

On the indispensability of theoretical terms and entities*

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Abstract

Some realists claim that theoretical entities like numbers and electrons are indispensable for describing the empirical world. Motivated by the meta-ontology of Quine, I take this claim to imply that, for some first-order theory T and formula $\delta(x)$ such that $T \vdash \exists x\delta \wedge \exists x\neg\delta$, where $\delta(x)$ is intended to apply to all and only empirical entities, there is no first-order theory T' such that (a) T and T' describe the δ :s in the same way, (b) $T' \vdash \forall x\delta$, and (c) T' is at least as attractive as T in terms of other theoretical virtues. In an attempt to refute the realist claim, I try to solve the general problem of *nominalizing* T (with respect to δ), namely to find a theory T' satisfying conditions (a)–(c) under various precisifications thereof. In particular, I note that condition (a) can be understood either in terms of syntactic or semantic equivalence, where the latter is strictly stronger than the former. The results are somewhat mixed. On the positive side, even under the stronger precisification of (a), I establish that (1) if the vocabulary of T is finite, a nominalizing theory can always be found that is recursive if T is, and (2) if T postulates infinitely many δ :s, a nominalizing theory can always be found that is no more computationally complex than T . On the negative side, even under the weaker precisification of (a), I establish that (3) certain finite theories cannot be nominalized by a finite theory.

1 Introduction

Scientific theories describe the behavior of ordinary things like tables, chairs, and measuring devices. They typically do so by postulating (the existence of) more exotic things like numbers, sets, and elementary particles. Entities of the first kind are usually called *empirical* or *observable*, and entities of the second kind *theoretical* or *non-observable*. A related distinction can be made between *concrete* and *abstract* entities, with elementary particles belonging to the first kind. It can also be made between *internal* and *external*

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entities, with experiences and sense data belonging to the first, and tables and chairs belonging to the second.

On a general empiricist view, entities of the second kind are postulated in order to explain the behavior of entities of the first kind. More importantly, scientific theories are empirically testable only insofar as they say something about entities of the first kind, i.e. only insofar as they have empirical consequences. One of the many virtues of this view is its ability to provide a rational explanation of the fact that scientists rarely evaluate a theory by merely looking at some formulation of it, but typically spend a lot of time deriving consequences thereof. Why do that? According to empiricism, the point of the exercise is to derive consequences that can be tested directly against empirical observations.

For the empiricist, belief in theoretical entities can only be justified by some kind of inference to the best explanation. This raises the question of whether postulating such entities is necessary for providing a good enough theory about the empirical world. In other words, are theoretical entities indispensable? In the case of numbers and sets, [Putnam \(1979, p. 347\)](#) famously argues that they are.

Importantly, however, the indispensability of theoretical entities should not be confused with the indispensability of theoretical *terms*. Using a theoretical term is arguably neither necessary nor sufficient for postulating a theoretical entity.¹ Obviously, when people say ‘there are no electrons’, they are using a theoretical term, but they are not postulating any theoretical entity. If anything, they are denying that some such entity exists. Not even when people say ‘electrons exist’ are they necessarily postulating any theoretical entity, as their utterance may be followed by ‘but they are all cats’. Presumably, the hypothesis that all electrons are cats does not sit well with received theories about electrons, but it is certainly coherent.² Likewise, when people say ‘numbers exist, but they are just inscriptions on a piece of paper’, although they may be advocating an untenable position, they are not postulating any abstract entities (their intention is not to say that some inscriptions are abstract). Moreover, if we can find a non-trivial formula in purely empirical terms satisfied by all empirical entities, we can postulate the existence of theoretical entities without using any theoretical terms by saying that not everything satisfies that formula. For instance, if we grant that every concrete object is either warmer or colder than some other concrete object, we can postulate the existence of abstract objects without using any abstract terms by saying ‘not everything is warmer or colder than something else’. The number zero, on most accounts, is not.

Now, suppose realists argue that we should believe in the existence of electrons because our best scientific theories say they exist. The relevant question for the anti-realists to

¹As was pointed out to me by an anonymous referee, this was noted already by [Quine \(1951, p. 13\)](#):

Anyway it is not evident why there should be a connection between constant predicates used and entities presupposed. Surely the mere occurrence of a predicate in the formulation of a theory is not sufficient in order that the theory presuppose a corresponding universal entity—a corresponding class or property. Nor is it necessary; for we are familiar with theories which imply that there are indenumerably many classes or properties (even of the first type), though the available predicates are necessarily denumerable.

²According to the Kripke-Putnam view of natural kind terms, the hypothesis is necessarily false. But even they would concede that it is not false *a priori*. I wish to thank an anonymous referee for bringing this point to my attention.

ask is therefore *not* whether it is possible to formulate an equally good theory without using the term ‘electron’. Rather, the relevant question is whether it is possible to find an equally good theory that does not say electrons exist. But an equally good theory about what? Obviously, if electrons exist, it is necessary to postulate electrons in order to describe them. This seems to be the main message in [Putnam \(1965\)](#). To avoid begging the question, we can ask whether it is necessary to postulate electrons in order to describe the behavior of everything that is *not* an electron. Likewise, we can ask whether it is necessary to postulate numbers in order to describe the behavior of everything that is not a number. If the answer is ‘No’, we may at least conclude that numbers are dispensable for describing the behavior of everything else in the universe.

