A LOGICAL MODELING OF SEVERE IGNORANCE

Contents

ABSTRACT. In the logical context, ignorance is traditionally defined recurring to epistemic logic S_4 [21]. In particular, an agent ignores a formula φ when s/he does not know neither φ nor its negation $\neg\varphi$: $\neg\mathbf{K}\varphi \wedge \neg\mathbf{K}\neg\varphi$ (where $\mathbf{K}$ is the epistemic operator for knowledge). In other words, ignorance is essentially interpreted as "lack of knowledge". This received view has - as we point out - some problems, in particular we will highlight how it does not allow to express a type of content-theoretic ignorance, i.e. an ignorance of φ that stems from an unfamiliarity with its meaning. Contrarily to this trend, in this paper, we introduce and investigate a modal logic having a primitive epistemic operator $\mathbf{I}$, modeling ignorance. Our modal logic is essentially constructed on the modal logics based on weak Kleene three-valued logic introduced by Krister Segerberg [34]. Such non-classical propositional basis allows to define a Kripke-style semantics with the following, very intuitive, interpretation: a formula φ is ignored by an agent if φ is neither true nor false in every world accessible to the agent. As a consequence of this choice, we obtain a type of content-theoretic notion of ignorance, which is essentially different from the traditional approach based on S_4. We dub it *severe ignorance*. We axiomatize, prove completeness and decidability for the logic of reflexive (three-valued) Kripke frames, which we find the most suitable candidate for our novel proposal and, finally, compare our approach with the most traditional one.

1. INTRODUCTION

The study of ignorance is certainly as old as the study of knowledge; however the formal study of the logic of ignorance is still a

young area of research. In the epistemological studies of ignorance the standard view is to define it as lack of knowledge (see for example [27], [24], [26], [25]). We believe that this is the reason why also the formal study of the logic of ignorance has been developed with reference to the formal study of the logic of knowledge. This tradition is mainly due to the work of Hintikka [21], who distinguishes two notions of lack of knowledge relative to an agent, namely "*a* (an agent) does not know that φ" ($\neg \mathbf{K}_a \varphi$) and "*a* does not know whether φ" ($\neg \mathbf{K}_a \varphi \wedge \neg \mathbf{K}_a \neg \varphi$). It seems that, according to Hintikka, only the latter explicates the notion of ignorance; indeed, he [21, p.12] formalizes ignorance (of an agent *a*) as $\neg \mathbf{K}_a \varphi \wedge \neg \mathbf{K}_a \neg \varphi$. Such regimentation has become standard in the logical literature on ignorance. For this reason, by the expression "ignorance as lack/absence of knowledge", we will refer to Hintikka's view throughout this article. From psychology to education studies, passing through philosophy and many other disciplines, a plenitude of deep analyses of knowledge and ignorance have been put forward ([20], [29], [28], [13], [1]) and the standard view in the literature describes ignorance in terms of lack, or absence, of knowledge.Therefore, it is not surprising that this is also the standard view in the logical treatment of ignorance. However, in more recent times, van der Hoek and Lomuscio [39] introduced a modal logic (**Ig**) where ignorance is modeled by a primitive modal operator, unrelated to (lack of) knowledge. The spirit behind **Ig** is expressing "ignorance as a first class citizen" [39, p.3]. However, despite their intention, their solution does not seem too far from Hintikka's lack of knowledge. Indeed, in their semantics for the operator **I** – for ignorance – an agent ignores φ if s/he has access to two (different) worlds, where φ is evaluated differently (true in one and false in the other). In their own words (again): "[the] formula $\mathbf{I}\varphi$ is to be read as «the agent is ignorant about φ, i.e. s/he is not aware of whether or not φ is true»". The semantics of **I** is then the same as in Hintikka [21, p.12], with the only difference that **Ig** "can not speak" about knowledge.

Similarly, the *Logic of Unknown Truths* (LUT) and the subsequent logics of ignorance proposed by Steinsvold [35] subordinate the concept of ignorance to that of knowledge. In these logics the black box (■) in fact stands for $\varphi \wedge \neg \mathbf{K}\varphi$; if the latter formula is true, and $\varphi \rightarrow \neg \mathbf{K}\neg \varphi$ holds, then also $\neg \mathbf{K}\varphi \wedge \neg \mathbf{K}\neg \varphi$ holds, which is again Hintikka's definition of ignorance.

Following the research trend opened in Fano and Graziani [11], this article intends to discuss the fact that *lack of knowledge* is just one way to look at ignorance and, taking up van der Hoek and Lomuscio's challenge, to introduce a logic which addresses the purpose of defining "ignorance as a first class citizen". In this paper, after discussing the consequences of defining ignorance as lack of

knowledge (in the epistemic logic S_4), we introduce and investigate
a modal logic having a primitive epistemic operator **I**, modeling ig-
norance. In particular, the idea we have in mind is that of modelling
a type of *content-theoretic ignorance*, so to say an ignorance of some-
thing that stems from an unfamiliarity with its meaning, i.e. a *severe*
notion of ignorance that implies a lack of awareness with respect
to a subject-matter. In our view, this type of ignorance constantly
affects the practice of science. For instance, consider the follow-
ing situation: Max Planck, in approaching the black body radiation
problem, knew that, in the theoretical predictions of the black body,
there was a divergence for high frequencies, in contrast with ex-
perimental data. However, he did not simply ignore which physical
phenomena constituted the cause, but, more importantly, he did not
have any idea (is ignorant) of what could be a bundle of causes. In
logical terms, it is not *merely* the case that Planck does not know the
truth value of a physical statement (that could be the cause), but he
does not know which kind of event could be a cause. In other terms,
when thinking about severe ignorance, we have in mind situations
where scientists are ignorant of the bundle of causes that might be
at the root of a phenomenon. Contrarily, Hintikka's ignorance, in
particular in the variant proposed by Steinsvold, appears related to
the lack of knowledge of single agents, such as, for instance, a phys-
ical statement that is, perhaps, known in the community of trained
physicists but, possibly, ignored by a non-physicist. To achieve
the goal of modeling severe ignorance, we base the semantics of
our (modal) logic on the presence of a third truth-value, whose be-
haviour is infectious, as severe ignorance ultimately is. Returning
to the example about Planck's ignorance, the infectivity of his igno-
rance depends on the fact that every scientific issue whose content
is theoretically connected to the explanation of the black body is
ignored severely at the same way that the explanation is. The most
natural examples of infectious logics are the so-called weak Kleene
logics, which can be intuitively introduced via a matrix where truth-
values $\{0,1\}$ are joined by a third truth-value $1/2$ whose behaviour
is infectious in the sense that a complex formula φ is evaluated to
the third value $1/2$ whenever any of its atomic formulas is evaluated
to $1/2$ (independently of the structure of φ). Our modal logic will
be essentially constructed following the ideas of the modal logics
based on (one of the) weak Kleene logics introduced by Prior [31]
and Segerberg [34]. Our philosophical approach keeps fixed the
classical account that ignorance, as well as knowledge, is an *epis-
temic* notion and, for this reason, the logical modeling we primarily
purse is an epistemic (modal) logic, whose privileged semantics is
a relational one (Kripke-style). As a byproduct of our analysis, we
discover that the non-classical propositional basis chosen (Bochvar

external logic) indeed already incorporates (some) connectives that can be interpreted as modalities, to be used (also) for the formal representation of the severe ignorance. Therefore, we will highlight the coincidence between the Kripke-style interpretation of the modality for ignorance and that of one connective in the enriched language of Bochvar logic.

The paper is organized into four parts: in Section 2 we introduce the standard (logical) approach to ignorance as "lack of knowledge". In Section 3, we introduce Bochvar external logic which we will use in order to give a modal approach to severe ignorance. In Section 4, it is introduced the logic **SI** of severe ignorance; an axiomatization with relative completeness is proved in Subsection 4.2. We conclude the paper with Section 5 where we make some remarks on the validity of certain formulas relevant to capture a severe notion of ignorance, and compare the differences between the standard view and the proposed logic for severe ignorance.

2. Ignorance as Lack of Knowledge

As mentioned, the traditional logical approach to ignorance is based on the idea of defining ignorance as "lack of knowledge". This is considered the *standard view* and is mainly due to the work of Hintikka [21] (see also [13] and [9]), according to whom, an agent ignores a formula φ, when s/he "does not know *whether* φ". This translates ignorance into a modal operator in the epistemic logic S_4 defined as follows:

$$(1) \qquad\qquad \mathbf{I}\varphi := \neg\mathbf{K}\varphi \wedge \neg\mathbf{K}\neg\varphi,$$

where **K** stands for the knowledge operator. It follows from (1) that, in the standard Kripke-style semantics for S_4, the formula $\mathbf{I}\varphi$ is true in a world w (under a certain evaluation v, in symbols $v(w, \mathbf{I}\varphi) = 1$) if and only if there exist two worlds w', w'', related to w, such that φ is true (false, respectively) in w' (under v) and φ is false (true, resp.) in w'' (under v). In words, an agent ignores a formula φ, in a world w, if and only if s/he has access to two worlds each of which assigns a different truth value to φ. Roughly speaking, "do not knowing whether φ" – ignoring φ according to Hintikka – means seeing (at least two different) worlds where φ is assigned with different truth-values. We might say that Hintikka's approach models ignorance as a *truth-theoretic* notion: "ignoring whether φ" is translated as "being unsure" about the truth value of φ, due to the existing conflict of evaluation in the related worlds.

It is useful to underline that the semantics for the ignorance modality **I** is the same, as above, also in the logic **Ig**, introduced by Van der Hoek and Lomuscio (see [39, Definition 2.1]) with the aim of treating ignorance as a primitive notion, not subordinated to knowledge.

Indeed, the main difference, with respect to Hintikka's approach, is that, in **Ig**, ignorance is not defined as "lack of knowledge" as **Ig** does not contain any primitive knowledge operator: it is modeled via the primitive operator **I**.

We wonder whether modeling ignorance as lack of knowledge (or via **I** in **Ig**) is the only way to logically address the notion of ignorance. Far from saying that it is not the correct way to analyze the concept, we simply claim that "lack of knowledge" is only *one way* to approach ignorance, whose features are exemplified by the logical laws in which **I** actually occurs. We recap the logical laws and the notable failures involving **I** in the following remarks.

Remark 1. It is immediate to check that the following formulae are logical truths in S_4 – where **I** is defined according to (1):

(1) $\models_{S_4} \mathbf{I}\varphi \leftrightarrow \mathbf{I}\neg\varphi$;
(2) $\models_{S_4} \mathbf{I}(\varphi \wedge \psi) \rightarrow \mathbf{I}\varphi \wedge \mathbf{I}\psi$;
(3) $\models_{S_4} \mathbf{I}\varphi \vee \mathbf{I}\psi \rightarrow \mathbf{I}(\varphi \vee \psi)$;
(4) $\models_{S_4} \mathbf{I}(\varphi \rightarrow \psi) \rightarrow (\mathbf{I}\varphi \rightarrow \mathbf{I}\psi)$;
(5) $\models_{S_4} \mathbf{I}(\varphi \wedge \psi) \rightarrow \mathbf{I}\varphi \vee \mathbf{I}\psi$;
(6) $\models_{S_4} \mathbf{II}\varphi \rightarrow \mathbf{I}\varphi$.

