Modelling the Psychological Structure of Reasoning

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Abstract

Mathematics and logic are indispensable in science, yet how they are deployed and why they are so effective, especially in the natural sciences, is poorly understood. In this paper, I focus on the how by analysing Jean Piaget's application of mathematics to the empirical content of psychological experiment; however, I do not lose sight of the application's wider implications on the why. In a case study, I set out how Piaget drew on the stock of mathematical structures to model psychological content, namely, the operations of thought involved in reasoning. In particular, I show how operations of thought form structured wholes that initially resisted modelling by either lattices or groups but could be modelled adequately by modifications of these mathematical structures. Piaget coined the term 'grouping' for the modified structure, I conclude that it represents a non-canonical application of mathematics to the empirical content of experimental psychology. I also touch on the role external factors played in Piaget's development of the grouping.

According to the genetic epistemology conceived by Piaget, the origin of intelligence lies in the biological organism and develops in stages over time, and, via the grouping, Piaget established a genetic relationship between two stages of reasoning. I show how this relationship explains why mathematics and logic fit the psychological content of reasoning whilst simultaneously making their successful deployment in the natural sciences more mysterious. Finally, I turn to the explanation Piaget envisaged for the unreasonable effectiveness of mathematics in the natural sciences and consider some consequences for naturalism and Pythagoreanism.

Keywords

applicability of mathematics; psychology of reasoning; genetic epistemology; constructivism; structuralism; grouping; naturalism; Pythagorean strategy; Jean Piaget

1 Introduction

Since the Scientific Revolution, the natural sciences have been enormously successful, and their success is in no small part due to the deployment of mathematics. In fact, most natural sciences are unthinkable without mathematics today. However, the deployment of mathematics in the natural sciences is still puzzling, and many scientists past and present have marvelled at the appropriateness and accuracy of mathematical formulations of the laws of nature (Steiner 2009, 13–14).

Wigner characterised the puzzle in 'The unreasonable effectiveness of mathematics in the natural sciences' (Wigner 1960). Some mathematics is developed specifically to fulfil particular scientific purposes, and, having originally been tailored for the job, it is perhaps not surprising that such mathematics fits the reality investigated by the natural science in question. The puzzle is positively miraculous, however, when mathematics fits the natural world hand-in-glove, although development of the mathematics was entirely internally motivated by free choices, invention, aesthetics, convenience of calculation, etc. rather than the natural world it fits. In other words, mathematicians can be likened to artists who create fantastic works of art without realist pretentions only to discover that some of their works nevertheless depict reality with uncanny accuracy.

Wigner (1960) considered the appropriateness and accuracy of mathematical formulations of the laws of nature to be an article of faith, which encourages and reassures scientists in their use of mathematics in the pursuit of scientific knowledge. However, he never explained the effectiveness of mathematics in the natural sciences, although he argued that scientists' faith is not unreasonable on empirical grounds. Steiner (2005, 632; 2009, 45–47) on the other hand argues that the illustrations Wigner sets out constitute individual instances of the successful application of mathematics to science, which do not amount to a thesis but require individual explanations. Focussing on discoveries rather than descriptions, Steiner (2009) contends that natural scientists in using mathematical analogies to discover hidden aspects of the natural world harbour Pythagorean convictions. In doing so, he concludes that natural scientists suffer from 'intellectual schizophrenia' since beauty, convenience of calculation are instrumental in shaping the mathematics they use to investigate the natural world yet anthropocentrism is contrary to naturalism.[1] Furthermore, he also argues that the continued success of their Pythagorean strategy is grounds for questioning the dualism of mind and matter inherent in naturalism.

Hamming rose to the philosophical challenge raised by Wigner. On the one hand, he played down the success of the natural sciences: 'Science in fact answers comparatively few problems' (Hamming 1980, 89), whilst rejecting evolution on the grounds that natural selection based on reproduction and survival is a gradual process and mathematical reasoning has not yet been around long enough for evolution to have had any significant impact (Hamming 1980, 89). On the other hand, he saw explanations in '[w]e see what we look for' i.e., mathematics directs scientists' attention, thus priming them for certain empirical discoveries rather than others (Hamming 1980, 87); and '[w]e select the kind of mathematics to use' (Hamming 1980, 89), i.e., scientists chose from the wealth of mathematical concepts the most fitting tool for the job. Nevertheless, Hamming (1980, 90) laments: 'From all of this I am forced to conclude both that mathematics is unreasonably effective and that all of the explanations I have given when added together simply are not enough to explain what I set out to account for. I think that we … must continue to try to explain why the logical side of science—meaning mathematics, mainly—is the proper tool for exploring the universe as we perceive it at present.' The large unexplained 'residue' (Hamming 1980, 82) is due to the mystery the Bourbaki conglomeration aptly expressed as follows: 'mathematics appears . . . as a storehouse of abstract forms—the mathematical structures; and it so happens—without our knowing why—that certain aspects of empirical reality fit themselves into these forms, as if through a kind of preadaptation.' (Bourbaki 1950, 231)

According to Shapiro (2000, 36), the deployment of mathematics raises at least two questions: 'How is mathematics applied in scientific explanations and descriptions? What is the (philosophical) explanation for the applicability of mathematics to science? Elaborating on the second question, he holds that the success of mathematics in the natural sciences justifies entertaining 'the hypothesis that there is a relationship between the subject-matter of mathematics (whatever it is) and the subject-matter of science (whatever that is as well), and

[1] Steiner (2009, 55–9) distinguishes between 'overt' and 'covert' anthropocentrism, and via 'intellectual schizophrenia', he wishes to draw attention to the covert anthropocentrism inherent in the Pythagorean behaviour of natural scientists, which runs contrary to the their naturalistic background beliefs.

that it is no accident that mathematics applies to material reality.' And, it is clear that '[a]ny philosophy of mathematics or philosophy of science that does not provide an account of this relationship is incomplete at best.' In essence, Steiner (2005; 2009) agrees but sees chronic intellectual schizophrenia or a revision of the subject matter of natural science as the only viable alternatives. In this paper, I focus on the first question by setting out and analysing how Jean Piaget deployed mathematics in the human rather than natural sciences; however, I do not lose sight of the second, metaphysical question.

To this end, I, first, introduce the specific question Piaget intended to address in the more general context of genetic epistemology, namely the development of propositional reasoning from reasoning with concrete objects and their representations (2). I then illustrate concrete-operational and propositional reasoning by a patriarchal genealogy and the determination of causality between two associated phenomena (3). Since both types of reasoning are manifestations of operations of thought, the latter are introduced in general terms next before being further differentiated into intra- and interpropositional operations (4). In both subsections of operations of thought (4.1 & 4.2), the essential operatory mechanisms are highlighted and compared with the operations of familiar mathematical structures. For both intra- and interpropositional operations, I then outline how Piaget demonstrated that the operations of order and algebraic structures can model some but not all the essential operatory mechanisms adequately. In both cases, I then set out how Piaget envisaged the grouping as a modification of groups or lattices, designed to incorporate all the essential operatory mechanisms. I conclude that the grouping is a novel formal instrument that Piaget constructed by adopting and adapting pe-existing mathematical structures to psychological content to forge a developmental rather than a foundational link between reasoning with classes and relations, on the one hand, and propositions, on the other (5.1 & 5.2), before considering some repercussions of this result on the unreasonable effectiveness of mathematics in the natural sciences (5.3).

2 Genetic Epistemology and the Development of Reasoning

Jean Piaget trained as a biologist, but he was also interested in questions that were traditionally the preserve of philosophy. He therefore had two souls in his breast: one scientific, one philosophical, and, for epistemology, the scientific rather than the philosophical soul prevailed. In the Preface to the 1st edition of *Introduction à L'Épistémologie Génétique* (1950), written in 1949, he confessed his own motivations for founding genetic epistemology:

> While studying zoology, two interests—one in problems of biological variation and adaptation, the other in logical and epistemological questions—made me dream of constructing a biological epistemology founded exclusively on the notion of development. Recourse to positive psychology seemed to be essential and, above all, to what could be called 'the embryology of reason', namely, the study of children's intelligence. (Piaget 1950, 1:5)

Piaget (1950, vol. 1, chap. Introduction) dreamt of emancipating epistemology as a science from the clutches of philosophy. And, from the viewpoint of biological epistemology, the origin of intelligence lies in the living organism. The two most general functions of living organisms, according to Piaget, are self-organisation and adaptation. Biologically, Piaget understood by ‘adaptation’ a process in which organisms transform over time under the influence of the external environment. Being open systems, organisms depend materially and energetically on their environments for their survival, and adaptation effects variations that are advantageous for the material and energetic exchanges with these environments. Although adaptation is a single process, Piaget discerned two complementary aspects: ‘assimilation’ denotes the process of incorporating material and energetic needs from environmental sources into the organism’s organization; to facilitate incorporation, however, the existing organisation often modifies itself, and ‘accommodation’ denotes the modifications often accompanying assimilation. In the process of adaptation both aspects must be kept in balance. Since adaptation brings forth new organisations on the basis of old, organisation is both prerequisite and product of adaptation. Organisation and adaptation are therefore two complementary aspects of a single biological mechanism, and, according to Piaget, intelligence is an instance of adaptation. Cognitively, assimilation and accommodation are kept in balance as aspects of external reality are incorporated into organisational structures originating in the activity of the cognizing subject, and these structures are modified to incorporate these aspects more adequately. Being an instance of adaptation, organisational structures are for intelligence, like its biological counterpart, prerequisite and product of intellectual adaptation. (e.g., Piaget 1952, pt. Introduction; Piaget 1971a; Piaget 1977; Piaget 2001, chap. I)

Concepts, relations, and propositions originate in the activity of the subject and are parts of the assimilatory schemes that are accommodated to external reality and bring intellectual adaptation to expression. In the embryology of intelligence, reasoning on concrete objects and their representations involving concepts and relations precedes hypothetico-deductive reasoning with propositions (e.g., Piaget 1957; Piaget and Grize 1972; Piaget 2001, chap. V). However, logicism shaped Piaget’s interpretation of logic. Consequently, propositional logic was an autonomous system of theorems founded solely on axioms and rules of inference. Although logicians at that time tacitly accepted some correspondence between calculi of classes, relations and propositions, propositional logic usually served as the foundation of calculi with classes and relations, despite the ambiguity of the correspondence. To Piaget’s mind, the theory of reasoning inherent in contemporary logic was thus in opposition to an embryology of reason (Piaget and Grize 1972, vol. 15, sec. 27).