More generally, we shall therefore be concerned with the following question: For a given theory and a given kind of entity definable in that theory, is it possible to find at least as good a theory about entities of that kind without postulating any other entities? Somewhat more precisely, we shall ask, for a given theory T and a given formula $\delta(x)$ such that $T \vdash \exists x\delta \wedge \exists x\neg\delta$, where $\delta(x)$ is intended to apply to all and only empirical entities, whether it is possible find a theory T' such that

- (1) a. T and T' describe the δ :s in the same way,
- b. T' entails that everything is δ ($T' \vdash \forall x\delta$), or at least does not entail that something is not ($T' \not\vdash \exists x\neg\delta$), and
- c. T' is at least as attractive as T in terms of other theoretical virtues (like simplicity and elegance).

Re-purposing a piece of terminology used in connection with the abstract/concrete-distinction, we shall say that T is *nominalizable* (with respect to δ) just in case there is such a theory T' , and that T' *nominalizes* T .

Our first task will be to make this notion more precise. Part (b) is taken care of once we have chosen a logical framework (spoiler: it will be classical first-order logic with identity). Part (a) is then a matter of generalizing the notion of logical equivalence for this framework. This is done in section 2. As a result, we obtain two generalized notions of equivalence, one syntactic and one semantic, with the latter strictly stronger (more demanding) than the former. As you might expect, part (c) is subject to a much wider range of possible precisifications. Minimally, for T' to be at least as attractive as T , we shall require that T' is recursive if T is. But recursive theories vary greatly in terms of their *computational complexity*, measuring (in a precise sense) how hard it is to decide whether something is an axiom of the theory. Thus, in addition, one might require T' to be no more computationally complex than T .³

Our next task will be to investigate nominalizability under these precisifications. Section 3 recapitulates some classical results and observations concerning the indispensability of theoretical terms and entities, and expands a bit on the distinction between the two. A case study of [Goodman and Quine \(1947\)](#) will illustrate the importance of formulating explicit success criteria for nominalization. Section 4 introduces some model theory for tackling the question of nominalizability under semantic equivalence, and proves that virtually all theories are nominalizable in a minimal sense. Section 5 proves that all theories

³Comparing the length of the shortest proofs of the empirical consequences of the two theories yields another interesting measure of attractiveness, but one I will not employ here. See, for instance, [Ketland \(2005\)](#).

postulating infinitely many empirical entities are nominalizable in a much stronger sense, and section 6 compares this result with those of Burgess and Rosen (1999), while section 7 proves that some theories are not so nominalizable. Section 8 summarizes our results and their philosophical implications.

2 Syntactic and semantic equivalence

We will assume that a *theory* is a set of sentences in a language of first-order logic with identity, not necessarily deductively closed. Unless otherwise stated, we will also assume that the *vocabulary* of such a language (its non-logical symbols) is a countable set of predicates not containing any constants or function symbols. The last assumption just makes it easier to state and prove certain results, but is no limitation in principle. The first assumption, on the other hand, is substantial. It is largely motivated by Quine’s so-called *meta-ontology*, which is the idea that the ontological commitments of a theory are best revealed by formalizing the theory in a first-order language with identity and deriving the existentially quantified consequences thereof. For an excellent exposition and defense of this idea, see Inwagen (1998).

Let L be a vocabulary, and let T_1 and T_2 be L -theories. Relative to given deductive system, T_1 and T_2 are said to be *syntactically* equivalent just in case the same L -sentences are deducible from them, and *semantically* equivalent just in case they are satisfied by the same L -models. By soundness and completeness, these notions coincide in the case of classical first-order logic.

Intuitively speaking, syntactically equivalent theories *prove the same things*, and semantically equivalent theories *are true under the same circumstances*. In either of those senses, non-equivalent theories may still be equivalent with respect to a limited range of objects and their properties. Such theories may agree about the distribution of *certain* properties and relations among objects, or about the distribution of properties and relations among *certain* objects, or about the distribution of *certain* properties and relations among *certain* objects.

As we shall see, when equivalence is limited to certain objects and properties, the syntactic and semantic notions come apart. In order to make this claim precise, we need to introduce some standard definitions, namely those of *relativization*, *reduct*, and *part*:

Definition 1 (Relativization). Let $\delta(x)$ be a formula.⁴ For any formula φ , its δ -relativization (written $[\varphi]_\delta$) is defined recursively:

- (i) $[P\bar{x}]_\delta = P\bar{x}$.
- (ii) $[\neg\varphi]_\delta = \neg[\varphi]_\delta$.
- (iii) $[\varphi \rightarrow \psi]_\delta = [\varphi]_\delta \rightarrow [\psi]_\delta$.
- (iv) $[\forall x\varphi]_\delta = \forall x(\delta(x) \rightarrow [\varphi]_\delta)$.

⁴As is customary, when we say ‘let $\delta(x)$ be a formula’, we mean ‘let δ be a formula with at most one free variable x ’. To avoid cluttering, we will thenceforth usually refer to the formula simply as ‘ δ ’.

Remark 1. We may regard the other standard operators as defined in terms of \neg , \rightarrow and \forall . In particular, with $\exists x\varphi$ defined as $\neg\forall x\neg\varphi$, it follows that $[\exists x\varphi]_\delta = [\neg\forall x\neg\varphi]_\delta = \neg[\forall x\neg\varphi]_\delta = \neg\forall x(\delta(x) \rightarrow [\neg\varphi]_\delta) = \neg\forall x(\delta(x) \rightarrow \neg[\varphi]_\delta)$, which is equivalent to $\exists x(\delta(x) \wedge [\varphi]_\delta)$.

Intuitively speaking, δ -relativized sentences only talk about objects satisfying δ . This claim is made precise by Lemma 2 below.

Definition 2 (Reducts and parts). Let $L \subseteq L'$ be vocabularies, let $\delta(x)$ be an L' -formula, and let \mathcal{M} be an L' -model.

