang 2012; Longino 2002; Kellert et al. 2006; Šešelja 2017) to non-classical logics (Brown and Priest 2004, 2015; Meheus 2002). In general, there are different ways in which inconsistencies can arise, namely internally (to a theory) and externally (between theories and between theory and observation). All of these can, in turn, arise in the context of theory pursuit and the context of theory acceptance.

The present article is the second in a two-part series aiming at extending this consideration of inconsistency toleration from the sciences to mathematics, in particular set theory. The first article (Antos) starts from the premise that for such a toleration to take place in mathematics one has to consider a weaker form of inconsistency, namely, one that does not present us with an outright contradiction of the form $P \wedge \neg P$ for a mathematical statement P in a formal theory. Antos, PAGE?? makes this idea concrete by introducing the notion of *inter-model inconsistency*, developed in the context of current set theory:

Inter-Model Inconsistency in Set Theory (MIST) The mathematical field of set theory is *inter-model inconsistent* if there are at least two first-order models of some appropriate axiomatization of set theory, M_1 and M_2, and at least one statement P such that

1. P holds in one model and $\neg P$ holds in the other, so either $M_1 \models P$ and $M_2 \models \neg P$ or $M_1 \models \neg P$ and $M_2 \models P$ (*Incompatibility Clause*);

2. P is considered as an important open problem by the respective mathematical community (here the community of set theory). A resolution of this problem can have fundamental consequences for the entire field and this is shown by concrete mathematical results. (*Open-problem Clause*); and

3. the models M_1 and M_2 (and possibly others like them) are recognized to represent the mathematical field of set theory. This recognition is grounded in the practice of the discipline meaning that extensive research is done which leads to plentiful and fruitful knowledge about M_1 and M_2, recognizing them as models of set theory that decide the statement P differently. (*Models of Set Theory Clause*)

We say that M_1, M_2 and P witness the inter-model inconsistency, if they fulfill the above clauses.

Antos, PAGE?? develops this notion by reflecting on the practice of modern set theory and argues that it presents us with a form of inconsistency by showing how it allows us to operate with contradictory truth values for the same mathematical statement in set theory without giving rise to a logical contradiction. Let me give a broad outline of the idea behind the development of MIST, an in-depth treatment can be found in Antos.

The starting point is the question of independent statements in set theory and the models in which they are studied. Statements that are independent from some set-theoretic axiomatization cannot be proven or disproven in that axiomatization. This gives rise to the open problem of independent sentences: Usually a problem in mathematics is called "open" if the truth value of the associated mathematical sentence is not known. With statements that are not independent this means that they have not been proven or disproven yet, so set theorists don't know if a sentence is true or false in models of ZFC. With independent statements this is slightly different, because, by being independent, there must be at least two models, one that decides it positively and one that decides it negatively. Still, the problem is called "open" if set theorists cannot decide which of these models tell us something about the real truth value of the statement.

At the center of the case study of Antos stands the following controversial observation drawn from set-theoretic practice: For the independent sentence of the Continuum Hypothesis (CH), which states that the size of the set of natural numbers is the first uncountable cardinal, both CH and ¬CH are considered as true statements in the practice of set theory (the same holds for other important sentences independent from the standard axiomatization of set theory, Zermelo-Fraenkel with the Axiom of Choice ZFC). This observation has a distinctly inconsistent flavour; however, is does not give rise to a logical inconsistency, as these statements are considered in models that have the same base theory but different extended theories that decide CH. Logical inconsistency does therefore not seem able to capture this inconsistent flavour of the practice. However, this "inconsistent practice", as Antos calls it, is quite persistent: It can be traced to concrete and varied mathematical work about different models, independent sentences and their overall mathematical relations. This work is a major area in modern set theory which influences the methods and results of the whole field of set theory. Moreover, it is the source for a fundamental re-evaluation of the foundations of the discipline, most recently being discussed in (but not restricted to) the universe/multiverse debate.[1] So the inconsistent practice of set theory can be made concrete mathematically and influences the foundational debate in a manner similar to the detection of an inconsistency as the ones discussed in philosophy for science. Antos's main claim is that addressing such a situation requires a new concept of inconsistency and she proposes MIST as the candidate notion. She shows that MIST can relate to the relevant situation in the practice of the discipline and the connected foundational debate when the usual notion of logical inconsistency is too strong to be applicable here.

In this paper I apply Antos' foundational work on MIST to the current debate in the philosophy of set theory. I investigate how current existing programs in the foundations of set theory can be seen as reactions towards the situation of *de facto* inter-model inconsistency toleration described by MIST proposing resolution strategies ranging from consistency preservation to inconsistency toleration. As these strategies are now targeted towards the new notion of inter-model inconsistency, they have to be suitably adapted and can differ from the usual forms as for example used in Bueno (2017).

In particular, I will elaborate on selected programs that appear in the so-called universe-multiverse debate, which has been extensively discussed in the last decade.[2] Briefly put, *uni-*

[1] We will go into detail about this debate and the programs it gives rise to in the next sections.

[2] See Martin (2001); Gitman and Hamkins (2010); Hamkins (2011); Arrigoni and Friedman (2012); Hamkins (2012b); Fuchino (2012); Arrigoni and Friedman (2013); Hamkins (2015); Barton (2016); Maddy (2017); de

versism is the view that there is one intended model of set theory, the universe, which we can approximate enough to decide the truth values of independent statements in a general manner. *Multiversism* rejects this view, stating that set theorists investigate all (or some larger part of) the models to get an accurate picture of set theory and in turn accept a pluralistic view of truth values of independent sentences because these can change with respect to the model the sentences are considered in.[3]

In the present article, I will argue how the programs developed in the universe/multiverse debate can be interpreted to answer the challenge provided by MIST. This contributes to the inconsistency debate in the philosophy of mathematics as it details inconsistency-toleration strategies that are already present in the field of (the philosophy of) set theory. It also provides new research avenues in the philosophy of set theory by revealing new challenges to existing programs and proposing a new way of how to address the situation of MIST.