The foundational approach to logic not being a suitable midwife for the birth of an embryology of reasoning, Piaget sought a genealogical rather than a foundational connection between propositional reasoning, on the one hand, and what he considered to be the more elementary manifestations of reasoning in classifications and seriations, on the other. In other words, he sought the ‘natural order of construction or descent’ (Piaget and Grize 1972, 15:21 my translation) of propositional reasoning and reasoning with classes and relations. He proceeded by adopting familiar structures from the storehouse of mathematics and adapting them to fit the empirical evidence of psychological experiments, and, in this paper, I outline his solution before characterising his application of mathematics to a socio-psychological reality and considering its significance for the unreasonable effectiveness of mathematics. In other words, I address both questions raised by Shapiro on the applicability of mathematics by illustrating how mathematics was actually applied in science with a psychological case study and by elaborating on its implications for the explanation of the applicability of mathematics to science in the wider context of a biological epistemology.

3 Two Manifestations of Human Reasoning

Concepts, relations, and propositions are involved in reasoning. Although, particular examples from each category can be considered in isolation, they manifest naturally in systems. Classifications, for example, systematically connect concepts with each other, and Piaget characterised them as follows:

3.1 Classifications

Classifications, known already in Greek antiquity as Porphyrian trees, are found today in biological taxonomies, genealogies, etc.

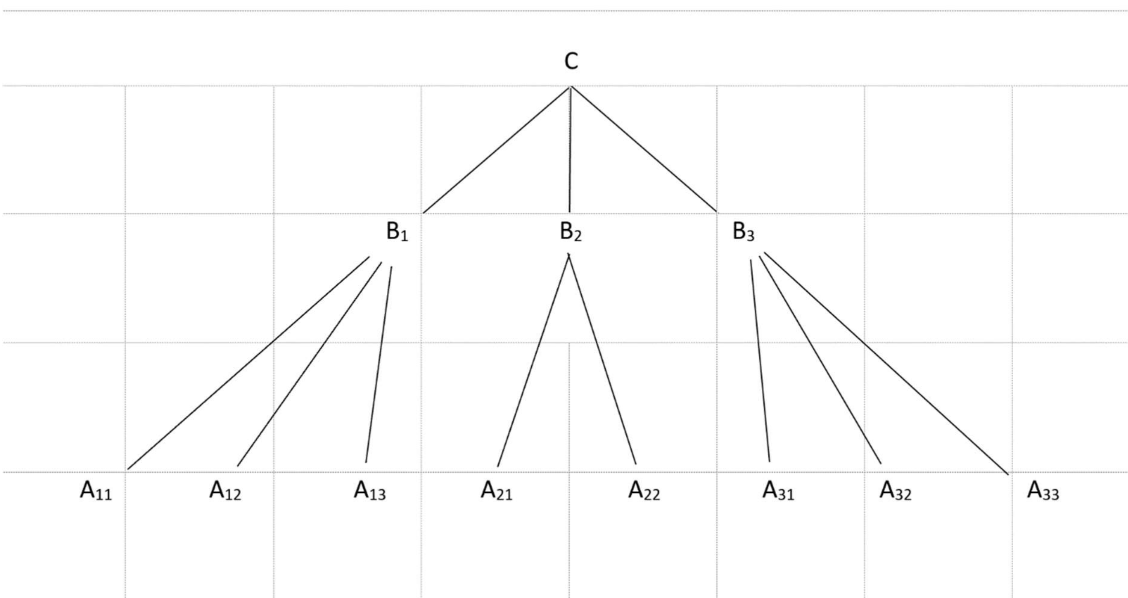

Figure 1 Patriarchal Genealogical Tree. In this genealogical tree, C is the father of B_1, B_2, and B_3, who are fathers of A_{11}, A_{12}, & A_{13}; A_{21}, & A_{22}; and A_{31}, A_{32}, & A_{33}, respectively. The sons of the As are the great-grandsons of C, but they are not indicated in the diagram. (Adaptation of Piaget and Grize 1972, vol. 15, figs. 3 & 5)

Piaget (Piaget and Grize 1972, vol. 15, sec. 9) highlights five salient characteristics of classifications:

1. Every class except the reference class and the most elementary class chosen is included in a class of superior order and includes classes of lower order. In the family tree in Figure 1, for example, the sons of A_{11} are grandsons of B_1 and great-grandsons of C; the grandsons of B_1 are therefore included in the great-grandsons of C, on one hand, and include the sons of A_{11} on the other.
2. Classes belonging to the same hierarchical level are disjoint. For example, the grandsons of B_1 are comprised of the sons of A_{11}, A_{12} and A_{13}; however, the subclasses of grandsons they form are disjoint since they cannot have two biological fathers.
3. Classes of the same rank can only be characterized dichotomously, that is, by the presence or absence of given properties. From the perspective of B_1, the sons of A_{11}, A_{12}, etc. are all grandsons, for example; however, from the perspective of A_{11} the same grandsons are either his sons or not, but not both. The grandsons of B_1 are thus

divided into two dichotomous classes by the presence or absence of the property 'son of A_{11}'. Despite all being grandsons of B_1, the sons of A_{12}, A_{13}, etc. are therefore distinct from the sons of A_{11} by virtue of not having the property 'son of A_{11}'.

4. Every item in the classification is nested in a series of classes belonging to the successive levels of the hierarchy. Clearly, in the patriarchal family tree, each male has a lineage even if he does not have any male offspring of his own. Going backwards through the generations, he must have a biological father, who also had a biological father, etc. Each male in the patriarchal family tree is therefore simultaneously someone's son, grandson, great-grandson, great-great-grandson, etc.
5. A classification orders the things classified, but only partially. The great-grandsons of C, for example, can be ordered according to degrees of kinship. The sons of A_{11} are also grandsons of B_1 and great-grandsons of C. From the perspective of A_{11}, then, the sons of A_{11} are brothers; the grandsons of B_1 but not sons of A_{11} are their first cousins; the great-grandsons of C but not grandsons of B_1, second cousins; etc. The intervention of ancestry, the vertical component of the family tree, therefore, imposes an order—reflected in degrees of kinship—on the otherwise unordered members of each generation, the horizontal component of the family tree. However, the order is only partial since there is no order among the brothers, first cousins, second cousins, etc. themselves.

The systems of classes constituting classifications structure discrete objects and their representations according to likenesses and differences. Although the objects and the properties classified are given, the classification is not. It is engendered by the subject acting on concrete objects and representations.

3.2 Propositional Reasoning

Piaget describes how adolescents reason hypothetic-deductively when attempting to grasp connections between phenomena as follows:

> Let us take as an example the implication $p \supset q$, and let us imagine an experimental situation in which a child between twelve and fifteen tries to understand the connections between phenomena which are not familiar to him but which he analyses by means of the new propositional operations rather than by trial and error. Let us suppose then that he observes a moving object that keeps starting and stopping and he notices that the stops seem to be accompanied by lighting of an electric bulb. The first hypothesis he will make is that the light is the cause (or an indication of the cause) of the stops, or $p \supset q$ (light implies stop). There is only one way to confirm the hypothesis, and that is to find out whether the bulb ever lights up without the object stopping, or $p\bar{q}$ ($p\bar{q}$ is the inverse of or negation of $p \supset q$). But he may also wonder whether the light, instead of causing the stop, is caused by it, or $q \supset p$ (now the reciprocal and not the inverse of $p \supset q$). To confirm $q \supset p$ (stop implies light), he looks for the opposite case which would disconfirm it; that is, does the object ever stop without the light going on? This case, $\bar{p}q$, is the inverse of $q \supset p$. The object stopping every time the light goes on is quite compatible with its sometimes stopping for some other reason. Similarly, $p\bar{q}$, which is the inverse of $p \supset q$, is also the correlative of $q \supset p$. If every time there is a stop the bulb lights up ($q \supset p$), there can be lights without stops. Similarly, if $q \supset p$ is the reciprocal of $p \supset q$, then $\bar{p}q$ is also the reciprocal of $p\bar{q}$. (Inhelder and Piaget 1969, 139)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$p\cdot q$	-	$p\cdot q$	-	-	$p\cdot q$	$p\cdot q$	-	$p\cdot q$	-	$p\cdot q$	-	$p\cdot q$	-	$p\cdot q$	-
$p\cdot\bar{q}$	-	$p\cdot\bar{q}$	-	$p\cdot\bar{q}$	-	-	$p\cdot\bar{q}$	$p\cdot\bar{q}$	-	-	$p\cdot\bar{q}$	$p\cdot\bar{q}$	-	-	$p\cdot\bar{q}$
$\bar{p}\cdot q$	-	$\bar{p}\cdot q$	-	$\bar{p}\cdot q$	-	$\bar{p}\cdot q$	-	-	$\bar{p}\cdot q$	-	$\bar{p}\cdot q$	-	$\bar{p}\cdot q$	$\bar{p}\cdot q$	-
$\bar{p}\cdot\bar{q}$	-	-	$\bar{p}\cdot\bar{q}$	$\bar{p}\cdot\bar{q}$	-	$\bar{p}\cdot\bar{q}$	-	$\bar{p}\cdot\bar{q}$	-	$\bar{p}\cdot\bar{q}$	-	-	$\bar{p}\cdot\bar{q}$	-	$\bar{p}\cdot\bar{q}$
$p * q$	o	$p \vee q$	$\bar{p}\cdot\bar{q}$	$p\|q$	$p\cdot q$	$p \supset q$	$\overline{p \supset q}$	$q \supset p$	$\overline{q \supset p}$	$p \equiv q$	$p\, w\, q$	$p[q]$	$\overline{p[q]}$	$q[p]$	$\overline{q[p]}$

