The Global Phase Is Real

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Abstract

Normalized vectors in the Hilbert space of the world can differ merely by a complex scalar multiple of unit modulus, also known as the global phase. According to orthodoxy, the global phase is a paradigm case of a redundancy in our mathematical description of the world, mere representational fluff of no physical significance. In this paper, I argue that this view is inconsistent with the fact that a physically substantive proposition turns on how rotations are represented within the linear formalism of quantum theory. I also explore strategies for resisting this conclusion and highlight an intimate connection between the reality of the global phase and the metaphysics of spacetime.

Normalized Hilbert space vectors can differ merely by a complex scalar multiple of unit modulus, also known as the *phase*. Differences in phase between two vectors are empirically detectable if those vectors characterize quantum states of subsystems of the world, in which case the phase is known as the *relative phase*. By contrast, *global* phase differences—differences between points in a Hilbert space that is intended to characterize the state space of the *entire* world—are widely regarded as redundancies in our mathematical description of the world, mere representational fluff of no physical significance. As opposed to relative phase differences, empirical predictions of quantum mechanics are insensitive to changes in global phase: given the Born rule, any two points in the Hilbert space of the world that differ merely by a complex scalar multiple must have exactly the same empirical content. This is usually taken to suggest that physical states correspond to (normalized) *rays* in the Hilbert space of the world—equivalence classes of normalized Hilbert space vectors under complex scalar multiplication—so that the space of physical states has the structure of a projective Hilbert space rather than that of a linear Hilbert space. In other words, the undetectability of the global phase in quantum mechanics motivates the view I call

**RAYS.** No two vectors that belong to the same ray in the Hilbert space of the world correspond to distinct physical states.

According to this view, the global phase is a mere redundancy in our mathematical description of the state space of the world; it is a mathematical degree of freedom that does not correspond to a physical degree of freedom.

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Rays is perhaps the most clear-cut case of orthodoxy common to physics and philosophy of physics; the goal of this paper is to challenge it. I will argue that the global phase is not a representational redundancy or a mere mathematical degree of freedom without a physical counterpart, but that it corresponds to a real physical parameter.

I focus on the Hilbert space $\mathcal{H}_s$ that characterizes the total spin degrees of freedom of the world: the infinite-dimensional Hilbert space spanned by vectors $|JM\rangle$ where for any non-negative integer $n$, $J = n/2$ is a possible value of total spin and any $M$ such that $-J \leq M \leq J$ is a possible spin component value along some spatial direction. For every fixed $J = n/2$, let $H_J$ be the finite-dimensional Hilbert space spanned by $|JM\rangle$; that is, the Hilbert space whose points all agree on the total spin $J$. Then, the total Hilbert space decomposes into the direct sum

$$\mathcal{H}_s = \mathcal{H}_0 \oplus \mathcal{H}_{1/2} \oplus \mathcal{H}_1 \oplus \mathcal{H}_{3/2} \oplus \ldots$$

(1)

Suppose that the world has $N$ basic constituents with spin. These can in principle be any kind of basic subsystem, but for simplicity I will henceforth refer to them as ‘particles’. Then $\mathcal{H}_s$ is given by an $N$-fold tensor product

$$\mathcal{H}_s = \overbrace{\mathcal{H}_{\text{ind}} \otimes \ldots \otimes \mathcal{H}_{\text{ind}}}^{\text{N times}}$$

(2)

whose factors $\mathcal{H}_{\text{ind}}$ are themselves infinite-dimensional Hilbert spaces of single-particle systems. Each $\mathcal{H}_{\text{ind}}$ is isomorphic to $\mathcal{H}_s$ in the sense that $\mathcal{H}_{\text{ind}}$ is spanned by $|jm\rangle$ where for any non-negative integer $n$, $j = n/2$ is a possible value of spin of some particle, and any $m$ such that $-j \leq m \leq j$ is a possible spin component value along some spatial direction.\(^1\) Moreover, the standard formalism for the addition of angular momenta implies a systematic relationship between the total spin of the world and the spin of the individual particles.\(^2\) For example, in a world consisting of two particles with spins $j$ and $j'$, the possible values of total spin $J$ are restricted to the range $|j - j'| \leq J \leq j + j'$ respectively, $M = m + m'$, and each basis vector of $\mathcal{H}_s$ can be decomposed according to

$$|JM\rangle = \sum_{m=-j}^{j} \sum_{m'=-j'}^{j'} |jmjm'\rangle\langle jmjm'|JM\rangle.$$

(3)

The total state space of the world $L^2(\mathbb{R}^{3N}) \otimes \mathcal{H}_s$ includes the position degrees of freedom given by the space $L^2(\mathbb{R}^{3N})$ of square-integrable functions on $\mathbb{R}^{3N}$. The spin and position degrees of freedom are independent of each other, in the sense that, for each vector $\psi$ in $L^2(\mathbb{R}^{3N}) \otimes \mathcal{H}_s$, there is another vector in $L^2(\mathbb{R}^{3N}) \otimes \mathcal{H}_s$ that differs from $\psi$ only with respect to $L^2(\mathbb{R}^{3N})$ and another vector that differs from $\psi$ only with respect to $\mathcal{H}_s$.\(^3\) In this sense, vectors in $\mathcal{H}_s$ characterize the spin states of the

\(^1\)A fortiori, each Hilbert space $\mathcal{H}_{\text{ind}}$ decomposes into a direct sum of the form (1) in which the terms are now taken to represent states of a single-particle world and the vectors in each term agree on the spin of the particle.

\(^2\)(Ballentine, 2000, pp. 185).