- (i) The L -reduct of \mathcal{M} (written $\mathcal{M}|L$) is the L -model with the same domain as \mathcal{M} such that, for any predicate $P \in L$, $P^{\mathcal{M}|L} = P^{\mathcal{M}}$.
- (ii) Provided that $\mathcal{M} \models \exists x\delta$, the δ -part of \mathcal{M} (written \mathcal{M}_δ) is the L' -model whose domain D consist of all objects satisfying δ in \mathcal{M} and such that, for any n -place predicate $P \in L'$, $P^{\mathcal{M}_\delta} = P^{\mathcal{M}} \cap D^n$.⁵

It is easy to establish that the L -reduct of a model satisfies an L -sentence just in case the model satisfies it, and that the δ -part of a model satisfies a sentence just in case the model satisfies its δ -relativization:

Lemma 1. Let $L \subseteq L'$ be vocabularies, let $\delta(x)$ be an L' -formula, let \mathcal{M} be an L' -model such that $\mathcal{M} \models \exists x\delta$, and let φ be an L -sentence. Then we have $\mathcal{M}_\delta|L \models \varphi$ iff $\mathcal{M} \models [\varphi]_\delta$.

Proof. Let X be the set of variables, let D be the set of objects satisfying δ in \mathcal{M} , and let φ be an L -formula. It is straightforward to show, by induction on the complexity of φ , that for any variable assignment $v : X \rightarrow D$, $\mathcal{M}_\delta|L, v \models \varphi$ iff $\mathcal{M}, v \models [\varphi]_\delta$. Hence, for any L -sentence φ , we have $\mathcal{M}_\delta|L \models \varphi$ iff $\mathcal{M} \models [\varphi]_\delta$. \square

It follows that models with identical L -reducts satisfy the same L -sentences, and that models with identical δ -parts satisfy the same δ -relativized sentences:

Lemma 2. Let $L \subseteq L'$ be vocabularies, let $\delta(x)$ be an L' -formula, and let \mathcal{M} and \mathcal{M}' be L' -models such that $\mathcal{M}, \mathcal{M}' \models \exists x\delta$ and $\mathcal{M}_\delta|L = \mathcal{M}'_\delta|L$. Then, for any L -sentence φ , $\mathcal{M} \models [\varphi]_\delta$ iff $\mathcal{M}' \models [\varphi]_\delta$.

Proof. We have $\mathcal{M} \models [\varphi]_\delta$ iff (by Lemma 1) $\mathcal{M}_\delta|L \models \varphi$ iff (by assumption) $\mathcal{M}'_\delta|L \models \varphi$ iff (by Lemma 1) $\mathcal{M}' \models [\varphi]_\delta$. \square

We shall say that two theories are *syntactically L -equivalent over δ* just in case they entail the same δ -relativized L -sentences, and *semantically L -equivalent over δ* just in case the models satisfying them have the same δ -relativized L -reducts. More precisely:

Definition 3 (Syntactic and semantic equivalence). Let T_1 and T_2 be theories in L_1 and L_2 , respectively, and let $L \subseteq L_1 \cap L_2$. Provided that $\delta(x)$ is an L -formula such that $T_1 \vdash \exists x\delta$ and $T_2 \vdash \exists x\delta$, we say that

⁵If L' should contain constants or function symbols, \mathcal{M}_δ is defined provided that $\mathcal{M} \models \delta(c)$ for any constant $c \in L'$, with $c^{\mathcal{M}_\delta} = c^{\mathcal{M}}$, and $\mathcal{M} \models \forall x_1 \dots \forall x_n (\delta(x_1) \wedge \dots \wedge \delta(x_n) \rightarrow \delta(f(x_1, \dots, x_n)))$ for any n -place $f \in L'$, with $f^{\mathcal{M}_\delta} = f^{\mathcal{M}} \cap (D^n \times D)$. If L' but not $L \subseteq L'$ should contain such symbols, and $\mathcal{M} \models \exists x\delta$, we define $\mathcal{M}_\delta|L$ as the L -model whose domain D consists of all objects satisfying δ in \mathcal{M} , with $P^{\mathcal{M}_\delta|L} = P^{\mathcal{M}} \cap D^n$ for any n -place predicate $P \in L$.

1. T_1 and T_2 are *syntactically L -equivalent over δ* just in case, for any L -sentence φ , $T_1 \vdash \varphi_\delta$ iff $T_2 \vdash \varphi_\delta$.
2. T_1 and T_2 are *semantically L -equivalent over δ* just in case $\{\mathcal{M}_\delta|L : \mathcal{M} \models T_1\} = \{\mathcal{M}_\delta|L : \mathcal{M} \models T_2\}$.

Remark 2. The intended application of this definition is of course when L is the set of all empirical terms, and δ is satisfied by all and only empirical entities. Then we shall say that T_1 and T_2 are syntactically/semantically *empirically equivalent* just in case T_1 and T_2 are syntactically/semantically L -equivalent over δ .