Table 1 16 Logical Operators of Interpropositional Operations. The true conjunctions comprise the columns of this table, and the columns are set out in pairs comprising the full complement of four conjunctions. Connecting the conjunctions in each column disjunctively gives rise to the disjunctive normal form of the logical operators in the bottom row. Except for *, *w*, p[q], and q[p] the binary operators are familiar. * represents the complete affirmation; w, exclusive disjunction; and p[q] as well as q[p] are affirmations of p and q conjointly with either q̄ or p̄, respectively.[2] (Based on Piaget and Grize 1972, 15:214: Table 100)

More generally, given any two observable phenomena represented by propositions p and q, it is not immediately obvious how they are in fact related to each other. Nevertheless, four associations of these phenomena can occur, represented by the conjunctions p·q, p̄·q, p·q̄ and p̄·q̄, and are directly accessible to observation. Individually, each observed association does not allow the relationship between the phenomena to be determined. Observation of p and q always occurring together, p·q, for example, means that p and q could be related in any of 8 ways (p·q occurs in 8 columns in Table 1). Through observation of all four possible associations of the phenomena, on the other hand, the relationship can be uniquely determined. Observation of associations p·q and p̄·q̄ occurring without exception but no cases of either p̄·q or p·q̄, for example, shows that the phenomena represented by p and q are equivalent; whereas observation of associations p·q, p̄·q, and p·q̄ but no cases of p̄·q̄ means that p∨q (see Table 1). Conversely, if p⊃q is postulated, p·q, p̄·q, and p̄·q̄ are observations that would support this hypothesis, whereas p·q̄ would falsify it. A framework of possible relations between phenomena based on combinations of observations thus mediates the discovery of the factual relationship between phenomena, and the development of this framework demarcates propositional from concrete reasoning (Inhelder and Piaget 1958, chaps. 16–17; Smith 1987, sec. Piaget's Logic: A Constructivist Interpretation).

The developmental problem is then the transition from concrete to propositional reasoning; in particular, the 'reversal of the direction of thinking between reality and possibility in the subjects' method of approach' (Inhelder and Piaget 1958, 251), i.e., how a framework of possible relations subsuming phenomena develops from an ability to classify, seriate, etc., these phenomena.

[2] Piaget pursued constructivist ends when modelling propositional reasoning and found it convenient to use the symbolism of propositional logic; however, he stresses that the symbols do not have the familiar logical meanings (Piaget and Beth 1966, 12:180–1; Apostel 1982). The formalism was only partially revised in the second edition to bring it more in line with logical conventions (Piaget and Grize 1972, 15:XVI; cf. Seltman and Seltman 1985). To facilitate referencing, I adopt Piaget's notation for the logical operators although it is partially antiquated and idiosyncratic.

4 Operations of Thought

Piaget understood propositions as meaningful categorical statements that are true or false, and writes p, q, r, … and $\overline{p}$, $\overline{q}$, $\overline{r}$, … for the affirmations and their negations. With respect to propositions he defines two types of operation. One type consists in combining propositions with others to form new propositions with well-defined truth conditions; for example, the conditional $p \supset q$ (if p then q) will be a new false proposition in the one case when p is true and q false; the conjunction $p \cdot q$ (p and q) will be a new true proposition only if p and q are both true, etc. Since these operations combine propositions as wholes using propositional connectives, he denotes them 'interpropositional operations' (Piaget and Grize 1972, 15:34). Another type of operation decomposes propositions into their constituent parts and transforms them into new propositions by modifying these parts; for example, in a proposition such as 'this rose is red', the subject 'this rose' can by replaced by other terms ('flag', 'all the roses', etc.), or the predicate 'red' by others ('yellow', 'black', etc..), or the connection between subject and predicate 'is' can be modified ('this rose excels this one in beauty', etc.). Since these operations are performed on the innards of propositions, Piaget denotes them 'intrapropositional operations'(Piaget and Grize 1972, 15:35).

I adopt Piaget's terminology in this paper; however, merely denoting their effects on propositions and their parts does not make any commitment to the psychological nature of operations. Psychologically, Piaget characterises operations as follows:

> operations are actions which are internalizable, reversible, and coordinated into systems characterized by laws which apply to the system as a whole. They are actions, since they are carried out on objects before being performed on symbols. They are internalizable, since they can also be carried out in thought without losing their original character of actions. They are reversible as against simple actions which are irreversible. In this way, the operation of combining can be inverted immediately into the operation of dissociating, whereas the act of writing from left to right cannot be inverted to one of writing from right to left without a new habit being acquired differing from the first. Finally, since operations do not exist in isolation they are connected in the form of structured wholes (Piaget 1957, 8; Piaget and Beth 1966, 12:172; Piaget 1971b, 21–2; see also Piaget and Grize 1972, 15:55; Piaget 2001, chap. 2)

4.1 Intrapropositional Operations

As already mentioned, classifications are characteristic of reasoning with concrete objects and their representations. Piaget's intention was to model such reasoning, and, to this end, he turned to structures in the storehouse of mathematics. Since operations of thought are by nature reversible, each operational transformation of an object can be returned to its original state by further operations. Having direct, inverse and identity operations, groups therefore suggest themselves immediately as potential candidates for modelling the structure of intrapropositional operations, and Piaget turned in particular to the additive groups of disjunctive parts and the group of equivalences derived by B. A. Bernstein from the algebra of Boolean classes first to assess how adequately they correspond to such classifications.

4.1.1 **Groups**

In the powerset P(E) of E, Piaget (Piaget and Grize 1972, 15:88–90) defined the operations $AwB=df(A\cup B)-(A\cap B)$ and $A\equiv B=df(A\cap B)\cup(\overline{A}\cap\overline{B})$, where A and B are subsets of E, and he showed that they form groups. Being reversible, the operations of these groups correspond to the intrapropositional operations; however, classifications structure objects qualitatively according to *genus proximum et differentia specifica*. The operations of the groups on the other hand transform any two classes into another class of the power set. Since the classes are being treated as if they were simply collections of unrelated objects, the collections engendered do not necessarily have any common properties and may not therefore constitute classes of a classification. {Sons of A11}w{grandsons of B1}=B1−A11, i.e., {first cousins of A11}, for example, transforms classes of the classification into others; whereas {sons of A11} w {sons of A21}forms a subset of the powerset, but it does not constitute a class of the classification because none of the properties structuring the genealogy qualitatively unite these elements into a single class.

Another inadequacy of these groups is that the operations only deal with the common or non-common parts of classes but not both together. Hierarchal inclusions of classes, on the other hand, are characteristic of classifications; each class therefore has both common and non-common parts with its subclasses. Uniting the sons of A11 with the grandsons of B1, for example, therefore does not alter the latter since they are already included; similarly, uniting them with themselves leaves the sons of A11 unaltered. However, $A11wA11=A\cup A-A\cap A=\emptyset$ and A11wB1=B1−A11, i.e., first cousins, and $A\equiv B=A$ when $A\subset B$, although $A\equiv A=A$.

4.1.2 **Lattices**

While the groups reflect the reversibility of operations of thought, they do not also reflect the specific hierarchical inclusions of classes since they only operate on either the common or non-common parts but not both. A lattice, on the other hand, can be characterised either as a partially ordered set $(L, \leqq)$, in which any two elements of L have both a least upper bound (join, supremum) and a greatest lower bound (meet, infimum), or algebraically as a set of elements equipped with two binary operations $(L, \vee, \wedge)$ that obey commutative, associative and absorptive laws (Rutherford 1966, secs. 3 & 4). Like lattices, classifications also impart partial order to the objects classified via the hierarchy of nesting classes, and uniting classes included in each other obeys the absorption and idempotent laws. Aspects of both characterisations of lattices thus correspond to classifications, and Piaget (Piaget and Grize 1972, 15:90–92) defines the operations of a lattice and draws attention to its inadequacies as follows:

For classes A and B, the Supremum = $dfA\cup B$, and the infimum = $dfA\cap B$.