\(^3\)Consider the three vectors

$$\Psi = c (\phi(x) \otimes |1,-1\rangle - \varphi(x) \otimes |1,1\rangle)$$

$$\Psi' = c' (\phi'(x) \otimes |1,-1\rangle - \varphi'(x) \otimes |1,1\rangle)$$

$$\Psi'' = c'' (\phi(x) \otimes |1,1\rangle - \varphi(x) \otimes |1,-1\rangle)$$
world.\textsuperscript{4}

My argument applies to all realist quantum theories in which spin states are characterized by $\mathcal{H}_s$. Different realist quantum theories will generally differ as to what, exactly, ‘spin states’ consist in. For definiteness, let me spell this out in the case of wavefunction realism, an ontology that underlies a range of solutions to the quantum measurement problem.\textsuperscript{5} According to this proposal, fundamental reality consists in a field on a high-dimensional arena—an entity that can be thought of as an assignment of a property to each location in this arena. If the world has spin degrees of freedom, the fundamental field is characterized by wavefunctions in $L^2(\mathbb{R}^{3N}) \otimes \mathcal{H}_s$ so that the properties assigned by the field to each location in the fundamental arena are represented by $\mathcal{H}_s$. Within wavefunction realism, ‘spin states’ are just these properties. Henceforth, I will use ‘quantum mechanics’ (QM) as an umbrella term for realist theories which use $\mathcal{H}_s$ to characterize spin states.

The goal of this paper is to argue for the negation of the restriction of \textsc{Rays} to $\mathcal{H}_s$:

\begin{enumerate}
\item \textsc{Spin-Rays}. No two vectors that belong to the same ray in $\mathcal{H}_s$ correspond to distinct spin states.
\item \textsc{Rays} implies \textsc{Spin-Rays}.\textsuperscript{6} So, the negation of \textsc{Spin-Rays} is sufficient for the negation of \textsc{Rays}. Moreover, \textsc{Spin-Rays} is the negation of \textsc{Spin-Vectors}. There are vectors that belong to the same ray in $\mathcal{H}_s$ and correspond to distinct spin states.
\end{enumerate}

According to \textsc{Spin-Vectors}, there are variations in the global phase of $\mathcal{H}_s$ that result in physical differences, and so the global phase of $\mathcal{H}_s$ is a real physical parameter.

\textsc{Spin-Vectors} does not imply that the physical states of the world have the structure of a linear Hilbert space. In particular, it is consistent with \textsc{Spin-Vectors} that the states of the spatial degrees of freedom are in one-to-one correspondence

where $\phi, \phi', \phi'' \in L^2(\mathbb{R}^{3N})$ and $c, c', c''$ are normalization constants. $\Psi$ and $\Psi'$ disagree merely in the position degrees of freedom, whereas $\Psi$ and $\Psi''$ disagree merely in the spin degrees of freedom.

Note that the independence (in the sense just specified) of the spin and position degrees of freedom is consistent with entanglement of position and spin degrees of freedom. Indeed, the three vectors above describe states in which position and spin degrees of freedom are entangled.

\textsuperscript{4}Throughout this paper I assume realism about spin: the thesis that facts about spin (including the spin of the world and its subsystems) are mind-independent as well as independent of our empirical means of discovery. A fortiori, I assume non-contextuality about spin: the thesis that attributions of spin are independent of measurement context; that is, independent of how spin measurements are (or might be) carried out. I borrow this formulation of non-contextuality from (Held, 2018).

\textsuperscript{5}The view was originally proposed in (Albert, 1996; Loewer, 1996); (Chen, 2019) reviews more recent developments.

\textsuperscript{6}Proof: we proceed by showing the converse. To illustrate the strategy, we first consider vectors known as ‘product states’. Suppose that $|J,M\rangle$ and $e^{i\theta}|J,M\rangle$ are vectors that correspond to distinct spin states of the world. Then for any $\Psi \in L^2(\mathbb{R}^{3N})$, $\psi \otimes |J,M\rangle$ and $\psi \otimes e^{i\theta}|J,M\rangle$ are vectors on the same ray in $L^2(\mathbb{R}^{3N}) \otimes \mathcal{H}_s$ that correspond to distinct physical states. This generalizes to vectors that are not necessarily product states. Let $\Psi \in L^2(\mathbb{R}^{3N}) \otimes \mathcal{H}_s$, and suppose that there is some $e^{i\theta}$ such that for every $|\phi\rangle \in \mathcal{H}_s$ that occurs in $\Psi$, $|\phi\rangle \in \mathcal{H}_s$ and $e^{i\theta}|\phi\rangle \in \mathcal{H}_s$ correspond to distinct spin states of the world. Denote by $\Psi'$ the vector resulting from $\Psi$ by replacing, for all $|\phi\rangle \in \mathcal{H}_s$ that occur in $\Psi$, every occurrence of $|\phi\rangle$ with $e^{i\theta}|\phi\rangle$. Then $\Psi$ and $\Psi'$ correspond to distinct physical states. But since $\Psi' = e^{i\theta}\Psi$, it follows that there are vectors on the same ray in $L^2(\mathbb{R}^{3N}) \otimes \mathcal{H}_s$ that correspond to distinct physical states.
with rays (rather than with vectors) in $L^2(\mathbb{R}^{3N})$. The ontological status of the phase of $L^2(\mathbb{R}^{3N})$ is unaffected by the argument in this paper.

The argument proceeds from a background premise about the representational significance of $\mathcal{H}_s$, namely, that it characterizes (perhaps redundantly) the spin states of the world:

**Hilbert.** Spin states of the world are characterized (perhaps redundantly) by $\mathcal{H}_s$.

**Hilbert** is neutral between Spin-Rays and Spin-Vectors: it merely implies the existence of a representation relation between vectors in $\mathcal{H}_s$ and spin states but is silent on whether this relation is many-to-one (as required by Spin-Rays) or one-to-one. The latter is required by the view that spin states have the linear structure of $\mathcal{H}_s$—something that is suggested (but not implied) by Spin-Vectors.

Moreover, throughout this paper I write as if QM is true in the actual world. But it is at best an open question whether QM (i.e. any of the theories which characterize the spin degrees of freedom in terms of $\mathcal{H}_s$) is actually true. In the interest of simplicity, I omit the relevant qualifiers; all substantive claims of this paper should be read as restricted to the metaphysical possibilities at which QM is true.