I will proceed as follows: In Section 2 I will argue how philosophical programs in the universe/multiverse debate can be categorized as consistency-preserving and inconsistency-tolerating strategies by suitably adapting these notions from the the philosophy of science literature. Here I will first detail the case of the program of Hugh Woodin's Ultimate L as an instance of a consistency-preserving strategy. Then I will discuss the more controversial case of inconsistency-toleration strategies by analyzing multiversism, in particular the set-theoretic multiverse proposed by Joel Hamkins in Section 3 and the hyperuniverse program by Sy-David Friedman in Section 4. While it could be argued that, according to the picture given so far, universism is tied to consistency preservation and multiversism to inconsistency toleration, I will complicate this picture in Section 5, where I use an objection due to Diderik Batens to sketch an arguably multiversist position, which does nonetheless put forward a consistency-preserving strategy. Section 6 concludes by summarizing the results of this paper, formulate open questions and propose some avenues of future research.

2 Philosophical positions as strategies to deal with de facto inter-model inconsistency

Antos shows that MIST is an apt description of the current status of set theory with regard to CH (and similarly important open statements). In particular, she argues that it is a state of *de facto* toleration of inter-model inconsistency, not dissimilar to analogous phenomena which have long been discussed in the philosophy of natural sciences Antos, PAGE??.

In this paper, I reinterpret several positions in the philosophy of set theory as reactions stemming from an evaluation of this state of affairs, i.e. as *normative* strategies to deal with the state of *de facto* inconsistency toleration. A principal indicator is that all these positions respond to the basic situation described by MIST by providing reasons to criticize at least one

Ceglie (2018); Antos et al. (2018); Friedman and Ternullo (2018); Ternullo (2019); Meadows (2020); Clarke-Doane (2020); Antos et al. (2020); Livadas (2020); Venturi (2020); Maddy and Meadows (2020).

[3] Multiversism is a very heterogeneous notion, bringing together different viewpoints under one label. For an overview, see Antos et al. (2015).

of the three clauses in order to deal with the state of inter-model inconsistency. They want to resolve the inter-model inconsistency by either getting rid in various ways of it or by arguing why the *de facto* state of inter-model inconsistency should be tolerated or even embraced.

In the philosophy of science literature these strategies have been categorized according to the goal they want to reach with respect to some apparent inconsistency: In the terminology proposed by Bueno (2017), on the one end of the spectrum, *consistency-preserving* strategies aim at getting rid of any and all inconsistencies; on the other end of the spectrum, *inconsistency-tolerant* strategies aim to maintain the state of inconsistency by dropping explosion as its most glaring drawback (Bueno 2017). I propose to carry over this categorization—modulo the weaker notion of inter-model inconsistency—to reinterpret various positions in the philosophy of set theory as different reactions to MIST that can be mapped onto the spectrum between (suitably adapted) consistency preservation and inconsistency toleration.

This operation results in affinities as well as differences with respect to the cases, mostly from the natural sciences, dealt so far in the literature. The notion of inconsistency toleration in the inconsistency in science literature is quite strict—understandably so, given its target of inconsistency via contradiction. Inconsistency-tolerating strategies are those which employ paraconsistent logics (these include paraconsistent compartmentalization, such as Chunk and Permeate and dialetheism); anything less, preserving classical logic, is understood to be part of consistency-preservation strategies (these include compartmentalization and information restriction). As Antos points out, there is no threat of exploding inferences in the MIST setting on the level of mathematical practice: MIST does not provide us with an outright contradiction (and can therefore seen to be "weaker" than the usual notion of logical inconsistency), as independent sentences such as CH or ¬CH are made true or false in different models of set theory. The inconsistency of MIST arises from the goals and conceptualizations of the practice as instantiated by the Open Problem Clause and the Models of Set Theory Clause in its definition. Therefore even a toleration of MIST does not *require* the use of an underlying paraconsistent logic for mathematical practice.[4]

At the same time, MIST still engender responses by agents in mathematics in a similar way in which inconsistency engenders responses by agents in the natural sciences. In particular, agents of the relevant set-theoretic community react to MIST by developing strategies to either make them consistent or by tolerating their inconsistency. Set theorists may, among the responses, propose to

1. adopt some models and reject the models incompatible with them;
2. defuse the semantic inconsistency by introducing a multiplicity of set concepts;
3. compartimentalize truth-values.

Importantly, my analysis builds on clearly distinguishing between the normative philosophical discourse and the practice of set theory over the last fifty years out of which MIST arose.

[4]Though it may *allow* for its introduction, see Section 4 for a development of this line of argument.

While these are all eminently discussed strategies, none of them have so far been *adopted* by the relevant set-theoretic community. As an example, a proponent of the universe view (the paradigmatic consistency-preserving strategy, see below) maintains that CH is resolved when a suitable extension of ZFC (call it $ZFCA$) will be found, such that it proves either CH or ¬CH. On my descriptive account, however, this doesn't resolve CH as long as the relevant set-theoretic community hasn't *adopted* $ZFCA$ (presumably by some consensus-forming mechanism). For the purpose of this paper, let's define *set-theoretic adoption* by using textbooks and accepted papers as a proxy: $ZFCA$ is *adopted* by the relevant set-theoretic community if standard set-theoretic textbooks or papers state theorems derived from the extended axiom system *unconditionally*.[5] As far as we can ascertain, ZFC is still unconditionally assumed throughout. Extensions of ZFC may be seen as set-theoretic as well in the sense that their models are seen to be models of set theory), however no single extension has been unconditionally accepted as the one "right" extension of ZFC.

