Mathematical proofs and metareasoning[1]

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Abstract: In this paper, informal mathematical proofs are conceived as a form of (guided) intentional reasoning. In a proof, we start with a sentence – the premise; this sentence is followed by another, the conclusion of an inferential step. Guided by the text, we produce an autonomous reasoning process that enables us to arrive at the conclusion from the premise. That reasoning process is accompanied by a metareasoning process. Metareasoning gives rise to a feeling of correctness, which makes us feel-know that the reasoning is correct. Guided by the proof, we go through small inferential steps, one at a time. In each of these cycles, we produce an autonomous reasoning process that "links" the premise to the conclusion. This enables, due to our metareasoning, to associate to the verbal conclusion a feeling of correctness. In each step/cycle of the proof, as a (guided) intentional reasoning process, we have a feeling of correctness. Overall, we reach a feeling of correctness for the whole proof. The main purpose of this work is to suggest that this approach allows us to address the issues of how does a proof functions, for us, as an enabler to ascertain the correctness of its argument, and how do we ascertain this correctness.

1. Introduction

Let us start the present work by considering different answers to the question of what a mathematical proof is.[2] This will be a way to introduce what is for us the core function of a mathematical proof. For our purpose, the following "answers" suffice:

A proof is a piece of discourse […] a proof is meant to convince its intended audience of the truth of its conclusion […] a proof should indicate not only that its conclusion is true (given the truth of the premises) but also why it is true. (Dutilh Novaes 2020, 225-6)

An ordinary mathematical proof consists of an argument to convince an audience of peer experts that a certain mathematical claim is true and, ideally, to explain why it is true. (Tall et al. 2021, 15)

A mathematical demonstration is a piece of discourse presupposing a putative audience; it puts forward a chain of reasoning for public scrutiny. (Dutilh Novaes 2013, 48)

The first statement about proofs seems to imply that they have different functions on an equal footing. In this case, to show the correctness of the conclusion (by convincing of the correctness of the reasoning), and to explain why the conclusion is true. We agree with Dutilh Novaes that mathematical proofs have many functions/roles and that besides establishing the truth of a mathematical claim,

[1] This paper is part of my work made in the context of the research project "the genesis of geometrical knowledge" whose PI is José Ferreirós from the University of Seville.
[2] In the present work, we only address proofs formulated in natural language and not the so-called formal proofs that rely on formal languages and deductive apparatus.

there are "other, equally, important roles for proofs" (Dutilh Novaes 2020, 223).[3] However, we consider that these roles are not at the same footing as that of proof as a public display of a "chain of correct reasoning". If a mathematical argumentation is not correct, we might want to call it, e.g., a tentative proof or a faulty proof, but, simply, it is not a proof. In a way, it is an issue about semantics; of what we accept/define as a mathematical proof. Concerning proofs in/with a natural language, we adopt Avigad's view on formal proofs: "a sequence of a thousand inferences in which a single one is incorrect is simply not a proof" (Avigad 2020, 5).

We would then prefer the second statement regarding mathematical proofs. We might say that, e.g., in contemporary mathematics, we ideally try to explain with the proof why a mathematical claim is true. But even if a proof is not explanatory, it is still a proof.[4]

However, in the present work, we will only address the core function of proof as stressed in the third statement above. We might re-state it as follows:

A proof is a piece of discourse that puts forward a chain of reasoning for public scrutiny, whose core function is to establish the truth of a mathematical claim.

But let us now consider the following excerpt:

Mathematics, considered as a social practice, is remarkably stable […] The notion of a mathematical proof appears to be particularly invariant: While the standards for what can count as evidence in, say, physics or biology have considerably changed over the last 2000 years, we can still evaluate an argument from e.g. Euclidean geometry and agree on its correctness. (Carl 2019, 316)

The stability of proofs comes from *our capacity*, even with "old" proofs, to "evaluate an argument" and ascertain its correctness. We will re-re-state the notion of mathematical proof as follows:

A proof is a piece of discourse that puts forward a chain of reasoning for public scrutiny, whose core function is to enable the audience to ascertain the correctness of the reasoning.

This might seem equivalent to the previous "definition", but it is not quite so. The proof does not establish the truth; it is an enabler for us to ascertain the truth. The reasoning is in us, not in the proof. As mentioned by Rav:

Mathematical texts abound in terms such as "it follows from … that" […] a mathematical proof in general only says that it follows, not why [it follows] […] why the consequent follows from the antecedents has to be figured out by the reader of a proof. (Rav 2007, 316-7)

In this work, we want to address the issue of how does a proof functions, for us, as an enabler to ascertain the correctness of its argument; and how do we ascertain this correctness. We will do so by considering some of the earliest proofs in ancient Greek geometry. In this way, we will see how "we can still evaluate an argument from e.g. Euclidean geometry and agree on its correctness" (Carl 2019, 316).

The present work is structured as follows. In section 2, we will look in some detail into an ancient geometric proof that is previous to those in Euclid's *Elements*. That is our first example of a mathematical proof. In section 3, we will present a schematic model of intentional reasoning as a cyclical process, in which metareasoning gives rise to a feeling of correctness associated with each reasoning process. We will address mathematical proofs as a form of (guided) intentional reasoning. We will return to the proof of section 2, considering this perspective on mathematical proof, and see how the proof functions, for us, as an enabler to ascertain the correctness of its argument; and how we ascertain its correctness. That makes it clear how we can still evaluate an ancient mathematical

[3] With respect to the different functions of a mathematical proof, see, e.g., Dutilh Novaes (2020, 222-8).

[4] On proof's explanatoriness, see, e.g., Dutilh Novaes (2018). For a "classic" on the issue see Steiner (1978).

proof. In section 4, we will extend the previous approach to two proofs included in Euclid's *Elements*. A simple single-level proof and a proof by contradiction (a reductio proof).

We want to stress that in this paper, we do not address the geometric proofs *qua* geometric proofs, but as early examples of mathematical proofs. An idea underlying the present work is that intentional reasoning as a cyclical process might be a general feature of informal mathematical proofs and that the feeling of correctness might be how we, individually, ascertain the correctness of the proof. For this to be even a possibility, it must be the case that extant early proofs can be addressed in these terms; this is the main reason we adopt ancient Greek mathematical proofs in the present work.[5, 6]

2. An ancient Greek proof by Hippocrates of Chios

The earliest substantial text with mathematical proofs that has survived, via an indirect source, to the present day was written by Hippocrates of Chios on the subject of the quadrature of lunes or lunules, somewhere during the second half of the fifth century (say, between 450-430 BCE).[7] Hippocrates is said to have taught geometry and have written the first collection of elements of geometry. These had not yet the axiomatic format that we know from Euclid, neither there is any reason to expect them to have had a coherent format. Most likely it consisted of a loose collection of known results and techniques – which were taken for granted – and newer developments that were obtained using these (Høyrup 2019a). Examples of these might have been the results obtained by Hippocrates himself.

According to Høyrup, "this collection is likely to have been connected to Hippocrates's teaching" (Høyrup 2019b, 36). In the same way, we might speculate that Hippocrates' own results might be part of his teachings. We might expect this teaching to have some aspects in common to those related to Euclid's *Elements*. According to Saito's reconstruction, a teacher would recite the general enunciation of a proposition and, afterward, draw a diagram on sand or a wax tablet; then, after the proof of the proposition, "the diagram was probably erased to make room for the next proposition to be learned" (Saito 2018, 8). Equivalently, we might conceive of Hippocrates teaching on his quadrature of lunes, as going sequentially from the simplest case to the more complex one.[8]

For the view being present in the present work, this historical sketch doesn't have to approach whatever might have occurred. However, it leads us into focusing ancient Greek geometrical proofs in terms of what we might call its verbal exposition to other people. The basic starting idea of the present work is that the proof might have specific roles for its intended audience, in relation to making explicit what is taken into account to arrive at the result, in partitioning the argumentation in small steps, and in enabling the cognition of the correctness of the steps. This would lead to a cognition of

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5 Another reason is to see how "we can still evaluate an argument from e.g. Euclidean geometry and agree on its correctness" (Carl 2019, 316).

6 It would be interesting to determine if this is, in fact, the general case in later or contemporary informal proofs and if there are exceptions to it, and in this case, how correctness is articulated by us when guided by the proof. These are subjects not addressed here.

7 The extant text, by Simplicius, is from the 6th century CE. It reports what Alexander of Aphrodisias wrote on a previous text (from around 200 CE) and also reports on a text by Eudemus (written in the late 4th century BCE, or possibly a later version of this text) (Høyrup 2019a). Here, we consider the part of Simplicius' text related to Eudemus' account. This is made in terms of the reconstruction by Becker of the Eudemian text. We will adopt Netz's translation to English of this reconstruction (Netz 2004).