Before we begin, let me clarify something important. If I’m right and the global phase is not a redundancy in our mathematical description of the world, then what exactly does it represent? The answer is that the global phase is a physical parameter: a physical feature of the world that makes for objective sameness and difference between the possibilities described by quantum theory. In this respect, the global phase would be like any other physical parameter: just like physical possibilities can differ merely in the mass of some particle, physical possibilities can differ merely in the value of the global phase. To be sure: there would remain a key difference between the global phase and physical parameters such as mass: as opposed to the latter, the global phase does not feature in the laws and so no two physical possibilities that differ merely in the global phase differ with regard to possible measurement results. But the thought that physical theories imply the existence of physical entities that are unobservable-in-principle should not be repugnant to anyone but the most hard-headed empiricists. Philosophers recently seem to have converged on the more moderate position that such unobservables should be posited if there are compelling physical reasons to do so. The point of this paper is to argue that this condition is satisfied for the global phase.

Here is the plan. Sections 1 and 2 contain the argument against Spin-Rays. Section 3 explains an important sense in which the conclusion is less radical than it seems. Finally, section 4 explores possible ways of blocking the argument and highlights an intimate connection between the reality of the global phase and the metaphysics of spacetime.

## 1 Rotational Symmetry

If Spin-Rays is true, there are two mathematically distinct but physically equivalent ways of characterizing rotational symmetry.
Before I explain, let me review some technical preliminaries. First of all, according to Spin-Rays, the state space of the world has the structure of a projective space: the space mathematically characterized by the quotient space \( \mathbb{P}(L^2(\mathbb{R}^3) \otimes \mathcal{H}_s) = (L^2(\mathbb{R}^3) \otimes \mathcal{H}_s \setminus \{0\})/\sim \), where \( \psi \sim \psi' \) iff there is a nonzero complex number \( c \) such that \( \psi = c\psi' \), for \( \psi, \psi' \in L^2(\mathbb{R}^3) \otimes \mathcal{H}_s \). For the sake of expository simplicity, the argument of this and the next section will assume that the state space of the world is represented by the space of spin states \( \mathcal{H}_s \) (so that, according to Spin-Rays, the spin states have the structure of \( \mathbb{P}(\mathcal{H}_s) \)); the argument straightforwardly generalizes to the full state space.

Second, recall that symmetries are automorphisms on the state space of the world. Given Spin-Rays, this means that symmetries are automorphisms of \( \mathbb{P}(\mathcal{H}_s) \)—henceforth projective automorphisms. More precisely: let \( \gamma : \mathcal{H}_s \setminus \{0\} \to \mathbb{P}(\mathcal{H}_s) \) be the canonical projection map and let

\[
\delta(a, b) = \frac{|\langle \psi, \psi' \rangle|^2}{||\psi||^2||\psi'||^2},
\]

for any \( a, b \in \mathbb{P}(\mathcal{H}_s) \) and \( \psi, \psi' \in \mathcal{H}_s \) such that \( a = \gamma(\psi) \) and \( b = \gamma(\psi') \). Then, a bijection \( R : \mathbb{P}(\mathcal{H}_s) \to \mathbb{P}(\mathcal{H}_s) \) is a symmetry iff \( \delta(Ra, Rb) = \delta(a, b) \), for any \( a, b \in \mathbb{P}(\mathcal{H}_s) \). Let \( \text{Aut}(\mathbb{P}(\mathcal{H}_s)) \) denote the group of projective automorphisms on \( \mathbb{P}(\mathcal{H}_s) \).

Third, according to a theorem by Wigner (1931), for every continuous symmetry \( R \in \text{Aut}(\mathbb{P}(\mathcal{H}_s)) \) there is a unitary operator \( O \) such that \( R = \hat{\gamma}(O) \), where \( \hat{\gamma} : U(\mathcal{H}_s) \to \text{Aut}(\mathbb{P}(\mathcal{H}_s)) \) is a group homomorphism such that \( \hat{\gamma}(O)(f) = \gamma(O(\psi)) \) for all \( f = \gamma(\psi) \) with \( \psi \in \mathcal{H}_s \), and \( U(\mathcal{H}_s) \) denotes the group of unitary operators on \( \mathcal{H}_s \). Let \( U(\mathbb{P}(\mathcal{H}_s)) = \hat{\gamma}(U(\mathcal{H}_s)) \subset \text{Aut}(\mathbb{P}(\mathcal{H}_s)) \) be the subgroup of unitary projective automorphisms on \( \mathbb{P}(\mathcal{H}_s) \).

In QM, rotational symmetry is implemented as a homomorphism \( T : SO(3) \to U(\mathbb{P}(\mathcal{H}_s)) \). There are two ways to characterize \( T \) in terms of unitary operators on \( \mathcal{H}_s \). First, in terms of a family of maps \( S_J : SO(3) \to U(\mathcal{H}_J) \), one for each term \( \mathcal{H}_J \) in the direct sum \( \mathcal{H}_s = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus ... \), subject to the constraint that, for each \( J \),

\[
S_J(g)S_J(g') = e^{i\phi_J(g,g')}S_J(gg'),
\]

where \( e^{i\phi_J(g,g')} \) is known as a cocycle and defined by

\[
\phi_J(g, g') = \begin{cases} 
\pi, & \text{if } J = n/2 \text{ for } n \text{ odd and the loop from } e \text{ to } g' \text{ to } gg' \text{ and back to } e \text{ is not contractible to the identity}; \\
0, & \text{otherwise}.
\end{cases}
\]

If \( J \) is an integer, \( S_J \) is a group homomorphism, also referred to as a linear representation of \( SO(3) \) on \( \mathcal{H}_J \); if \( J \) is a half-integer, \( S_J \) is referred to as a projective representation of \( SO(3) \) on \( \mathcal{H}_J \). Since \( e^{i\phi_J(g,g')} = 1 \) for all \( g, g' \in SO(3) \) only if \( J \) is integer, \( S_J \) is not a group homomorphism (and thus not a linear representation) if \( J \) is half-integer.

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\(^7\)Here, I am largely following (Schottenloher, 2008, Chpt. 3).