Whereas the supremum is defined for any classes of a classification, the infimum is defined for pairs of classes included in each other but not for the dichotomous subclasses. The infimum of B_1 and A_{11}, and B_1 and A_{11}', for example, are A_{11} and A_{11}'; however, for disjoint classes A_{11} and A_{11}', A and B_1', etc. no infimum exists. Classifications therefore correspond better to join-semilattices rather than lattices; however, even this correspondence is not ideal. Although each pair of classes has a supremum, many pairs of classes can have the same supremum, and similarly for the infimum if it exists. The great-grandsons of C are, for example, the supremum of the grandsons of B_1 and the grandsons of B_1's brothers, B_1';

however, it is also the supremum of B_1 and C, B_1' and C, as well as B_1 and A_{21}, B_1 and A_{21}', B_2 and A_{11}, etc. The transformation being many-to-one, an inverse operation that would return the original classes from the infimum or supremum cannot therefore be defined. Hence the reversibility of operations of thought is not reflected in the lattice operations.

In summary, the groups reflect the reversibility of intrapropositional operations; however, they do not adequately account for the qualitative inclusions of classes characteristic of classifications. Join-semilattices, on the other hand, reflect the latter but cannot account for the reversibility of intrapropositional operations. According to Piaget, 'The problem is then to characterize a structure that reconciles the reversibility proper to the group and the system of limited inclusions proper to the lattice.' (Piaget and Grize 1972, 15:92 my translation) The grouping is Piaget's solution to this problem and it fulfils both requirements as follows: 'One can indeed conceive of a 'grouping' either as a lattice made reversible by virtue of a play of dichotomies or hierarchical complementaries (A and A', B and B', etc.)[3], or as a group whose mobility is restricted by the intervention of inclusions involving special identities $A \cup A=A$ and $A \cup B=B$, as well as a principle of contiguity.' (Piaget and Grize 1972, 15:92 my translation)[4]

4.2 Interpropositional Operations

According to Piaget, propositional reasoning also has an operational nature. The first step in unveiling its structure is the discovery of all meaningful connections between two propositions, i.e., all binary operators with distinct logical meanings. Mathematically, there are 2^{2^2} combinations of true or false affirmations and negations of two bivalent propositions; however, Piaget (1972, 15:215) points out that mathematical combinations do not necessarily have logical significance. Via corresponding classes, he therefore first demonstrates that each of the disjunctive combinations of conjunctions in the columns of Table 1 represents a distinct, logically meaningful operator. However, reasoners form hypotheses concerning the relationship between phenomena and infer observable consequences, while remaining able to change their minds in light of unexpected, perhaps contradictory, evidence. Rather than being static combinatorics, reasoning is thus the ability to move dynamically from one operator to another, and operations of thought are the source of its dynamism.

4.2.1 Essential Operatory Mechanisms

Along with negation, disjunction, conjunction, conditional and biconditional are widely accepted as the basic propositional connectives of propositional logic. Nevertheless, it is not a rigid convention, and logicians have shown enormous ingenuity in reducing their number to as few as one. In using the disjunctive normal form, Piaget essentially reduces propositional connectives to combinations of disjunctions and conjunctions. Favouring particular logical operators over others is to a large extent a question of preferences, and practical reasons for choosing the normal forms abound. However, Piaget's motivation is geared toward

[3] Piaget is referring to a hierarchy of *relative* complementaries here arising from nested inclusions of classes $A \supset B \supset C$…, namely, $A \cup A'=B$, $B \cup B'=C$…

[4] Several attempts have been made at formalizing the grouping (Piaget and Grize 1972, 15:92 footnote 1); however, they were felt to be wanting (Piaget and Grize 1972, 15:XIV–XVI). The formalisations of the grouping are not set out in this paper since the process of applying mathematics rather than the specific details of the result are directly relevant.

determining the ‘essential operatory mechanisms’ (Piaget and Grize 1972, 15:253) inherent in propositional reasoning.

Via the normal forms, Piaget (Piaget and Grize 1972, vol. 15, sec. 39 C) brought to light a unified, reversible system of transformations operating on the 16 logical operators set out in Table 1. Just as the relation between phenomena p and q changes as observations of new associations of the phenomena occur, disjunctive compositions of the conjunctions pq, $\bar{p}q$, $p\bar{q}$ and $\bar{p}\bar{q}$ transform logical operators relating p and q into each other; for example, $(p{\equiv}q){\vee}\bar{p}q{=}p{\supset}q$; $(pwq){\vee}\bar{p}\bar{q}{=}p|q$; etc. For Piaget, the direct operation composes conjunctions disjunctively with logical operators, and it generates the 16 logical operators; for example, $o{\vee}pq{=}pq$; $pq{\vee}\bar{p}q{=}p[q]$; $(pq{\vee}\bar{p}q){\vee}\bar{p}\bar{q}{=}p{\supset}q$; etc. However, on its own the direct operation does not allow all operators to be transformed into each other; the outcome of disjunctively composing any of the conjunctions with the complete affirmation is, for example, again the complete affirmation. Nevertheless, each operator has a dual expression; for example, $p \supset q = pq \vee \bar{p}q \vee \bar{p}\bar{q} = \overline{p\bar{q}}$; $p \vee q = pq \vee \bar{p}q \vee p\bar{q} = \overline{\bar{p}\bar{q}}$; etc. Furthermore, $pq \vee \bar{p}q \vee \bar{p}\bar{q} = \top \cdot \overline{p\bar{q}}$; $pq \vee \bar{p}q \vee p\bar{q} = \top \cdot \overline{\bar{p}\bar{q}}$, where ⊤ is the shorthand for the complete affirmation. Just as ruling out the possibility of observing an association of phenomena changes the hypothesized relation between them, conjunctively composing negations of the conjunctions transforms logical operators by eliminating conjunctions in the disjunctive normal form. Via the dual expression Piaget thus defined a second operation, the inverse operation, as conjunctions of negations of the four conjunctions. With both direct and inverse operations in combination, all 16 binary operators can thus be transformed into each other.

Moreover, just as it is possible to find counterexamples for any relation hypothesized, a judicious implementation of direct and inverse operations transforms a logical operator into the complete negation; for example, $pq \cdot \bar{p}\bar{q} = o$; $(p \supset q) \cdot p\bar{q} = \overline{p\bar{q}} \cdot p\bar{q} = o$, etc. In fact, in Table 1, the columns are organised in complementary pairs with respect to the complete complement of four conjunctions; and, whilst the complete affirmation is the outcome of composing the logical operators in these pairs of columns disjunctively via the direct operation, the complete negation is the outcome of composing them via the inverse operation. Excluded middle and non-contradiction, two of the laws of thought, thus come to expression in these operations; and the law of identity, the third law of thought, is reflected in the general identity operation $\vee o$, which Piaget defined as the transformation that leaves any logical operator unaltered on its application and is composed of the direct and inverse operations.

Like operations of groups, the direct, inverse and identity operations are reversible. However, $pq{\vee}pq{=}pq$; $pq{\vee}[pq{\vee}p\bar{q}]{=}[pq{\vee}p\bar{q}]$; $pq{\cdot}(p{*}q){=}pq$; etc. are also compositions that leave the operator unchanged although they are not the general identity. Moreover, they limit associativity. Associativity works for disjunctions of the conjunctions $[pq{\vee}p\bar{q}]{\vee}\bar{p}q{=}pq{\vee}[p\bar{q}{\vee}\bar{p}q]$, for example, but $[pq \vee pq] \cdot \bar{p}\bar{q} \neq pq \vee [pq \cdot \bar{p}\bar{q}]$, since $pq \cdot \bar{p}\bar{q} = (o)$, whereas $pq{\vee}(o){=}pq$. Piaget called such compositions ‘special identities’, and they are reminiscent of absorptions and idempotence in lattices. In short, there are group-like and lattice-like operations in the essential operative mechanisms inherent in the system of interpropositional transformations of logical operators. However, operations from these two different mathematical structures are not good bedfellows as the difficulties with associativity reveal. Piaget therefore sought to reduce the system of interpropositional operations to the operations of either groups or lattices.

4.2.2 **Groups**

By translating the intrapropositional operations of the groups already considered in the context of classifications into interpropositional operations, Piaget (Piaget and Grize 1972, 15:306–10) assessed the reduction of the system of interpropositional operations to the operations of groups. In particular, he used the exclusive disjunction (w) to translate into interpropositional operations the union of non-common parts of two classes and the equivalence (≡) to translate the union of their common parts, and he showed that these operations form groups. However, individually these groups do not adequately correspond to the whole system of interpropositional operations. In essence, the problem is that the operands of each individual group are too limited. All possible combinations of the four conjunctions $p{\cdot}q$, $\bar{p}{\cdot}\bar{q}$, $\bar{p}{\cdot}q$ and $p{\cdot}\bar{q}$ constitute the 16 logical operators, and Piaget divides them into common—$p{\cdot}q$, $\bar{p}{\cdot}\bar{q}$—and non-common—$\bar{p}{\cdot}q$, $p{\cdot}\bar{q}$—parts of two propositions. Some of the 16 logical operators are composed of both common and non-common parts; for example, the conditional $p{\supset}q{=}p{\cdot}q{\lor}\bar{p}{\cdot}q{\lor}\bar{p}{\cdot}\bar{q}$, its converse $p{\subset}q{=}\ p{\cdot}q{\lor}p{\cdot}\bar{q}{\lor}\bar{p}{\cdot}\bar{q}$ and disjunction $p{\lor}q{=}p{\cdot}q{\lor}p{\cdot}\bar{q}{\lor}p{\cdot}\bar{q}$. However, such operators cannot be operands of either the group defined by exclusive disjunction, $pwq{=}\bar{p}{\cdot}qwp{\cdot}\bar{q}$, or the group defined by equivalence $p{\equiv}q{=}p{\cdot}qw\bar{p}{\cdot}\bar{q}$ since either the non-common or the common parts of two propositions are in play but not both together. In short, exclusive disjunction and equivalence are operations that constitute two separate groups, but individually the groups cannot account for the whole system of interpropositional logical operators because their operands are limited to either the common or non-common parts of two propositions.