8 Regarding the extant edited version of Hippocrates' work, Høyrup considers that the text by Alexander, reported in this text, "draws on Hippocrates's teaching, being based either on lecture notes of his or on students' notes" (Høyrup 2019a, 158). Regarding the Eudemian text, also reported in the text, his view is that "the Eudemos-version may instead go back to what Hippocrates published more officially" (Høyrup 2019a, 158). We might then conceive of the Eudemian text that we will consider in the present work as a written rendering of somewhat loose writing related to the oral presentation of the proofs.

the correctness of the proof and, because of this, of the result obtained.[9,10]

For our purpose, it is sufficient to address the first quadrature as given in the Eudemian account of Hippocrates' text. It is as follows:

(2) So he made his starting point by assuming, as the first among the things useful to the quadratures, that both the similar segments of the circles, and their bases in square, have the same ratio to each other [...] [(4)] he first proved by what method a quadrature was possible, of a lunule having a semicircle as its outer circumference. (5) He did this after he circumscribed a semicircle about a right-angled isosceles triangle and, about the base, <he drew> a segment of a circle, similar to those taken away by the joined <lines>. (6) And, the segment about the base being equal to both <segments> about the other <sides>, and adding as common the part of the triangle which is above the segment about the base, the lunule shall be equal to the triangle. (7) So the lunule, having been proved equal to the triangle, could be squared. (8) In this way, taking the outer circumference of a semicircle as the <outer circumference> of the lunule, he readily squared the lunule. (Netz 2004, 248-9)[11]

Let us imagine Hippocrates seated with a wax tablet in his hands with his students seated in front of him. He might have started, e.g., by mentioning that he would address the quadrature of lunes (i.e., the issue of finding a figure – whose area we know – that has the same area as a lune). After the "enunciation" of what is going to be proved, Hippocrates would mention the geometric knowledge useful to arrive at the intended result. Some of this knowledge is so well-known that Hippocrates would leave it implicit. In this way, simple arithmetic of areas (additivity and subtractivity of areas) and the Pythagorean rule would not be mentioned explicitly. He only mentions that both the similar segments of the circles and their bases in square have the same ratio to each other. According to Høyrup, this principle, like the Pythagorean rule, and the simple arithmetic of areas, "were known since well above a millennium [BCE] in Near Eastern practical and scribal geometry" (Høyrup 2019a, 178). He would then proceed to the drawing of a diagram (for the case of a lunule having a semicircle as its outer circumference). According to the reconstructed text, Hippocrates would start by circumscribing a semicircle about a previously drawn right-angled isosceles triangle. This gives rise to two segments of circle, each having a side of the triangle as its base (see figure 1).

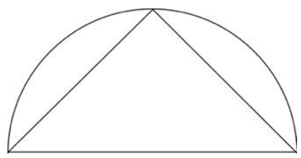

Figure 1. Initial drawing with a semicircle circumscribing an isosceles triangle

Then Hippocrates might have completed a square with sides equal to the base of the triangle. Afterward, he would use the lower corner of this square as the center of another circle, and he would draw about the base of the triangle a segment of a circle (see figure 2).

[9] We do not consider that mathematical proofs might not have gone through historical changes regarding their "cognitive functioning" (i.e., their diverse cognitive roles in relation to, e.g., clarifying the presupposition taken into account, the correctness of the steps, the use of diagrams in the proof, the regimentation of the language adopted in the proof, the kind of argumentation strategies adopted, the role of formal languages, being "explanatory", etc.). In this way, here, we only endeavor to capture important aspects of mathematical proofs that can be found in its earliest examples. If and how these might have evolved in time is a subject we do not tackle here.

[10] As the reader will soon see, we only focus on a geometric proof in relation to its being an oral or written piece of linguistic argument. This is only a partial approach to geometric proof. Here, we do not try to give a global perspective on ancient Greek geometric proof. In particular, we do not address the issue of the role of diagrams in the proof; on this, see, e.g., Dal Magro and García Pérez (2019).

[11] The numbering of each sentence is not part of the ancient text (or the contemporary reconstruction). It was made by Netz to ease reference (Netz 2004, 248). We leave it since it provides a clear separation between the different parts of the text.

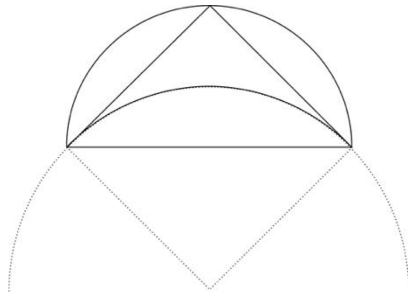

Figure 2. Completion of the drawing of the lune

This segment of circle is similar to the two previous segments of circle (since this segment makes up the same part of a circle as the other two). Also, since we have a right-angled isosceles triangle, according to the Pythagorean rule, the base in square is equal to (the sum of) the sides in square (this is left implicit since, as mentioned, is taken to be well-known by the intended audience). After the completion of the diagram, Hippocrates starts his argumentation proper. This is given very concisely in part (6) of the text above:

The segment about the base being equal to both <segments> about the other <sides>, and adding as common the part of the triangle which is above the segment about the base, the lunule shall be equal to the triangle. (Netz 2004, 249)

Here, Hippocrates uses the above-mentioned principle together with the Pythagorean rule and concludes that the segment of circle about the base is equal to the sum of the segments of circles about the sides. With the aid of the diagram, Hippocrates "adds" the part of the triangle which is above the segment to the two small segments of circle. This "adding" can be simply made by pointing in the diagram to the three regions that must be considered as a whole area (see figure 3 left). Here, Hippocrates is using the simple arithmetic of areas. This area is equal to that resulting from "adding" the part of the triangle which is above the segment of circle to this segment of circle (see figure 3 right). In this way, Hippocrates proves that the area of the lune is equal to the area of the triangle. This concludes Hippocrates' "exposition" as rendered in the extant written text.

Figure 3. Two ways of doing the "visual operation" of adding areas of figures

As Høyrup mentions, the argumentation is made on a single level, being directly based on the geometric knowledge useful to arrive at the intended result. (Høyrup 2019a, 179). Also, the argument is, really, a sequence of arguments, each laying the ground for the next one. In a didactic oral presentation, we can expand the condensed written rendering of the arguments as a sequence of steps, each one having a single level (i.e., relying directly on the geometric knowledge admitted without proof). One could start by mention the Pythagorean rule; something like the enunciation of proposition 47 of book 1 of Euclid's *Elements*: "in right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle" (Heath 1956, 349). Figure 4 shows the example that is relevant in Hippocrates' proof, that of a right-angled isosceles triangle. In this case, the area (i.e., the side in square) of the two smaller squares is the same.

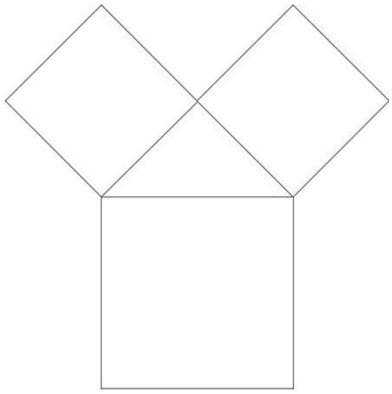

Figure 4: The Pythagorean rule for the case of a right-angled isosceles triangle

In the second step, we would recall the principle adopted at the beginning of the presentation: both the similar segments of the circles and their bases in square have the same ratio to each other. We can imagine an auxiliary figure like figure 5, which gives a visual rendering of this for the case at hand, that of a right-angled isosceles triangle.

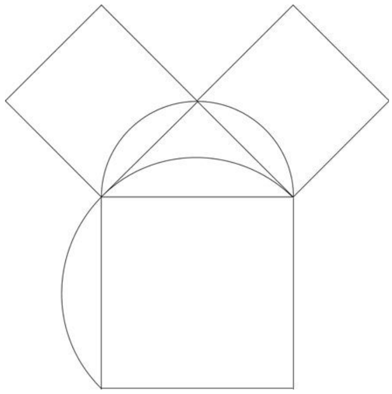

Figure 5: Pythagorean rule plus the "similar segments principle"

With the aid of figure 5, we conclude in step three that "the segment about the base [is] equal to both <segments> about the other <sides>" (Netz 2004, 249). That is the crucial result obtained by Hippocrates.