\(^8\)Although strictly speaking, Wigner’s theorem allows that symmetries be implemented as antiunitary transformations, operators that implement continuous symmetries (such as rotations) are unitary rather than antiunitary (Weinberg, 1995, p. 51).
An alternative way to characterize $T$ in terms of unitary operators on $\mathcal{H}_s$ is by ‘lifting’ the projective representations of SO(3) to linear representations of SU(2), the double cover of SO(3). That is: there is a group homomorphism $S' : SU(2) \rightarrow U(\mathcal{H}_s)$ which is reducible to representations $S'_J$ of SU(2) on each term $H_J$ such that

$$S'_J(g)S'_J(g') = e^{i\phi_J(\pi(g), \pi(g'))}S_J(\pi(g)\pi(g'))$$

for any $g, g' \in SU(2)$, where $\pi : SU(2) \rightarrow SO(3)$ is the covering homomorphism. Since $\hat{\gamma}$ maps any two unitary operators on $\mathcal{H}_s$ that differ merely by an overall phase to the same projective automorphism on $\mathcal{P}(\mathcal{H}_s)$, the linear representation $S'$ of SU(2) and the family $S_J$ of linear and projective representations of SO(3) agree about which unitary projective automorphisms count as rotations: exactly those which lie in the codomain of $T$. This is illustrated in figure 1.

![Figure 1: Illustration of two alternative ways to characterize the projective automorphism $T$ in terms of unitary operators on $\mathcal{H}_s$: first, by way of a family $S_J$ of linear and projective representations of SO(3); second, in terms of a linear representation $S'$ of SU(2). $\pi$ is the covering homomorphism and $\hat{\gamma}$ the canonical projection map on unitary operators. Solid lines indicate group homomorphisms.](image-url)

Here is a different way to put this. Suppose someone were to insist that, instead of $T$, rotational symmetry should be implemented as the homomorphism $\hat{\gamma} \circ S' : SU(2) \rightarrow U(\mathcal{P}(\mathcal{H}_s))$. Whereas $T$ is faithful, $\hat{\gamma} \circ S'$ is not: if for any $\varphi \in \mathbb{R}$ we denote by $g_{2\varphi}$ the element of SU(2) that corresponds to a rotation by $\varphi$ about some axis, then although $g_{2\pi} \neq g_{4\pi}$, $\hat{\gamma} \circ S'$ maps both group elements to the same projective automorphism: $\hat{\gamma} \circ S'(g_{2\pi}) = \hat{\gamma} \circ S'(g_{4\pi})$. Implementing rotations in terms of $\hat{\gamma} \circ S'$ therefore results in exactly the same projective automorphisms as does implementing rotations in terms of $T$. For convenience, let me introduce the following abbreviations:

- **Projective-SO(3)**. Rotational symmetry is characterized in terms of the family $S_J$ of maps from SO(3) into the unitary operators on $\mathcal{H}_s$.
- **Linear-SU(2)**. Rotational symmetry is characterized in terms of the linear representation $S'$ of SU(2).

If **Spin-Rays** is true, the only way in which Projective-SO(3) and Linear-SU(2) could differ physically is by having different implications as regards which

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9More precisely: $S'$ can be written as $S'(g) = S'_0(g) \oplus S'_1(g) \oplus S'_2(g) \oplus \ldots$ for any $g \in SU(2)$, where for each $J$, $S'_J : SU(2) \rightarrow H_J$ is a group homomorphism.
unitary projective automorphisms count as rotations of points in \( \mathbb{P}(H_s) \). But as we have just seen, there is no such difference: both Projective-SO(3) and Linear-SU(2) classify exactly the same unitary projective automorphisms as rotations. In this sense, Spin-Rays implies that there cannot be physical differences between Projective-SO(3) and Linear-SU(2).

As I now argue, this is not true: Projective-SO(3) and Linear-SU(2) differ about a physically substantive proposition.

2 The Univalence Superselection Rule

The central observation is that Projective-SO(3) implies what is known as the univalence superselection rule, whereas Linear-SU(2) does not. For the purposes of this paper, I will follow the statement of this rule given in (Giulini, 2003, p. 278):

Univalence. There are no spin states corresponding to superpositions of vectors in \( H_J \) and \( H_{J'} \), for any \( J \neq J' \) such that \( |J - J'| = n/2 \) for \( n \) odd.

Say that a QM world is a metaphysical possibility at which QM is true. QM worlds at which Univalence is true differ from QM worlds at which Univalence is false: there are fewer spin states at the former than there are at the latter. Univalence is therefore a physically substantive proposition, and whether Univalence is true at a QM world is a physically substantive matter.

However, as is widely recognized, Univalence is implied by Projective-SO(3) but not by Linear-SU(2). Here, it will suffice to review the basic argument; the proof can be found in the literature. First, if rays in \( H_J \oplus H_{J'} \) represent spin states, then \( H_J \oplus H_{J'} \) carries a (possibly projective) representation \( S_{J,J'} : SO(3) \to U(H_J \oplus H_{J'}) \), such that the (projective) representations \( S_J \) and \( S_{J'} \) are subrepresentations of \( S_{J,J'} \). This is the case only if \( \phi_J \) and \( \phi_{J'} \) differ from each other at most by an overall phase. However, \( \phi_J \) and \( \phi_{J'} \) differ from each other by at most an overall phase only if \( H_J \) and \( H_{J'} \) are either both integer or both half-integer-spin subspaces; that is, only if there is an even number \( n \) such that \( |J - J'| = n/2 \). It follows that, for \( J \neq J' \) such that \( |J - J'| = n/2 \) for odd \( n \), there are no spin states represented by superpositions of vectors in \( H_J \) and \( H_{J'} \), which is just Univalence.

There is no analogous argument for Linear-SU(2). The fact that \( S' \) is a linear representation implies that there are only trivial cocycles: one can always choose an overall phase such that the cocycle in \( S'(g)S(g') = e^{i\theta(g,g')}S'(gg') \) is unity for all \( g, g' \in SU(2) \). If the projective automorphisms that capture rotational symmetry are characterized in terms of the linear representation \( S' \) of SU(2), Univalence cannot be inferred.

It follows that Projective-SO(3) and Linear-SU(2) disagree about Univalence, and thus about a physically substantive proposition. But Spin-Rays implies

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\(^{10}\)This is consistent with the third type of superselection rule (SSRIII) given in (Earman, 2008, p. 384) on the assumption that the spin states of the world are necessarily pure states.