Remark 2. The following formulae do not hold, in general, in S_4 – where **I** is defined according to (1):

a) $\not\models_{S_4} \mathbf{I}\varphi \wedge \mathbf{I}\psi \rightarrow \mathbf{I}(\varphi \wedge \psi)$;
b) $\not\models_{S_4} \mathbf{I}\varphi \vee \mathbf{I}\psi \rightarrow \mathbf{I}(\varphi \vee \psi)$;
c) $\not\models_{S_4} \mathbf{I}\varphi \rightarrow \mathbf{II}\varphi$;
d) $\not\models_{S_4} \mathbf{I}\varphi \rightarrow \mathbf{I}(\varphi \vee \neg\varphi)$;
e) $\not\models_{S_4} \mathbf{I}\varphi \rightarrow \mathbf{I}(\varphi \wedge \neg\varphi)$;
f) $\not\models_{S_4} \mathbf{I}\varphi \rightarrow \varphi$.

We just show a simple counterexample for (a). Consider a Kripke model $\mathcal{M} = \langle W, R, v \rangle$ with $W = \{w, s\}$, $R = \{(w,w), (s,s), (w,s)\}$,

$$
\begin{array}{c}
s \;\circlearrowright \\
\uparrow \\
w \;\circlearrowright
\end{array}
$$

where evaluation v is defined as follows: $p \in w$, $p \notin s$ and $q \notin w$, $q \in s$. It then follows that $v(w, \mathbf{I}p) = 1$, $v(w, \mathbf{I}q) = 1$. On the other hand, $v(w, p \wedge q) = v(s, p \wedge q) = 0$, thus $v(w, \mathbf{I}(p \wedge q)) = 0$.

Intuitively, to falsify a) it is sufficient to consider a model with two different related worlds, each of which makes one formula true and the other false, respectively. In this way, each formula is ignored but the conjunction is not, since is false in every world.

We are convinced that the set of formulas listed in Remarks 1 and 2 – although they might not constitute an exhaustive list – tells

something relevant about the notion of ignorance that Hintikka had in mind (more detailed comments on this can be found in Section 5). Let us analyze, through some examples, the applicability (as well as limits of applicability) of Hintikka's notion of ignorance.

Suppose that Magnus and Jan[1] are about to play a single chess match. It is plausible to think that a rational agent ignores (does not know) whether Magnus is going to win (although it is very unlikely to happen, he might lose or the match could end in a draw); similarly, s/he ignores whether Jan is going to win. On the other hand, our rational agent *does not* ignore whether Magnus *and* Jan is going to win, as s/he knows that the same chess match can not have two different winners. This shows a case in which the ignorance of two conjuncts does not translate in the ignorance of their conjunction, as it happens to be the case in S_4 (see Remark 2-a).

Observe, however, that ignorance as lack of knowledge behaves according to the principle that ignoring a conjunction implies ignoring both the conjuncts and the disjuncts (Remark 1), which shades some confusion between "and" and "or" when referring to notions that are ignored.

Nevertheless, we believe that, sometimes, lack of knowledge is understood in a way which is not exemplified by the behaviour of **I** in S_4 (and **Ig**). We try to clarify what we mean, giving some examples relative to the behaviour of **I** with respect to the conjunction.

Suppose that one of the authors of this paper has just concluded to examine a student, who aimed to pass his/her exam in modal logic. During the exam, she was asked to answer some questions (obviously, in a finite number), each of which with the precise goal to verify whether she *is ignorant* – hopefully, is not ignorant – of the main topics which, together, form the program of the entire course. Unfortunately, due to her deficient answers, the examiner has collected enough evidence to conclude that she is ignorant of all the main topics, say $\varphi_1, \ldots, \varphi_n$, characterizing the course. The rational examiner is so brought to conclude that the student is ignorant of the whole subject of the exam, which can be exemplified as the conjunction $\varphi_1 \wedge \cdots \wedge \varphi_n$, and thus can not pass the exam. In other words, she is ignorant of the program $\varphi_1 \wedge \cdots \wedge \varphi_n$ of the exam. It seems reasonable to think that the above exemplified notion of ignorance is indeed lack of knowledge (the examiner is ultimately testing if the student "knows $\varphi_1, \ldots, \varphi_n$") and it seems reasonable also to think that the ignorance of each of the statements $\varphi_1, \ldots, \varphi_n$ implies the ignorance of the conjunction $\varphi_1 \wedge \cdots \wedge \varphi_n$ (how could this not be the case?!). However, ignorance as lack of knowledge, modeled in S_4, and closure with respect to conjunction can not stand

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[1]Names refer to real professional players: the (current) world-number one Magnus Carlsen and our colleague (and Grandmaster) Jan Michael Sprenger.

together.

Another weakness regarding the standard view (as discussed in recent literature, see [23], [18]) is that the so called *Factivity Principle* (usually intended relative to knowledge as $\mathbf{K}\phi \to \phi$), does not work in the standard view framework, i.e. it does not hold that if an agent ignores ϕ then ϕ is true. This fact is also highlighted in our Remark 2, where we prove that factivity of ignorance does not hold in S_4 (in contrast with the factivity of knowledge which clearly holds).

It is also possible to design other examples allowing us to stress that there are cases where ignorance is severe and does not coincide with lack of knowledge. Let us consider the discovery, happened at the beginning of November 2021, of the new Omicron variant of Coronavirus. The group of South-African scientists who isolated the variant communicated immediately their discovery, however it is reasonable to think that the sentence "Omicron is a variant of concern" was ignored by everyone at the time (and perhaps in the following days). This kind of ignorance is severe (in our sense), since it is natural that it spreads over sentences containing the previous. For instance, also any implication of the form "if the Omicron variant is of concern then there will be more deaths due to it" is genuinely to be ignored. This example seems to be convincing on the infectiousness of severe ignorance. More precisely, the lesson to learn from the above discussion is that the notion of ignorance is more subtle and problematic than it might appear at first look. Modeling it as "lack of knowledge" is surely a possibility, which has both qualities and flaws, depending on the context of applicability.

The aim of the present work is to propose a logical modeling of severe ignorance, a notion that differs from Hintikka's lack of knowledge. This change of perspective significantly impacts on the formulas holding/not holding in this new logic with respect to S_4 (see Section 5 for a comparison and further discussion). Indeed, when ignorance is conceived as severe, then the failure of certain formulas, such as (2) and (4) in Remark 1, comes with no surprise; similarly, it is not surprising that a formula like a) in Remark 2 holds in this new system. Intuitively, one could think that a way to address a severe ignorance is possible also in S_4, by recurring to the so-called "second-order ignorance" (also referred to as Rumsfeld ignorance by Fine [13]), rendered by applying $\mathbf{I}$ twice to a formula. However, applying $\mathbf{II}$ does not validate the fact that ignoring two formulas implies ignoring their conjunction, as witnessed by the following.

Remark 3. The following formulas are not logical truths of S_4:

(1) $\not\models_{S_4} \mathbf{II}\varphi \wedge \mathbf{II}\psi \to \mathbf{II}(\varphi \wedge \psi)$;

(2) $\not\models_{S_4} \mathbf{II}(\varphi \wedge \psi) \to \mathbf{II}\varphi \wedge \mathbf{II}\psi$.

The same counterexample introduced in Remark 2 serves also for (1). Indeed, observe that $v(w, \mathbf{I}p) = v(w, \mathbf{I}q) = 1$ and $v(s, \mathbf{I}p) = v(s, \mathbf{I}q) = 0$, thus $v(w, \mathbf{II}p) = v(w, \mathbf{II}q) = 1$, hence $v(w, \mathbf{II}p \wedge \mathbf{II}q) = 1$. However, since $v(w, \mathbf{I}(p \wedge q)) = v(s, \mathbf{I}(p \wedge q)) = 0$ and there exists no world $x \in W$ such that wRx and $v(x, \mathbf{I}(p \wedge q)) = 1$, then $v(w, \mathbf{II}(p \wedge q)) = 0$.

A simple counterexample to (2) is given by the following. Consider a Kripke model $\mathcal{M} = \langle W, R, v \rangle$ with $W = \{w, r, s, t\}$, $R = \{(w,w), (r,r), (s,s), (t,t), (w,s), (s,r), (w,r), (w,t)\}$
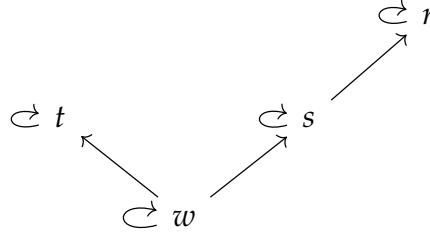
Evaluation is defined as follows: $p, q \in s$, $p, q \in t$, $p, q \in r$ and $q \in s$, $p \notin s$. It follows that $v(s, \mathbf{I}(p \wedge q)) = 1$ and $v(t, \mathbf{I}(p \wedge q)) = 0$, thus $v(w, \mathbf{II}(p \wedge q)) = 1$. However, $v(x, \mathbf{I}q) = 0$, for every $x \in W$, thus $v(w, \mathbf{II}q) = 0$. $\square$

The possibility of nesting the modality $\mathbf{I}$ (i.e. having formulas such as $\mathbf{II}\varphi$, $\mathbf{III}\varphi$, etc.), which is allowed in S_4, as we just saw, presents also a remarkable disadvantage. Although $\mathbf{I}\varphi \rightarrow \mathbf{II}\varphi$ is not a theorem of S_4, it is not difficult to check that the formula $\mathbf{II}\varphi \rightarrow \mathbf{III}\varphi$ is a theorem. More in general, abbreviating with $\mathbf{I}^n$ the n-times application of the modality $\mathbf{I}$, in S_4 the formula $\mathbf{I}^n\varphi \rightarrow \mathbf{I}^{n+1}\varphi$ holds, for $n \geq 2$. This quite problematic phenomenon is usually referred to as the *black hole* of ignorance (see [13]).

3. BOCHVAR EXTERNAL LOGIC B_e

Given a similarity type ν, the absolutely free algebra $\mathbf{Fm}$ of type ν over a countably infinite set X of generators will be called the *formula algebra* of type ν; its members will be called *formulas*. Members of X will be called (propositional) *variables* and referred to by the symbols $p, q, \ldots$. We denote algebras by $\mathbf{A}, \mathbf{B}, \mathbf{C} \ldots$ and the respective universes by $A, B, C \ldots$ We understand a *logic* (of type ν) as a pair $\mathsf{L} = \langle \mathbf{Fm}, \vdash_{\mathsf{L}} \rangle$, where $\mathbf{Fm}$ is the formula algebra (of type ν), and $\vdash_{\mathsf{L}}$ is a substitution-invariant consequence relation over $\mathbf{Fm}$ ($\vdash \; \subseteq \mathcal{P}(Fm) \times Fm$).