4.2.3 **Ring**

Although not able individually, the groups defined by exclusive disjunction and equivalence might be able to account for all the logical operators in combination; Piaget (Piaget and Grize 1972, vol. 15, sec. 36 IV) therefore investigated a ring involving the operations of these groups, but he rejected it because of a lack of operational unity as follows.

Piaget showed that (w) and (·) as the additive and multiplicative binary operations, respectively, and negation (‾) as the unary operation form a ring. Furthermore, all 16 logical operators of interpropositional operations can be expressed in terms of these operations; for example,

$(p{\lor}q){\leftrightarrow}(pwq)w(p{\cdot}q)$

$(p|q){\leftrightarrow}\ (pwq)w(\bar{p}{\cdot}\bar{q})$

$(p{\supset}q){\leftrightarrow}(pw\bar{q})w(\bar{p}{\cdot}q)$

$(q{\supset}p){\leftrightarrow}(pw\bar{q})w(p{\cdot}\bar{q})$

etc.

However, the operations of this ring do not truly reflect the unity inherent in the system of transformations of interpropositional operations:

1. Exclusive disjunction (w) and conjunction (·) are not always associative. For example, $(p{\cdot}\bar{p})wq{\leftrightarrow}(owq){\leftrightarrow}q$ applies regardless of the relationship between p and q; however, the outcome of $p{\cdot}(\bar{p}wq)$ depends on how p and q are related: if p and q are completely disjoint, ($\bar{p}$

wq)=(p≡q) and qwq↔o, therefore p·(o)↔o; if p and q have common and non-common parts (p∨q), then p·(p·q); if p⊃q and pwq≡p; then [p·($\overline{p}$wq)]↔p, etc.

2. Equivalences are not always preserved under the same transformations. In particular, (wp) and (·$\overline{p}$) or (·p) and (w$\overline{p}$) are pairs of direct and inverse operations; however, reversing pwp using (·$\overline{p}$), for example, would retrieve p but for the equivalent expression p·$\overline{p}$ (pwp↔p·$\overline{p}$), the outcome is (p·$\overline{p}$)·$\overline{p}$=o, so that p↔o. Similarly, reversing p·p↔p using (w$\overline{p}$) would give pw$\overline{p}$=U hence the absurdity p↔U.

3. Rather than a unique identity operation, two identity operations are inherent in the pairs of operations (wp) and (·$\overline{p}$) or (w$\overline{p}$) and (·p). On the one hand, (pwp)↔o and (pwo)↔p; (o) is therefore the identity operation for the group of non-common parts; on the other hand, (p·p)w($\overline{p}$·$\overline{p}$)↔U and p·U↔p; U is therefore the identity operation for the group of common parts. Although a proposition p is left unchanged by adding nothing (pwo) or extracting the part it has in common with the whole system (p·U), Piaget argued that the equivalence between (wo) and (·U) is only apparent and partial.

4.2.4 **Lattice**

Besides reversibility, the special identities are also seminal characteristics of the system of interpropositional operations. Inclusion relations, q≧p, and self-inclusions, p∨p=p and p·p=p, are characteristics of lattices. In contrast to groups, the operations of the lattice therefore correspond to the special identities in the system of interpropositional operations. Changing tack, Piaget (Piaget and Grize 1972, vol. 15, sec. 37) therefore used non-exclusive disjunction (∨) and (·) conjunction as the fundamental binary operations to define a lattice, and he assessed the adequacy of this lattice as a model of the system of interpropositional operations as follows.

The lattice describes the external contours of the structure inherent in the system of interpropositional operations without entering into the internal details of the transformations themselves. One essential detail of rationality is reversibility. Whilst the lattice contains the negations of all propositions, it does not contain the negation of operations.

The negation of an operation is its inverse; e.g., N(p⊃q)= $\overline{(p \supset q)}$ = p · $\overline{q}$, and (p⊃q)·(p·$\overline{q}$)=o. Although it is a single operation, it can be analysed into two components, which also constitute operations in their own right: the reciprocal operation, R, corresponds to leaving the operators unaltered but substituting affirmations for negations and vice versa, e.g., R(p⊃q)=$\overline{p}$⊃$\overline{q}$=q⊃p; on the other hand, the correlative, C, leaves the affirmations and negations unaltered while substituting conjunctions for disjunctions and vice versa; e.g., C(p⊃q)=C($\overline{p}$∨q)=$\overline{p}$·q. These are the operations that are instrumental in determining the causal relationship between phenomena (see 3.2); however, the lattice is only partially reversible since the operations (·) and (∨) support the correlative but not negation and reciprocal.

A consequence of the limited reversibility of the operations of the lattice is that it is not possible to return unequivocally to any one of the two constituent propositions p or q once they are composed into an upper (p∨q) or lower bound (p·q). The operation (p∨q) composing p and q into the supremum, for example, does not specify unequivocally the actual relationship between p and q since only one of the three conjunctions (p·q), (p·$\overline{q}$) or ($\overline{p}$·q) need be true. The following possibilities therefore exist: a) exclusive disjunction when

$(p\cdot\overline{q})w(\overline{p}\cdot q)$; b) inclusive disjunction, when $(p\cdot q)\vee(p\cdot\overline{q})\vee(\overline{p}\cdot q)$; c) implication $(p\supset q)$, when $(p\cdot q)\vee(\overline{p}\cdot q)$; d) implication $(q\supset p)$, when $(p\cdot q)\vee(p\cdot\overline{q})$; e) equivalence $(p\equiv q)$, when $(p\cdot q)$. $(p\vee q)\cdot\overline{p}$. An inverse operation consisting in negating p would therefore effect multiple disparate outcomes: either $(\overline{p}\cdot q)$ in cases a), b), and d) or (o) in cases c) and e). Recovering either of the propositions constituting the infimum $p\cdot q$ through an inverse operation would be equally equivocal since it is only the common part of two propositions, which could be related in 8 different ways. In short, the lattice, in contrast to the groups and ring, models the special identities, but it does not typically reflect the full reversibility characteristic of rationality.

4.2.5 **Grouping of Interpropositional Operations**

The crux of the problem is to unite the inclusions and self-inclusions due to special identities with reversibility. However, the structures Piaget has so far considered can only adequately model one of these essential operatory mechanisms at a time. Nevertheless, Piaget (1972, 15:317) drew attention to a special case of the lattice in which a univocal inverse is possible. If for propositions p and q the supremum and infimum are $(p\vee q)\leftrightarrow q$ and $(p\cdot q)\leftrightarrow p$, respectively, there is an inclusion relation between p and q, namely, p is included in q $(q\geqq p)$. Moreover, $(p\vee q)$ is dichotomously partitioned into the common and non-common parts of p and q, i.e., $(p\cdot q)\vee(\overline{p}\cdot q)$. $p\cdot q$ and $\overline{p}\cdot q$ are thus relative complements in $(p\vee q)$, and, in contrast to the general lattice, negating one of the conjunctions contained in the supremum would indeed constitute an unambiguous inverse operation, namely, its relative complement. For example, negating $\overline{p}\cdot q$ in $(p\vee q)$ is equivalent to finding the common part of $(p\vee q)$ and $\overline{(\overline{p}\cdot q)}$, i.e., $(p\vee q)\cdot\overline{(\overline{p}\cdot q)}\leftrightarrow(p\cdot q)$, which is the relative complement of $\overline{p}\cdot q$ in $(p\vee q)$. Conversely, negating $p\cdot q$ in $(p\vee q)$ is equivalent to finding the common part of $(p\vee q)$ and $\overline{(p\cdot q)}$, i.e., $(p\vee q)\cdot\overline{(p\cdot q)}\leftrightarrow(\overline{p}\cdot q)$, which is the relative complement of $p\cdot q$ in $(p\vee q)$. Since $(p\vee q)=((p\cdot q)\vee(\overline{p}\cdot q))=((\overline{p}\cdot q)\vee(p\cdot q))$, $\vee(\overline{p}\cdot q)$ and $\cdot\,\overline{(\overline{p}\cdot q)}$ as well as $\vee(p\cdot q)$ and $\cdot\,\overline{(p\cdot q)}$ are pairs of reversible operations, and this special case provided Piaget with the road map for uniting the essential operatory mechanisms into a single grouping (Piaget and Grize 1972, vol. 15, sec. 38) as follows:

1. The direct operation is the disjunctive composition ($\vee$) of conjunctions $p\cdot q$, $\overline{p}\cdot q$, $p\cdot\overline{q}$, and $\overline{p}\cdot\overline{q}$; e.g., $(o)\vee(p\cdot q)$; $(p\cdot q)\vee(p\cdot\overline{q})$; etc.
2. The inverse operation is a negation of these conjunctions composed conjunctively; e.g., $(p\vee q)\cdot\overline{(\overline{p}\cdot q)}$; $(q\supset p)\cdot\overline{(p\cdot\overline{q})}$;etc.
3. The general identity operation $\vee o$ leaves the elements it is composed with unaltered, e.g., $(p\cdot q)\vee(o)=(p\cdot q)$, and it is the product of the direct and inverse operations; e.g., $(p\cdot q)\cdot\overline{(p\cdot q)}=o$.
4. The special identities are:
   a. Tautology: $(p\cdot q)\vee(p\cdot q)=(p\cdot q)$
   b. Reabsorption: $(p\cdot q)\vee[(p\cdot q)\vee(p\cdot\overline{q})]=[(p\cdot q)\vee(p\cdot\overline{q})]$
   c. Absorption: $(p\cdot q)\cdot(p*q)=(p\cdot q)$
5. Associativity is limited to disjoint elements after tautifications and absorptions due to the special identities. (Piaget and Grize 1972, 15:335)

Piaget summarises his attempts to reconcile the essential operatory mechanisms as follows:

> The structured whole specific to bivalent logic[5] is therefore neither the group, too narrow to embrace operations of self-inclusion, nor the lattice, too broad to account for reversibility, but the grouping that reconciles the reversibility with the inclusions of parts in the whole. (Piaget and Grize 1972, 15:319 my translation).