Multiversism, on the other hand, capitalizes on the rich experience set theorists have acquired of incompatible models in the context of theory pursuit in order to argue that acceptance should give way to adoption: incompatible models aren't transient, they are here to stay.

In the following subsections, I will review three positions in the universe-multiverse debate; reinterpret them as strategies responding to MIST; and situate them on the consistency preservation–inconsistency toleration spectrum.

2.1 Ultimate L: Eliminate forcing

Inspired by Gödel's work on the constructible universe L, Hugh Woodin seeks to build a model (called "Ultimate L") that aims at resolving CH; is immune to forcing; and can be shown to exhibit the features of V, i.e. it holds that "Ultimate $L = V$", without, however, excluding large cardinals, like L does.[6] As major hypothesis in the program have not been proven yet, research on Ultimate L is still ongoing. But were it completed, the mathematical structure of Ultimate L shows it to be a desirable candidate for the intended model of set theory in the universist sense.[7]

This program responds to MIST by attacking two of its clauses full-on: If one can show that such an Ultimate L model exists, it resolves CH therefore eliminating the Open Problem Clause of MIST. CH is then not seen to be an open problem anymore, removing the goal of finding the truth value of CH from set-theoretic practice. Moreover, it drastically changes the Models of Set Theory Clause: Instead of considering both models M_1 and M_2 as each producing equally accurate knowledge about independent sentences and therefore seeing them as models of set theory, only the model Ultimate L is a model of set theory as it is a very good approximation

[5] A theorem P is derived conditionally if it is stated relative to some specific axiom that is assumed additionally to standard ZFC; otherwise it is derived unconditionally.

[6] The fact that Gödel's L does not allow for "large" large cardinals such as the ones over and including measurable cardinals is a fundamental results by D. Scott (1961).

[7] This has been recently argued for in Woodin (2017); a philosophical discussion and introductory presentation of the contents is given in (Rittberg 2015). My present discussion is based on these accounts.

to the intended model of set theory. Models that disagree with it on important statements such as CH should not be regarded as equally contributing to set-theoretic knowledge. Note that does not mean that universist that argue along this line deny the existence of models that decide CH differently. They might simply regard the other models as a form of "non-standard" models of set theory. So they reject parts of the Models of Set Theory Clause but still accept the Incompatibility Clause. Interestingly, and much more strongly, the Ultimate L project aims at producing a model that will decide[8] all sentences that can be shown to be independent by the technique of forcing (this is meant by 'being immune to forcing'). Therefore, the Open Problem Clause and parts of the Models of Set Theory Clause would fail for all these independent sentences because, very roughly, the models disagreeing with the Ultimate L model on these sentences could no longer be produced by forcing.

This scenario, however, assumes that showing the existence of Ultimate L will lead to its adoption by the community, therefore obviating the lack of consensus on how to extend ZFC that I described above.[9] It has been questioned whether this will ever happen, even if an Ultimate L model can be shown to exist;[10] this is, however, a slightly different debate, as it is about the adoption of a specific resolution strategy. Here I want to answer a different question: Is such a resolution simply a reaction towards the phenomenon of independence or is it, as I claim, more, namely a resolution strategy for the problem posed by MIST?[11]

At first, it may seem like the former is the case; after all, the Ultimate L program is a universist program, much like the standard answer towards the problem posed by independent statements given already by Gödel. Gödel (1947) developed a program called the "search for new axioms" to solve the problem on how set theory can give a foundation for mathematics that is as complete as possible: The idea is to search for new axioms that can be added to the standard axiomatization of ZFC, thus producing a new axiomatic system from which previously independent statements can be proven (or disproven) (see Feferman 1996). If this program was successful, the resulting axiomatization would decide all interesting mathematical statements such as CH (even if Gödel's first incompleteness result tells us that there would still be statements independent of the resulting axiomatization).

It can very well be contested that the 'search for new axioms' program is a reaction to MIST: First, it would be historically inaccurate, as the program was developed long before the invention of forcing—and MIST is very much tied to the availability of general model-building techniques, like forcing. Second, it addresses the issue with clear reference to independence, incompatible models are mainly seen as a side-product to proving independence. This is not

[8] They are either decided directly in the model or as a consequence of the adoption of large cardinals, cf. Rittberg (2015, 128, 132).

[9] Note that Ultimate L gives rise to specific axioms that Woodin claims should be adopted when Ultimate L is accepted. This is not straightforward as Ultimate L is a model for different possible extensions of ZFC that can decide CH differently and it therefore requires an additional argument to accept the axioms Woodin favours (for details, see Rittberg 2015, 145–146). To make the discussion easier to follow we will collapse these two acceptance steps into one.

[10] See Hamkins' objection to the dream template solution, also discussed in the next section.

[11] It is an open questions whether the proponents themselves would recognize their programs as resolution strategies to inter-model inconsistency. (Thanks to a anonymous reviewer for suggesting to clarify this point.)

surprising as techniques that allowed to build models flexibly and quickly only became available three decades after Gödel proved his incompleteness theorems.