Kleene's three-valued logics – introduced by Kleene in his *Introduction to Metamathematics* [22] – are traditionally divided into two families, depending on the meaning given to the connectives: *strong Kleene* logics – counting strong Kleene and the logic of paradox [30]

– and *weak Kleene* logics, namely Bochvar logic [3] and paraconsistent weak Kleene – PWK in brief (sometimes referred to as Hallden's logic [19]).

Kleene logics are traditionally defined over the algebraic language $\mathcal{L}_K : \neg, \vee$ of type $(1, 2)$; $\varphi \wedge \psi$, $\varphi \to \psi$ and $\varphi \leftrightarrow \psi$ are abbreviations for $\neg(\neg\varphi \vee \neg\psi)$, $\neg\varphi \vee \psi$ and $(\varphi \to \psi) \wedge (\psi \to \varphi)$, respectively. The language $\mathcal{L}_K$ is usually referred to as the *internal language* (and $\neg, \vee$ as the *internal connectives*). Enlarging $\mathcal{L}_K$ with a new unary connective (and the constants $0, 1$), one obtains $\mathcal{L}_{K^e} : \neg, \wedge, J_2, 0, 1$ (of type $(1, 2, 1, 0, 0)$). Let us denote, with an abuse of notation (that hopefully does not create confusion) by **Fm** the formula algebra over the algebraic language $\mathcal{L}_{K^e}$.

Semantics of the language $\mathcal{L}_{K^e}$ is given by the three-elements algebra $\mathbf{WK}^e = \langle \{0, 1, 1/2\}, \neg, \wedge, J_2, 0, 1 \rangle$ displayed in Figure 1 (semantics of $\vee$ and $\to$ is recalled in Figure 2).

	¬
1	0
1/2	1/2
0	1

∨	0	1/2	1
0	0	1/2	1
1/2	1/2	1/2	1/2
1	1	1/2	1

	J_2
1	1
1/2	0
0	0

FIGURE 1. The algebra $\mathbf{WK}^e$.

∧	0	1/2	1
0	0	1/2	0
1/2	1/2	1/2	1/2
1	0	1/2	1

→	0	1/2	1
0	1	1/2	1
1/2	1/2	1/2	1/2
1	0	1/2	1

FIGURE 2. Semantics of $\vee$ and $\to$ in the algebra $\mathbf{WK}^e$.

Historically, the truth-value 1/2 has been read as "meaningless" and used for interpreting meaningless sentences in several contexts (for a discussion on the use of meaningless sentences in logic and philosophy, see [12] and [37]). Contrarily to the usual trend, in this paper, we will read the value 1/2 simply as *indeterminate* or *undefined* (similar choices, even though with remarkable differences, have been advanced for weak Kleene logics, for instance in [2], [36], [4], [10])

The language $\mathcal{L}_{K^e}$ is significantly rich and allows to define the so-called *external* formulas.[2] Intuitively, a formula α is *external* when it

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[2]The idea of considering the external language is originally due to Bochvar [3], who wanted to move non-classical to get rid of set-theoretic and semantic

is evaluated to $\{0, 1\}$ (which is the universe of a Boolean subalgebra of $\mathbf{WK}^e$) from any homomorphism $h\colon \mathbf{Fm} \to \mathbf{WK}^e$. In other words, an external formula is one such that can not be evaluated to $1/2$ (see [14, p. 208]).

Via J_2, it is possible to define more connectives (which will be very useful for our analysis): $J_3\varphi := \neg J_2\neg\varphi \to J_2\varphi$, $J_1\varphi := \neg J_3\varphi$ and $J_0\varphi := \neg(J_1\varphi \vee J_2\varphi)$ interpreted (in $\mathbf{WK}^e$) as follows.

φ	$J_3\varphi$
1	1
1/2	0
0	1

φ	$J_1\varphi$
1	0
1/2	1
0	0

φ	$J_0\varphi$
1	0
1/2	0
0	1

Intuitively, connectives J_0, J_1, J_2, J_3 allow to speak not only about a statement φ but also about its truthfulness, falseness and more.

Bochvar (external) logic B_e is the logic induced by the matrix $\langle \mathbf{WK}^e, \{1\}\rangle$, i.e.

$\Gamma \models_{\mathsf{B}_e} \varphi$ if and only if, for every homomorphism $h\colon \mathbf{Fm} \to \mathbf{WK}^e$,

$$\text{if } h[\Gamma] \subseteq \{1\} \text{ then } h(\varphi) = 1.$$

In words, B_e is the logic preserving only the truth-value 1 ("true").[3]

Definition 4. A variable p is *open* in a formula φ when there is at least one occurrence of it which does not fall under the scope of J_i, with $i \in \{0, 1, 2, 3\}$. It is *covered* if it is not open, namely it occurs in φ and all occurrences fall under the scope of J_i, for some $i \in \{0, 1, 2, 3\}$.

The intuition behind external formulas is made precise by the following.

Definition 5. A formula $\varphi \in Fm$ is called *external* if all the variables occurring in φ are covered or, if a variable is open in φ then φ falls under the scope of J_i, for some $i \in \{0, 1, 2, 3\}$.

Examples of external formulas are: $J_1 p \vee J_2 q$, $J_1(p \vee q)$, etc.

A Hilbert-style axiomatization of B_e has been introduced by Finn and Grigolia [14, pp. 236-236]. In order to present it, let

$$\varphi \equiv \psi := \bigwedge_{i=0}^{2} J_i\varphi \leftrightarrow J_i\psi,$$

and α, β, γ denote external formulas.

Axioms

(A1) $(\varphi \vee \varphi) \equiv \varphi$;
(A2) $(\varphi \vee \psi) \equiv (\psi \vee \varphi)$;
(A3) $((\varphi \vee \psi) \vee \chi) \equiv (\varphi \vee (\psi \vee \chi))$;
(A4) $(\varphi \wedge (\psi \vee \chi)) \equiv ((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$;
(A5) $\neg(\neg\varphi) \equiv \varphi$;
(A6) $\neg 1 \equiv 0$;
(A7) $\neg(\varphi \vee \psi) \equiv (\neg\varphi \wedge \neg\psi)$;
(A8) $0 \vee \varphi \equiv \varphi$;
(A9) $J_2\alpha \equiv \alpha$;
(A10) $J_0\alpha \equiv \neg\alpha$;
(A11) $J_1\alpha \equiv 0$;
(A12) $J_i\neg\varphi \equiv J_{2-i}\varphi$, for any $i \in \{0,1,2\}$;
(A13) $J_i\varphi \equiv \neg(J_j\varphi \vee J_k\varphi)$, with $i \neq j \neq k \neq i$;
(A14) $(J_i\varphi \vee \neg J_i\varphi) \equiv 1$, with $i \in \{0,1,2\}$;
(A15) $((J_i\varphi \vee J_k\psi) \wedge J_i\varphi) \equiv J_i\varphi$, with $i,k \in \{0,1,2\}$;
(A16) $(\varphi \vee J_i\varphi) \equiv \varphi$, with $i \in \{1,2\}$;
(A17) $J_0(\varphi \vee \psi) \equiv J_0\varphi \wedge J_0\psi$;
(A18) $J_2(\varphi \vee \psi) \equiv (J_2\varphi \wedge J_2\psi) \vee (J_2\varphi \wedge J_2\neg\psi) \vee (J_2\neg\varphi \wedge J_2\psi)$;
(A19) $\alpha \rightarrow (\beta \rightarrow \alpha)$;
(A20) $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$;
(A21) $\alpha \wedge \beta \rightarrow \alpha$;
(A22) $\alpha \wedge \beta \rightarrow \beta$;
(A23) $(\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \gamma) \rightarrow (\alpha \rightarrow \beta \wedge \gamma))$;
(A24) $\alpha \rightarrow \alpha \vee \beta$;
(A25) $\beta \rightarrow \alpha \vee \beta$;
(A26) $(\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \vee \beta \rightarrow \gamma))$;
(A27) $(\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \neg\beta) \rightarrow \neg\alpha)$;
(A28) $\alpha \rightarrow (\neg\alpha \rightarrow \beta)$;
(A29) $\neg\neg\alpha \rightarrow \alpha$.

Deductive rule

$$[\text{MP}] \quad \frac{\varphi \qquad \varphi \rightarrow \psi}{\psi}$$

Observe that the axiomatization contains a set of axioms (A19-A29), which, together with the rule of *modus ponens*, yields a complete axiomatization for classical logic (relative to external formulas). Upon defining the notion of derivation ($\vdash_{B_e} \varphi$) in the usual way, Finn and Grigolia proved weak completeness for B_e.

Theorem 6. [14, Theorem 3.4] $\vdash_{B_e} \varphi$ *if and only if* $\models_{B_e} \varphi$.

It is natural to wonder whether B_e can be provided with a more synthetic Hilbert-style axiomatization and/or with a different style axiomatization (natural deduction, Gentzen-style, etc.). Actually a stronger version of Theorem 6 can be proved (the details of the

proof are displayed in the Appendix, where we also show that B_e is algebraizable).

Theorem 7. $\Gamma \vdash_{B_e} \varphi$ *if and only if* $\Gamma \models_{B_e} \varphi$.

Theorem 8 (Deduction Theorem). $\Gamma \vdash_{B_e} \varphi$ *if and only if there exist formulas* $\gamma_1, \ldots, \gamma_n \in \Gamma$ *such that* $\vdash_{B_e} J_2\gamma_1 \wedge \cdots \wedge J_2\gamma_n \to J_2\varphi$.

Proof. ($\Rightarrow$) By induction on the length of the derivation of φ (from Γ).

($\Leftarrow$) We reason by contraposition and suppose that $\Gamma \nvdash_{B_e} \varphi$, thus $\Gamma \nvDash_{B_e} \varphi$ (by Theorem 7), i.e. there is a homomorphism $h \colon \mathbf{Fm} \to \mathbf{WK}^e$ such that $h(\gamma) = 1$, for every $\gamma \in \Gamma$ and $h(\varphi) \neq 1$. Then, for every subset of formulas $\{\gamma_1, \ldots, \gamma_n\} \subseteq \Gamma$, $h(J_2\gamma_i) = 1$, for every $i \in \{1, \ldots, n\}$ and $h(J_2\varphi) = 0$, hence $h(J_2\gamma_1 \wedge \cdots \wedge J_2\gamma_n \to J_2\varphi) = 0$, i.e. $\nvDash_{B_e} J_2\gamma_1 \wedge \cdots \wedge J_2\gamma_n \to J_2\varphi$, thus $\nvdash_{B_e} J_2\gamma_1 \wedge \cdots \wedge J_2\gamma_n \to J_2\varphi$ (by Theorem 6). $\square$

4. The logic SI of Severe Ignorance

The logic of (severe) ignorance we are going to introduce consists of a modal logic, whose propositional basis is B_e.

Let $\mathbf{Fm_I}$ be the formula algebra constructed over a numerable infinite set of propositional variables *Var* in the language $\mathcal{L_I} : \neg, \vee, J_2, \mathbf{I}, 0, 1$ of type $\langle 1, 2, 1, 1, 0, 0 \rangle$.