Syntactic or semantic equivalence *simpliciter* can now be seen as a limiting case of syntactic or semantic L -equivalence over δ , namely when $L_1 \cup L_2 \subseteq L$ and δ is trivial (e.g. $x = x$). As mentioned earlier, the syntactic and semantic notions coincide in the limit. In general, they do not. The semantic notion is strictly stronger:

Lemma 3. *For any vocabularies $L \subseteq L'$, L' -theories T_1 and T_2 and L' -formula $\delta(x)$ such that $T_1 \vdash \exists x\delta$ and $T_2 \vdash \exists x\delta$: if T_1 and T_2 are semantically L -equivalent over δ , then they are syntactically L -equivalent over δ .*

Proof. Assume that $T_1 \vdash \exists x\delta$, $T_2 \vdash \exists x\delta$, and that $\{\mathcal{M}_\delta|L : \mathcal{M} \models T_1\} = \{\mathcal{M}_\delta|L : \mathcal{M} \models T_2\}$. Let φ be an L -sentence, and suppose that $T_1 \vdash [\varphi]_\delta$. Let \mathcal{M} be a model of T_2 . By assumption, $\mathcal{M}_\delta|L$ can be extended to a model \mathcal{M}' of T_1 such that $\mathcal{M}' \models [\varphi]_\delta$. Since $\mathcal{M} \models \exists x\delta$, it follows by Lemma 2 that $\mathcal{M} \models [\varphi]_\delta$. Hence, by completeness, $T_2 \vdash [\varphi]_\delta$. The other direction is symmetrical. \square

Here is a counterexample, with trivial δ but $L \subset L_1 \cup L_2$, showing that the relation of strength is strict:

Example 1. Let P and Q be unary predicates and R binary, let $L_1 = \{P, Q, R\}$ and $L = L_2 = \{P, Q\}$, let T_1 be an L_1 -theory saying that R is a bijective relation between the P :s and the Q :s, namely

$$T_1 = \{\forall x(Px \rightarrow \exists!y(Qy \wedge Rxy)), \forall y(Qy \rightarrow \exists!x(Px \wedge Rxy))\}$$

and let T_2 be the set of all L -consequences of T_1 . By construction, T_1 and T_2 are syntactically L -equivalent. Let \mathcal{M} be an L -model where $P^\mathcal{M}$ is countably infinite and $Q^\mathcal{M}$ is uncountable. Let $Th(\mathcal{M})$ be the set of true L -sentences in \mathcal{M} . By downward Löwenheim-Skolem, $Th(\mathcal{M})$ has a countable model \mathcal{M}' , one in which both $P^{\mathcal{M}'}$ and $Q^{\mathcal{M}'}$ are countably infinite. Since \mathcal{M}' can be expanded to a model of T_1 , \mathcal{M}' is a model of T_2 . And since $T_2 \subseteq Th(\mathcal{M})$, so is \mathcal{M} . But \mathcal{M} cannot be expanded to a model of T_1 . Hence, T_1 and T_2 are syntactically but not semantically L -equivalent.

Here is another counterexample, with $L = L_1 = L_2$ but non-trivial δ :

Example 2. Let $L = \{O, P, Q, R, S\}$, where O, P, Q are unary and R, S are binary. Let T_1 be an L -theory saying that all P :s and Q :s are O :s, that R is a bijective relation between the P :s and the non- O :s, and that S is a bijective relation between the Q :s and the non- O :s. Define $T_2 = \{[\varphi]_{Ox} : T_1 \vdash [\varphi]_{Ox}\}$. By construction, T_1 and T_2 are syntactically L -equivalent over Ox . Let \mathcal{M} be an L -model where the P :s are uncountable and the Q :s

are countably infinite, R and S are empty, and where everything is O . By downward Löwenheim-Skolem, $Th(\mathcal{M})$ has a countable model \mathcal{M}' , one in which both the P :s and the Q :s are countably infinite, and where R and S are empty. Let \mathcal{M}^* be the reduct of \mathcal{M}' to $\{O, P, Q\}$. Then \mathcal{M}^* can be extended to a model \mathcal{M}^+ of T_1 . Clearly, $\mathcal{M}_{Ox}^+ = \mathcal{M}'$. Since \mathcal{M}^+ is a model of T_2 , it follows by Lemma 2 that \mathcal{M}' is a model of T_2 . But for any L -sentence φ , we have $\mathcal{M} \models \varphi$ iff $\varphi \in Th(\mathcal{M})$ iff $\mathcal{M}' \models \varphi$. Hence, \mathcal{M} is a model of T_2 , but cannot be extended to a model of T_1 . Thus, T_1 and T_2 are syntactically but not semantically equivalent over Ox .

3 Conceptual and ontological parsimony

As we argued earlier, using theoretical terms is neither necessary nor sufficient for postulating theoretical entities. It follows that dispensing with theoretical terms is neither necessary nor sufficient for dispensing with theoretical entities. One might say that the first is a matter of *conceptual* parsimony, whereas the latter is a matter of *ontological* parsimony. Unfortunately, the distinction between terms and entities is not always respected. A blatant example is provided by Colyvan (2019):

The first thing to note is that ‘dispensability’ is not the same as ‘eliminability’. If this were not so, every entity would be dispensable (due to a theorem of Craig). What we require for an entity to be ‘dispensable’ is for it to be eliminable and that the theory resulting from the entity’s elimination be an attractive theory.