Piaget thus managed to reconcile all essential operatory mechanisms in the interpropositional grouping; moreover, his model is based on a reversible special case of the lattice, namely a complemented distributive lattice, aka a Boolean algebra (Rutherford 1966, sec. 12). In other words, Piaget's model of the essential operatory mechanisms of propositional reasoning is based on Boolean algebra.

5 Piaget's Application of Logic and Mathematics to Psychology

Having characterised the grouping, its development, and the role it plays in the embryology of reasoning, it is time to consider it as an application of mathematics. Science being a human endeavour, an application of mathematics not only depends on the empirical content but also the situation. For convenience, I therefore divide my reflections into general historical and biographical considerations before analysing in detail Piaget's application of mathematics to psychological content.

5.1 Context

Piaget was not a logician, and his familiarity with contemporary developments in logic were limited. According to Grize:

> Except for Russell—and for all that not the author of the Principia mathematica but of the Introduction to mathematical philosophy in its French version—he hardly knows more than Edmond Goblot's (1918) and Charles Serrus' (1945) works. Those distinguished scholars wrote manuals but never treatises for logicians. Moreover, we must mention a phenomenon that is typical of French-speaking countries, and France specifically, i.e. their late development of mathematical logic. (Grize 2013, 149)

Piaget was thus acquainted with a particular interpretation of logic, an interpretation that viewed logic as a universal language rather than an algebra (van Heijenoort 1967; van Heijenoort 1992), was developed for the foundations of mathematics, and highlighted axiomatisation (Read 1995; Shapiro 2000, pt. III). From the viewpoint of biological epistemology, this interpretation of logic represented a reversal of the embryology of reasoning. Whereas the calculi of classes and relations were founded on propositional logic, reasoning with classes and relations is more elementary than reasoning with propositions and occurs much earlier in the embryology of reasoning. The challenge he saw was therefore to explain the development of more advanced forms of reasoning from more elementary ones, whilst also accounting for a reversal of the roles in the context of justification. In other words, his problem was the reconciliation of the foundational priority of propositional logic with its developmental posteriority. Piaget's limited understanding of contemporary logic thus framed

[5] For Piaget (Piaget and Grize 1972, 15:20), bivalent logic as '*the formal theory of deductive operations*' is based on the interpropositional operations of thought modelled by the grouping.

the problem for the embryology of reasoning; however, his approach to the problem has biographical origins.

Piaget originally trained as a biologist, and, to establish a developmental relationship between reasoning with relations, classes and propositions, he proceeded in analogy to comparative anatomy (Piaget 1950, vol. 1, sec. Introduction §2). Seriations, classifications, and propositions are ostensibly disparate kinds of reasoning, yet Piaget showed that they share a 'common anatomy', the grouping the essential operatory mechanisms form. The grouping is then to the different kinds of reasoning what a common anatomy is to different species. By proceeding in analogy to zoology, Piaget thus cleared a view on the natural descent of reasoning, which culminated in him showing how the grouping of interpropositional operations translates and synthesises the intrapropositional operations on relations and classes into a single structure[6] (Piaget and Grize 1972, 15:XV–XVI, 343–5). By means of the grouping Piaget could thus interpret in analogy with comparative anatomy the correspondence between propositional reasoning and reasoning with classes and relations in terms of natural descent.

There is a much broader context still, which has historical-biographical components. Just as post-enlightenment philosophers shook their heads in disbelief at theological explanations, especially of miracles, contemporary scientists are sceptical of philosophical explanations (e.g., Hamming 1980, 81). Some philosophers perhaps realising they were fighting a losing battle, changed their colours and rallied around science's flag in the first half of the 20th century, and most contemporary philosophers profess to be naturalists of some description. Naturalism is not clearly defined, but it typically involves a rejection of the supernatural by regarding reality as being completely exhausted by nature and scientific methods as the only means of investigating it (Papineau 2009; Rysiew 2017). Moreover, the success of the sciences measured in terms of agreement amongst its practitioners is considered to be due to their emancipation from philosophy (Shapin 2008, 163–4). In the early years of Piaget's intellectual development two souls resided in his breast, but rather than succumbing to the enchantment of philosophy and making concessions to science, Piaget inaugurated a new science based on the growth of knowledge to shed light on age-old epistemological questions. With biological epistemology, he thus dreamt of emancipating yet another science from the speculation of philosophy. (Piaget 1950; Piaget 1972, chap. Introduction; Jean Piaget 1981, pt. VIII)

Having touched on the historical-biographical setting of the application of mathematics, I go on to characterise Piaget's application of mathematics in the next section after briefly recalling the results.

5.2 Characterisation

According to biological epistemology, reasoning develops in an invariant sequence of stages over time. The stages are due to the equilibria operations of thought achieve during cognitive development, whilst rationality in particular is due to the reversibility inherent in the

[6] In essence, intrapropositional operations involving classes and relations correspond to algebraic and order structures, which have incompatible inverse operations. In the interpropositional grouping, however, INRC-group synthesized these inverses into a structure. However, characterizing the interpropositional grouping as a single structure is, according to Grize, a question of interpretation since the lattice and group structures coexist independently of each other in the interpropositional grouping. (Grize 2013)

structured wholes operations of thought form in these states of equilibrium. Whether mental or physical, actions like uniting or separating objects transform the reality they act on, and the outcomes of these transformations can be returned to their original states by means of further operations. Mathematically, groups form reversible structures, and the direct, inverse and identity operations of groups correspond to the reversibility inherent in operations of thought. On the other hand, classifications and seriations as well as propositional reasoning are manifestations of operations of thought in states of equilibrium; however, inclusions and self-inclusions are among the operations of these systems as well. Such transformations correspond to the operations of semi-lattices and lattices, but they are not commensurate with the reversible operations of groups or rings. The grouping is designed to incorporate both the reversibility of operations of thought and the self-inclusions and inclusions in the systems of transformations operations of thought form in equilibrium. Piaget thus envisaged the grouping either as a relaxation of the strict reversibility of groups through augmentation with inclusions and self-inclusions or a tightening of the lattice-like operations through the introduction of reversibility, in short, a modification of either lattice or group structures.

The case study shows that the grouping is indeed a mathematical structure constructed by adapting structures found in the storehouse of mathematics to the specific empirical content of the embryology of reasoning. However, Piaget was drawn to the group as a model because reversibility is a characteristic of rationality and the operations of a group are reversible; on the other hand, he was drawn to lattices because of the self-inclusions and inclusions of operations of thought. Mathematics might determine what we look for in the material world, as Hamming pointed out (see 1), but the case study shows that perusal and selection of structures in the storehouse of mathematics is also guided by empirical criteria.

Moreover, the structures short-listed did not prove to be equally amenable to the modifications required. Relaxing reversibility to incorporate self-inclusion, inclusion, and contiguity into the algebraic structures (the groups and ring) proved to be a bigger obstacle than tightening the operations of lattices to make them reversible. In fact, incorporating reversibility into the lattice simply involved introducing a hierarchy of relative complements via nested inclusions—$A \cup A' = B$, $B \cup B' = C$, etc. for classes $A \subset B \subset C$ etc. or via the relative complements $(p \cdot q)$ and $(\overline{p} \cdot q)$ derived from $p \vee q = q$ and $p \cdot q = p$ due to the inclusion relation $q \geqq p$ for interpropositional operations. Mathematically, the former represents a join semi-lattice whereas the latter simply represents a special case of lattices in general, namely a complemented distributive lattice. Both therefore represent modifications of lattices; however, the modification required to tighten the operations of lattices with reversibility to accommodate interpropositional operations is simply a specialisation of a structure already extant in the storehouse of mathematics. From the point of view of a lattice, it is thus evident that mathematics' preoccupation with itself is the source of the mathematical structure Piaget used to model the essential operatory mechanisms of propositional reasoning (Piaget and Beth 1966, vol. 12, sec. 44; Bilová 2001; Steiner 2005, sec. Canonical Nonempirical Applications; Burris and Legris 2018). Hence, the construction of the interpropositional grouping at least is a non-canonical empirical application of mathematics according to Steiner (2005)'s classification. Moreover, being a non-canonical empirical application of mathematics to a socio-psychological reality rather than the subject matter of the natural sciences, the construction of the interpropositional grouping also suggests that restrictions of the unreasonable effectiveness of mathematics to the natural sciences can be lifted, as Hamming (1980, 82–3) intimated.