For the sake of simplicity and convenience of exposition, we will introduce the logic of ignorance SI via its relational semantics, and provide only afterwards a Hilbert-style axiomatization for it. We are indeed convinced that the interpretation of the operator $\mathbf{I}$ catches adequately the intuition behind the notion of severe ignorance that we are about to introduce: a formula φ is ignored (in a world) when it is indeterminate (i.e. evaluated to $1/2$) in all the related worlds.

We will axiomatize, prove completeness and decidability for the logic SI with respect to all *reflexive* frames.

4.1. **Semantics.** The semantics of the logics of ignorance consists of a relational (Kripke-style) structure where formulas, in each world, are evaluated into $\mathbf{WK}^e$. We introduce these structures according to the current terminology adopted in many-valued modal logics (see, for instance, [8, 15, 16]).

Definition 9. A *weak three-valued Kripke model* $\mathcal{M}$ is a structure $\langle W, R, v \rangle$ such that:

(1) W is a non-empty set (of possible worlds);
(2) R is a binary relation over W ($R \subseteq W \times W$);
(3) v is a map, called *valuation*, assigning to each world and each variable, an element in $\mathbf{WK}^e$ ($v \colon W \times \mathbf{Fm_I} \to \mathbf{WK}^e$).

Non-modal formulas will be interpreted as in B_e, i.e. we assume that v is a homomorphism, in its second component, with respect to $\neg, \vee, J_2, 1, 0$. The reduct $\mathcal{F} = \langle W, R \rangle$ of a model $\mathcal{M}$ is called *frame*.

Notation: for ordered pairs of related elements, we equivalently write $(w, s) \in R$ or wRs.

The semantical interpretation of the epistemic modality **I** in a weak three-valued Kripke model is given in the following.

Definition 10. Let $\langle W, R, v \rangle$ be a weak three-valued Kripke model, and $w \in W$. Then

(1) $v(w, \mathbf{I}\varphi) = 1$ iff $v(s, \varphi) = 1/2$ for every $s \in W$ such that wRs.
(2) $v(w, \mathbf{I}\varphi) = 0$ iff $v(w, \varphi) \neq 1/2$ and there exists $s \in W$ such that wRs and $v(s, \varphi) \neq 1/2$.
(3) $v(w, \mathbf{I}\varphi) = 1/2$ iff $v(w, \varphi) = 1/2$ and there exists $s \in W$ such that wRs and $v(s, \varphi) \neq 1/2$.

Observe that there is no special assumption behind the accessibility relation R in weak three-valued Kripke structures: it is simply interpreted as an *epistemic accessibility relation*. Accordingly, the rationale behind the interpretation of **I** is that a formula is being ignored in case it is neither true nor false – it is indeterminate – in every world s an agent has epistemic access to from w. Recall that the notion of ignorance we aim at modeling with this semantics is severe. To further clarify our goal imagine, for instance, the following situation. Charles Darwin was aware, in 1859, of the existence of a form of hereditariness; however he did not know exactly the functioning mechanism of such process. Moreover, in every scenario accessible to his mind in that period, the cause of hereditariness was not determined. So, if we think a formula φ exemplifying the mechanism of hereditariness, then, in 1859, it held that Darwin was (severely) ignorant of φ, because φ was indeterminated in every possibile scenario. Other mechanisms, although not entirely known, were not (severely) ignored by Darwin himself at that time. For instance, we can not say that he was ignorant of the so called "missing links". Although he could not find them, he had an idea of how to search them, thanks to the analysis of fossils.

Accordingly, it is false that a formula φ is being ignored (in a world w) when there is a (related) world where φ is either true or false, and the same happens in the actual world w.

We say that a formula φ is *valid* in a model $\mathcal{M} = \langle W, R, v \rangle$ – we will write $\mathcal{M} \models \varphi$ – if, for every $w \in W$, $v(w, \varphi) = 1$.

A comment on this choice is in order. The introduced semantics of **I** relies on the presence of the third truth-value $1/2$ to be read as "indeterminate". In particular, severe ignorance, thought as a

content-theoretic notion (in contrast with the truth-theoretic notion modeled by the standard view in S_4), is rendered thanks to the infectious behaviour of $1/2$. For this reason, it is natural to take $1/2$ as not designated, since the evaluation of a formula to $1/2$ (in every related world) is a good reason for its ignorance.

We say that a formula φ is valid in a frame $\mathcal{F} = \langle W, R \rangle$ (and write $\mathcal{F} \models \varphi$) if it is valid in every model having $\mathcal{F}$ as frame. A frame will be called *reflexive* if its accessibility relation is reflexive. From now on, we will write Kripke model instead of weak three-valued Kripke model. When a formula is valid in every *reflexive* model we will write $\models_{\mathrm{SI}}$: **SI** is the *global* modal logic on the class of all Kripke frames (see e.g. [8]), i.e.

- $\Gamma \models_{\mathrm{SI}} \varphi$ iff, for every Kripke model $\mathcal{M}$,
  if $\mathcal{M} \models \gamma$, for every $\gamma \in \Gamma$, then $\mathcal{M} \models \varphi$.

Remark 11. Notice that, given a model $\mathcal{M} = \langle W, R, v \rangle$ and a world $w \in W$, the truth of the formula $\mathbf{I}\varphi$ at the world w ($v(w, \mathbf{I}\varphi) = 1$) is equivalent to the fact that, for every world s related to w, there exists a propositional variable p occurring open in φ such that $v(s, p) = 1/2$. □

The above consideration is due to the peculiar behavior of the truth-value $1/2$ in weak Kleene and gives already a gist of the severity of ignorance obtained via the introduced semantics of $\mathbf{I}$. Indeed a (complex) formula φ is being ignored when a part of it (occurring open) is actually being ignored (as evaluated to $1/2$ in every related world), independently of the logical form of φ (exceptions hold for external formulas).

Remark 12. Let $\mathcal{F} = \langle W, R \rangle$ a reflexive frame. Then, for any valuation v and formula $\varphi \in Fm_{\mathbf{I}}$, $v(w, \mathbf{I}\varphi) \neq 1/2$, for every $w \in W$. Indeed, suppose by contradiction that $v(w, \mathbf{I}\varphi) = 1/2$, then $v(w, \varphi) = 1/2$ which, since $(w, w) \in R$ implies that $v(w, \mathbf{I}\varphi) = 1$, a contradiction.

In words, formulas whose main connective is $\mathbf{I}$ are external, according with the intuition that "the content of φ" is being ignored can not be indeterminate (but only either true or false). A part from this, the choice of defining the logic **SI** as that of all reflexive frames is mainly motivated by the fact that accessibility is interpreted in epistemic sense, thus is natural to think that every world is (epistemically) accessible to itself.

The following provides the behaviour of the epistemic modality $\mathbf{I}$ in **SI**.

Proposition 13. *The following formulas are valid in* **SI***:*

(1) $\models_{\mathrm{SI}} \mathbf{I}\varphi \leftrightarrow \mathbf{I}\neg\varphi;$
(2) $\models_{\mathrm{SI}} \mathbf{I}\varphi \wedge \mathbf{I}\psi \rightarrow \mathbf{I}(\varphi \wedge \psi);$

(3) $\models_{SI} \mathbf{I}(\varphi \wedge \psi) \to \mathbf{I}\varphi \vee \mathbf{I}\psi$;
(4) $\models_{SI} \mathbf{I}\varphi \vee \mathbf{I}\psi \leftrightarrow \mathbf{I}(\varphi \vee \psi)$;
(5) $\models_{SI} \mathbf{I}\varphi \to \mathbf{I}(\varphi \vee \neg\varphi)$;
(6) $\models_{SI} \mathbf{I}\varphi \to \mathbf{I}(\varphi \wedge \neg\varphi)$;
(7) $\models_{SI} J_1\varphi \to \mathbf{I}\varphi$;
(8) $\models_{SI} \mathbf{I}\varphi \to J_1\varphi$;
(9) $\models_{SI} \mathbf{II}\varphi \to \mathbf{I}\varphi$;
(10) $\models_{SI} \mathbf{II}\varphi \to \mathbf{I}^n\varphi$, for $n \geq 2$.

Proof. We just show the validity of some of the listed formulas.
(1) We verify only one direction (as the other is analog). Let $\mathcal{M} = \langle W, R, v \rangle$ be a model such that $v(w, \mathbf{I}\varphi) = 1$, for some $w \in W$; then $v(s, \varphi) = 1/2$, for every $s \in W$ such that wRs, hence $v(s, \neg\varphi) = 1/2$, i.e. $v(s, \mathbf{I}\neg\varphi) = 1$.
(3) Let $\mathcal{M} = \langle W, R, v \rangle$ be a model such that $v(w, \mathbf{I}(\varphi \wedge \psi)) = 1$, for some $w \in W$. Then $v(s, \varphi \wedge \psi) = 1/2$, for every $s \in W$ such that wRs, which implies that $v(s, \varphi) = 1/2$ or $v(s, \psi) = 1/2$ (or both), whence $v(w, \mathbf{I}\varphi) = 1$ or $v(w, \mathbf{I}\psi) = 1$. Since neither $v(w, \mathbf{I}\varphi) = 1/2$ nor $v(w, \mathbf{I}\psi) = 1/2$ (by Remark 12) then $v(w, \mathbf{I}\varphi \vee \mathbf{I}\psi) = 1$.
(9) Observe that, by Remark 12, $v(w, \mathbf{II}\varphi) \neq 1/2$, for every $w \in W$ and every valuation v. In particular, this implies that $v(w, \mathbf{II}\varphi) = 0$, since $v(w, \mathbf{II}\varphi) = 1$ would imply $v(w, \mathbf{I}\varphi) = 1/2$, in contrast with Remark 12.
(10) holds for the same reason of (9).   □

As noticed in [39], the essential feature of any notion of ignorance is captured by formulas (1) and (3) in Proposition 13. It is indeed very reasonable to think that an agent is ignorant about a formula if and only if is about its negation. Moreover, ignorance transfers from a conjunction to at least one constitutive part of it. Severe ignorance meets the minimal desiderata.

As a remarkable difference with S_4 (and **Ig**), in this new semantics for **I**, being ignorant of two (or more) formulas implies being ignorant also of their conjunction.

Not surprisingly, the converse (which holds in S_4) does not characterize severe ignorance (see Remark 14). It is indeed reasonable to think that being ignorant of a book (a conjunction of statements), does not mean to be ignorant of any single statements in the book, but, perhaps, some relevant parts of it. Moreover, (4) holds in virtue of the infectivity of the third truth-value.

We will discuss the significance of all the mentioned logical laws in Section 5.

Some notable failures are collected in the following.

Remark 14. The following formulas are not valid in SI:

(1) $\not\models_{SI} \mathbf{I}(\varphi \wedge \psi) \to \mathbf{I}\varphi \wedge \mathbf{I}\psi$;
(2) $\not\models_{SI} \mathbf{I}(\varphi \to \psi) \to (\mathbf{I}\varphi \to \mathbf{I}\psi)$;

(3) $\not\models_{\mathrm{SI}} (\mathbf{I}\varphi \to \mathbf{I}\psi) \to \mathbf{I}(\varphi \to \psi)$;
(4) $\not\models_{\mathrm{SI}} \mathbf{I}\varphi \to \mathbf{II}\varphi$;
(5) $\not\models_{\mathrm{SI}} \mathbf{I}\varphi \to \varphi$.