But the same does not hold for more recent universist programs. The main contenders in the last decades were the aforementioned Ultimate L program and the program based on Ω-logic, which Woodin pursued before Ultimate L (Woodin 2001a,b).[12] Mathematically, Ultimate L developed out of the Ω-logic project and both projects share a relevant commonality for our account of MIST: Instead of trying to solve CH via a suitable axiomatization, they approach the problem from the model-perspective. In both programs, the focus is on building a model of ZFC that is as close as possible to the intended universe of set theory. As Rittberg (2015, 139) points out, Ω-logic does this in a localized manner, building the model up step-by-step, whereas Ultimate L builds the whole model in a general manner. The principles according to which this model-building takes place have been described above, namely, removing the problems introduced by forcing and containing desirable properties, such as large cardinals, etc. In the end, each of Woodin's programs would produce a model that is as close as possible to the intended model of set theory, so that the way in which CH is resolved is determined by the model. A very surprising fact is that the models produced by the two programs do not agree on the truth value of CH: pursuing Ω-logic will decide CH negatively, Ultimate L will decide it positively.[13]

These recent programs thus show a transition of focus in universist strategies from the syntactic level, on which independence and axiomatizations are the concepts driving research, towards semantics, where the central concepts are models. In this way we can reinterpret the recent programs as direct reaction to MIST, rejecting the Open Problem and Models of Set Theory Clauses as described above. These programs then firmly embrace the stance of consistency preservation as their goal is to eliminate not only undesirable models, but to make the resulting model impervious to the very technique that gives rise to the diversity of models in the first place.

3 The set-theoretic multiverse: accept incompatible models

Let us consider a very different reaction towards MIST, the set-theoretic multiverse view proposed by Joel Hamkins (2012a).[14] Hamkins takes the existence of model-building techniques

[12] Note that both programs are incomplete, meaning that they rest on conjectures that have not been proven yet. The main open problem is the proof of the completeness of Ω-logic, known as the Ω-conjecture (see Rittberg 2015, 139-140 for an overview).

[13] At least the version Woodin favours, see again Rittberg (2015, 146).

[14] As mentioned above, there are a wide variety of different multiverse notions. As they all rely on the multiverse, i.e. a collection of models built through techniques like forcing, they are all closely related to the situation described by MIST. It is, however, an open question how to categorize them on the consistency preservation-inconsistency toleration spectrum. Here we discuss one such case, Hamkins' multiverse, that is quite clearly on the side of inter-model inconsistency toleration. However, the reader should be aware that I do not claim that all multiverse proposals amount to inter-model inconsistency toleration strategies. Indeed, in Section 5 I will present a multiverse position advancing an inter-model consistency-preserving strategy.

and the results they delivered over the decades as evidence that the universe view does not provide a viable project for set theory anymore. Instead, he argues for a Platonistic view of all the models of set theory, i.e. he argues for the claim that all of them are ontologically on a par. He connects this view to the existence of different concepts of set instantiated by these models:

> I shall argue for [...] *the multiverse view*, which holds that there are diverse distinct concepts of set, each instantiated in a corresponding set-theoretic universe, which exhibit diverse set-theoretic truths. Each such universe exists independently in the same Platonic sense that proponents of the universe view regard their universe to exist. [...] Often the clearest way to refer to a set concept is to describe the universe of sets in which it is instantiated, and [...] I shall simply identify a set concept with the model of set theory to which it gives rise. (Hamkins 2012a, 416–7)

Hamkins also gives an argument as to why CH will never be decided in the way the universe view works towards:

> [We] have an informed, deep understanding of the CH and ¬CH worlds and of how to build them from each other. Set theorists today grew up in these worlds, and they have flicked the CH light switch many times in order to achieve various set-theoretic effects.[15] Consequently, if you were to present a principle and prove that it implies ¬CH, say, then we can no longer see as obviously true, since to do so would negate our experiences in the set-theoretic worlds having CH. Similarly, if were proved to imply CH, then we would not accept it as obviously true, since this would negate our experiences in the worlds having ¬CH. (Hamkins 2012a, 430)[16]

This quote is interesting from the viewpoint of MIST because it not only endorses the Models of Set Theory Clause, it even strengthens it. Not only does Hamkins see it as an accurate description of current set theory, he bases a prediction of its future development on this. This shows that Hamkins' multiverse view can be read as a reaction towards MIST in a very strong sense; his whole picture of the multiverse as well as his arguments for such a picture depend on the existence of models disagreeing on important independent sentences and the way in which they are used in set-theoretic practice.

Still, it does not follow that Hamkins would agree with the concept of inter-model inconsistency as expressed by MIST. In fact, I don't expect that he would, as he explicitly rejects the Open-problem Clause for the statement CH:

> On the multiverse view, consequently, the continuum hypothesis is a settled question; it is incorrect to describe the CH as an open problem. The answer to CH consists of the expansive, detailed knowledge set theorists have gained about the extent to which it holds and fails in the multiverse, about how to achieve it or its negation in combination with other diverse set-theoretic properties. Of course,

[15] Hamkins is here referring to the 'switch'-like behavior of CH further explained in Antos, PAGE??.
[16] See also Hamkins (2015).

> there are and will always remain questions about whether one can achieve CH or its negation with this or that hypothesis, but the point is that the most important and essential facts about CH are deeply understood, and these facts constitute the answer to the CH question. (Hamkins 2012a, 429)

This argument exploits the tension captured by MIST and resolves it by holding on to the Models of Set Theory Clause in order to abandon the Open-problem Clause: The rich experience and knowledge of models which are inter-model inconsistent accumulated by set theorists regarding CH implies that CH is not an open question anymore. Instead, it is solved (at least for some sense of "solved") by the set theorists' knowledge of its behavior over the multiverse. This means that we can interpret Hamkins' multiversism as a resolution strategy which responds to the inconsistency described by MIST.

When we compare this multiversism with the resolution provided by the universist picture in the last section, we can see that there are significant differences in their respective approaches towards MIST: Universism ultimately aims at resolving the open problem posed by independent sentences. If such a resolution can be achieved (here: by constructing Ultimate L), the proponents of Universism give reasons (or predict) why this resolution should (or will) be adopted by the community as an answer, therefore resolving the Open Problem Clause. The Models of Set Theory Clause is, in turn, adjusted by reducing it to only include models that accord with the answer to the open problems. On the other side, Hamkins' multiversism also resolves the Open Problem Clause—however, by providing a very different answer. Hamkins defines the answer to CH to be the combined knowledge about the truths-in-a-model of CH and how these truth values change when moving from one model to the next. This answer enables him to embrace the Models of Set Theory Clause in full, by strengthening it to claim that the models are not only models of different set theories, but models of different concepts of set.