Three points are in order. First of all, only *terms* are eliminable in the sense relevant to the application of Craig’s theorem in the context of this quote, as it refers to the following fact:

Theorem 1. *For any vocabularies $L \subseteq L'$, every recursive L' -theory is syntactically L -equivalent to a recursive L -theory.*

The theorem follows from the fact that the L -consequences of a recursive L' -theory are recursively enumerable, together with the aforementioned theorem of Craig (1953):

Theorem 2 (Craig). *Every recursively enumerable theory is equivalent to a recursive theory.*

Craig’s theorem can be established by the following handy argument:⁶

Proof. Let $\varphi_0, \varphi_1, \varphi_2, \dots$ be the recursive enumeration of a theory T , and define the sequence $\psi_0, \psi_1, \psi_2, \dots$ by recursion:

$$\begin{aligned}\psi_0 &= \varphi_0 \\ \psi_{n+1} &= \psi_n \wedge \varphi_{n+1}\end{aligned}$$

Let $T' = \{\psi_0, \psi_1, \psi_2, \dots\} = \{\varphi_0, \varphi_0 \wedge \varphi_1, \varphi_0 \wedge \varphi_1 \wedge \varphi_2, \dots\}$. Clearly, T' is recursively enumerable and equivalent to T . Moreover, it is easily shown by induction that the

⁶A similar argument is found in Putnam (1965, pp. 253-254), but with $T' = \{\varphi_0, \varphi_1 \wedge \varphi_1, \varphi_2 \wedge \varphi_2 \wedge \varphi_2, \dots\}$.

sentences in the sequence $\psi_0, \psi_1, \psi_2, \dots$ are strictly increasing in length, and that any number n is smaller than the length of ψ_n . Thus, in order to decide whether a given sentence σ of length n is a member of T' , it is enough to go through the enumeration of T' from ψ_0 to ψ_n . Hence, T' is recursive. \square

This brings me to my second point, which is to acknowledge that Colyvan is correct in denying that recursive axiomatizability is sufficient for attractiveness, and in distinguishing between dispensability and mere eliminability. Although the strategy for eliminating theoretical terms suggested by Craig's theorem preserves recursive axiomatizability, it certainly does not preserve every potentially attractive feature of the original theory. As Field (1980, p. 8) puts it, the resulting theories "are obviously uninteresting, since they do nothing whatever toward explaining the phenomena in question in terms of a small number of basic principles." In particular, the strategy does not preserve finite axiomatizability. There are many cases of finite L' -theories having no syntactically L -equivalent finite L -theory. To take a simple example: for any finite consistent theory with only infinite models, there is no finite theory in the empty vocabulary entailing the same sentences in that vocabulary.

But, as for my third and final point, it is not obvious that theoretical terms *are* eliminable. That depends on whether the relevant notion of equivalence is syntactic or semantic. More precisely, Theorem 1 does not hold for semantic equivalence. In Example 1, it easily follows that *no* L -theory is semantically L -equivalent to T_1 , since such a theory would have to be equivalent to T_2 *simpliciter*. Although T_1 and T_2 have the same L -consequences, they are not true under the same L -circumstances. T_2 , but not T_1 , has a model where there are more Q :s than P :s. Other examples to the same effect have been offered by van Benthem (1978, p. 324), Melia (2000, pp. 459-461), Ketland (2004, p. 297), and Johannesson (2020, pp. 492-493). From the point of view of semantic equivalence, this means that theoretical terms are sometimes indispensable. In order to describe certain properties and relations among objects, it is sometimes necessary to talk about other properties and relations as well.

At any rate, these results only pertain to the indispensability of theoretical *terms*. What about the indispensability of theoretical *entities*? In order to describe the properties and relations among certain objects, is it sometimes necessary to talk about other objects as well? From the point of view of syntactic equivalence, and in spite of my first point concerning the quote earlier, Craig's theorem does make theoretical entities dispensable in the following sense:

Theorem 3. *For any vocabulary L , (recursive) L -theory T and L -formula $\delta(x)$ such that $T \vdash \exists x\delta$, there is a (recursive) L -theory T' syntactically L -equivalent to T over δ such that $T' \vdash \forall x\delta$.*

Proof. Let $S = \{[\varphi]_\delta : T \vdash [\varphi]_\delta\} \cup \{\forall x\delta\}$. We need to show that S and T are syntactically L -equivalent over δ . By construction, we have $S \vdash [\varphi]_\delta$ if $T \vdash [\varphi]_\delta$. For the other direction, assume that $T \not\vdash [\varphi]_\delta$. Then there is a model $\mathcal{M} \models T, \neg[\varphi]_\delta$, and thus $\mathcal{M} \models T, [\neg\varphi]_\delta$. Since $\mathcal{M} \models \exists x\delta$, it follows by Lemma 2 that $\mathcal{M}_\delta \models S, [\neg\varphi]_\delta$, and thus $\mathcal{M}_\delta \models S, \neg[\varphi]_\delta$. Hence, $S \not\vdash [\varphi]_\delta$.

Finally, since S is recursively enumerable if T is recursive, S is equivalent to a recursive L -theory T' according to Theorem 2. \square

But the corresponding result does not hold for semantic equivalence. In Example 2, the set of Ox -relativized consequences of T_1 is not semantically equivalent to T_1 over Ox . If we are allowed to use additional predicates, however, it is easy to construct a theory T semantically L -equivalent to T_1 over Ox such that $T \vdash \forall xOx$. What T_1 says about the O :s is that there are as many P :s as there are Q :s. To achieve the same thing without postulating any non- O :s, we only need to introduce a new binary predicate U , with T saying that U is a bijective relation between the P :s and the Q :s, that R and S are empty, and that everything is O . One might say that the gain in ontological parsimony comes at the conceptual cost of having to introduce a new predicate U defined on the O :s.

In the next section, we shall indeed prove that this holds more generally. But before we do that, it is instructive to compare our current predicament with that of Goodman and Quine (1947). One of their goals is to be able to say that

- (2) There are more cats than dogs

without postulating any abstract set-theoretic objects. The standard set-theoretic rendering of (2) is

- (3) There is an injective but no bijective function from the set of all dogs to the set of all cats.