\(^{12}\)More precisely: \( S_J \) and \( S_{J'} \) are subrepresentations of \( S_{J,J'} \) iff there is a function \( \theta : SO(3) \to \mathbb{R} \) such that \( \phi_J(g,g') = \phi_{J'}(g,g') + \theta(g) + \theta(g') - \theta(gg') \). For a proof of this biconditional, see (Giulini, 2003, p. 284).
that \textsc{Projective-SO}(3) and \textsc{Linear-SU}(2) do not disagree as regards any physically substantive proposition. So \textsc{Spin-Rays} is false.

It will be useful to state our argument in the form of the following syllogism:

\begin{enumerate}
\item[A1.] If \textsc{Spin-Rays} is true then \textsc{Projective-SO}(3) and \textsc{Linear-SU}(2) imply all and only the same physically substantive propositions.
\item[A2.] \textsc{Projective-SO}(3), but not \textsc{Linear-SU}(2), implies \textsc{Univalence}.
\item[A3.] \textsc{Univalence} is physically substantive.
\end{enumerate}

C. \textsc{Spin-Rays} is false.

As noted in the introduction, C implies both \textsc{Spin-Vectors} as well as the negation of \textsc{Rays}. Holding fixed premises A1-A3, \textsc{Hilbert} therefore implies \textsc{Spin-Vectors}: if the spin states of the world are to be represented in terms of \(\mathcal{H}_s\), the only consistent way to do so is as prescribed by \textsc{Spin-Vectors}. In short: given \textsc{Hilbert}, the fact that a physically substantive proposition turns on how rotations are represented within the linear formalism of QM implies that the global phase is a real physical parameter.

Those who wish to maintain the view that the global phase is not a physical parameter are therefore forced to reject \textsc{Hilbert}: the thesis that these states can be represented (perhaps redundantly) in terms of the linear Hilbert space \(\mathcal{H}_s\). Rejecting \textsc{Hilbert} removes the significance of the linear apparatus of quantum mechanics as a mathematical means for representing the spin states of the world, and \textit{a fortiori} makes it illegitimate to infer the existence or non-existence of spin states based on how rotational symmetry is modeled in terms of unitary operators on \(\mathcal{H}_s\). In particular, if \textsc{Hilbert} is denied, the fact that there is no (possibly projective) representation \(S_{JJ'}\) of \textsc{SO}(3) on \(\mathcal{H}_J \oplus \mathcal{H}_{J'}\) which has \(S_J\) and \(S_{J'}\) as subrepresentations cannot make a difference as to whether \(\mathcal{H}_J \oplus \mathcal{H}_{J'}\) represents spin states, simply because it is already settled that no spin states are represented by \(\mathcal{H}_s\) or any of its subspaces. Denying \textsc{Hilbert} thus severs the link between the linear representation theory of \textsc{SO}(3) and \textsc{SU}(2) and facts about which spin states exist.

This is a significant departure from orthodoxy. The conjunction of \textsc{Spin-Rays} and \textsc{Hilbert}—the view that, although spin states have the structure of a projective Hilbert space, they can be represented (albeit redundantly) in terms of a linear Hilbert space—is explicit in virtually every textbook on quantum mechanics and enjoys nearly axiomatic status in debates about the philosophical foundations of QM. But our argument shows that this conjunction is inconsistent: those who represent spin states in terms of \(\mathcal{H}_s\) are required to regard spin states as having the structure of a linear Hilbert space. Representing spin states of the world \textit{directly} in terms of a projective space (rather than by way of a linear Hilbert space) is not optional, but necessary for the view that the global phase is not a real physical parameter.

It may seem that our conclusion conflicts with common practice of using linear Hilbert spaces to characterize spin states—a practice which has resulted in incredibly successful calculations and experimental predictions. But it does not. The overwhelming majority of such calculations and predictions concern \textit{subsystems} of the world rather than the world as a whole. Insofar as \(\mathcal{H}_s\) is used to represent the spin states of a subsystem, the physical significance of the phase in \(\mathcal{H}_s\) as the empirically detectable \textit{relative} phase is not in doubt. What we learn from our argument is something else: the proviso, virtually omnipresent in presentations of QM, that physical states in
general and spin states in particular are represented by rays in Hilbert space must be
given up. Since relative phases are physically significant, this proviso can only apply
to physical systems that are not subsystems, i.e. that each amount to a possible
world in their own right. Our argument shows that, if the spin states of such a world
are represented in terms of a linear Hilbert space, then the phase in this space is a
real physical parameter. My claim is therefore not that the widespread use of linear
structure by physicists and philosophers is mistaken. Rather, I urge that we endorse
the other end of the conditional: we were not wrong to represent spin states of the
world in terms of linear Hilbert spaces; but we were wrong to think that these states
have projective structure.

3 Ontology and Equivalence

There is a sense in which our conclusion should be less surprising than it might first
appear. To explain how, let me put the result of our argument slightly differently. Consider the following three variants of QM.

**Linear-Projective QM.** The space of spin states has the structure of
the linear Hilbert space $\mathcal{H}_s$, and rotational symmetry transformations
are implemented in terms of the family $S_J$ of linear and projective
representations of $SO(3)$ on $\mathcal{H}_s$.

Another theory agrees with Linear-Projective QM (or LPQM, for short) as regards
state space structure, but disagrees about the nature of rotations:

**Linear QM.** The space of spin states has the structure of a linear Hilbert
space $\mathcal{H}_s$, and rotational symmetry transformations on the spin states of
the world form the linear representation $S'$ of $SU(2)$ on $\mathcal{H}_s$.

LPQM, but not Linear QM (or LQM for short), implies **Univalence**. So while
LQM and LPQM both imply the existence of integer spin states in each subspace
$\mathcal{H}_J$, LPQM (but not LQM) is inconsistent with the existence of spin states that
correspond to superpositions of vectors in integer- and half-integer-spin subspaces of
$\mathcal{H}_s$. Finally:

**Projective QM.** The space of spin states has the structure of the
projective space $\mathbb{P}(\mathcal{H}_s)$, and rotational symmetries are implemented in
terms of the homomorphism $T$ into the unitary projective automorphisms
on $\mathbb{P}(\mathcal{H}_s)$.