On this view, Hamkins' multiverse does not opt for a consistency preservation strategy, because he resolves the Open Problem Clause in a manner which still recognizes it as a valid goal of the practice by offering an answer to the open problem posed by independent sentences. This answer is, however, ongoing and perhaps opened anew: It could very well be that at some point in time a new model-building technique is developed which allows for the introduction of new models of set theory to the practice.[17] Then the problem of CH would be open again, as set theorists investigate how CH behave over these new models; an answer to CH could only be provided once this investigation is completed.

I evaluate Hamkins' multiversism as involving a form of inter-model inconsistency toleration (albeit a weaker one than the hyperuniverse offers, see next Section). To illustrate this, let us look again at the usual case of strong inconsistency toleration as described in Bueno (2017, 230). Here inconsistency is said to be tolerated by changing the underlying logic to one that removes the main problem, namely, the explosion derived from the inconsistency. In the case of MIST, concerned set theorists are not threatened by explosion; in fact a logical contradiction

[17] Note that there are already a variety of model-building techniques beside forcing, like those of inner model theory, see for example Mitchell (2010) and Steel (2010). However, as forcing is by far the most fruitful of these, we investigate the programs in concentrating on forcing as the most relevant example.

never arises directly. Instead the work with contradictory truth values of independent sentences like CH happens in different models and the situation that P and $\neg P$ hold in one model never arises (if it did, the situation would be more accurately described by the usual notion of logical inconsistency). Therefore, set theorists don't have to choose a paraconsistent logic to avoid explosion in their mathematical work. However, a similar problem arises: With MIST, *all* independent sentences that give rise to MIST become true (in the respective models) and they also become false (in the respective models) within the mathematical field of set theory. If one at the same time commits to there being a general answer to CH, another kind of inconsistency, an inter-model inconsistency, arises: If it is accepted as part of set-theoretic knowledge that CH is generally true, this contradicts the truth values of CH in models that decide it as false (and *vice versa*, if CH is generally false). To solve this problem, as we have seen, already, Hamkins does not choose a different underlying logic; instead he re-defines what makes a problem *open* or *solved*.

Hamkins detaches the openeness of CH from its unique truth value: he considers the problem to be "solved" if set theorists know enough about how the relativized truth value behaves. This is possible because for Hamkins there is no absolute set-theoretic background. He makes this concrete by offering the picture of many distinct concepts of set, each instantiated by a (forcing-)extension in a corresponding set-theoretic universe.

When we take this into account, we see that Hamkins' rejection of the Open-problem Clause of MIST involves his own take of what "being open" means for independent statements rather than the usual one. Again, there is an analogy between logical inconsistency via contradiction and inter-model inconsistency: In order to answer the problem, some basic understanding is changed (respectively, the logic and the notion of "open problem"); however, the original inconsistency remains untouched (the contradiction in the logical case, the incompatible models and their use in practice in the case of MIST). The change in understanding allows us to "preserve" the inconsistency and work with it in a fruitful manner.

4 The hyperuniverse: Compartmentalize inconsistent models

The hyperuniverse program was first introduced in Arrigoni and Friedman (2013) and further pursued in Antos et al. (2018). It presents a different form of multiverse view than Hamkins', both in philosophical and mathematical respects. First, it is not as broad as Hamkins' multiverse, where roughly all models that can be considered to be set-theoretic enough are considered. Instead, the hyperuniverse program uses two types of multiverses, one connected to the idea of producing different models by "lengthening" them in height[18] (but not allowing "thickenings" produced by adding sets as for example forcing does); and the other multiverse being the collection of all countable transitive models of ZF (therefore allowing forcing extensions), the so-called hyperuniverse.

[18]In Friedman and Ternullo (2018, 164) this is called the *core structure*.

How these multiverses interact is a quite technical issue (related to V-logic) not relevant for the purpose of this paper, so I will skip this issue. Instead, I want to highlight that the reaction strategy towards MIST, implicit in this program, differs from the others already discussed. The hyperuniverse program does not accept all possible models (or even the ones contained in its multiverses) as equal. Instead, it tries to identify general principles which seem especially desirable to adopt. These principles can be called axioms; however, they usually are not first-order principles and can therefore not be adopted outright as axiom extensions of ZFC. Instead, they give us a partition of the hyperuniverse into models that satisfy these principles—and models that do not. Set theorists can then focus their study on the members of the hyperuniverse that satisfy such a general principle and find out what first-order consequences of the general principles hold in these models. These first-order consequences will then express possible extensions of ZFC that, if adopted, will solve formerly independent statements.

This overview of the program shows that the hyperuniverse can indeed be interpreted as a further reaction towards MIST: Its investigation involves two ontological structures (the two multiverses), and it is carried out by identifying properties of the models in the hyperuniverse that then (can) give rise to first-order axiom candidates for extending ZFC.[19] However, in comparison to Hamkins' multiverse, it seems on first glance that the hyperuniverse view proposes a consistency-preservation strategy: The models of the hyperuniverse that are selected according to some general principle could decide a statement independent from ZFC in a unique way. Then, set theorists can decide to only work with these "preferred" models and therefore eliminate the Models of Set Theory Clause of MIST in a similar manner as the universist.