We then go into the "second part" of the proof. Here, we take advantage of the well-known geometric knowledge about the additivity of areas. That can be made "directly on" figure 2 constructed during the initial part of the demonstration/presentation, as described above using figures 3 left and 3 right. The linguistic rendering of this procedure is given in the proof as: "adding as common the part of the triangle which is above the segment about the base" (Netz 2004, 249). And, when doing this, we finally conclude that "the lunule shall be equal to the triangle" (Netz 2004, 249). We would end the presentation of the proof by taking stock on the result just obtained with what we might conceive as a commentary: "(7) So the lunule, having been proved equal to the triangle, could be squared. (8) In this way, taking the outer circumference of a semicircle as the <outer circumference> of the lunule, he readily squared the lunule" (Netz 2004, 249). This "commentary" has two parts. The first calls attention to the fact that something extraordinary was just done: we squared the lune. That must be understood in its historical context. The quadrature of the lune is a particular version of the more general problem of the quadrature of the circle; i.e., finding the exact area of a circle, which, in practice, corresponds to finding a rectilinear figure with the same area of a circle. Hippocrates, for the first time, showed that the area of a circular-shaped figure – a lune – was exactly equal to the area of a rectilinear figure – a right-angled isosceles triangle. As Knorr mentioned, "an important principle has been established: that curvilinear figures, specifically those associated with circular arcs, are not different in kind from rectilinear figures as far as their quadrability is concerned" (Knorr 1985, 40). The second part of the commentary calls attention to the fact that the result is not general; it relates to a very particular lune. One in which the outer circumference of a semicircle is the outer circumference of the lune. Here, Hippocrates concludes his presentation of his proof to his students. He might then go on and present some other result included in his *Elements*.

3. Intentional reasoning as a cyclical process

To better frame the discussion of this section, we might start with a couple of "slug lines": "language is not thought, and vice versa" (Jackendoff 1996, 6); "Language and thought are not the same thing" (Fedorenko and Varley 2016, 132).

One way in which the differentiation of thought and language can be understood is by taking into account the brain regions in which the neural correlates of different cognitive processes are occurring. The point is to determine if regions associated with language are active during the different cognitive processes, in a way that we might consider that language has a role in these processes. There are robust findings that led us to consider that language is not determinant in many cognitive processes in which, at first, we might expect language to have a relevant role. Fedorenko and Varley give a clarifying assessment of the situation:

Language processing relies on a set of specialized brain regions […] These regions are not active when we engage in many forms of complex thought, including arithmetic, solving complex problems, listening to music, thinking about other people's mental states, or navigation the world. Furthermore, all these nonlinguistic abilities further appear to remain intact following damage to the language system, suggesting that linguistic representations are not crucial for much of human thought. (Fedorenko and Varley 2016, 145)

Since in this work our interest centers on mathematical proofs and their "associated" reasoning, we will consider results that bear directly on the approach being developed here; we will start by mentioning a few results regarding the issue of the role of language in what we can broadly name as conditional type of reasoning (Monti et al. 2007; Monti, Parsons, and Osherson, 2009; Coetzee and Monti 2017). In a set of experiments, participants had to assess the correctness of simple or more complex arguments that corresponded to those of propositional logic. One example of an argument is as follows:

If the block is either round or large then it is not blue
The block is round

The block is not blue (Monti et al. 2007, 1006).

The experimental results indicate that the role of language is only that of decoding the verbally presented argument into "non-linguistic structural representations" (Monti et al. 2007, 1009-10); The neural subtract of deductive reasoning proper is located in regions not related to language (Coetzee and Monti 2017). What we might call conditional reasoning is made without resort to language.

This kind of result if generalized would imply that language has no "active" role in reasoning; it would be only a means to communicate the result of reasoning or to, somehow, express linguistically the reasoning process (that would be independent of language). That is not the case.

Work on, everyday, argumentative reasoning, found a neural basis for this reasoning that largely differs from the neural basis previously associated with logical reasoning (Prado et al. 2020). This result is not incompatible with the ones mentioned just above since it has been noticed that the neural network that underpins logical reasoning is task-dependent (Goel and Waechter 2018).

From these results, we might expect that an eventual role of language in reasoning might depend on the particular reasoning being made and that the results by Monti and co-workers do not, by themselves, serve to conclude that, in the case of more complex reasoning, the language's only role is that of "decoding verbally presented information" (Coetzee et al. 2021, 22).

There is, in fact, research suggesting that language does have a role in complex reasoning. For example, Baldo et al. (2015) made a series of studies comparing the performance on complex reasoning tasks of a group of language-impaired (aphasic) stroke patients with another group of patients with their language faculties intact. The findings of these studies are as follows:

The aphasic group as a whole was disproportionately impaired on reasoning tasks relative to the non-aphasic group, but the two groups showed comparable performance on other cognitively demanding tasks that did not involve reasoning. (Baldo et al. 2015, 8)

These results point to language having a facilitating role in reasoning. The results in Baldo et al. (2015) suggest that the capacity to formulate and comprehend language "is most strongly related to reasoning performance" (Baldo et al. 2015, 9). That, together with other previous studies mentioned in Baldo et al. (2015), points to a possible role of inner speech, or more generally, talking to oneself, in facilitating reasoning (Baldo et al. 2015, 9).[12] The authors' position is that "language can 'facilitate' and is 'supportive' of higher-level reasoning capacity" (Baldo et al. 2015, 11).

In what follows, we will develop an approach to ancient Greek geometrical proofs based on this view; but, for that to be possible, we need a model of how speaking to oneself can have a facilitating or supporting role in reasoning.

Here, we will adopt Frankish's model of intentional reasoning (Frankish 2018). By intentional reasoning, we mean a "deliberate, intentionally controlled reasoning" (Frankish 2018, 10), in which we work out in a sequence of steps or "actions", e.g., a mathematical problem (Frankish 2018, 10).

Frankish conceives intentional reasoning as a cyclical process (Frankish 2018, 12-4). Let's say that we begin with a written (or spoken) sentence made of linguistic expressions and even symbolic ones (e.g., from mathematics). In Frankish's model, "we start with [the] sentence, interpret it as a step in an argument, form a belief about the next step, add that sentence, and so on" (Frankish 2018, 13). Figure 6 illustrates the cycles. Forming a belief and producing a new sentence would be the result of autonomous, non-conscious reasoning. That implies that, while the reasoning cycle is intentional – we are intentionally "immersed" in the cycle –, "the processes that guide and support this reasoning will be autonomous" (Frankish 2018, 10). In this way, "intentional reasoning is not wholly intentional, but guided and mediated by autonomous reasoning" (Frankish 2018, 10).

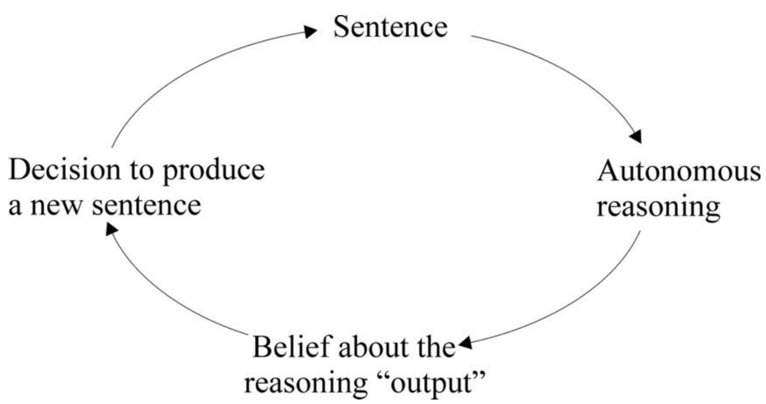

Figure 6. Frankish's model of intentional reasoning as a cyclical process

[12] For our purpose, we will treat inner speech and overt speech as equivalent. That is, grating a facilitating role to speaking to oneself, we will consider that doing this overtly or covertly does not affect our "performance" when following a mathematical proof, like the ones considered in the present work. However, it must be noticed that inner speech and outer speech are not the same. Current research indicates that, for adults, inner speech is impoverished at phonological and phonetic levels (Stephan, Saalbach, and Rossi 2020, 2).

What is the role of language then? In Frankish's view, "even if utterances were merely passing information, they would still be essential components in a larger, temporally extended reasoning process, since there may be no internal channels to pass the information" (Frankish 2018, 13). In this way, when we read or utter a sentence (as a step of an argument), "[our] language comprehension system interprets it, and its content is globally broadcast to other mental subsystems" (Frankish 2018, 14). Also, in his view, "it is not the case that the utterances are simply [broadcasting] information. They are passing it on in a certain context […] Each utterance is interpreted in the context of preceding utterances" (Frankish 2018, 15).[13] Finally, Frankish also considers that the process of linguistic articulation of the result of autonomous reasoning "profoundly shapes the overall reasoning process" (Frankish 2018, 15). This is so because "different choices of utterance may take the process in completely different directions" (Frankish 2018, 15).[14, 15] That fits well with both views we mentioned above. We might still have non-linguistic reasoning, like Monti and co-workers consider that is the case with conditional reasoning (which guides and mediates the intentional reasoning). But these moments of autonomous reasoning are embedded in an intentional and linguistically driven cycle, where we find the facilitating or supportive role of language.[16]

There is, in our view, a crucial aspect of the cyclical process proposed by Frankish not developed in his work, which has enormous consequences when addressing a mathematical proof in terms of intentional reasoning.