The argument of this paper shows that Projective QM (or PQM, for short)
cannot be expressed in linear terms without contradiction. PQM thus requires a
representational apparatus that specifies the projective structure among spin states
in **intrinsically** projective ways, that is, without recourse to linear structure. Our
argument therefore implies a breakdown of the familiar idea that a physical theory
can be expressed by any of the formulations to which it is mathematically equivalent.
Our representational assumptions tie physical theories to specific mathematical
representations more closely than we might have been inclined to believe.
But this predicament is more common than it might appear. Antecedently, we may have thought that the linear formalisms of LPQM and LQM stand to a (putative) intrinsically projective formalism of PQM like electrodynamics in terms of the four-vector potential stands to electrodynamics in terms of the field strength tensor: the latter results from the former by ‘quotienting out’ the redundant gauge degrees of freedom. However, our argument shows that this is not true. Whereas in familiar cases, the gauge degrees of freedom removed by the transition to an intrinsic formulation are already manifestly lacking in physical significance, the phase degree of freedom removed by the transition to an intrinsically projective formalism does have physical significance: distinct implementations of rotational symmetry that differ merely at the level of linear structure differ about how many spin states there are. Unlike in the case of gauge theories, the presence of a mathematically equivalent formalism which does not involve the global phase does not imply that the global phase is unphysical. This is an instance of a fairly widespread and familiar phenomenon: for example, although the Lagrangian function does not occur in the (mathematically and empirically equivalent) formalism of Hamiltonian mechanics, it would be obviously incorrect to infer from this that the Lagrangian function is not physically significant within Lagrangian mechanics.

Another instance of this phenomenon can be found in the metaphysics of spacetime. Newtonian theories of space posit absolute locations. Just like the global phase, absolute locations are in-principle empirically undetectable—a fact which is widely taken to give support to theories which do not involve such a posit, such as Galilean theories of spacetime. However, the existence of such alternatives to Newtonian theories does not license the conclusion that absolute locations are unphysical (or ‘pure gauge’) parameters within Newtonian theories: according to these theories, absolute locations encode physically significant and empirically detectable facts about acceleration. An interpretation of the mathematical formalism of Newtonian theories according to which absolute locations are mere mathematical redundancies would be physically and empirically inadequate. As in the case of the global phase, the relationship between Newtonian theories and their relevant counterparts is not one of ‘quotienting out’ an unphysical gauge parameter; it is simply the relationship between empirically equivalent theories that make incompatible claims about what the world is like.\footnote{This echoes a similar point made in (North, 2021, p. 198).}

4 Routes of Escape

There are ways to resist our argument; but as I now argue, they are not without costs.

To begin with, let me dispense with three tempting but ineffective responses to our argument. A first response takes inspiration from Steven Weinberg, who acknowledges the disagreement between (what I refer to as) PROJECTIVE-SO(3) and LINEAR-SU(2) as regards Univalence and concludes that “the issue of superselection rules is a bit of a red herring”, taking this disagreement to show that “one cannot settle the question [of superselection rules] by reference to symmetry principles” (Weinberg, 1995, pp. 90-1). However, what is at issue in our argument is not whether Univalence is true, or
which strategy is best for inferring a superselection rule. Rather, what does the work in our argument is the thesis that Univalence is physically substantive (premise A3), something which is not denied by Weinberg. Neither does our argument require that there is a QM world at which Univalence is false: it is conceivable that QM metaphysically necessitates Univalence—perhaps because it is implied by another component of QM, or because it is promoted to an axiom of the theory. In such a scenario, there would nonetheless remain a clear sense in which Projective-SO(3), but not Linear-SU(2), is responsible for the truth of Univalence at every QM world—something that one could cash out in terms of sufficiently fine-grained notions such as essence or ground.\textsuperscript{14}

Second, some readers may be inclined to follow Erich Joos in insisting that the difference between Linear-SU(2) and Projective-SO(3) shows that SO(3) is the “wrong group” and that SU(2) must instead be regarded as “the proper quantum rotation group” (Joos, 2003, p. 58n13). But our argument does not involve a claim about which of SO(3) and SU(2) is the ‘proper’ quantum rotation group. Indeed, as we saw in section 1, implementing rotational symmetry in terms of the homomorphism $\hat{\gamma} \circ S'$ on SU(2) results in exactly the same unitary projective automorphisms as does implementing rotational symmetry in terms of the homomorphism $T$ on SO(3). Our argument is therefore neutral as regards which group should be used to mathematically systematize rotations of QM worlds.

Third, it may be pointed out that QM is essentially used as a theory of subsystems of the world rather than of the world as a whole. The thought is that our argument falsely assumes the truth of QM of the world as a whole rather than merely its truth (or descriptive adequacy) of subsystems of the world; and restricted to subsystems, our argument would merely establish the entirely uncontroversial reality of the relative phase.

But our argument assumes no such thing. As noted in the introduction, the substantive claims of this paper are to be understood as restricted to the metaphysical possibilities in which QM is true (of the world as a whole). For this objection to be effective against our argument, the objector would thus have to argue for the stronger claim that it is \textit{metaphysically impossible} for QM to be true. This move would not just be \textit{ad hoc}; it would also amount to replacing one radical view about modal reality (that the global phase is real at every possibility where QM is true) with another radical view about modal reality (that QM is necessarily false). But there is a bigger point here that is worth addressing: recall that I use the label ‘QM’ for any quantum theory in which the spin states of the world are characterized in terms of $H_s$. There is evidence that some of the major contenders for a theory of quantum gravity—such as loop quantum gravity—have this feature.\textsuperscript{15} So it is not obvious that the actually true theory of the world is not one in which the spin states of the world are characterized in terms of $H_s$, much less that such a theory is necessarily false.

Since premises A1 and A2 follow immediately from the mathematical apparatus of QM, resistance to our argument is likely to converge on premise A3: the thesis that Univalence is physically substantive. A first strategy might be to claim that

\textsuperscript{14}(Fine, 1994; Rosen, 2010; Fine, 2012; Audi, 2012).