The overall situation is however more intricate: This way of consistency preservation within the hyperuniverse program only works if, while considering just *one* such higher-order principle, we further assume that the first-order consequences in the models are not inter-model inconsistent in the way described by MIST; or if, while considering *several* such higher-order principles, we further assume that we can unify these principles towards one and that the resulting first-order consequences in the members of the hyperuniverse are not incompatible. The proponents of the hyperuniverse program argue for the second of these options: They discuss several such principles and a lot of mathematical work has gone into unifying two of them, the Inner Model Hypothesis and $\sharp$-generated V (see for example the discussion in Friedman and Ternullo 2018, 175). Moreover, they state explicitly that the overall goal of the program is to find an optimal such general principle that contains the essence of the various other principles to allow us to make a clear partition of the hyperuniverse:

> Of course there are members of [the hyperuniverse denoted by] $\mathbb{H}$ which do not satisfy the [Inner Model Hypothesis]. Consequently, [the negation of Projective Determinacy] is a statement holding only in a portion of $\mathbb{H}$, something which accounts for our idea that $\mathbb{H}$-axioms are 'local' axioms. This is inevitable if one wishes to be conceptually faithful to the multiverse phenomenon.

[19] One example for such a property that can be formulated as a higher-order principle is the Inner Model Hypothesis, see for example Friedman et al. (2008).

> However, there is a global corrective to this 'pluralistic' view. The programme strives for the identification of an 'optimal' maximality principle ($\mathbb{H}$-axiom). Now, suppose that P were such a principle; we would then exclude any member of $\mathbb{H}$ which would not satisfy P and therefore P could be taken to be the 'new' $\mathbb{H}$-axiom we are searching for, derivable from the maximal iterative conception and with intrinsically justified first-order consequences. (Friedman and Ternullo 2018, 176)

It is yet completely open if such a unifying principle exists. This has been criticized by Woodin as a drawback of the program because it fails to give a hard mathematical criterion for the failure of the program.[20] However, from a multiverse perspective this does not seem so problematic: Even if the synthesis of the general principles stalls at some point, we would get a separation of the hyperuniverse into possibly not-disjoint parts: set theorists would still be able to identify preferred models in the hyperuniverse and, therefore, gain additional insight into a possible structure of the multiverse. This structure would be something like a *compartmentalization* into different "chunks" of the hyperuniverse. Of interest to our purposes is that this multiverse would, in general, still give rise to MIST because it would encompass inter-model inconsistency where each one of these models belongs to some preferred compartment of the hyperuniverse. This means that there could be models which are inter-model inconsistent that are equally accepted by the multiversist, because there are higher-order principles that identify them as being desirable representatives of set theory.

This possibility, I submit, is a stronger form of inter-model inconsistency toleration than Hamkins' because it would *not reject* any of the clauses of MIST. In contrast to Hamkins, the hyperuniverse program is explicitly developed in order to provide a way of deciding independent statements, such as CH, in the usual sense. The program embraces the Open Problem Clause without defusing it and without seeing the open question as already solved (like Hamkins does). At the same time, it strengthens the Models of Set Theory Clause: On the one hand, it selects specific models as models of set theory, therefore shrinking the domain of relevant models for this clause (like universist programs do). On the other hand, there are higher-order principles which pick out all these models equally as models of set theory, but which, unlike in universist programs, could fall into *incompatible* compartments, leading to a much stronger sense of how these models "contradict" each other.

These higher-order principles in the hyperuniverse permit us to make this point more precise in terms of mathematical content: The general principles that have been investigated in this project until now all aim at expressing aspects of a very desirable feature of set-theoretic models, namely maximality.[21] This can be spelled out in different ways; the Inner Model Hypothesis, for example, formulates maximality with respect to inner models; height potentialism expresses the idea of having every higher models with respect to ordinals; etc. The hyperuniverse program

[20] Communication at the second Set Theoretic Pluralism Network Symposium, University of Bristol, June 20-25, 2017.

[21] The issue of maximality in set theory is a heavily discussed one, spanning from Maddy's principle of MAXIMIZE (Maddy 1988a,b, 1997, 2007) to questions of potentialism and actualism (Hamkins and Linnebo 2019). For a collection of papers that discuss the hyperuniverse and maximality, see Antos et al. (2018).

now argues that this kind of desirability transfers to the models chosen by the principles, claiming that the first-order consequences that hold in these models can be seen as intuitively true:

> [T]he methodology envisaged here precisely aims to provide an alternative notion of 'intuitively true' as based on the acceptance of the intuitive truth of maximality principles concerning V. Therefore, in our view, the 'meaningfulness' of the consequences of a maximality principle is guaranteed by the meaningfulness of the principle itself. (Friedman and Ternullo 2018, 176)

We can therefore reformulate the Models of Set Theory Clause of MIST in the following way: We want to consider models M_1 and M_2 which give rise to inter-model inconsistency if they are recognized to represent the mathematical field of set theory, not only because extensive research is done which leads to plentiful and fruitful knowledge about M_1 and M_2 but also because there are intrinsic reasons that justify this recognition of them as models of set theory, by being consequences of respective higher-order principles. This re-formulation rests on mathematically expressible principles such as the Inner Model Hypothesis.

So, the hyperuniverse program turns out to be a very curious proposition when interpreted in the MIST framework: Depending on its mathematical outcome (the *synthesis* of different principles), it can either provide a resolution to MIST via consistency preservation in a universist manner; or it can lead to a multiverse with an underlying toleration of inter-model inconsistency as it gives the set theorists reasons why certain models—which are inter-model inconsistent (the ones in different "compartments" of the hyperuniverse)—should all be considered proper models of set theory, namely, by satisfying different intrinsically justified higher-order principles. In contrast to consistency-preserving strategies in the fashion of the Ultimate L program, it is *not* part of the hyperuniversist methodology to aim at providing one model as the intended model of set theory. Certainly, a consistency-preserving synthesis can be one outcome of the program if all of the higher-order principles just turn out to synthesize in this way. However, this outcome will be accidental, as it were, as the overall methodology of the program in considering the most diverse principles, axioms and models is a pluralistic one. We therefore see this program as *permitting* a full-blown inter-model inconsistency toleration strategy.