Their strategy is to dispense with set-theory in favor of mereology. Unlike the set-theoretical notion of *membership* (expressed by the binary predicate \in), the mereological notion of *parthood* is to be defined exclusively on a domain of concrete objects. This domain will have to include more than just cats and dogs, however. In addition, it will at least include the parts of all cats and dogs and the mereological sums thereof. But unlike the set of all cats and the set of all dogs, these are all supposed to be concrete objects. On this domain of concrete objects, they also use the binary predicate ‘is bigger than’, and define the unary predicate ‘is a bit’ in terms of it by declaring that something is a bit just in case it is as big as the smallest individual among all cats and dogs. Their mereological rendering of (2) then becomes

- (4) Every individual that contains a bit of each cat is bigger than some individual that contains a bit of each dog.

Do they succeed? Unfortunately, they are not very explicit about their success criteria. In one sense, (2) cannot be expressed in first-order logic at all, not even using set-theory. More precisely, with $L = \{\text{cat}, \text{dog}\}$,

- (5) There is no vocabulary $L' \supseteq L$ and L' -theory T such that, for any L -model \mathcal{M} , $|\text{cat}^{\mathcal{M}}| > |\text{dog}^{\mathcal{M}}|$ iff \mathcal{M} can be expanded to a model of T .

Assuming T to be such a theory, it is enough to consider a model of T with uncountably many cats and countably infinitely many dogs. By Löwenheim-Skolem, the theory of that model has a countable model, which is also a model of T , but one where the number of cats and dogs are both countably infinite.

Clearly, their goal is not to be able to express (2) in this sense. But in what sense,

then? Even when restricted to countable models, (5) still holds.⁷ In order to formulate a satisfiable success criterion along these lines, we need to restrict it to finite models, or at least to models with finitely many cats and dogs. If we do that, it is indeed possible to find a theory satisfying it, even without extending the vocabulary:

- (6) There is an L -theory T such that, for any L -model \mathcal{M} with finite $\text{cat}^{\mathcal{M}}$ and $\text{dog}^{\mathcal{M}}$, $|\text{cat}^{\mathcal{M}}| > |\text{dog}^{\mathcal{M}}|$ iff $\mathcal{M} \models T$.

The following theory does the job:

$$T = \{\neg(\exists_{\leq n} x \text{cat}(x) \wedge \exists_{\geq n} x \text{dog}(x)) : n \in \mathbb{N}\}$$

However, although such a theory requires no mereological notions, it needs to be infinite.⁸ What they are seeking is a *statement*. Presumably, statements are finite. Thus, in order to obtain a finite theory expressing (2), we need to extend the vocabulary:

- (7) There is a vocabulary $L' \supseteq L$ and a finite L' -theory T such that, for any L -model \mathcal{M} with finite $\text{cat}^{\mathcal{M}}$ and $\text{dog}^{\mathcal{M}}$, $|\text{cat}^{\mathcal{M}}| > |\text{dog}^{\mathcal{M}}|$ iff \mathcal{M} can be expanded to a model of T .

For instance, with P a unary and R a binary predicate, and $L' = L \cup \{P, R\}$, we can formulate a finite L' -theory T saying that every P is a cat, not all cats are P , and that R is a bijective relation between the P :s and the dogs. Then T defines P and R exclusively on concrete objects. Moreover, every L -model with finitely many cats and dogs has more cats than dogs just in case it can be expanded to a model of T .

Obviously, introducing two new predicates for each case like (2) is not a sustainable solution. Perhaps the mereological approach is better in that regard. Whether it also provides a witness to (7) depends on how much mereological background theory it requires (i.e. whether that theory is finite), but I shall not delve further into it. Either way, the more interesting question is whether the mereological theory as a whole nominalizes its set-theoretic counterpart in the sense of (1). In other words, to the extent that concrete objects can be described in a certain way by a first-order theory also postulating abstract objects, can they be so described by a first-order theory postulating no abstract objects? This is the question to which we now turn.

4 Nominalizability under semantic equivalence

Suppose we have an L -theory T and an L -formula $\delta(x)$ such that $T \vdash \exists x \delta \wedge \exists x \neg \delta$. If L is finite, and T has a model with finitely many δ :s, it is a trivial matter to find an

⁷In that case, since T has models where the number of dogs are of any finite cardinality, it follows by compactness that it has a model with infinitely many dogs. By downward Löwenheim-Skolem, it has a countable such model.

⁸Otherwise, we can let φ be the conjunction of its axioms, and let φ' be the result of replacing every occurrence of dog in φ with $\neg \text{cat}$. It follows that, for any finite L -model \mathcal{M} , $\mathcal{M} \models \varphi'$ iff there are more cats than non-cats in \mathcal{M} . Let $q \in \mathbb{N}$ be the quantifier rank of φ' , and let ψ be the sentence $\exists_{\geq q} x \text{cat}(x) \wedge \exists_{\geq q} x \neg \text{cat}(x)$. Clearly, the quantifier rank of $\varphi' \wedge \psi$ is also q . Let \mathcal{M} be a finite model with $q+1$ cats and q non-cats. By assumption, $\mathcal{M} \models \varphi' \wedge \psi$. Hence, $\varphi' \wedge \psi$ is satisfiable. By the small model property of monadic logic with identity, it follows that $\varphi' \wedge \psi$ has a model with at most $2q$ elements, contradicting our assumption.