In Section 5, we will argue that it is not a problem for the severe notion of ignorance not to have distribution of $\mathbf{I}$ over conjunction and implication (formulas (2) and (3)). On the other hand, since severe ignorance is here conceived as a content-theoretic notion, it is obvious to expect the failure of the *factivity* (5).

4.2. **Axiomatization and Completeness.** We introduce a Hilbert-style axiomatization for **SI** for which we prove completeness and decidability.

Axioms

- the axioms of B_e;

(**I**) $J_1\varphi \to \mathbf{I}\varphi$.

Deduction rules

- the rule [MP].

By $\vdash_{\mathrm{SI}}$ we intend the derivability relation of the deductive system defined by the above axioms and inference rule. When no danger of confusion is occurring we will drop the subscripts **SI** and B_e to $\vdash$.

Remark 15. Since **SI** is a linguistic and axiomatic extension of B_e (no deduction rule is added), then it holds that $\Gamma \vdash \varphi$ if and only if there exist formulas $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that $\vdash J_2\gamma_1 \wedge \cdots \wedge J_2\gamma_n \to J_2\varphi$.

Our proof of completeness consists in an adaptation of the strategy, devised by Segerberg in [34], for modal logics based on the external version of Paraconsistent Weak Kleene (H_0).

Definition 16. A set of formulas $\Sigma \subset Fm_\mathbf{I}$ is *maximal* iff for all formulas $\varphi \in Fm_\mathbf{I}$, either $\varphi \in \Sigma$, or $\neg\varphi \in \Sigma$, or $J_1\varphi \in \Sigma$.

Recall that a set of formulas Σ is inconsistent in case $\Sigma \vdash \varphi$, for every $\varphi \in Fm_\mathbf{I}$. Σ is consistent if it is not inconsistent.

Remark 17. A useful operative notion of consistency (for sets of formulas) is given by the following: a set of formulas $\Sigma \subset Fm_\mathbf{I}$ is *consistent* iff there is no formula $\varphi \in Fm_\mathbf{I}$ such that $\Sigma \vdash J_i\varphi$ and $\Sigma \vdash \neg J_i\varphi$, for any $i \in \{0, 1, 2\}$ (it is immediate to check that this corresponds to the above notion of consistency).

We denote by $\mathcal{X}$ the set of all maximal and consistent sets of formulas, whose basic properties are recalled in the following.

Lemma 18. *For every $X \in \mathcal{X}$, the following hold:*

(1) If $\varphi \to \psi \in X$ and $\varphi \in X$ then $\psi \in X$;
(2) $\varphi \wedge \psi \in X$ if and only if $\varphi, \psi \in X$;
(3) $\varphi \in X$ if and only if $J_2\varphi \in X$;
(4) $J_2\varphi \in X$ if and only if $\neg J_2\varphi \notin X$.

Proof. Straightforward.   □

Lemma 19. *Let Σ be a consistent set of formulas. $\Sigma \cup \{\varphi\}$ is inconsistent if and only if $\Sigma \vdash \neg\varphi$ or $\Sigma \vdash J_1\varphi$.*

Proof. ($\Rightarrow$) Let $\Sigma \cup \{\varphi\}$ be inconsistent and that $\Sigma \nvdash \neg\varphi$. Then $\Sigma \cup \{\varphi\} \vdash 0$. So, there exist formulas $\gamma_1, \ldots, \gamma_n \in \Sigma$ such that $\vdash J_2\gamma_1 \wedge \cdots \wedge J_2\gamma_n \wedge J_2\varphi \to J_20$, hence $\vdash J_2\gamma_1 \wedge \cdots \wedge J_2\gamma_n \to \neg J_2\varphi$, thus $\Sigma \vdash \neg J_2\varphi$. Since $\vdash_{B_e} \neg J_2\varphi \leftrightarrow J_1\varphi \vee J_0\varphi$, $\Sigma \vdash J_1\varphi \vee J_0\varphi$. By assumption, $\Sigma \nvdash \neg\varphi$ which implies $\Sigma \nvdash J_0\varphi$, hence $\Sigma \vdash J_1\varphi$.
($\Leftarrow$) Let $\Sigma \vdash \neg\varphi$ or $\Sigma \vdash J_1\varphi$. Suppose $\Sigma \vdash \neg\varphi$ is the case, hence $\Sigma \vdash J_0\varphi$ (since $\neg\varphi \vdash_{B_e} J_0\varphi$). On the other hand, $\Sigma \cup \{\varphi\} \vdash J_2\varphi$, hence $\Sigma \cup \{\varphi\} \vdash J_0\varphi \wedge J_2\varphi$, and it is immediate to check that $\vdash J_0\varphi \wedge J_2\varphi \to 0$, thus it is inconsistent. The proof is analog in case $\Sigma \vdash J_1\varphi$.   □

Lemma 20. *Let Σ be a consistent set of formulas. The following are equivalent:*

(1) $\Sigma \vdash \varphi$;
(2) for every $X \in \mathcal{X}$, if $\Sigma \subseteq X$ then $\neg\varphi \notin X$ and $J_1\varphi \notin X$.

Proof. (1) $\Rightarrow$ (2). Suppose $\Sigma \vdash \varphi$. Let $X \in \mathcal{X}$ such that $\Sigma \subseteq X$ and, by contradiction, that $\neg\varphi \in X$ or $J_1\varphi \in X$. Observe that $X \vdash \varphi$, so $X \vdash J_2\varphi$ (since $\varphi \vdash_{B_e} J_2\varphi$). Suppose $\neg\varphi \in X$ is the case. Then $X \vdash \neg J_2\varphi$ (since $\neg\varphi \vdash_{B_e} \neg J_2\varphi$), in contradiction with the fact that X is consistent (see Remark 17). Differently, $J_1\varphi \in X$ is the case. Thus $X \vdash J_1\varphi \wedge J_2\varphi$, again in contradiction with the consistency of X (since $\vdash J_2\varphi \wedge J_1\varphi \to 0$).
(2) $\Rightarrow$ (1). We reason by contraposition and suppose that $\Sigma \nvdash \varphi$. Consider an enumeration $\psi_1, \psi_2, \psi_3, \ldots$ of the formulas in Fm_I. Define:

$$\Sigma_0 = \begin{cases} \Sigma \cup \{\neg\varphi\} & \text{if consistent,} \\ \Sigma \cup \{J_1\varphi\} & \text{otherwise.} \end{cases}$$

$$\Sigma_{i+1} = \begin{cases} \Sigma_i \cup \{\psi_i\} & \text{if consistent, else} \\ \Sigma_i \cup \{\neg\psi_i\} & \text{if consistent, else} \\ \Sigma_i \cup \{J_1\psi_i\}. \end{cases}$$

$$\Sigma^* = \bigcup_{i\in\mathbb{N}} \Sigma_i.$$

Observe that Σ^* is maximal, by construction. We want to show that Σ^* is also consistent. We first claim that Σ_0 is consistent. If $\Sigma_0 = \Sigma \cup \{\neg\varphi\}$ then it is consistent by construction. Differently,

$\Sigma_0 = \Sigma \cup \{J_1\varphi\}$, which means that $\Sigma \cup \{\neg\varphi\}$ is inconsistent. Hence, by Lemma 19, $\Sigma \vdash \neg\neg\varphi$ or $\Sigma \vdash J_1\neg\varphi$. However, $\Sigma \nvdash \neg\neg\varphi$ (since, by assumption, $\Sigma \nvdash \varphi$), so $\Sigma \vdash J_1\neg\varphi$, which implies $\Sigma \vdash J_1\varphi$. By Lemma 19, $\Sigma_0 = \Sigma \cup \{J_1\varphi\}$ is consistent if and only if $\Sigma \nvdash \neg J_1\varphi$ and $\Sigma \nvdash J_1 J_1 \varphi$. Now, since Σ is consistent and $\Sigma \vdash J_1\varphi$, then $\Sigma \nvdash \neg J_1\varphi$. Moreover, since Σ is consistent $\Sigma \nvdash J_1 J_1 \varphi$ (as $\vdash J_1 J_1 \varphi \leftrightarrow 0$). This shows that Σ_0 is consistent.

We claim that Σ_{i+1} is consistent, given that Σ_i is. So, suppose that $\Sigma_i \cup \{\varphi\}$ and $\Sigma_i \cup \{\neg\varphi\}$ are inconsistent. Then, by Lemma 19, $\Sigma_i \vdash \neg\varphi$ or $\Sigma_i \vdash J_1\varphi$, and, $\Sigma_i \vdash \neg\neg\varphi$ or $\Sigma_i \vdash J_1\neg\varphi$. By consistency of Σ_i, the only possible case is that $\Sigma_i \vdash J_1\varphi$ and $\Sigma_i \vdash J_1\neg\varphi$, from which follows the consistency of $\Sigma \cup \{J_1\varphi\}$ (indeed, if it is not consistent then, by Lemma 19, $\Sigma_i \vdash \neg J_1\varphi$, in contradiction with the consistency of Σ_i). This shows that Σ^* is maximal and consistent ($\Sigma^* \in \mathcal{X}$) and, by construction, $\neg\varphi \in \Sigma^*$ or $J_1\varphi \in \Sigma^*$. $\qquad\square$

We now define the accessibility relation between elements in $\mathcal{X}$.

Definition 21. Let $X, Y \in \mathcal{X}$. We define a relation $\mathcal{R}$ as follows:

- $X\mathcal{R}Y$ if and only if for every formula $\varphi \in Fm_{\mathbf{I}}$, if $\mathbf{I}\varphi \in X$ then $J_1\varphi \in Y$.

Lemma 22. *Let $X \in \mathcal{X}$. For every formula $\varphi \in Fm_{\mathbf{I}}$, $\mathbf{I}\varphi \in X$ if and only if $J_1\varphi \in Y$, for every $Y \in \mathcal{X}$ such that $X\mathcal{R}Y$.*

Proof. The left to right direction is obvious. For the other, assume that $J_1\varphi \in Y$, for every $Y \in \mathcal{X}$ such that $X\mathcal{R}Y$. Consider the set $\Sigma = \{J_1\psi \mid \mathbf{I}\psi \in X\}$. We claim that $\Sigma \vdash J_1\varphi$. To this end, let $\Sigma \subseteq Z$, for some $Z \in \mathcal{X}$ and observe that this implies that $J_1\varphi \in Z$. Indeed, for $\mathbf{I}\psi \in X$ then $J_1\psi \in \Sigma$, thus $J_1\psi \in Z$; so $X\mathcal{R}Z$ (by Definition 21), whence, by hypothesis, $J_1\varphi \in Z$. Now, since Z is consistent, $J_3\varphi \notin Z$ and $J_1 J_1 \varphi \notin Z$ then, by Lemma 20, we have $\Sigma \vdash J_1\varphi$.