This opens the question whether a paraconsistent strategy is of relevance to this program: As pointed out in Section 3, the mathematical work in set theory does not require one to adopt paraconsistent logic, because even recognizing MIST as a correct description of set-theoretic practice with regard to statements like CH does not give rise to a logical contradiction in a formal theory. We saw that Hamkins does not need to employ paraconsistent logic in his philosophical program because he redefines what it means to answer the question of CH. However, this resolution is not contemplated in hyperuniversist program because the latter aims to solve CH in the traditional manner, namely by settling the question whether CH is true or false as truth *simpliciter*. The interpretation of the hyperuniverse program as a resolution strategy responding to MIST therefore offers a challenge to the hyperuniverse program: it has to be clarified how the program proposes to handle the possibility of tolerating inter-model inconsistency. As it stands now, we do not see that this situation has been satisfactorily addressed.

5 The algebraic multiverse

In the previous sections, all multiversist programs presented were connected with inconsistency-tolerating strategies. This raises the question if there is a necessary connection between the two. In this section I am going to argue for the denial of this hypothesis by providing a counterexample: a multiverse position with an associated consistency-preserving strategy, which I call the *algebraic multiverse*. This position is not novel: it was gestured towards already shortly after the introduction of forcing (see below) and has been amply referenced in Hamkins (2012b).[22]

The following background observation motivates the algebraic multiverse: the situation described by MIST seems *odd* from the vantage point of mathematicians working in mainstream fields like topology, modern algebra and geometry. From the perspective of the practice of these mathematical fields, the oddity stems from the Open Question Clause, or, more precisely, the consideration that there are *any* independent set-theoretic sentences which satisfy this Clause, including CH. To make this line of argument plausible, let's first consider the Parallel Postulate as purported analogue to CH.

After having been considered an open problem for many centuries, the issue of the truth of the Parallel Postulate was *dissolved* by the discovery and adoption of non-Euclidean geometries. The 19th century discovery that non-Euclidean geometries are consistent and, therefore, that the Parallel Postulate holds in some geometries and not in others, has shown that *there is no fact of the matter* whether the Parallel Postulate 'is true or not' in geometry overall or what the 'true geometry' is. The question of the truth value of the Parallel Postulate does simply not arise anymore. (Let's call this the *geometry-template*.)

The practice of group theory can be described in a similar manner: Take the property of commutativity as purported analogue to CH. Commutativity holds for some groups (called abelian groups) and not for others (called non-abelian groups). But nobody would describe the existence of these groups as giving rise to 'inter-model inconsistency in group theory', the comparison continues. Commutativity is a property that applies to some groups, but not others. No "open question" about the truth value of commutativity arises or whether abelian groups are 'more group-theoretic' than non-abelian ones. (Let's call this the *algebra-template*.)

If transferred to set theory, these templates show a way to respond to MIST by attacking the Open Problem Clause full-on: CH is not true or false; or its truth value indeterminate; or

[22] In the context of this paper, the position is inspired by an objection to the MIST framework advanced by Diderik Batens (University of Ghent), whom I thank for pressing me on this point. He presented this algebraic interpretation of set theory as a *description* of the mathematical field of set theory rivalling the description provided by MIST; however, the evidence Antos and I give to show that CH fulfills the Open-Clause is also a reason to think that it is a *false* description. Still, here I want to show that the view is a viable candidate when reissued as a multiversist view reacting to MIST, which advances reasons to dissolve the inter-model inconsistency in a consistency-preserving way.

decided in an idiosyncratic way over the multiverse. Rather, there is no answer to the question, because there is *no fact of the matter* to the question being asked. The algebraic multiverse reacts to the situation described by MIST by denying that there is a meaningful question to be asked in the first place: while one can say that since CH is true in some models of set theory and false in others, *there is no fact of the matter* whether CH "is true or not" in set theory overall or what the "true set theory" is. Like the Parallel Postulate, we should not consider CH to be an open problem. Similarly, dual models in set theory should be considered in analogy to dual algebraic structures. Because of this, the Open Problem Clause fails with respect to CH (and independent sentences like it).

Interestingly, publications contemplating the algebraic universe emerged shortly after the introduction of forcing. Comparisons between the then-recent development of set theory and the situation of non-Euclidean geometry or group theory in abstract algebra can respectively be found in Mostowski (1967); Kalmar (1967) and continue to be referenced in the current debate (see for example Hamkins 2012a, 418):

> [We] now think that an axiom system defines the common properties of all models of it, standard and non-standard, a point of view adopted long ago in Algebra. Soon we will speak of 'an arithmetic of natural numbers' or 'a set theory', just as we now speak of a group or a ring. (Kalmar 1967, 191)

> This shows that the incompleteness of set-theory is caused by other circumstances than the incompleteness of arithmetic. It is comparable rather to the incompleteness of group theory or of similar algebraic theories. (Mostowski 1967, 94)

These analogies were meant to provide a diagnosis of what was happening in modern set theory with an eye towards the future: Set theory was going through a similar development as geometry, when non-Euclidean geometries were developed; or algebra, when it turned abstract and gave rise to group theory and ring theory.[23] On this picture, set theory would finally provide set theorists with the mathematical tools necessary to get rid of more traditional elements that are tied to conceptions of 'the one intended model of set theory' or conceptions of similar universist flavours.

And yet I claim that the algebraic multiverse joins universism in rejecting any substantive form of inconsistency, including inter-model inconsistency. In Section 2.1, I showed that Ultimate L employs a consistency-preserving strategy by working towards making the Models of Set Theory Clause fail with regard to CH (and perhaps other independent statements). The algebraic multiverse adopts a consistency-preserving strategy too, but, in contrast to universism, it completely rejects the notion that any independent set-theoretic notion, including CH, satisfies the Open-Problem Clause, because there is no fact of the matter in need of being answered.