L -theory T' semantically equivalent to T over δ such that $T' \not\vdash \exists x\neg\delta$. This is due to the well known fact that every finite L -model can be described up to isomorphism by a single L -sentence. Thus, let σ be an L -sentence describing the finite δ -part \mathcal{M}_δ of some L -model $\mathcal{M} \models T$, and let $T' = \{\varphi \vee \sigma : \varphi \in T\}$. Since $\sigma \vdash \forall x\delta$, any model of σ is a model of T' , it follows by soundness that $T' \not\vdash \exists x\neg\delta$. As for semantic equivalence, every model satisfying T will obviously satisfy T' . For the other direction, let \mathcal{M}' be an arbitrary L -model, and assume that $\mathcal{M}' \models T'$. If $\mathcal{M}' \models \sigma$, then $\mathcal{M}' = \mathcal{M}'_\delta$ and $\mathcal{M}'_\delta \cong \mathcal{M}_\delta$, in which case \mathcal{M}'_δ can be extended to a model of T . If, on the other hand, $\mathcal{M}' \not\models \sigma$, then we have $\mathcal{M}' \models T$.

Thus construed, however, T' still carries a conditional commitment to the existence of non- δ :s in the sense that, for some L -sentence φ (namely $\neg\sigma$) such that $T' \not\vdash \neg[\varphi]_\delta$, we have $T' \vdash [\varphi]_\delta \rightarrow \exists x\neg\delta$. Presumably, a nominalist will not happily commit to the claim that, if there are such-and-such concrete objects (which, according to his theory, there very well may be), then abstract objects exist. In this section, we shall therefore be occupied with the general problem of finding a nominalizing theory T' such that $T' \vdash \forall x\delta$. To do that, we shall employ certain model-theoretic results formulated in terms of the following notions:

Definition 4 (Elementary, pseudo-elementary, and relativized pseudo-elementary classes). Let L be a vocabulary, and let \mathcal{C} be a class of L -models. We say that

- (i) \mathcal{C} is (*finitely/recursively*) *elementary* just in case, for some (finite/recursive) L -theory T , we have $\mathcal{C} = \{\mathcal{M}|L : \mathcal{M} \models T\}$.
- (ii) \mathcal{C} is (*finitely/recursively*) *pseudo-elementary* just in case, for some vocabulary $L' \supseteq L$ and (finite/recursive) L' -theory T , we have $\mathcal{C} = \{\mathcal{M}|L : \mathcal{M} \models T\}$.
- (iii) \mathcal{C} is (*finitely/recursively*) *relativized pseudo-elementary* just in case, for some vocabulary $L' \supseteq L$, (finite/recursive) L' -theory T and unary predicate $P \in L' - L$, we have $\mathcal{C} = \{\mathcal{M}_{Px}|L : \mathcal{M} \models T, \exists xPx\}$.

Remark 3. In the terminology of [Tarski \(1954\)](#), a class of models is finitely pseudo-elementary iff it is \mathbf{PC} , pseudo-elementary iff it is \mathbf{PC}_Δ , finitely relativized pseudo-elementary iff it is \mathbf{PC}' , and relativized pseudo-elementary iff it is \mathbf{PC}'_Δ . In the extended terminology of [Makkai \(1964\)](#), a class of models is recursively pseudo-elementary iff it is $\mathbf{PC}_{\Delta\text{rec}}$. I prefer to use the more descriptive terminology.

The following lemma offers an alternative definition of relativized pseudo-elementary classes, more suitable for our purposes:

Lemma 4. *A class \mathcal{C} of L -models is (*finitely/recursively*) relativized pseudo-elementary just in case, for some vocabulary $L' \supseteq L$, (*finite/recursive*) L' -theory T and L' -formula $\delta(x)$, we have $\mathcal{C} = \{\mathcal{M}_\delta|L : \mathcal{M} \models T, \exists x\delta\}$.*

Proof. Left to right follows from taking $\delta(x)$ to be Px . Right to left follows from extending L' with a new unary predicate P , and extending T with $\forall x(Px \leftrightarrow \delta)$, in which case $\{\mathcal{M}_{Px}|L : \mathcal{M} \models T, \forall x(Px \leftrightarrow \delta), \exists xPx\} = \{\mathcal{M}_\delta|L : \mathcal{M} \models T, \exists x\delta\}$. \square

Example 1 demonstrates that not all pseudo-elementary classes are elementary. But certain non-elementary classes are not pseudo-elementary either. For instance, the class of all finite models of a vocabulary is not pseudo-elementary. Neither, as we saw in the previous section, is the class of all models with more cats than dogs. This is precisely what (5) says. In both cases, a slightly modified argument will demonstrate that they are not relativized pseudo-elementary either. The following remarkable theorem of Makkai (1964, p. 176, Theorem 2(a)) says that this holds in general:

Theorem 4 (Makkai). *Every relativized pseudo-elementary class is pseudo-elementary.*

Albeit in a less than minimal sense, Theorem 4 immediately entails that all theorems are nominalizable from the point of view of semantic equivalence:

Corollary 1. *For any vocabulary L , L -theory T and L -formula $\delta(x)$, there is a theory T' semantically L -equivalent to T over δ such that $T' \vdash \forall x\delta$.*

Proof. Let L be a vocabulary, T an L -theory and $\delta(x)$ an L -formula. By Lemma 4 and Theorem 4, there is $L' \supseteq L$ and L' -theory T' such that $\{\mathcal{M}|L : \mathcal{M} \models T'\} = \{\mathcal{M}_\delta|L : \mathcal{M} \models T, \exists x\delta\}$. Since δ is an L -formula, we get $T' \vdash \forall x\delta$, from which it also follows that $\{\mathcal{M}_\delta|L : \mathcal{M} \models T', \exists x\delta\} = \{\mathcal{M}_\delta|L : \mathcal{M} \models T, \exists x\delta\}$. \square

But in order to establish general nominalizability in any stronger sense, we also need to prove that the nominalizing theory can be recursive if the original is. We shall at least be able to prove that this holds when the original theory is formulated in a finite vocabulary. Arguably, this is no serious limitation. To do that, we use another theorem of Makkai (1964, p. 176, Theorem 2(b)):