Let us return to Frankish's mentions to one aspect of the reasoning cycle: belief. He gave two examples of intentional reasoning as a cyclical process: a long division and an everyday speech-based reasoning process. In the case of the long division:

We begin by writing out the figures in a certain format […] and autonomous processes interpret them as posing a simpler division problem [...] autonomous processes then generate a belief about the solution to this subproblem and a decision to write down further symbols expressing it. (Frankish 2018, 15)

Notice that according to Frankish, the result of an autonomous reasoning process is not so much a "solution" but a belief about the solution.

Elsewhere, Frankish mentions that "judging that p [is the case] results in my forming the belief that p [is the case]" (Frankish 2018, 7). He also mentions that "a judgment produces a settled belief" (Frankish 2018, 1). We could re-frame this and say that an autonomous reasoning process produces not so much a solution but a belief that a solution is the case.

That is why Frankish mentions the decision "to communicate a belief" (Frankish 2018, 7), and that "speech production starts with a belief, whose content is the message to be communicated" (Frankish 2018, 7). So, in Frankish's account, it is not so much the solution p that is expressed but

[13] This is particularly the case with written sentences in a geometric proof, like the ones we are considering here.

[14] Evidently, this is also the case when the different choices of written sentences made by the author of a proof, lead us in our reasoning process.

[15] There are many views on the role of language in reasoning (see, e.g., Clark 2006, Carruthers 2012, Lupyan 2016, Dove 2018, Hinzen et al. 2019). For our purpose, we do not have to make precise this role. It suffices the following: a) consider that there is a role (even if left somewhat undefined) as experimental research suggests; b) whatever the precise role(s) might be it is such that humans engage in intentional reasoning as a cyclical process along the lines of the schematic model adopted here (an adaptation of Frankish's model).

[16] It might be the case that autonomous reasoning cannot be considered totally non-linguistic in the case of geometric reasoning. As we know, geometric objects rely on linguistic definitions (e.g., "a line is breadthless length" (Heath 1956, 153)). Several studies regarding abstract concepts show that the processing of abstract concepts depends more on language areas of the brain when compared to concrete concepts (see, e.g., Wang, Conder, Blitzer and Shinkareva 2010; Shallice and Cooper 2013; Binder 2016). That points to the possibility that geometric reasoning, dependent on abstract concepts as it is, might not be completely non-linguistic. However, this does not affect the views of this work. For us, the different "modules" of the schematic model that we develop here (e.g., autonomous reasoning) are like "black boxes"; their internal workings are not addressed in the present work.

the belief that p is the case.[17] It is rather unclear if there are reasons to suppose that what is expressed corresponds not to p but to the belief that p is the case. In the long division case, it is supposed that we write new mathematical symbols corresponding to further steps in the division. Do these sentences express p or the belief that p is the case? They seem to be an expression of the output of autonomous reasoning – in this case, a particular arithmetic result – and not an expression of the belief that this arithmetic result is the case. We find this approach in terms of belief a bit cumbersome and even confusing.

In our view, in the speech-based reasoning example, Frankish, unnoticing, touches upon what we take to be a better way to frame intentional reasoning. This example consists of speech-based everyday reasoning. It goes as follows: "suppose I have been invited to a party with colleagues from work. I don't find myself strongly disposed to respond one way or the other, but I need to give an answer, so I engage in intentional reasoning" (Frankish 2018, 13). The reasoning cycle goes as follows:

I begin by questioning myself […] "Do I really want to go?" […] I hear my own utterance, my language comprehension system interprets it, and its content is globally broadcast to other mental subsystems. My mindreading faculty interprets me as requesting information about the party or an evaluation of it, and further automatic processes throw up a prediction, based on experience that Henry will be there. This message is selected for expression […] and I utter the words: "Henry will probably be there". Again, this utterance is heard and interpreted. […] Again, a response is selected and articulated: He'll want to talk about budget cuts' […] My affective response – let us suppose – is strongly negative, and I conclude by uttering, "I can't face that; I won't go". (Frankish 2018, 14)

While previously, Frankish mentioned that his model "applies to speech-involving reasoning too" (Frankish 2018, 13), he, inadvertently, does not use the term "belief" in his example; instead, Frankish mentions an "affective response".

That connects with what we consider to be a better way to frame intentional reasoning without resort to a notion of belief. It has been noticed that different cognitive processes are accompanied by metacognitive processes. Concerning reasoning, we also have what we might call metareasoning. Metareasoning provides monitoring of the reasoning process. It gives an "alarm signal" if anything goes wrong, enabling us to correct (control) incorrect reasonings.[18] According to Fiedler, Ackerman, and Scarampi, "metacognition is ubiquitous because virtually all cognitive operations are monitored and controlled, before, during, and after their execution" (Fiedler, Ackerman, and Scarampi 2019, 90). A crucial aspect of metacognition is that it leads to a "subjective feeling" (Fiedler, Ackerman, and Scarampi 2019, 96). In the case of metareasoning, we have what we might call a feeling of rightness (Ackerman and Thompson 2017). An important aspect of these "subjective feelings" is that they have different degrees or intensities, like other affects.[19] We can go from a low feeling of rightness to a strong feeling of rightness. We can even have a feeling of error.[20, 21]

[17] Frankish's notion of belief is developed in Frankish (2004). Here, we do not have to address it in any detail. In the schematic model of intentional reasoning we adopt, there is no need for the notion of belief; instead, we incorporate metareasoning and its consequences into the model.

[18] It is generally agreed upon that metareasoning is dependent on heuristic cues (see, e.g., Ackerman 2019). This issue is not essential for the present work, so we will not address it.

[19] According to Efklides, "'affect' is a generic term for emotions and other mental states that have the quality of pleasant-unpleasant, such as feelings, mood, motives, or aspects of the self, e.g., self-esteem" (Efklides 2006, 3). Also, according to Efklides, a particular characteristic of metacognitive feelings or affects is that they "have a dual character, that is, a cognitive and an affective one" (Efklides 2006, 3). For our purpose, it suffices to say that they are part of "explicit" metacognition – i.e., conscious metacognition – that is subjectively experienced as a feeling to which a descriptive label can be attached to. For example, the feeling of rightness can be accompanied by the verbal label "I fell it is right".

[20] According to Ackerman and Thompson, for the time being, "it is not clear whether a feeling of error is qualitatively different from a low feeling of rightness or whether they represent two ends of a single dimension of certainty" (Ackerman and Thompson 2017, 612).

[21] Even if we take inspiration from Frankish's speech-based reasoning example with its "affective response", our approach is quite different and based on metareasoning.

According to this view, the output of an autonomous reasoning process is not only, e.g., a linguistic expression; it is this conscious expression accompanied by a conscious affect – a weaker or stronger feeling of rightness or correctness.[22]

We think that when taking metareasoning into account we should re-frame Frankish's model of intentional reasoning, since in our view it is conflating two aspects of the reasoning in one notion. It is not the case that the output of the autonomous reasoning process is, e.g., "I belief p"; it is "p" plus a feeling of correctness – i.e., the feeling that the reasoning is correct. As we have mentioned, Frankish hints at this when, in the case of everyday speech-based reasoning, he mentions an "affective response" accompanying, or previous to, the linguistic expression of the conclusion – the linguistic "output" of the reasoning: "I can't face that; I won't go".

We will reformulate the schematic model of intentional reasoning as a cyclical process taking into account that the conclusion of a reasoning process comes along with what we might call a feeling of correctness. Our schematic model is presented in figure 7 bellow.[23, 24]

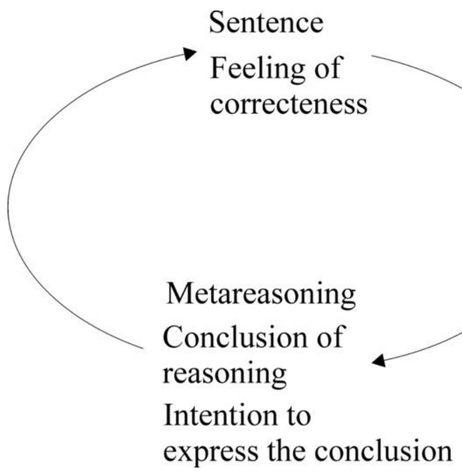

Figure 7. Schematic model of intentional reasoning as a cyclical process

Let us return to Hippocrates' proof by taking into account the present model of intentional reasoning as a cyclical process. If we imagine Hippocrates as making the proof for the first time, we would address it from the perspective of our schematic model as it is. But we want to address the proof as presented orally or in written form to other people. We suggest that the proof might be addressed as a form of guided intentional reasoning. The proof guides us into reproducing/recreating a cycle of reasonings. The initial phase previous to the proof proper is what we might call the setting of the stage. There is a sort of enunciation of what the proof is intended to show; the geometric knowledge useful for the proof is mentioned and the figure that will aid our reasoning is drawn. Then we start

[22] Since we will be considering the feeling of rightness in the context of mathematical proofs, we prefer to adopt the term feeling of correctness.