\textsuperscript{15}In loop quantum gravity, the space of spin network states on a graph $\Gamma$ is represented by what Rovelli (2011) refers to as ‘graph space’, the Hilbert space given by $H_{\Gamma} = L^2[G^{\Lambda}/G^N]$, where $G$ is the rotation group that acts irreducibly on $L^2[G^{\Lambda}/G^N]$. There is thus an analogous argument to the effect that the global phase in $L^2[G^{\Lambda}/G^N]$ is a real physical parameter.
the world lacks spin states at every metaphysical possibility where QM is true. If there are no spin states of the world, then *a fortiori* there are no spin states that correspond to superpositions of vectors from integer and half-integer spin subspaces of $H_s$. According to this response, *Univalence* would be vacuously true if true and therefore would not amount to a physically substantive proposition, *contra* A3. However, since it is a straightforward implication of the quantum-mechanical account of the addition of angular momenta that composite systems have spin states whenever any of their constituents do, this objection implies that the world does not contain systems with spin at any metaphysical possibility at which QM is true. This is implausible: the actual world contains systems with spin, and for all we know, QM is actually true. So for all we know, it is possible that QM is true and the world contains systems with spin.

Some readers may complain that I have misrepresented the objection. Of course, the objector is not claiming that no QM world could contain systems with spin; rather, they reject a specific instance of the thesis that composite systems have spin states whenever any of their constituents do: namely, the claim that the world has spin states whenever any of its constituents do. This means that, by ‘QM’, the objector means a different theory (or class of theories) than I do; call this theory QM$. It is the theory that agrees with QM in all claims except that, when QM ascribes spin states to the world, QM$ implies that the world lacks spin states. But the thesis that the world lacks spin states at every possibility where QM$ is true is consistent with our argument, which relies on the distinct assumption that the world has spin states at every possibility at which QM is true.16,17

A more promising way to resist A3 denies either that it is possible for the world to be in half-integer-spin states or that it is possible for the world to be in half-integer-spin states. This strategy is effective in blocking our argument because, even if there are spin states of the world corresponding to superpositions of vectors in integer- and half-integer subspaces of $H_s$, since either no integer-spin states or no half-integer-spin states are possible, neither are the states corresponding to superpositions of the two. If no such states are possible, the fact that PROJECTIVE-SO(3) and LINEAR-SU(2) differ as regards *Univalence* does not make for differences among the possibilities at which QM is true, *contra* A3.

It may be complained that this response is objectionably *ad hoc* in that it seems fine-tuned to block the conclusion of our argument. But it is worth noting that the underlying strategy is familiar from debates in adjacent areas of metaphysics. Substantivalism, the view that there are spacetime points that are the bearers of geometric properties and relations, allows for seemingly pernicious modal variations: for each distribution of geometric properties and relations over spacetime points, there is another possibility that differs merely in a permutation of the spacetime

16At this point, the objector may be tempted to deny that QM is possibly true. But this is just the third ineffective objection we just considered.

17It is worth noting that QM$ implies a somewhat strange lack of uniformity in the representational capacities of the formalism that expresses the theory. QM$ implies that Hilbert space vectors such as (3) can straightforwardly be read as representing (perhaps redundantly) the spin states of an isolated system composed of two particles as long as this system has worldmates; but remove the worldmates, and (3) has no physical content whatsoever. QM, by contrast, is a uniform interpretation of its mathematical formalism.
points that instantiate this distribution. Sophisticated substantivalists block this implication by laying down a modal claim to the effect that the results of the relevant permutations are not metaphysical possibilities, so that for any distribution of spatial and geometric properties over spacetime points, it is metaphysically necessary that these spacetime points instantiate this distribution. To be sure: the observation that the strategy of denying the possibility of either half-integer-spin or integer-spin states is similar to sophisticated substantivalism does not demonstrate that this strategy is not *ad hoc*. The point is rather it is no more (and no less) *ad hoc* than sophisticated substantivalism.

Nonetheless, those attracted to this response may be inclined to seek a more principled foundation for their claim that the relevant spin states are impossible. One natural option is to adopt the stronger claim that, among all the spin states that there are, the possible spin states are all and only those which assign zero total spin to the world. This version of the response can claim some independent plausibility: if attributions of non-zero spin to physical systems essentially capture how they are disposed to interact with other physical systems (such as other spin systems or with electromagnetic fields), non-zero spin values could not possibly be instantiated by the world since (by definition) there are no further physical systems for the world to interact with.

Moreover, this strategy aligns with certain popular doctrines in the metaphysics of spacetime. Rotations of the world as a whole are a special kind of permutation of spacetime points. If worlds which result from such permutations are metaphysically impossible, then the possible spin states of the world are all and only those which are unaffected by rotations. This condition is satisfied only by spin states whose mathematical surrogates lie in the trivial one-dimensional subspace invariant under SU(2): the subspace of Hₙ in which the total spin operator has eigenvalue zero and on which every element of SU(2) acts as the identity. So, none of the spin states which assign non-zero spin to the world are possible states unless rotations of the world generate distinct possibilities. But this latter claim is denied by spacetime relationalists: according to them, there are no spatial points, so there is nothing relative to which the physical constituents of the world (taken together) can be rotated. The claim is also denied by sophisticated substantivalists: those who accept that there are spatial points, but deny that for each distribution of geometric properties and matter over space, there is another possibility that differs merely in a permutation of the spatial points that instantiate this distribution.

More precisely, if we let U be the group homomorphism from SU(2) into the unitary operators on the Hilbert space L²(ℝ³ᴺ) ⊗ Hₙ of the world, then for each g ∈ SU(2) the operator U(g) = U_q(g) ⊗ U_s(g) is a tensor product with two factors: a unitary operator U_q(g) that implements rotations in the spatial degrees of freedom and a unitary operator U_s(g) that implements rotations of the spin degrees of freedom. According to the relationalist and the sophisticated substantivalist we are