In comparative anatomy, the focus lies on the relative positions of bones comprising an organ rather than the forms of the bones themselves and their function. The degree of abstraction involved in discovering a common anatomy is therefore high. Analogously, by focussing on the essential operatory mechanisms rather than the contents they operate on, the degree of abstraction involved in determining the grouping is also very high. However, analogies are rarely complete, and there is a difference between the two that is reminiscent of the challenge physicists faced as they began to delve the depths of subatomic world at the end of the 19th century. Through skilful dissection, the arrangements of bones in organs can be revealed; no matter how skilled the anatomist, however, cognitive structures in the black box of the mind escape sense perception, even introspection for that matter, in much the same way as the subatomic world. The grouping thus models cognitive structures with the help of mathematics, but the socio-psychological reality would not have any positive contours without the mathematical model. Steiner (Steiner 2009, 3–4) argues that mathematics provided an essential framework for discovering rather than describing the laws of the subatomic world; similarly, the case study shows that mathematics has more than just a descriptive role to play in mediating the cognitive structures of the mind.

Despite the socio-psychological reality only appearing in mathematical garb, it nevertheless does not mean that we only find what we look for, at least not in the sense of those who insinuate that there is nothing in the human psyche that has not first been put there by psychologists. Piaget was looking for a fitting structure in the storehouse of mathematical structures, but the case study shows that he could not smuggle them unaltered into the human psyche. The essential operatory mechanisms resisted incorporation in the mathematical structures he initially selected, and the groupings represent modifications of the mathematical structures originally selected that are designed to incorporate an extra-mathematical content. In other words, the groupings are not simply projections of preformed mathematical structures into the human psyche.

On the other hand, the grouping is not simply a description of an extra-mathematical reality. The operations of the ring generate all 16 logical operators of propositional reasoning; however, Piaget rejected the ring as a suitable model of interpropositional operations because of problems with associativity, equivalences under transformations, and identity operations. Reasoning is notoriously fallible but rather than measuring the adequacy of the description solely on the basis of the empirical referent, Piaget thus rejected the ring as a suitable model because its operations do not form a coherent system of transformations. Just as consistency is the only criterion for the existence of mathematical theories (Shapiro 2000, 156), Piaget's assumption thus seems to be that a model cannot represent the psychological structure of reasoning unless it is coherent. In analogy with mathematics, coherence is thus partly constitutive the grouping.

The mathematical analogies are not exhausted with coherence, but, before highlighting more, a brief excursion into biological epistemology will shed light on how Piaget understood the interaction of mathematics and the empirical findings of psychological experiment. The case study demonstrates how Piaget attempted to incorporate empirical content into preformed mathematical structures, but, due to the resistance of the psychological content to this application of mathematics, he modified these structures, and the structures thus modified —a join semi-lattice for classifications and a complemented distributive lattice for interpropositional reasoning—correspond to the essential operatory mechanisms more adequately than the unmodified structures. In the theory of knowledge, correspondence is fraught with difficulties (e.g., G. E. Moore 1962, chaps. 14–15), difficulties which are only compounded when one side of the correspondence relation is no longer accessible to sense experience. Piaget (e.g., 2001, chap. 1), in contrast, understood the former process as assimilation and the latter, as accommodation, in analogy with the two complementary aspects of biological adaptation; he thereby substitutes the cognitive equivalent of adaptation for correspondence, namely equilibrium in the mental processes of assimilation and accommodation. In terms of biological epistemology, Piaget's construction of the grouping thus illustrates assimilation and accommodation in the process of cognitive adaptation, and for propositional reasoning in particular preadaptation since a Boolean algebra is simply a special case of a lattice.

Returning to the mathematical analogies, the grouping alone, like a common anatomy in comparative anatomy, only provides evidence of common ancestry, yet intrapropositional precede interpropositional operations developmentally in the embryology of reasoning. For Piaget, the natural descent of the latter from the former is indicated structurally by the synthesis of incompatible intrapropositional operations into a single interpropositional structure. Negation N and reciprocal R are incompatible inverse operations on classes and relations at the intrapropositional level of reasoning; however, they comprise the INCR group at the interpropositional level together with the correlative C, which applied to the conditional operator form a framework of hypotheses that was shown to be instrumental in determining causality (see above). Furthermore, the interpropositional grouping contains models of intrapropositional groupings (Piaget and Grize 1972, vol. 15, secs. 28, 32 & 41). Intrapropositional groupings are thus conserved, albeit in a propositional form, in interpropositional grouping like substructures in superstructures in mathematics. Again, mathematics, in particular coherence and the relations of mathematical structures to each other, is serving Piaget as an analogy in his embryology of reasoning (see Piaget 1970, chap. 2). However, his use of mathematical analogies it is not covert. Piaget (Piaget and Grize 1972, 15:25) recognised the difficulties involved in distinguishing between artificial and natural structures, especially when they come to expression in the same mathematical formalism. He argued that a theoretical construct that represents extra-theoretical contents, on the one hand, and increases the coherence of the overall theory it is part of, on the other, is less artificial. Piaget thus distinguished two criteria for the naturalness of theoretical constructs, and the grouping, by representing a socio-psychological reality, on the one hand, whilst introducing a unifying coherence into the contentual diversity of intra- and interpropositional reasoning, on the other, is a natural structure according to Piaget's own criteria, despite its formal appearance.

5.3 Consequences

The case study could at first blush be construed as a confirmation of faith in 'the empirical law of epistemology' (Wigner 1960) since the storehouse of mathematics again had a fitting structure in stock, at least for the interpropositional operations. However, Steiner (2005, 632) argues that the observations presented by Wigner do not actually constitute a thesis since each illustration of the successful application of mathematics in the natural sciences is individual and requires its own tailor-made explanation. Nevertheless, the case study shows that Piaget used mathematical analogies not only to discover the grouping but also to discern direction in the development of reasoning. A Pythagorean strategy is thus evident in Piaget's approach to biological epistemology. Using an anthropocentric tool but being a naturalist at heart, Piaget therefore appears to suffer from the 'intellectual schizophrenia' Steiner (2009,

73) considered symptomatic of natural scientists who also employed a similar strategy, especially in delving the depths of the subatomic world.

Clearly, the case study raises the metaphysical question, and for a Piagetian explanation of how the storehouse of mathematics can have a structure in stock that miraculously fits a socio-psychological reality in the embryology of reasoning, it is helpful to refer to an anecdote Piaget was fond of relating over his first acquaintance with the research programme of the Bourbaki (e.g., Piaget and Beth 1966, 12:168)[7]. In a small colloquium on mathematical and mental structures, participants were struck by structural parallels in the embryology of reasoning and the mother structures, which were thought to constitute the architecture of mathematics, especially since the discoveries were made completely independently of each other. From a biological-epistemological perspective, logic and mathematics originate in coordinations of actions performed on arbitrary objects, and the operational structures in the embryology of reason represent intermediate stages in their construction. Once constructed, however, it is also possible to take logic and mathematics as given, and, abstracting from any particular mathematical content, analyse their manifold manifestations for underlying structures. According to Piaget (Piaget and Beth 1966, vol. 12, sec. 48), the mother structures of the Bourbaki are thus quasi-inductive generalisations of structures inherent in diverse logical and mathematical contents.[8] In the embryology of reasoning, the mother structures and the natural structures of operations of thought thus have a common origin in the coordinations of actions on arbitrary objects.

The grouping adequately characterises the structures of interpropositional operations of thought in the embryology of reasoning; on the other hand, it is a special case of a lattice, a special case, that is, of a structure resulting from ‘canonical nonempirical applications’ of mathematics (Steiner 2005, sec. III). In the framework of biological epistemology, however, the fact that a lattice fits a socio-psychological reality is no longer unreasonable because the operational structures discovered in the embryology of reasoning and logico-mathematical structures discovered in the contents of mathematics are part of the same developmental continuum of operations of thought. In Piaget’s words, a ‘genetic relationship’ (Piaget and Beth 1966, 12:189) exists between the operational structures in the embryology of reasoning and the mother structures.

In biology, similar structures in different species are explained by evolutionary convergence due to a common environment or homology, similar structures due to recent common ancestors. On the one hand, logico-mathematical structures and operational structures are different kinds of structures; on the other hand, they share a common ancestry in the coordination of actions on arbitrary objects in thought’s infancy. The preadaptation of logico-mathematical structures to a socio-psychological reality, the lattice to the structure of propositional reasoning for instance, is therefore no less mysterious than homologous structures in different species.

[7] Piaget (Piaget and Beth 1966, 12:168) relates this anecdote to dispel the insinuation *Nil est in intellectu quod non prius fuerit in psychologo* of many critical of psychology.

[8] Insofar as it pursues beauty, convenience, etc. mathematics is like music—anthropocentric. The investigations of the Bourbaki can therefore be likened to the retrospective analysis of music for the rules of harmony guiding its composition.

This result is interesting in several respects. Contrary to Wigner, Steiner (2005, 632; 2009, 45–47) claimed that explanations of preadaptation of mathematics have to be tailored individually to each application of mathematics to reality. However, the genetic relationship provides a generic explanation for applications of mathematical structures in the embryology of reason. In at least one field of application, Steiner's position is therefore overly conservative.