[23] In the figure, the relative position of the different components is not intended to imply that as neurological processes their temporal order corresponds to what is depicted. Even if we only had to address issues related to the component of metareasoning we do not know, as a process, how it blends with the other processes. We know that metacognition relies on domain-general resources but also on domain-specific components (Vaccaro and Fleming 2018; see also Rouault et al. 2018). Accordingly, "recent neuroimaging studies have highlighted both domain-general and domain-specific neural substrates" (Vaccaro and Fleming 2018, 2). Different metareasoning processes can have a different neural underpinning. This suggests that these processes might "blend" differently with the corresponding reasoning processes (which can also rely on different domain-specific components; see, e.g., Bartley et al. 2018).

[24] We do not include an independent component corresponding to the intention to express verbally the conclusion ("solution") of an autonomous reasoning process. We take the intention (as conceived within Levelt's model of language production, like Frankish does; see Levelt (1989)) to be already part of the setting into motion of the autonomous reasoning; i.e., the reasoning begins with a linguistic expression, and is such that it has as an "output" another linguistic expression (or what we might call an augmented linguistic expression; i.e., a linguistic expression augmented with, e.g., symbols and drawings of mathematical nature).

our proof.

The structure of the proof fits within the schematic model proposed. One of the main aspects of intentional reasoning as a cyclical process is that we do not jump to a final conclusion from an initial linguistic expression (sentence); there is in Frankish's words a "deliberative mastication" (Frankish 2018, 11). From sentence 0 we go to sentence 1 through an autonomous reasoning process, and so on until the final conclusion is reached in, say, sentence n. The oral or written proof guides us through cycles in which we face the premise and the conclusion of a reasoning process; we suggest that this leads us into producing an autonomous reasoning process that connects the linguistic expressions or sentences. We produce or reproduce the reasoning process that links the two sentences.[25] In this way, at the beginning of Hippocrates' proof, we would have the sentence corresponding to the geometric principle adopted as a starting point of the proof: both the similar segments of the circles, and their bases in square, have the same ratio to each other. We must also take the Pythagorean rule to be implicitly formulated (at least it is "in the back" of the students' minds). Also, these principles are to be taken into account for the case of the particular figure that was drawn (since we are not in the context of the same mathematical practice, we might also adopt an auxiliary figure).

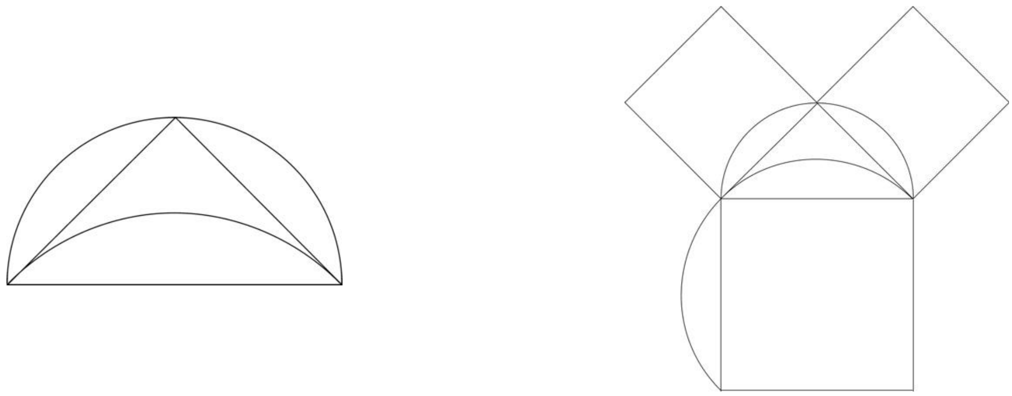

Figure 8. Drawing of a lune and a triangle as constructed during the proof (left); auxiliary figure (right)

With the aid of the above figures, we confirm the correctness of the conclusion as given, e.g., in the text: "the segment about the base [is] equal to both <segments> about the other <sides>" (Netz 2004, 249). That is, we produce or reproduce an autonomous reasoning process that leads us into feeling that the conclusion is correct. We suggest that this is the crucial aspect provided by framing mathematical proofs as a form of (guided) intentional reasoning. By dividing the proof into subproofs, one step at a time, we can reproduce the reasoning that connects each step, and our metareasoning generates a feeling of rightness or correctness associated to each step.

Having concluded a cycle of intentional reasoning we go into another, having the conclusion of the previous step as the starting linguistic expression for the next one. In this case, we take into account the previous conclusion and the geometric knowledge about the additivity of areas, again in the context of the particular figure being considered. We are also given a prescription of what to do, to enable the autonomous reasoning process. We must "[add] as common the part of the triangle which is above the segment about the base" (Netz 2004, 249). Following the "instructions", we are led into an autonomous reasoning process that enables us to confirm the correctness of the presented conclusion: "the lunule shall be equal to the triangle" (Netz 2004, 249).[26] The autonomous reasoning

[25] We want to clarify why we say that we produce or reproduce the reasoning process that links the two sentences. According to the present approach, we are guided, e.g., by the written text of the proof, which works for us as a written rendering of an intentional reasoning process (a cyclical one). For each step/cycle, we are "induced" into generating/producing an autonomous reasoning process that "links" the premise with the conclusion. It might be the case that when producing the reasoning we are, actually, reproducing the "original" reasoning by the author of the proof.

[26] Again, we must remark that our intention is not to provide a complete model of geometric proof. It is evident that there is a "visual character" to this reasoning, but our interest is in the role of metareasoning in each cycle of the proof as a

process is accompanied by a metareasoning process that gives rise to a feeling of correctness. We confirm by producing/reproducing the autonomous reasoning that it is correct.

We propose that the form given to mathematical proofs is such that it enables them to function as a form of (guided) intentional reasonings as cyclical processes so that we can have a strong feeling of correctness in each step of the proof. That is, accepting the correctness of sentence 0 as given, we can move through the proof, step by step, generating strong feelings of correctness associated with sentences 1, 2, and so on, until the final step where we reach the intended conclusion of the proof.

There might be important differences between the schematic model of intentional reasoning as a cyclical process and Frankish's model. Frankish's choice of words in the speech-based reasoning example seems to imply that a person imagines himself/herself at the party and notes his/her affective response to the scenario. The affective response would be the evidential base of the judgment that he/she does not want to go. If this is the case, this is a completely different role from that of the feelings of correctness one experiences in working out a proof. This latter "feeling" is not part of the evidential base for judging the correctness of the proof. Metareasoning is an autonomous process that accompanies the also autonomous process of reasoning (even if both are "embedded" in an intentional reasoning process). The metacognitive "feeling" in its dual nature, of a cognitive and an affective one, is what makes us fell-know that the inferential step is correct (even if we might be wrong). The "evidence" is hidden in the reasoning and metareasoning processes. We only have the premise and the conclusion. Guided by the text, we must go through the reasoning process ourselves; it is the "output" of the metareasoning process, the feeling-knowing of correctness, which, as a "subjective" (i.e., conscious) mental phenomenon, makes us "aware", so to speak, of the correctness of the inferential step.

It is important to notice that having a strong feeling of the correctness of each step, and because of this of the whole proof, does not imply that we are right. Our metareasoning might be leading us astray. We can fell-know that a reasoning process is correct, i.e., we can have a strong feeling of correctness associated with the reasoning process, but, ultimately, be completely wrong. In fact, it is known that there are important biases regarding the feelings of rightness generated by our metareasoning processes (see, e.g., Fiedler, Ackerman, and Scarampi 2019).

In this case, if, ultimately, accepting, e.g., Hippocrates' proof as correct depends on our metareasoning then can we be sure that the proof is "really" correct? Here, we only suggest that the "objectivity" of the correctness of the proof could be the result of a robust intersubjectivity: a strong feeling of correctness, shared by many, that results from group discussions that enable to arrive at this strong feeling of correctness for most of the participants, or even to improve the proof (e.g., if they find reasoning gaps, unclear passages, things left implicit, etc.), in a way that this shared strong feeling is finally reached. There are, however, issues regarding the effectiveness of group discussions (see, e.g., Silver, Mellers, and Tetlock 2021; Bang and Frith 2017). So, there is no simple answer.