Via axiom (**I**) we have that $\Sigma \vdash \mathbf{I}\varphi$. Therefore, by Remark 15, there exist formulas $J_1\psi_1, \ldots, J_1\psi_n \in \Sigma$ such that $\vdash J_2 J_1 \psi_1 \wedge \cdots \wedge J_2 J_1 \psi_n \to J_2 \mathbf{I}\varphi$, i.e. $\vdash J_1\psi_1 \wedge \cdots \wedge J_1\psi_n \to J_2\mathbf{I}\varphi$, hence $\vdash \mathbf{I}\psi_1 \wedge \cdots \wedge \mathbf{I}\psi_n \to J_2\mathbf{I}\varphi$. Observe that $\mathbf{I}\psi_1, \ldots, \mathbf{I}\psi_n \in X$, hence by Lemma 18, $J_2\mathbf{I}\varphi \in X$, thus $\mathbf{I}\varphi \in X$, as desired. $\qquad\square$

We are ready to define the concept of *canonical model* (keeping the usual nomenclature in modal logic).

Definition 23. A *canonical model* is a weak three-valued Kripke model $\mathcal{M} = \langle \mathcal{X}, \mathcal{R}, v \rangle$ such that $\mathcal{R}$ is as in Definition 21, and v is defined as follows:

- $v(X, p) = 1$ if and only if $p \in X$;
- $v(X, p) = 0$ if and only if $\neg p \in X$;
- $v(X, p) = 1/2$ if and only if $J_1 p \in X$,

for every $X \in \mathcal{X}$.

The following result extends the construction of the universal model to all reflexive frames.

Lemma 24. *Let $\mathcal{M} = \langle \mathcal{X}, \mathcal{R}, v \rangle$ be the universal model with $\mathcal{R}$ reflexive. Then, for every formula $\varphi \in Fm_1$ and every $X \in \mathcal{X}$, the following hold:*

(1) $v(X, \varphi) = 1$ if and only if $\varphi \in X$;

(2) $v(X, \varphi) = 0$ if and only if $\neg\varphi \in X$;

(3) $v(X, \varphi) = 1/2$ if and only if $J_1\varphi \in X$.

Proof. The claim is proved by induction on the length of φ. The basis follows from Definition 23. As for the inductive step, we show only the cases of $\varphi = J_2\psi$ and $\varphi = \mathbf{I}\psi$, for some $\psi \in Fm_1$ (the others are routine). As regards the former, suppose that $\varphi = J_2\psi$, for some $\psi \in Fm_1$. For any valuation v (and any $X \in \mathcal{X}$), $v(X, J_2\psi) \neq 1/2$ (in accordance with the fact that $J_1 J_2 \varphi \notin X$), thus we only have to consider two cases:

(a) $v(X, J_2\psi) = 1$ iff $v(X, \psi) = 1$, thus, by induction hypothesis, $\psi \in X$ and, by Lemma 18, $J_2\psi \in X$.

(b) $v(X, J_2\psi) = 0$ iff either $v(X, \psi) = 0$ or $v(X, \psi) = 1/2$. Consider, first, the case $v(X, \psi) = 0$; by induction hypothesis, $\neg\psi \in X$ and, since X is consistent, $J_2\psi \notin X$, thus by Lemma 18, $\neg J_2\psi \in X$. In the second (sub)case, $v(X, \psi) = 1/2$, thus, by induction hypothesis, $J_1\psi \in X$. Since X is maximal (and consistent) then $\psi \notin X$, thus, by Lemma 18, $J_2\psi \notin X$, whence $\neg J_2\psi \in X$.

Consider now the case of $\varphi = \mathbf{I}\psi$, for some $\psi \in Fm_1$. Preliminarly, observe that there is no valuation v such that $v(X, \varphi) = 1/2$ Indeed, suppose, by contradiction, that there is a valuation v such that $v(X, \mathbf{I}\psi) = 1/2$. This implies that $v(X, \psi) = 1/2$, moreover there exists $Y \in \mathcal{X}$ such that $X\mathcal{R}Y$ and $v(Y, \psi) \neq 1/2$. By induction hypothesis, $J_1\psi \in X$ and since $X\mathcal{R}X$, by Lemma 22, $\mathbf{I}\psi \in X$. This implies (since $X\mathcal{R}Y$) that $J_1\psi \in Y$ (again, by Lemma 22) and, by the previous case (recalling that J_1 is defined via J_2), that $v(Y, \psi) = 1/2$, a contradiction. Observe, moreover, that this is in accordance with the fact that $J_1 \mathbf{I}\psi \notin X$ (for any $X \in \mathcal{X}$). Indeed, if $J_1\mathbf{I}\psi \in X$ then $J_1 J_1 \varphi \in X$, against the fact that X is consistent. Therefore, we only have to consider the two cases below.

(i) $v(X, \mathbf{I}\psi) = 1$ if and only if $v(Y, \psi) = 1/2$, for every $Y \in \mathcal{X}$ such that $X\mathcal{R}Y$. By induction hypothesis, $J_1\psi \in Y$, thus by Lemma 22 $\mathbf{I}\psi \in X$.

(ii) $v(X, \mathbf{I}\psi) = 0$ if and only if $v(X, \psi) \neq 1/2$ and $v(Y, \psi) \neq 1/2$, for some $Y \in \mathcal{X}$ such that $X\mathcal{R}Y$. By induction hypothesis, $J_1\psi \notin Y$, thus by Lemma 22 , $\mathbf{I}\psi \notin X$, hence $\neg\mathbf{I}\psi \in X$ or $J_1\mathbf{I}\psi \in X$. But the latter is never the case, since X is consistent, whence $\neg\mathbf{I}\psi \in X$. $\quad\square$

We are now ready to prove (strong) completeness for the logic $\vdash_{SI}$ with respect to the class of all reflexive models.

Theorem 25 (Completeness). *$\Gamma \vdash_{\mathrm{SI}} \varphi$ if and only if $\Gamma \models_{\mathrm{SI}} \varphi$.*

Proof. ($\Rightarrow$) It is immediate to check that all axioms are sound and that the rule (MP) preserves validity.
($\Leftarrow$) We reason by contraposition and suppose that $\Gamma \nvdash_{\mathrm{SI}} \varphi$. This implies that Γ is consistent; by Lemma 20 $\Gamma \subseteq X$, for some $X \in \mathcal{X}$ such that $\neg\varphi \in X$ or $J_1\varphi \in X$. Let $\mathcal{M} = \langle \mathcal{X}, \mathcal{R}, v \rangle$ be the canonical model with $\mathcal{R}$ reflexive. Since $\Gamma \vdash_{\mathrm{SI}} \gamma$, for each $\gamma \in \Gamma$, then, by Lemma 20, $\neg\gamma \notin X$ and $J_1\gamma \notin X$, therefore $\gamma \in X$, which implies $v(X, \gamma) = 1$, by Lemma 24. On the other hand, since $\neg\varphi \in X$ or $J_1\varphi \in X$, then $v(X, \varphi) \neq 1$, i.e. $\Gamma \not\models_{\mathrm{SI}} \varphi$. $\qquad\qquad\square$

The completeness strategy applied insofar allows to prove decidability for the logic **SI** (Theorem 28).

Definition 26. Let $\varphi \in Fm_{\mathrm{I}}$. The set $\mathrm{Sub}(\varphi)$ of subformulas of φ is the smallest set of formulas such that:

(1) $\varphi \in \mathrm{Sub}(\varphi)$;
(2) if $\varphi = \neg\psi$, or $\varphi = J_2\psi$, or $\varphi = \mathbf{I}\psi$ (for some $\psi \in Fm_{\mathrm{I}}$) then $\psi \in \mathrm{Sub}(\varphi)$;
(3) if $\varphi = \psi \vee \chi \in \mathrm{Sub}(\varphi)$, then $\psi, \chi \in \mathrm{Sub}(\varphi)$.

Let $\mathcal{M} = \langle W, R, v \rangle$ be a model. We say that a model has cardinality n (with $n \in \mathbb{N}$), if W has cardinality n ($\mid W \mid = n$).

Lemma 27. *Let φ a formula such that $\mid \mathrm{Sub}(\varphi) \mid = n$. The following are equivalent:*

(1) $\vdash_{\mathrm{SI}} \varphi$;
(2) $\mathcal{M} \models_{\mathrm{SI}} \varphi$ for all models with cardinality 2^n.

Proof. (i) $\Rightarrow$ (ii) is obvious.
(ii) $\Rightarrow$ (i). We reason by contraposition and suppose that $\nvdash_{\mathrm{SI}} \varphi$. Define the following relation on the set $\mathcal{X}$: $X \equiv Y$ if and only if, for all $\psi \in \mathrm{Sub}(\varphi)$, $\psi \in X$ iff $\psi \in Y$. It is immediate to check that $\equiv$ is an equivalence relation on $\mathcal{X}$. Since $\varphi \in \mathrm{Sub}(\varphi)$, then clearly $\mid \mathcal{X}_{/\equiv} \mid \leq 2^n$. Define the binary relation ρ on the set $\mathcal{X}_{/\equiv}$ (whose elements are denoted by $[X], [Y], [Z], \dots$) as follows:

$[X]\rho[Y]$ if and only if, for all ψ such that $\mathbf{I}\psi \in \mathrm{Sub}(\varphi)$, if $\mathbf{I}\psi \in X$, then $J_1\psi \in Y$.

Consider the structure $\mathcal{N} = \langle \mathcal{X}_{/\equiv}, \rho, w \rangle$, where w is defined as follows:

- $w([X], p) = 1$ if and only if $p \in \mathrm{Sub}(\varphi)$ and $p \in [X]$;
- $w([X], p) = 0$ if and only if $p \in \mathrm{Sub}(\varphi)$ and $\neg p \in [X]$;
- $w([X], p) = 1/2$ if and only if $p \in \mathrm{Sub}(\varphi)$ and $J_1 p \in [X]$,

for every $[X] \in \mathcal{X}_{/\equiv}$. It is immediate to check that $\mathcal{N}$ is a model of **SI**. Moreover, let $\mathcal{M}$ be the canonical model defined in Definition 23: it is not difficult to prove that $v(X, \psi) = w([X], \psi)$, for every

formula $\psi \in \mathrm{Sub}(\varphi)$ (the proof of this claim runs by induction on the length of the formula ψ). Since $\nvdash_{\mathbf{SI}} \varphi$, then there exists $X \in \mathcal{X}$ and some v (in the canonical model) such that $v(X, \varphi) \neq 1$ (this follows from the proof of Theorem 25). Then, by the previous claim, $w([X], \varphi) \neq 1$ and the cardinality of the model $\mathcal{N}$ is at most 2^n. $\square$

As a direct consequence of Lemma 27 one gets

Theorem 28. *The logic* $\mathbf{SI}$ *is decidable.*

5. Conclusion and Comparison with other approaches

We have introduced *severe* ignorance as a *content-theoretic* notion. In particular, we have focused on the logical modeling of such notion, assumed as primitive ("as a first class citizen"), i.e. disconnected from knowledge, via a modal logic based on a three-valued propositional logic. The intuition behind our proposal is that being ignorant of φ means that φ is indeterminate (is assigned with the third value $1/2$) in all the worlds accessible to an agent. To the best of our knowledge, the unique existing system considering **I** as a primitive modality is the logic **Ig**, by Van der Hoek and Lomuscio [39]. However, as discussed in Section 2, in **Ig** the semantics of **I** coincides with the interpretation of ignorance as "lack of knowledge" in S_4, although no (term-definable) modality expressing knowledge can be defined in **Ig**. Being conscious of this relevant difference between **Ig** and the standard view in S_4, we will identify them with respect to the behaviour of the modality for ignorance **I** in the following discussion.