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18Cf. (Pooley, 2013).
19Indeed, that the total angular momentum of the world is zero is an implication of the Machian approach to relationalism pioneered by Julian Barbour (Barbour and Bertotti, 1982); cf. (Pooley and Brown, 2002).
20Here it is assumed that the relevant views in the metaphysics of spacetime are necessarily true if true. But the conclusion of our argument can also be blocked by the weaker claim that these views, if true, are true at least at every metaphysical possibility where QM holds.
imagining, the only possible physical states of the world are those represented by fixed points of $U(g)$ for all $g \in SU(2)$. Since a vector $\Psi = \phi_q \otimes \varphi_s$ in $L^2(\mathbb{R}^{3N}) \otimes \mathcal{H}_s$, with $\phi_q \in L^2(\mathbb{R}^{3N})$ and $\varphi_s \in \mathcal{H}_s$, is a fixed point of $U(g)$ for all $g \in SU(2)$ only if $\varphi_s$ is a fixed point of $U_s(g)$ for all $g \in SU(2)$, our relationalist and sophisticated substantivalist objectors claim that the only possible spin states are represented by fixed points of $U_s(g)$ in $\mathcal{H}_s$. By contrast, the (unsophisticated) substantivalist claims that the possible states of the world are represented (perhaps redundantly) by the full Hilbert space $L^2(\mathbb{R}^{3N}) \otimes \mathcal{H}_s$.\(^{21}\)

It is worth noting that one of the standard arguments against (unsophisticated) substantivalism—roughly, that if substantivalism were true, the differences between permutation-related worlds would be undetectable, together with a general norm to prefer theories that do not posit undetectable structure—has a straightforward counterpart that speaks against the reality of the global phase.\(^{22}\) From this point of view, Spin-Rays and some form of relationalist or sophisticated substantivalist metaphysics of spacetime form a natural package.

Neither the stronger nor the weaker versions of this response are without costs. The claim that the world has integer spin at every possibility at which QM is true implies that, at every such possibility, the spins of all constituents of the world add up to an integer value; similarly, the claim that the world has half-integer spin at every possibility at which QM is true implies that, at every such possibility, the spins of all constituents of the world add up to a half-integer value. Both are strong modal theses that severely constrain which matter distributions are metaphysically compossible with QM. For example, the claim that the world has zero spin at every possibility at which QM is true implies that no such possibility consists of an uneven number of half-integer spin particles, since there are no possible arrangements of their spin components that result in a joint state with zero spin.\(^{23}\)

Equivalently: none of their joint states falls into the trivial one-dimensional representation of $SU(2)$. But for all we know, the world might be both quantum-mechanical and contain an odd number of half-integer-spin particles; that this possibility is ruled out by the objector’s view is a serious strike against it.

\(^{21}\)It is worth noting that there is a coherent intermediate position according to which the possible states of the world are those whose mathematical surrogates are fixed points of $U_s(g)$ but not necessarily of $U_s(g)$, for all $g \in SU(2)$. Since this position would not block the conclusion of our argument, I will not discuss it in detail. Here, I merely note that it is not easy to make sense of this view. Our physical grasp of rotations is via $U(g)$, rotations that affect all degrees of freedom at once, as it were, rather than of rotations of subspaces of $L^2(\mathbb{R}^{3N}) \otimes \mathcal{H}_s$: the only rotations of the spin degrees of freedom that can be physically operationalized are those that involve rotations of the total Hilbert space of the system in question—i.e. those implemented by $U(g)$.

\(^{22}\)More specifically, this follows from (1) the fact that systems composed of an even number of half-integer-spin systems have integer spin, together with (2) the fact that systems composed of one integer and one half-integer spin system have half-integer spin.

To show (1), consider two systems with half-integer spin $n/2$ and $n'/2$, respectively. Since $n$ and $n'$ are odd, there are positive natural numbers $m, m'$ such that $|n - n'| = 2m$ and $n + n' = 2m'$. So, the total spin of the composite system has values $m, m + 1, ..., m' - 1, m'$, all of which are integer. So, every system composed of an even number $k$ of half-integer spin systems is composed of $k/2$ integer-spin systems. Since the composite system of any number of integer-spin systems has integer spin, any composite system consisting of an even number of half-integer-spin systems has integer spin. To see (2), note that if $j$ is integer and $j'$ is half-integer, then $|j - j'|$, $|j - j'| + 1, ..., j + j' - 1, j + j'$ are all half-integer.
To be sure: as noted above, some of those who subscribe to the thesis that the world does not possess nonzero spin at any metaphysical possibility at which QM is true are already committed to a fairly strong restriction on modal reality. According to them, it is impossible for the world to be (non-trivially) rotated—either because some version of relationalism is true or because of an independent modal constraint along the lines of sophisticated substantivalism. However, to channel Bas van Fraassen (1980, p. 72), it is not a principle of metaphysics that one might as well hang for a sheep as for a lamb. That one’s preferred metaphysical theory implies one strong modal thesis does not reduce the costs of any of its further modal implications. Nonetheless, faced with the menace of having to accept the reality of the global phase, some may feel that such costs are well worth bearing.

5 Conclusion

That the global phase has the status of a mere gauge parameter is virtually axiomatic in physics and philosophy. The goal of this paper was to argue against this view: if (as is standard), spin states are represented in terms of linear Hilbert spaces, the global phase is not a redundancy in the mathematical description of the world but a real physical parameter. Differently put: if the spin states of the world have projective structure, then it is illegitimate to represent them in linear terms.

We also saw that the most attractive strategy for blocking this conclusion implies that no world at which QM is true contains an odd number of half-integer-spin constituents. This is a significant result in its own right: the conjunction of HILBERT and SPIN-RAYS implies a substantive restriction on which arrangements of matter are metaphysically compossible with QM.

Moreover, our discussion suggests an intimate connection between the metaphysics of spacetime and the reality of the global phase. If the denial of the weaker versions of the response to our argument is held fixed—as would be consistent with the ambition to seek a more principled account of why the differences between PROJECTIVE-SO(3) and LINEAR-SU(2) do not make for differences in the possibilities at which QM is true—our argument implies that proponents of some non-sophisticated form of substantivalism who are committed to the linear formalism of QM are required to accept that the global phase is a real physical parameter in possibilities at which QM is true. Similarly, those who wish to maintain both orthodoxy about the global phase and the linear formalism of QM are forced not just to deny (unsophisticated) substantivalism, but also to accept the thesis that no world at which QM is true contains an odd number of half-integer-spin particles. From this perspective, our argument can be regarded as laying down a novel challenge for deniers of (unsophisticated) substantivalism who subscribe to orthodoxy about the global phase: they must either give up the linear formalism of QM or accept that QM is metaphysically incompossible with the existence of an odd number of half-integer spin particles.

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