4. Two proofs from Euclid's *Elements*

How geometric results were proved changed from the time of Hippocrates to that of Euclid, roughly one and a half centuries later (see, e.g., Mueller 2006). In the case of Hippocrates, as we have seen, general geometrical principles were used without being proved. With Euclid, we do not have that anymore. Euclid's assumptions are much simpler than those admitted by Hippocrates. In Euclid's *Elements*, we find definitions like that of point, straight line, or circle (Heath 1956, 153); we also have postulates, like the ones that license the construction of a straight line, or a circle (Heath 1956, 154); there are also common notions that make more precise general principles that were applied intuitively by Hippocrates, in particular in the context of addition of areas. For example, we can take Hippocrates to use the following common notion: "if equals be added to equals, the wholes are equal" (Heath 1956, 155).

Since we only admit a few very basic assumptions, even the simplest geometric results must be

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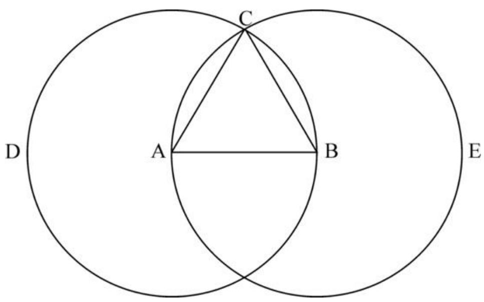
form of guided intentional reasoning, not in the workings of the autonomous reasoning process.

proved. This implies that, contrary to the case of Hippocrates, the proofs have generally more than a single level. In this way, besides relying on the admitted assumptions, a proof may rely on previously proved propositions. This can create a quite complex structure. For example, a proof of a result that seems quite simple like that of constructing, from a point, a perpendicular to a given line is made in Euclid's *Elements* in proposition 12 of book 1.[27] For that purpose, Euclid not only takes into account admitted assumptions like postulates 1 and 3 (that license the construction of a straight line and a circle, respectively) but also previously proved propositions; in this case, proposition 8. But proposition 8 relies on proposition 7, which relies on proposition 5, which relies on propositions 3 and 4. Proposition 4 stands on its own; it only relies on the admitted assumptions, not on other previous propositions. Proposition 3 relies on proposition 2, which depends on proposition 1, which stands on its own (Heath 1956, pages 241-271).[28] This is the general case for the proofs in Euclid's *Elements*.

To address Euclidean geometric proofs in terms of (guided) intentional reasoning, we think it is clearer if we start by addressing a proposition that does not depend on other previously proved propositions.

The simplest case is that of the first proposition in the *Elements* of Euclid (I.1). The Euclidean text is as follows:

On a given finite straight line to construct an equilateral triangle.



Let *AB* be the given finite straight line.
Thus it is required to construct an equilateral triangle on the straight line AB.

With center *A* and distance *AB* let the circle *BCD* be described; [Post. 3]
again, with center *B* and distance *BA* let the circle *ACE* be described; [Post. 3]
and from the point C, in which the circles cut one another, to the points *A, B*
let the straight lines *CA, CB* be joined. [Post. I]
Now, since the point *A* is the center of the circle *CDB, AC* is equal to *AB.* [Def. I5]
Again, since the point *B* is the center of the circle *CAE, BC* is equal to *BA.* [Def. I5]
But *CA* was also proved equal to *AB;* therefore each of the straight lines *CA, CB* is equal to *AB.*
And things which are equal to the same thing are also equal to one another; [C. N. 1]
therefore *CA* is also equal to *CB.*
Therefore the three straight lines *CA, AB, BC* are equal to one another.
Therefore the triangle *ABC* is equilateral; and it has been constructed on the given finite straight line *AB.*
(Being) what it was required to do. (Heath 1956, pp. 241-242)

The structure of the Euclidean proof has points in common to that of Hippocrates, but also differences.

[27] We must notice that we are not simply constructing a perpendicular; we are doing that while proving that the constructed line is, actually, perpendicular to the given line.

[28] While Euclid constructs a perpendicular from the admitted assumptions by relying on an intricate assemblage of proofs, Hippocrates simply takes this result for granted using Oinopides' construction without proof, or even older practitioners' methods (Høyrup 2019a, 165).

In the first, we have an explicit enunciation of the proposition that will be demonstrated. It is not clear if we might have that with Hippocrates or some more vague introduction. In his case, what is mentioned as the starting point is the geometric principle being assumed and necessary for the proof. This does not occur with Euclid. There are no principles assumed; Euclidean assumptions are much more basic (definitions, postulates, and common notions), and they are used throughout the proof, usually without being mentioned. Then, in both proofs, we have the construction of a geometric figure, and it is afterward that we have what here we address as the written rendering of a cyclical process of intentional reasoning. The reader goes through autonomous reasoning and metareasoning processes while following the proof; i.e., having the verbal premise(s) and the verbal conclusion, the reader – or the listener of an oral presentation – will go through an autonomous reasoning process that gives rise to a strong feeling of correctness.

Analyzing the structure of the proof in I.1 in terms of intentional reasoning, we start with a simple cycle that leads from sentence 0 ("point *A* is the center of the circle *CDB*") (as a premise)[29] to the conclusion in sentence 1 ("*AC* is equal to *AB*"). We would then start a new cycle with sentence 2 ("point *B* is the center of the circle *CAE*") as a new premise for another autonomous reasoning,[30] which has as its verbal conclusion sentence 3 ("*BC* is equal to *BA*"). Sentence 3 will be one of the premises of a third autonomous reasoning process; this reasoning process is made by taking sentence 4 ("*CA* was also proved equal to *AB*") also as a premise. The conclusion of this cycle is sentence 5 ("each of the straight lines *CA, CB* is equal to *AB*"). Sentence 5 will be one of the premises for a new cycle, together with sentence 6 ("things which are equal to the same thing are also equal to one another"). The conclusion of this cycle is sentence 7 ("*CA* is also equal to *CB*"). In the next cycle, we have three premises and only sentence 7 is explicit. This reasoning cycle starts with sentence 7 ("*CA* is also equal to *CB*") but also takes into account previous conclusions now as premises: sentence 3 ("*BC* is equal to *BA*"), and sentence 4 ("*CA* was also proved equal to *AB*"). The conclusion is sentence 8 ("three straight lines *CA, AB, BC* are equal to one another"). In the final cycle, sentence 8 becomes the premise for another (guided) autonomous reasoning process, which must lead to the conclusion ("the triangle *ABC* is equilateral") (as sentence 9).[31]

The proof of I.1 consists of 6 cycles of intentional reasoning. In each of these cycles, our metareasoning generates a strong feeling of correctness regarding the autonomous reasoning process being made.

The proof of I.1 is a bit more general than the schematic model proposed by Frankish (2018) and further elaborated in the present paper. As we have just seen, in general, the verbal conclusion of a cycle is not, just by itself, taken to be the premise of the following cycle. This is the case only with cycle 6, from sentence 8 to sentence 9. More generally, we add explicitly (or implicitly, like when going from sentence 7 to sentence 8) previous conclusions as premises to enable a new reasoning cycle (or even an assumption, like common notion 1 in cycle 4). However, even if a bit more complex, we still have a structure of a (cyclical) intentional reasoning process.

We will now consider another type of proof that was pervasive in ancient Greek mathematics; it is called in Latin "reductio ad absurdum" (see, e.g., Netz 1999, Heath 1981). We will call it simply a reductio proof.

The form of a reductio proof is, schematically, as follows: to prove P, assume not P, derive a "contradiction"; i.e., derive an obviously false result, like a geometric statement and its opposite (e.g., $AB = CD$ and $AB \neq CD$). Conclude that P is true (Cunningham 2012, 93).

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[29] We do not include "since" as part of the premise; it works as an explicit indicator that what follows is the premise. We also will not include "therefore" as part of the conclusion; it works as an explicit "connector", linking the premise to the conclusion (see Netz 1999, 115-6).

[30] More appropriately, it is a guided autonomous reasoning process since the reader has at his/her disposal also the verbal conclusion of the reasoning. In the modern version of the text, sometimes it is indicated what assumption is used during the reasoning. In this case, it is the definition of circle ([Def. 15]). This definition is used in the context of the figure to arrive at the conclusion. Here, we do not try to address this complex reasoning combining linguistic definitions and visual cues.

[31] While it is not indicated in the text, we must take the reasoning to be made by taking into account the definition of equilateral triangle (Def. 20; see Heath (1956, 154)).

Ancient Greek proofs that adopt a reductio strategy have a very fixed argumentative structure, as described by Netz (1999, 140). Simplifying, when we intend to prove a geometrical statement P, we use "for if" to introduce the negation of P (i.e., not P) and a consequence of not P. A sequence of "arguments" follows (i.e., the verbal expressions corresponding to premises and conclusions), leading to some property, "which is impossible/absurd" (Netz 1999, 140). Afterward, we find the crucial arguments in the reductio proof. First, we have "therefore not (not P)"; second, we have "therefore P".