Piaget was a naturalist, but his Pythagorean strategy was crowned with success; the case study might therefore be construed as a plaidoyer for intellectual schizophrenia—even in the psychology of reasoning! However, intellectual schizophrenia is based on the background belief that the natural world and the human mind are distinct, unrelate natural kinds, and mathematics, belonging to the human mind, is an anthropocentric pursuit, which cannot naturally have any inherent relevance for the natural world. Background beliefs on logic, mathematics and the human mind on the other hand vary. Since a categorical difference is not always assumed, the success of Piaget's Pythagorean strategy might not therefore appear to be quite as miraculous as in the natural sciences. Nevertheless, the genetic relationship Piaget's biological epistemology established between mathematics and the embryology of reason actually exacerbates intellectual schizophrenia for the natural sciences. From the viewpoint of biological epistemology, logic and mathematics are constructions originating in the coordinations of actions performed on arbitrary objects. Thus, not only is their development motivated by aesthetics, calculational convenience, etc., they also originate in the activity of the subject. In short, mathematics and logic originate and continue to develop independently of external reality in the activity of the subject. However, it will be recalled that causality is not directly observable but is established via the interpropositional grouping by interpreting observable associations of phenomena with the help of the framework of all conceivable relations between two observable phenomena (see above). In fact, logico-mathematical structures are instrumental in discovering the most fundamental concepts in the natural sciences (e.g., Inhelder and Piaget 1958). In other words, mathematics and logic are not just enormously successful but indispensable in the natural sciences. For biological epistemology, there is thus a difference between products of the mind in the embryology of reason and objects of the natural world much like the one that made the enormous effectiveness of mathematics in the natural sciences appear so miraculous to Wigner. Moreover, the difference rather than being based on background beliefs has a scientific foundation. Via its small victory, biological epistemology has thus raised the stakes for naturalists on the metaphysical problem concerning the applicability of logico-mathematical structures to the natural world.

Mathematics has long been a riddle for Western philosophy, and traditional battle lines are thought to have reached a permanent standoff (Shapiro 2000, 256). Before alluding to Piaget's envisaged solution to this riddle, I will first briefly outline how biological epistemology has brought movement into the entrenched battle lines despite exacerbating the metaphysical problem.

Inspired by the manifest success of the natural sciences, especially in the most recent centuries, 'naturalism' broadly denotes the orientation of philosophy on natural science, and the philosophy of mathematics is no exception.[9] Through phylogenetic and ontogenetic investigation of the growth of knowledge, Piaget developed a scientific method for investigating epistemological questions (Piaget 1950, vol. 1, chap. Introduction); in other words, biological epistemology represents a methodological naturalization of epistemology.[10] Moreover, it is a theory of knowledge founded on fundamental biological functions rather than inherited adaptations, which arguably navigates the opposing trends of specialisation in evolutionary speciation and non-centricity of objective knowledge (Engels 1989). Using this methodology, Piaget developed a constructivist theory of knowledge,[11] which differs from classical and contemporary positions in the philosophy of mathematics. According to Piaget's constructivism, mathematics and logic are mental constructs. Although it is not possible to rule out Platonic proclivities entirely, construction is both necessary and sufficient to gain access to the denizens of such Platonic realms. Since assuming the mind-independent existence of logico-mathematical entities is less parsimonious, Platonism is dispensable[12] in a constructivist theory of logical and mathematical entities. However, without logico-mathematical structures, it would be impossible to discover the properties of observable phenomena; in other words, they are indispensable[13] for natural science. On the other hand, construction of logico-mathematics entities over time in the mind is consistent with empiricism. Whereas the applicability of logic and mathematics to the natural world does not pose problems for empiricists, the loss of necessity and certainty in logical and mathematical knowledge is the price they have to pay. According to constructivism, however, the origin of mathematical and logical entities lies in coordinations of actions performed on arbitrary objects of the natural world. Despite being applicable to objects populating the natural world, they are therefore not derived from properties of these objects. Furthermore, logico-mathematical constructions are necessary and self-evident (Piaget and Beth 1966, vol. 12, sec. 49). In short, constructivism *a la* Piaget is not partisan to the entrenched philosophical battles lines and could as neutral arbitrator help resolve the standoff in the 'rational adjudication in the dispute on realism in ontology and its opposite' (Shapiro 2000, 256). In fact, it has already changed the battle lines since it persuades empiricists on empirical grounds to abandon their trenches and rally around the rationalists' flag in a common front. In

[9] According to Shapiro (2000, chap. 10, sec. 4), for example, epistemology can shed light on ontology due to the mutual dependency of the ontologies and epistemologies of mathematical entities. Although he bemoans that there is no generally accepted school, he nevertheless endorses in principle the contribution cognitive psychology could make to the philosophy of mathematics.

[10] In 'Science and Epistemology—A Genetic Epistemological Perspective', I argue that biological epistemology represents a methodological naturalism of epistemology that navigates objections typically levelled at naturalized epistemologies well.

[11] 'Constructivism' (e.g., Howard. E. Gruber and Vonèche 1977, xxxvii; Smith 2009), 'dialectical constructivism' (e.g., Campbell 2009), and 'interactionism' (e.g., Ferrari 2009) are several terms used to characterise Piaget's theory. Since '*[t]here is no structure apart from construction'* (Piaget 1970, 140 author's italics) according to Piaget, 'constructive structuralism' would also perhaps be another appropriate terminology.

[12] The use of 'dispensable' here corresponds to Colyvan (2001, 77)'s definition, and it is ironic that an epistemology properly naturalized actually undermines the indispensability argument supporting the mind-independent existence of logical and mathematical entities (see (Colyvan 2001, chap. 2) for the Quinean backdrop for the indispensability argument).

[13] The use of 'indispensable' here is not the opposite of Colyvan's definition of 'dispensable' i.e., '"couldn't get by without it"'(Colyvan 2001, 12). It refers to a necessary condition for the discovery of unobservable aspects of the natural world, and it is again ironic that discovery of this stricter form of indispensability is due to a naturalised epistemology being able to distinguish between mathematical and material entities.

doing so, however, constructivism does not make the effectiveness of mathematics and logic in the natural sciences any less mysterious.

Piaget divined self-organisation and adaptation as functional similarities between living organisms, which, as open systems, are organisations of material processes successfully adapted for survival in their natural environments, and cognition, in which operations of thought organise themselves into structures, which are adapted to their cognitive environments. In analogy with comparative physiology, Piaget thus unveiled functional as well as structural similarities (Piaget 1950, vol. 1, sec. Introduction §2). Moreover, the coordinations of actions performed on arbitrary objects also depend on the underlying biological subject. Mathematics' genealogy can therefore be traced back beyond voluntary actions of the subject to the interactions of the biological subject with the natural world. Extending the homology analogy to comparative physiology, then, logico-mathematical structures are constructions that completely transcend the natural world on the one hand whilst having roots in interactions with the natural world via the biological subject. Since the workings of the mind are functionally continuous with the self-organisation of living organism and in contact with the natural world via the biological organism, it is, from a naturalistic viewpoint, not implausible that mathematics and logic can develop as non-empirical canonical applications independently of the natural world and transcend it in all directions yet not lose touch with it altogether. In several works, Piaget (e.g, 1952; 1971a; 1977, pt. Introduction) tried to explicate the connections between living organisms and cognitive processes; however, he was acutely aware of the fact that his attempts remained speculative due to substantial gaps in biological knowledge.

6 Conclusion

The grouping is a modification of pre-existing mathematical structures to psychological content, and the interpropositional grouping, especially, represents a non-canonical application of mathematics to a psychological structure. Piaget's deployment of mathematics in biological epistemology exhibits traits of a Pythagorean strategy, but, besides mathematical analogies, Piaget, commensurate with his training, also employed biological analogies. In particular, analogies with biological adaptation on the one hand and comparative anatomy and physiology together with homology on the other were instrumental in his scientific explanation of cognition and the applicability of mathematics to the embryology of reason and the natural world, respectively. In other words, Piaget pursued a Georges-Curvierian as well as a Pythagorean strategy. Moreover, both biological and mathematical analogies were crowned with success since the effectiveness of mathematics in the psychogenesis of reasoning no longer appears unreasonable thanks to the genetic relationship between the subject-matter of both.

However, Piaget's success in applying mathematics to a socio-psychological reality actually makes the effectiveness of mathematics in the natural sciences appear even more unreasonable, especially for naturalists. By extending the homology analogy via comparative physiology, Piaget proposed an explanation for this mystery too, but he was not able to fully realise his dream of a biological epistemology due to substantial gaps in biological knowledge. Nevertheless, his empirically founded speculations show that not only mathematical analogies but analogies of other kinds can be fruitful in formulating scientific hypotheses. Furthermore, his hypothesis though speculative shows that the success of the

Pythagorean strategy of natural scientists does not necessitate abandoning naturalism since it draws attention to a natural property of mathematics that could account for the success of mathematical analogies, perhaps even those used in the subatomic realm.

Moreover, background beliefs are unavoidable in the acquisition of scientific knowledge since science has to begin in *medias res* and the mind is not a *tabula rasa*. Being claims to knowledge, consequential naturalists like Piaget must eventually subject these background beliefs to scientific inquiry as well;[14] however, these beliefs might turn out to be false. The intellectual schizophrenia of having naturalist convictions while pursuing a Pythagorean strategy might not therefore be as pathological as it first appears—indeed it might even be a quite reasonable adaptation to science's predicament, namely, acquiring true beliefs on the basis of fallible ones.

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[14] Psychogenetic studies of the development of specific concepts always presuppose the frame of reference provided by the current state of knowledge. When genetic epistemology also takes this framework into account it truly becomes epistemology. Piaget sets out this transition from special to general epistemology in (Piaget 1950, vol. 1, chap. Introduction: esp. §6-7).

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