We make a comparison, in Figure 3, between $\mathbf{SI}$ and S_4 (and thus also **Ig**) in terms of logical truths explicitly involving **I** (all the listed formulas have been mentioned in the previous sections). The aim is to show the existing difference between approaching ignorance as lack of knowledge (standard view in S_4) and the type of content-theoretic ignorance analyzed here, according to the three-valued modal logic $\mathbf{SI}$.

As already discussed in Section 2, $\mathbf{SI}$ and S_4 present remarkable differences, with respect to the behaviour of the modality **I**. Regarding, for instance, conjunctive statements, in our proposal, an agent who is ignorant of all the chapters of a book then is ignorant of the whole book (formula 2), which does not happen to be the case in S_4. In the latter, perhaps, it does not make sense to express sentences like "ignoring a book". Indeed, one could say that "an agent does not know the content of a book", and not that "an agent does not know *whether* the content of a book". The converse implication (3) holds in S_4, while does not in $\mathbf{SI}$.

A remarkable difference distinguishes S_4 and $\mathbf{SI}$ relatively to the behaviour of **I** with respect to disjunctive statements, too. Severe

	Formulae	SI	S_4
1	$\mathbf{I}\varphi \leftrightarrow \mathbf{I}\neg\varphi$	✓	✓
2	$\mathbf{I}\varphi \wedge \mathbf{I}\psi \to \mathbf{I}(\varphi \wedge \psi)$	✓	✗
3	$\mathbf{I}(\varphi \wedge \psi) \to \mathbf{I}\varphi \wedge \mathbf{I}\psi$	✗	✓
4	$\mathbf{I}\varphi \vee \mathbf{I}\psi \to \mathbf{I}(\varphi \vee \psi)$	✓	✗
5	$\mathbf{I}(\varphi \vee \psi) \to \mathbf{I}\varphi \vee \mathbf{I}\psi$	✓	✓
6	$\mathbf{I}(\varphi \to \psi) \to (\mathbf{I}\varphi \to \mathbf{I}\psi)$	✗	✓
7	$\mathbf{I}\varphi \to \mathbf{II}\varphi$	✗	✗
8	$\mathbf{II}\varphi \to \mathbf{I}\varphi$	✓	✓
9	$\mathbf{I}(\varphi \wedge \psi) \to \mathbf{I}\varphi \vee \mathbf{I}\psi$	✓	✓
10	$\mathbf{I}\varphi \to \mathbf{I}(\varphi \vee \neg\varphi)$	✓	✗
11	$\mathbf{I}\varphi \to \mathbf{I}(\varphi \wedge \neg\varphi)$	✓	✗
12	$\mathbf{I}\varphi \to J_1\varphi$	✓	n.d.
13	$J_1\varphi \to \mathbf{I}\varphi$	✓	n.d.
14	$\mathbf{I}\varphi \to \varphi$	✗	✗
15	$\mathbf{II}\varphi \to \mathbf{I}^n\varphi \ (n \geq 2)$	✓	✓

FIGURE 3. A comparison of the logical truths in the logics **SI** and the standard view of **I** S_4 (see Definition (1)).

ignorance is characterized by the principle that a disjunction is ignored if and only if one of the two disjuncts are ignored. This shall not appear strange in scientific contexts that inspire our notion of severe ignorance. Indeed, to make an example, Kepler, before investigating the astronomical data collected by Tycho Brahe, was ignorant of (as anyone else) the laws that today go under his name. After he discovers the first law, we might think that he still was ignorant of the others, and we might say that he also was ignorant

of the disjunction (of the three laws), because such disjunction contains scientific terms (the second and third law) which Kepler could not imagine nor understand.

This does not happen to be the case in S_4, because lack of knowledge is different from severe ignorance. In a toy example: suppose that I do not know whether my aunt yesterday went to the cinema but I know that she went out for dinner. Thus, I do not know whether she went our for dinner (only) or also to the cinema (maybe before or after cinema), but surely I do not ignore that she went out to dinner or to the cinema.

The distribution of **I** over implication (6), which does not hold in **SI** while it does S_4, also differentiates enough the two systems under analysis. This gives the occasion, once more, to illustrate the sense of severe ignorance (in the scientific context). To exemplify the failure of (6), we might reasonably think that in 1914, Einstein was ignorant of the fact that the curvature of space-time is the cause of the anomaly affecting the perihelion shift of Mercury. At the time, the implication is scientifically ignored, however scientists were conscious of the anomaly in Mercury's perihelion. This is a good reason why **I** should not distribute over implication, in case it models severe ignorance.

Formulas 7 and 8 witness that the two logics have the same behaviour with respect to the relationship between first-order ($\mathbf{I}\varphi$) and second-order ignorance ($\mathbf{II}\varphi$). Not surprisingly, the latter implies the former but not the other way round.

Formula 9 is also in common between S_4 and **SI**. As we already commented in Section 4, it expresses the very intuitive principle that being ignorant of a conjunction implies being ignorant of at least one of the conjuncts, a principle that must be common (together with 1) to any possible notion of ignorance.

Formulas 10-13 witness the main difference due to the choice of different propositional basis. Indeed $\varphi \vee \neg\varphi$ ($\varphi \wedge \neg\varphi$, respectively) is true in every world, of every model of S_4 (false, respectively), hence can not be ignored. On the contrary, in a three-valued setting, those formulas can be indeterminate (when φ is indeterminate) and, consequently ignored. The validity of this formula tells us that the agent who is ignorant of φ is ignorant also that $\varphi \vee \neg\varphi$ coincides with the truth (something that is possible only in nonclassical cases). This confirms that the notion of severe ignorance in **SI** stands quite far from lack of knowledge.

Formulas 12 and 13 are worthy of receiving comments. Due to the choices of the semantics where φ is ignored when φ is undefined in all related worlds and that of focusing on reflexive models, the formula $\mathbf{I}\varphi$ turns out to be logically equivalent to $J_1\varphi$. This

seems to suggest that the severe ignorance could be perhaps modeled just recurring to the propositional level. In particular, this could possibly be achieved via the linguistic fragment $\neg, \wedge, J_1$ of Bochvar external logic (it is not difficult to check that, in this fragment, J_0 and J_2 are not definable). It is worth to remind that the choice to model (severe) ignorance as a modality via a Kripke-style semantics is primarily due to a philosophical stance (not a logical one), which traditionally frames ignorance (and knowledge) as an epistemic notion. What we ultimately get by our analysis is the coincidence between the modality **I** (with the intended semantics) and J_1, highlighting the modal behavior of the external connectives in Bochvar logic, which, to best of our knowledge, has not been observed insofar.[4] We believe that the reason of such coincidence relies on the choice of defining **SI** as the logic of all reflexive models, which is again founded on the philosophical assumption of interpreting **I** as en epistemic modality. Indeed, it is not difficult to check that the equivalence between **I** and J_1 does not hold in all models (see Remark 12).

Formula 14 expresses the "factivity of ignorance" (the analog of the usual notion of factivity for the modality **K** for knowledge). The importance of this property for the notion of ignorance has been recently discussed in literature, where some authors look for logics of ignorance where it holds (see [23] and [18]). In the context of severe ignorance, as a content-theoretic notion, we are not surprised that the formula does not hold. However, we highlight that a modal approach, based on a three-valued logic, can be adopted also for logics of ignorance admitting the factivity, by choosing a different set of designated values ($\{1, 1/2\}$).

Finally, both logics suffer the phenomenon that Fine [13] calls the "black hole of ignorance". In his paper, Fine shows that second-order ignorance and higher-orders of ignorance are tightly tied together: once second-order ignorance is present, an agent is doomed to the black hole of higher-order levels of ignorance. This is captured by formula 15.

We are conscious that much logical and epistemological work remains to be done and that also the choice of **SI** to model severe ignorance presents some difficulties. For instance, it could be argued that it is quite odd that being ignorant of a formula φ implies being ignorant of also $\varphi \wedge \psi$, when φ and ψ are totally unrelated formulas (this happens to be the case in **SI**). Nevertheless, the present exploration highlights that interesting aspects of ignoring are not successfully captured by the standard logical approach to ignorance, based on lack of knowledge. Interestingly, disconnecting

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[4]We thank an anonymous reviewer for suggesting further clarification on this aspect, on which we will focus in future works.

ignorance from knowledge allows for the logical modelling of severe ignorance, a notion which is common in the everyday practice of science. We have decided to introduce a modal logic grounded on a peculiar non-classical propositional basis. A choice essentially motivated by the willingness of modeling a severe notion of ignorance. Clearly, many other options are available, in the realm of non-classical logics: a possibility that we leave for further research.

APPENDIX

In this Appendix we provide the details of the proof of strong completeness for Bochvar (external) logic B_e (Theorem 7). Moreover, we also show that B_e is algebraizable. We start with a preliminary lemma.

Lemma 29. *The following elementary facts holds in* B_e:

(1) $\vdash_{B_e} J_0(\varphi \vee \psi) \leftrightarrow J_0\varphi \wedge J_0\psi$;
(2) $\vdash_{B_e} J_1(\varphi \vee \psi) \leftrightarrow J_1\varphi \vee J_1\psi$;
(3) $\vdash_{B_e} J_2(\varphi \vee \psi) \leftrightarrow (J_2\varphi \wedge J_2\psi) \vee (J_2\varphi \wedge J_2\neg\psi) \vee (J_2\neg\varphi \wedge J_2\psi)$;
(4) *if* α *is an external formula, then* $\vdash_{B_e} \alpha \leftrightarrow J_2\alpha$;
(5) $\vdash_{B_e} \neg J_2\psi \leftrightarrow (J_0\psi \vee J_1\psi)$.

Proof. Immediate by using Theorem 6.   □

Lemma 30. *Let* $\Gamma \cup \{\varphi\} \vdash_{B_e} \psi$. *Then* $\Gamma \vdash_{B_e} J_2\varphi \to J_2\psi$.

Proof. By induction on the length of the derivation of ψ (from Γ).   □

The class of *Bochvar algebras* is introduced by Finn and Grigolia [14, pp. 233-234] as algebraic semantics for B_e.