Here, a pause is necessary. As mentioned by Dutilh Novaes, the step of going from reaching an absurd result to conclude that P is the case (which in the Euclidean proof is made in two steps, as we have just seen), "strongly relies on a number of assumptions, and if these are not in place then the argument does not go through" (Dutilh Novaes 2016, 2625). These are related to what can be named the culprit problem and the act of faith problem. The first problem is described by Dutilh Novaes as follows:

We start with the initial assumption, which we intend to prove to be false, but along the way we avail ourselves of auxiliary hypotheses/premises. Now, it is the conjunction of all these premises and hypotheses that leads to absurdity, and it is not immediately clear whether we can single out one of them as the culprit to be rejected. (Dutilh Novaes 2016, 2614)

To avoid the culprit problem, we must assume that "we can isolate the culprit" (Dutilh Novaes 2016, 2615). For that: "what is required is that all auxiliary assumptions/ premises used in the argument have a higher degree of certainty to us than the initial assumption that is singled out to be rejected" (Dutilh Novaes 2016, 2625).

The second problem can the stated as follows: "How does one go from it being a bad idea to maintain the hypothesis to it being a good idea to maintain its contradictory?" (Dutilh Novaes 2016, 2615). According to Dutilh Novaes, "A reductio ad absurdum also starts with the tacit assumption of an exhaustive enumeration of cases: for a given proposition A, either A is the case or its contradictory is the case, and not both" (Dutilh Novaes 2016, 2615). Accordingly:

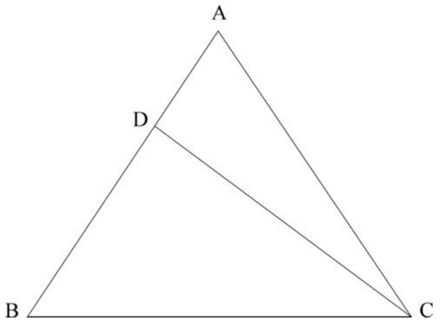
If we can be sure that the enumeration of cases is truly exhaustive (i.e., excluded middle holds in the relevant situation), and that we will not end up in a situation of aporia where all options lead to absurdity, then we can safely conclude not-p after showing that p leads to absurdity. (Dutilh Novaes 2016, 2625)

Here, we want to suggest that the culprit problem and the act of faith problem are avoided in Euclidean geometry and are dealt with, each one, one step at a time, in the final stages of the argumentative structure of Euclidean reductio proofs. As mentioned in the argumentative structure there is one point in which we reach some property which is impossible/absurd. Afterward, we deal with the culprit problem: therefore not (not P). There is an autonomous reasoning process that identifies not P as the culprit (and not some other assumption or set of assumptions). Then we face the act of faith problem: therefore P; i.e., the rejection of not P leads to an autonomous reasoning process that enables us to conclude that P is the case. We will consider an example of a reductio proof to see how this is done and how a reductio proof can be addressed from the perspective of proof as a form of (guided) intentional reasoning.

We will consider the first reductio proof in the *Elements*; that of proposition 6 of book 1 (I.6). The text of proposition I.6 reached us as follows:[32]

If in a triangle two angles be equal to one another, the sides which subtend the equal angles will also be equal to one another.

[32] Here, like in the case of I.1, we are considering an English translation of the standard edition of the *Elements*. On this see, e.g., Vitrac (2012).



Let *ABC* be a triangle having the angle *ABC* equal to the angle *ACB;*
I say that the side *AB* is also equal to the side *AC.*
For, if *AB* is unequal to *AC,* one of them is greater. [C.N.][33]
Let *AB* be greater; and from *AB* the greater let *DB* be cut off equal to *A*C the less; [I.3]
let *DC* be joined. [Post. 1]
Then, since *DB* is equal to *A*C, and *BC* is common, the two sides *DB, BC* are equal to the two sides *A*C, *CB* respectively;
and the angle *DBC* is equal to the angle *ACB;*
therefore the base *DC* is equal to the base *AB,* and the triangle *DBC* will be equal to the triangle *ACB, [I.4]*[34]
the less to the greater:
which is absurd [C.N. 5].
Therefore *AB* is not unequal to *AC*;
it is therefore equal to it. (Heath 1956, 255-6)
Thus, if a triangle has two angles equal to one another then the sides subtending the equal angles will also be equal to one another. (Which is) the very thing it was required to show.[35]

The demonstration starts with a triangle *ABC* with two equal angles, *ABC* and *ACB*. What we want to show is that the corresponding sides are equal; i.e., $AB = AC$. We start our proof by assuming the "opposite" property $AB \neq AC$ (that they are "opposite" follows from the "solution" to the act of faith problem). The first intentional reasoning process starts with this initial assumption (our sentence 0) and leads us to the conclusion: "one is greater" (our sentence 1). To reach the conclusion use is made of the previously unmentioned common notion: if two quantities are unequal, then one is greater or lesser than the other (see, e.g., Joyce 1998).

After this first cycle of intentional reasoning that includes an autonomous reasoning process, we have what we might call the "insertion" of another construction. This is made in "sentence 2": "Let *AB* be greater; and from *AB* the greater let *DB* be cut off equal to *A*C the less; let *DC* be joined". The new figure shows that the triangle *DBC* is smaller than the triangle *ACB*.[36, 37] This "property" will be important later since it will lead to a contradiction with another property that will be deduced later.

The intentional reasoning process resumes with a cycle going from sentence 3 ("*DB* is equal to *A*C, and *BC* is common") to sentence 4 ("the two sides *DB, BC* are equal to the two sides *A*C, *CB* respectively; and the angle *DBC* is equal to the angle *ACB*"). The (guided) autonomous reasoning

[33] According to Joyce (1998), there are other properties of quantities or magnitudes (angles, lengths, areas, etc.) that are considered in I.6, besides the ones that are made explicit as common notions. In Joyce's view, here, Euclid uses a common notion that was not included in the initial assumptions. It might be something like this: if two quantities are unequal, then one is greater or lesser than the other.
[34] Notice that the proof in I.6 is not a single-level proof since it relies on other previously proved propositions besides the general assumptions (definitions, postulates, and common notions).
[35] These last sentences are from the translation by Fitzpatrick (2008, 13) since in Heath's translation it is left incomplete.
[36] On diagrams in reductio proofs, see, e.g., Dal Magro and Valente (2021).
[37] In a way, in the context of sentence 2 and the corresponding drawing arises a reasoning process leading to the conclusion that the triangle *DBC* is smaller than the triangle *ACB*. Since, only later, this conclusion is taken into account, it might well be the case that this reasoning process only occurs at this later stage. In this way, we will leave it as implicit somewhere during the cycles, and we will not consider that, at this point, we have a cycle of the reasoning process.

process, leading from premise 3 to conclusion 4, uses properties of the triangles as depicted in the figure and given in the text. The conclusion as given in sentence 4 becomes a premise for the next cycle of intentional reasoning, leading to sentence 5 ("the base *DC* is equal to the base *AB,* and the triangle *DBC* will be equal to the triangle *ACB*"). The (guided) autonomous reasoning process is to be made by resort to proposition I.4 (as indicated by the inclusion in the text of [I.4]). That is our third cycle of intentional reasoning (including the cycle from sentence 0 to sentence 1). In the previous conclusion, we reached the result that the triangle *DBC* is equal to the triangle *ACB*. This part of the conclusion, together with the previously obtained result, as a consequence of the construction due to sentence 2, that the triangle *DBC* is smaller than the triangle *ACB*, work as the premisses for the next cycle of intentional reasoning (cycle number 4). The verbal conclusion of this reasoning, in sentence 6, is that "the less [is equal] to the greater". This conclusion is the start of another autonomous reasoning process that leads to the conclusion, as stated in sentence 7, that the previous result is absurd. During the autonomous reasoning processes, our metareasoning generated a strong feeling of correctness for each one. At this point, we might feel confident that so far, our proof is correct.

We will now consider the two final cycles, which are the most complex ones since in the first, we address, implicitly, the culprit problem and, in the second, the act of faith problem.

After arriving, in sentence 7, at the conclusion that an absurd result was reached, we go into a new cycle of intentional reasoning, taking this absurdity as the premise. The conclusion of this autonomous reasoning process is given in sentence 8: "*AB* is not unequal to *AC*"[38] (i.e., we deny the initial assumption that *AB* is unequal to *AC*). In this cycle of (guided) intentional reasoning, we reason to the rejection of the initial assumption $AB \neq AC$. It is here that the culprit problem is faced. Since there are no auxiliary assumptions besides the initial assumption,[39] we use this knowledge in the reasoning and conclude that the absurd result that triangle *DBC* = triangle *ACB* and triangle *DBC* ≠ triangle *ACB* follows from the initial assumption. Since there is one reasoning cycle to address the culprit problem, the metareasoning during this cycle generates a strong feeling of the correctness of the conclusion, in this way "dissipating" any doubts we might have in this inferential step concerning the culprit problem.