While the abstract nature of MIST allows us to see that algebraic multiversism is related to universism in the way the inter-model consistency of the mathematical field of set theory is pre-

[23] See Gray (2008) for an historical account of the rise of modern mathematics.

served, the mathematical implications of the algebraic multiverse are very similar to Hamkins' multiverse. Indeed, I speculate that the multiversist practice arising from Hamkins' program would perhaps be indistinguishable from the practice arising from algebraic multiversism: The study of the behavior of CH over the multiverse or how different models are connected to each other seems an algebraic goal alright. And yet: The philosophical reasoning put forward by Hamkins' couldn't be more different than the one advanced by algebraic multiversism: Hamkins does acknowledge the question of the truth of CH as an open problem and provides an (idiosyncratic) answer how this question can be answered. The algebraic multiverse, instead, rejects the meaningfulness of that very question and therefore offers no answer. Summarizing, we can say that while the algebraic multiverse and Hamkin's multiverse differ quite dramatically in their philosophical treatment of the Open-Question Clause, it may amount to a 'difference without a difference' in mathematical practice.

We already presented one possible argument for the algebraic multiverse as a resolution to MIST: if the development of abstract mathematics from assertoric theories to algebraic theories is a template for the development of modern mathematics generally, then "the transformation of set theory into a modern, sophisticated field of mathematics" enabled by forcing (Kanamori) should be accompanied by an algebraic turn, too. The algebraic multiverse offers just such a view of set theory. The algebraic multiverse could also be appealing to anti-realists for its deflationary nature, which contrast with the Platonist commitment of both Ultimate L and Hamkins' multiverse.[24] On the algebraic multiverse view, the independence phenomenon is interpreted in its most straightforward manner as a dissolution of truth simpliciter and its replacement with truth-in-a-model. As philosopher Stephan Körner put it in an early exposition of such a view:

> The discovery of the independence of the continuum postulate and the axiom of choice from the other postulates of the most widely used set-theories greatly increases the multiplicity of non-equivalent set-theories. Each of them is true in a different possible world, which there is no reason to regard as the actual world or as intersubjective intuition. (Körner 1967, 128)

6 Conclusion and outlook

This article was devoted to show that several positions in universe-multiverse debate can be reinterpreted as strategies reacting to an inherent tension in current set theory as described by MIST. Using the framework of the inconsistency debate in philosophy of science, I have mapped these positions in philosophy of set theory on to the spectrum from, suitably adapted, inter-model consistency-preservation to inter-model inconsistency-toleration strategies. According to my analysis, from the perspective of our weaker notion of inconsistency, Woodin's Ultimate L

[24] While Hamkins' multiverse view "holds that there are diverse distinct concepts of set, each instantiated in a corresponding set- theoretic universe, which exhibit diverse set-theoretic truths", the view also holds that "each such universe exists independently in the same Platonic sense that proponents of the universe view regard their universe to exist" (Hamkins 2012b, 416-17).

program provides an inter-model consistency-preserving strategy; Hamkins' multiverse program a weak and Friedman's hyperuniverse a full inter-model inconsistency-tolerating strategy, at least in a specific configuration. This mapping is achieved by analyzing how and which specific clauses of Antos's definition of MIST are dropped to reject, resolve or embrace the inconsistent practice described by MIST. Additionally, we also introduced a multiversist strategy which allows to deploy an inter-model consistency preserving strategy, thereby showing the richness of the philosophical landscape when viewed from the MIST perspective.

Importantly, this reinterpretation turns on distinguishing between a *description* of the current state of set theory (as given by MIST) and *normative strategies* advanced as implicit responses to MIST, such as the philosophical programs proposed in the universe-multiverse debate. On this reading, MIST is not only compatible with, but is indeed a presupposition of these programs.

Two aims have guided this enterprise: First, to give a new framework for how to think about the universe-multiverse debate in philosophy of set theory; second, to expand the inconsistency debate in philosophy of science by including a case study in the formal rather than natural sciences; and, third, to transfer conceptual resources from the philosophy of science to philosophy of mathematics to aid a rapprochement of the two fields.

As an outlook, it would be interesting to see how MIST can be applied to other areas of mathematics as well. This would require a more general formulation of MIST (or a more specific *re*formulation) as it is at the moment tailored to the situation in set theory. For example, from the discussion in Antos, PAGE?? it is reasonable to assume that the historical development of non-Euclidean geometries might have included a phase dominated by some kind of inter-model inconsistency.

A final open question concerns the fact that, for the most part, set theorists seem to be mostly unresponsive to the situation described by MIST, they remain *de facto* inter-model inconsistency tolerant. This can be construed as an argument against MIST it that it puts pressure on the problem-response schema of this paper: Do we not owe an explanation for this unresponsivness after all? A tentative direction of inquiry is to offer a more fine-grained view of the propositional attitudes involved in set-theoretic practice (the talk about e.g. "use" and "adoption" in this paper did not take place at the level of propositional attitudes). The scholarship in the *epistemology of science* has been interested in the propositional attitudes of scientists towards their theories and hypotheses for some time. One difference in this literature is that between the attitudes of "believing" and "accepting", where it is irrational to jointly *believe p* and *believe non-p*, but it is not irrational to jointly *accept p* and *accept non-p*.[25] Proposals of this kind could offer a template for explaining how the inter-model inconsistency is *de facto* tolerated at the level of propositional attitudes without the need to appeal to elaborate normative philosophical programs.

[25] Cf. Cohen (1992); see also Bratman (1992); Buckareff (2010) on acceptance and belief as different propositional attitudes.

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