Definition 31. A Bochvar algebra $\mathbf{A} = \langle A, \vee, \wedge, \neg, J_0, J_1, J_2, 0, 1 \rangle$ is an algebra of type $\langle 2, 2, 1, 1, 1, 1, 0, 0 \rangle$ satisfying the following identities and quasi-identities:

(1) $x \vee x \approx x$;
(2) $x \vee y \approx y \vee x$;
(3) $(x \vee y) \vee z \approx x \vee (y \vee z)$;
(4) $(x \wedge (y \vee z) \approx ((x \wedge y) \vee (x \wedge z))$;
(5) $\neg(\neg x) \approx x$;
(6) $\neg 1 \approx 0$;
(7) $\neg(x \vee y) \approx \neg x \wedge \neg y$;
(8) $0 \vee x \approx x$;
(9) $J_2 J_i x \approx J_i x$, for every $i \in \{0, 1, 2\}$;
(10) $J_0 J_i x \approx \neg J_i x$, for every $i \in \{0, 1, 2\}$;
(11) $J_1 J_i x \approx 0$, for every $i \in \{0, 1, 2\}$;
(12) $J_i(\neg x) \approx J_{2-i}x$, for every $i \in \{0, 1, 2\}$;
(13) $J_i x \approx \neg(J_j x \vee J_k x)$, for $i \neq j \neq k \neq i$;
(14) $J_i x \vee \neg J_i x \approx 1$, for every $i \in \{0, 1, 2\}$;
(15) $((J_i x \vee J_k x) \wedge J_i x) \approx J_i x$, for $i, k \in \{0, 1, 2\}$;

(16) $x \lor J_i x \approx x$, for $i \in \{1, 2\}$;

(17) $J_0(x \lor y) \approx J_0 x \land J_0 y$;

(18) $J_2(x \lor y) \approx (J_2 x \land J_2 y) \lor (J_2 x \land J_2 \neg y) \lor (J_2 \neg x \land J_2 y)$;

(19) $J_i x \approx J_i y \implies x \approx y$, where $0 \leq i \leq 2$.

We denote by $\mathcal{BA}_3$ the class of Bochvar algebras. $\mathcal{BA}_3$ forms a quasi-variety which is not a variety [14]. Recall that a class $\mathcal{K}$ of algebras is an algebraic semantics for a logic L provided that: $\Gamma \vdash_L \varphi$ iff $\{\tau(\gamma) : \gamma \in \Gamma\} \models_{Eq(\mathcal{K})} \tau(\varphi)$, where $\tau = \{\varphi_i(p) \approx \psi_i(p)\}$ is a formula-equation transformer and $Eq(\mathcal{K})$ denotes the usual equational consequence relation relative to the class $\mathcal{K}$.

Theorem 32. *$\mathcal{BA}_3$ is an algebraic semantics for* B_e. *In particular,* $\Gamma \vdash_{B_e} \varphi$ *iff* $\{\gamma \approx 1 : \gamma \in \Gamma\} \models_{Eq(\mathcal{BA}_3)} \varphi \approx 1$.

Proof. ($\Rightarrow$) By induction on the length of the derivation of φ (from Γ), by checking that axioms (A1)-(A29) are evaluated to 1 in every Bochvar algebra **A** and that the rule (MP) preserves this property. ($\Leftarrow$) We reason by contraposition. Suppose $\Gamma \nvdash_{B_e} \varphi$ and provide a counterexample to such inference by constructing the Lindenbaum-Tarski algebra. Let Δ be the smallest set of formulas including Γ and closed under $\vdash_{B_e}$ (from now on we will simply write $\vdash$ instead of $\vdash_{B_e}$). For any pair of formulas, define

$$\varphi \sim \psi \text{ if and only if } \varphi \equiv \psi \in \Delta.$$

We claim that:

(1) $\sim$ is a congruence on **Fm**;

(2) $[1]_\sim = \Delta$;

(3) The quotient algebra $\mathbf{Fm}_{/\sim}$ is a Bochvar algebra.

(1) It is easy to check that $\sim$ is an equivalence relation. To show that it is a congruence, we check the compatibility of $\sim$ with the operations in the type of $\mathcal{L}_{K^e}$.

[$\neg$] Suppose $\varphi \sim \psi$, then $\varphi \equiv \psi \in \Delta$, i.e. $\bigwedge_{i=0}^{2} J_i \varphi \leftrightarrow J_i \psi \in \Delta$, which is equivalent to $\bigwedge_{i=0}^{2} J_{2-i} \varphi \leftrightarrow J_{2-i} \psi \in \Delta$. Hence, in virtue of (A12), $\bigwedge_{i=0}^{2} J_i(\neg\varphi) \leftrightarrow J_i(\neg\psi) \in \Delta$, i.e. $\neg\varphi \equiv \neg\psi \in \Delta$, showing that $\neg\varphi \sim \neg\psi$.

[J_2] Suppose $\varphi \sim \psi$, thus $\varphi \equiv \psi \in \Delta$, i.e. $\bigwedge_{i=0}^{2} J_i \varphi \leftrightarrow J_i \psi \in \Delta$. In particular $\Delta \vdash J_2 \varphi \leftrightarrow J_2 \psi$. In virtue of (A9), we have $\vdash J_2 \varphi \leftrightarrow J_2 J_2 \varphi$ and $\vdash J_2 \psi \leftrightarrow J_2 J_2 \psi$, from which $\Delta \vdash J_2 J_2 \varphi \leftrightarrow J_2 J_2 \psi$, i.e. $J_2 J_2 \varphi \leftrightarrow$

$J_2 J_2 \psi \in \Delta$ (as Δ is closed under consequences of $\vdash$). Analog reasoning, using (A10) and (A11), shows that $J_0 J_2 \varphi \leftrightarrow J_0 J_2 \psi \in \Delta$ and $J_1 J_2 \varphi \leftrightarrow J_1 J_2 \psi \in \Delta$, from which $J_2 \varphi \sim J_2 \psi$.

[$\vee$] Suppose $\varphi_1 \sim \psi_1$ and $\varphi_2 \sim \psi_2$. Then $\Delta \vdash J_0 \varphi_1 \leftrightarrow J_0 \psi_1$ and $\Delta \vdash J_0 \varphi_2 \leftrightarrow J_0 \psi_2$, hence $\Delta \vdash (J_0 \varphi_1 \wedge J_0 \varphi_2) \leftrightarrow (J_0 \psi_1 \wedge J_0 \psi_2)$. Applying Lemma 29-(1), we have $\Delta \vdash J_0 (\varphi_1 \vee \varphi_2) \leftrightarrow J_0 (\psi_1 \vee \psi_2)$, from which $J_0 (\varphi_1 \vee \varphi_2) \leftrightarrow J_0 (\psi_1 \vee \psi_2) \in \Delta$. Analog reasoning (using Lemma 29-(2,3)) allows to conclude $\varphi_1 \vee \varphi_2 \sim \psi_1 \vee \psi_2$.

(2) [$\subseteq$] Let $\psi \in [1]_\sim$. Then $\psi \equiv 1 \in \Delta$, i.e. $\bigwedge_{i=0}^{2} J_i \psi \leftrightarrow J_i 1 \in \Delta$. In particular, $J_2 \psi \in \Delta$ (since $\vdash J_2 1$) and $J_1 \psi \leftrightarrow 0 \in \Delta$ (as $\vdash J_1 1 \leftrightarrow 0$), from which we deduce that ψ is an external formula, so, by Lemma 29-(4), $\psi \in \Delta$.

[$\supseteq$] Let $\psi \in \Delta$. Observe that $1 \in \Delta$ (since $\vdash 1$), hence, by Lemma 30, $\Delta \vdash J_2 \psi \leftrightarrow J_2 1$, thus $J_2 \psi \leftrightarrow J_2 1 \in \Delta$. Moreover, $\Delta \vdash \neg J_2 \psi \leftrightarrow 0$ and, by Lemma 29, $\vdash \neg J_2 \psi \leftrightarrow (J_0 \psi \vee J_1 \psi)$, so $\Delta \vdash (J_0 \psi \vee J_1 \psi) \leftrightarrow 0$, from which $\Delta \vdash J_0 \psi \leftrightarrow 0$ and $\Delta \vdash J_1 \psi \leftrightarrow 0$; therefore $\Delta \vdash J_0 \psi \leftrightarrow J_0 1$ and $\Delta \vdash J_1 \psi \leftrightarrow J_1 1$ (as $\vdash J_0 1 \leftrightarrow 0$ and $\vdash J_1 1 \leftrightarrow 0$). This shows that $\psi \equiv 1 \in \Delta$, i.e. $\psi \in [1]_\sim$.

(3) It is routine to check that $\mathbf{Fm}_{/\sim}$ is indeed a Bochvar algebra. To provide a counterexample to the inference $\Gamma \nvdash_{B_e} \varphi$, consider the Bochvar algebra $\mathbf{A} = \mathbf{Fm}_{/\sim}$ and the homomorphism $h \colon \mathbf{Fm} \to \mathbf{A}$, $h(\varphi) = [\varphi]_\sim$. Since $\Gamma \subseteq \Delta$ and $\Delta = [1]_\sim$, then $h(\gamma) = 1^{\mathbf{A}}$, for each $\gamma \in \Gamma$, but $h(\varphi) \neq 1^{\mathbf{A}}$ (since $\varphi \notin \Delta$). $\square$

Theorem 7 follows from Theorem 32 by observing that $\mathcal{BA}_3$ is the quasi-variety generated by $\mathbf{WK}^e$ ([14, Theorem 3.3]).

It is natural to wonder whether the quasi-variety of Bochvar algebras is simply *an* algebraic semantics for B_e. Actually the relationship between B_e and the class $\mathcal{BA}_3$ is tighter. Recall that a logic $\vdash$ is algebraizable with $\mathcal{K}$ as equivalent algebraic semantics (where $\mathcal{K}$ is a class of algebras of the same language as the logic $\vdash$) if there exists a map τ from formulas to sets of equations, and a map ρ from equations to sets of formulas such that the following conditions hold, for any pair of formulas φ, ψ and set of equations E.

(ALG1) $\Gamma \vdash \varphi$ iff $\tau[\Gamma] \models_{Eq(\mathcal{K})} \tau(\varphi)$;
(ALG2) $E \models_{Eq(\mathcal{K})} \varphi \approx \psi$ iff $\rho(E) \models_{Eq(\mathcal{K})} \rho(\varphi, \psi)$;
(ALG3) $\varphi \dashv\vdash \rho(\tau(\varphi))$;
(ALG4) $\varphi \approx \psi =\models\models_{Eq(\mathcal{K})} \tau(\rho(\varphi, \psi))$.

Examples of algebraizable logics include, among many others, classical, intuitionistic logic, all substructural logics and global modal logics. Not all logics though are algebraizable: examples of non-algebraizable logics can be found in the realm of Kleene logics, such as the Logic of Paradox (see [32]), Paraconsistent weak Kleene (see [5]) and Bochvar internal logic (see [6, 7]).

The above definition of algebraizable logic can be drastically simplified: $\vdash$ is algebraizable with equivalent algebraic semantics $\mathcal{K}$ if and only if it satisfies either ALG1 and ALG4 (or, else ALG2 and ALG3).[5]

Theorem 33. *The logic* B_e *is algebraizable with* $\mathcal{BA}_3$ *as equivalent algebraic semantics.*

Proof. Consider $\tau = \{\varphi \approx 1\}$ and $\rho = \{\varphi \equiv \psi\}$. $\square$

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