In cycle 7, the final cycle of our (guided) intentional reasoning, we take sentence 8 ("*AB* is not unequal to *A*C") as the premise for another reasoning process (the seventh reasoning process). The act of faith problem is faced during this reasoning process. As mentioned by Fitzpatrick, here we use a previously unmentioned common notion: if two quantities are not unequal then they must be equal (Fitzpatrick 2008, 13). This common notion implies that, for quantities, there are no other cases/options that might lead to an act of faith problem. If two lengths are not unequal then these lengths must be equal; there is no other option. The reasoning process taking into account this evident property of quantities leads to the conclusion: "it is therefore equal to it"; i.e., $AB = AC$.[40] By addressing just the act of faith problem in this last cycle, we generate a strong feeling of correctness that "dissipates" any doubts we might have concerning the act of faith problem.[41]

As we see, reductio proofs fit nicely within a perspective on proofs as a form of guided intentional reasoning.[42]

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[38] We have not included the connective "therefore".

[39] All the other "assumptions" are taken to be "true" (definitions, postulates, common notions) or proved to be true (propositions I.3 and I.4).

[40] We do not face any risk of having a situation of aporia where all options lead to absurdity. We have determined that if the triangle had the sides *AB* and *AC* unequal, there would be an absurd consequence; that of two triangles being at the same time equal and different. The only alternative that remains is that the sides *AB* and *AC*, of the triangle, are equal. If this also leads to absurd consequences, then pure geometry would not be consistent.

[41] It is not necessary that we are conscious of these problems, to intuitively address them in different cycles of intentional reasoning. That ancient Greek mathematicians made this suggests that they had a grasp of some "difficulties" that had to be addressed before arriving at the conclusion. By having two cycles of reasoning, one for each "difficulty", they improved the feeling of correctness for each one; in this way, providing a stronger feeling that, overall, the proof is correct.

[42] This is not to say that there are no other approaches that give an enlightening perspective on mathematical proofs, in general, and reductio proofs, in particular. We are thinking, in particular, about the dialogical conception of mathematical proof (Dutilh Novaes 2018, 2020) and the view of mathematical proof as audience-reflective argumentation (Ashton 2021). However, it is beyond the scope of the present work to address the fitting of the approach developed here with

5. Further comments

Let us return to what we mentioned in the introduction; we were going to see how we still can evaluate an ancient mathematical proof and agree on its correctness. To be more precise, our purpose was, in relation to ancient Greek proofs, to see how does a proof functions, for us, as an enabler to ascertain its correctness; and how do we, actually, ascertain this correctness. The mathematical proof works as a guided intentional reasoning process (GIRP); or better, cycles of these. Each cycle corresponds to what we usually call an inferential step that goes from a premise to a conclusion (both given as linguistic expressions – sentences). In each cycle, these expressions are "connected" by an autonomous reasoning process that produces/reproduces a reasoning process that leads from the premise to the conclusion. This "format" of the proof enhances the metareasoning processes that accompany the reasoning processes. We have a sequence of "small" reasoning steps, each accompanied by a metareasoning process. It is our metareasoning that creates a feeling-knowing of correctness associated with the verbal conclusion of an autonomous reasoning process. At the end of an inferential step (i.e., a cycle of intentional reasoning), we have a verbalized conclusion arising from our reasoning "machinery" and it comes with a weaker or stronger feeling of its correctness. Feeling-knowing that the conclusion is correct, we move on with the proof, adopting this verbal conclusion as the starting point for another cycle of reasoning that must lead to the next verbal conclusion presented in the proof. That goes on until the last conclusion that ends the proof is reached. Having a strong feeling of correctness associated with every "inferential step" enables us to assess the correctness of the whole mathematical proof.

As mentioned at the end of section 3, that one individual might have a strong feeling of correctness associated with a mathematical proof does to imply that he/she is not wrong, since there are important biases regarding the feelings of rightness generated by our metareasoning processes. So, how can we reach the idea that a proof is "objectively" correct? According to the view proposed here, the "objectivity" of the correctness of a proof must be built on top of individual strong feelings of correctness. We hypothesize that the "objectivity" of the correctness of mathematical proofs could be the result of a robust intersubjectivity: a strong feeling of correctness, shared by many. But this is an issue to be addressed elsewhere.

6. Coda: Is an approach in terms of GIRPs really about mathematical proofs?

Here, we will consider the position of a hypothetical reader, Skeptic. She/he might argue that the approach presented in the present work is not really about mathematics since it pays no attention to distinctive features of mathematical proof such as the use of notation that partially encodes rigor in its syntax, the use of diagrams, and so on. As it is, Skeptic might say that the approach to mathematical proofs as a GIRP could have been illustrated just as well with, say, legal reasoning or the verbal logic problems from general intelligence tests.

The first part of an answer would be that this is quite so. Here, we focus on commonalities, not specificities; that is, on aspects of mathematical proofs that can be shared with human reasoning with/about different human practices, activities, etc. The "distinctive features of mathematical proof" can then be addressed from within this approach in terms of GIRPs.

What this approach brings to the fore is the role of metacognition also in mathematical reasoning and, more specifically, in mathematical proofs. It is a metareasoning process related to the mathematical reasoning process that gives rise to a feeling of correctness. And yes, this is a common feature of human reasoning: "metacognition is ubiquitous because virtually all cognitive operations are monitored and controlled, before, during, and after their execution" (Fiedler, Ackerman, and Scarampi

other approaches.

2019, 90). Regarding our experiencing of metareasoning as a "subjective feeling" we must notice that this is not an "emotional" response. We know that the reasoning is correct because we feel that the reasoning is correct. What we mean is that this "feel" as an affect can be more of a cognitive nature and less of an affective one. As mentioned, a particular characteristic of metacognitive feelings or affects is that they "have a dual character, that is, a cognitive and an affective one" (Efklides 2006, 3). So, we do not have to think of metacognition as giving rise to bursts of emotional responses while we follow a proof. If it makes us more comfortable, we might think about the "output" of metareasoning as a "purer" strong "knowing" that the inferential step is correct: we fell-know its correctness.

Now, to the second part of the answer. Mathematical specificities are to be found in the reasoning process and accompanying metareasoning process that occurs in each cycle of a GIRP. Nowadays, geometric cognition and reasoning are still poorly understood.[43] This is not to say that we must wait for science to open the "black boxes" of (geometric) autonomous reasoning and metareasoning to see geometric specificity at work in a geometric proof as a GIRP.[44, 45] We have seen an example of mathematical specificity in the present paper. Skeptic would have to agree that reductio proofs are a hallmark of mathematics not to be found in other human practices (e.g., in legal reasoning). And there is a good reason for this that is made clear by addressing mathematical proofs as GIRPs. As we have seen, to be able to adopt a reductio proof we must face the culprit problem and the act of faith problem. If these are not avoided, "the argument does not go through" (Dutilh Novaes 2016, 2625). It is a specificity of mathematics (geometry, in this case) that we do not have these problems. As we have seen, the culprit problem is faced in one of the cycles of the geometric proof as a GIRP. Since there are no auxiliary assumptions besides the initial assumption, the problem simply does not arise (as mentioned, since there is one reasoning cycle to address the culprit problem, the metareasoning during this cycle generates a strong feeling of the correctness of the conclusion). In the same way, there is another cycle in which the act of faith problem is "dissipated". It simply does not arise due to the specificity of geometry. That is an example where Skeptic can find a distinctive feature of mathematical proof at work "directly" on the mathematical proof as a GIRP.

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[43] For some views on the issue see, e.g., Hohol and Miłkowski (2019), Ferreirós and García-Pérez (2020) and Hohol (2020).

[44] In fact, we can already rely on important "insights" from inquiries into, e.g., the role of diagrams and diagrammatic reasoning in proofs like the ones considered in the present paper. A small list of some of the relevant works on these matters is: Freksa, Barkowsky, Falomir, and van de Ven (2019); Dal Magro and García Pérez (2019); Magnani (2013); Giardino (2013); Manders (2008); Giaquinto (2007).

[45] An interesting research venue for the case of Euclidean proofs might result by bringing together a metareasoning-related investigation to an approach in terms of cognitive artifacts like linguistic formulae and lettered diagrams, which, in the view of Hohol and Miłkowski (2019), constitute mutual constraints for mathematical reasoning.

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