Theorem 5 (Makkai). *Every finitely relativized pseudo-elementary class is recursively pseudo-elementary.*

The reason for the restriction to finite vocabularies is so we can use a result due to Craig and Vaught (1958, p. 292, Theorem 2.1):

Theorem 6 (Craig and Vaught). *Provided that L is finite, every recursively elementary class of infinite L -models is finitely pseudo-elementary.*

Remark 4. In their terminology, this means that every recursive theory⁹ with only infinite models is *finitely axiomatizable (in the semantic sense) using additional predicates*. The result is a generalization of Kleene (1952). Roughly speaking, given a finite vocabulary L and a recursive L -theory T , the proof strategy is to formalize the arithmetized syntax and semantics of L -formulas as a finite extension of Robinson arithmetic (sufficient for representing all recursive relations on the natural numbers) added with a statement to the effect that all axioms of T are true.

Under the assumption that all vocabularies are finite, we use this theorem to show that every recursively pseudo-elementary class is finitely relativized pseudo-elementary. Since, as we shall see in section 7, not every recursively pseudo-elementary class is finitely pseudo-elementary, this result is interesting in its own right. Extending the terminology of Craig and Vaught (1958), it means that every recursive theory is finitely axiomatizable (in the semantic sense) using additional predicates and entities:

⁹Craig and Vaught (1958) assume, by definition, that all theories have finite vocabularies.

Lemma 5. *For any finite vocabulary L and recursive L -theory T , there is finite theory T' in a vocabulary $L' \supseteq L \cup \{P\}$ with a unary predicate $P \notin L$ such that $\{\mathcal{M}_{P_x}|L : \mathcal{M} \models T', \exists xPx\} = \{\mathcal{M}|L : \mathcal{M} \models T\}$.*

Proof. Let L be a finite vocabulary and T a recursive L -theory. Let $P \notin L$ be a new unary predicate, let $L^+ = L \cup \{P\}$, and let

$$T^+ = \{[\varphi]_{P_x} : \varphi \in T\} \cup \{\exists_{\geq 1}x\neg Px, \exists_{\geq 2}x\neg Px, \exists_{\geq 3}x\neg Px, \dots\}$$

We show that $\{\mathcal{M}_{P_x}|L : \mathcal{M} \models T^+, \exists xPx\} = \{\mathcal{M}|L : \mathcal{M} \models T\}$. For left to right, assume that $\mathcal{M} \models T^+, \exists xPx$. By Lemma 1, $\mathcal{M}_{P_x}|L \models T$, and obviously $(\mathcal{M}_{P_x}|L)|L = \mathcal{M}_{P_x}|L$. For the other direction, assume that $\mathcal{M} \models T$. Extend \mathcal{M} to a model \mathcal{M}' by adding infinitely many new elements and interpreting P as the elements of \mathcal{M} . Since $\mathcal{M}'_{P_x} = \mathcal{M}$, it follows by Lemma 1 that $\mathcal{M}' \models T^+, \exists xPx$, and obviously $\mathcal{M}'_{P_x}|L = \mathcal{M}|L$.

Since T^+ only has infinite models, it follows by Theorem 6 that there is finite theory T' in a vocabulary $L' \supseteq L^+$ such that $\{\mathcal{M}|L^+ : \mathcal{M} \models T'\} = \{\mathcal{M}|L^+ : \mathcal{M} \models T^+\}$. Hence, $\{\mathcal{M}_{P_x}|L : \mathcal{M} \models T', \exists xPx\} = \{\mathcal{M}_{P_x}|L : \mathcal{M} \models T^+, \exists xPx\} = \{\mathcal{M}|L : \mathcal{M} \models T\}$. \square

Still under the assumption that all vocabularies are finite, we then show that every recursively relativized pseudo-elementary class is finitely relativized pseudo-elementary:

Lemma 6. *For any finite vocabulary L , recursive L -theory T and L -formula $\delta(x)$, there is a finite vocabulary $L' \supseteq L$, finite L' -theory T' and L' -formula $\delta'(x)$ such that $\{\mathcal{M}_{\delta'}|L : \mathcal{M} \models T', \exists x\delta'\} = \{\mathcal{M}_\delta|L : \mathcal{M} \models T, \exists x\delta\}$.*

Proof. Let L be a finite vocabulary, T a recursive L -theory, and $\delta(x)$ an L -formula. By Lemma 5, there is a vocabulary $L' \supseteq L$ with a unary predicate $P \in L' - L$ and a finite L' -theory T' such that

$$(8) \quad \{\mathcal{M}_{P_x}|L : \mathcal{M} \models T', \exists xPx\} = \{\mathcal{M}|L : \mathcal{M} \models T\}.$$

We need to show that $\{\mathcal{M}_{P_x \wedge \delta}|L : \mathcal{M} \models T', \exists x(Px \wedge \delta)\} = \{\mathcal{M}_\delta|L : \mathcal{M} \models T, \exists x\delta\}$. For left to right, assume that $\mathcal{M} \models T', \exists x(Px \wedge \delta)$. Let $\mathcal{M}' = \mathcal{M}_{P_x}|L$. It follows by (8) that $\mathcal{M}' \models T, \exists x\delta$. And since $\mathcal{M}'_\delta = \mathcal{M}_{P_x \wedge \delta}|L$, we also have $\mathcal{M}'_\delta|L = \mathcal{M}_{P_x \wedge \delta}|L$. For the other direction, assume that $\mathcal{M} \models T, \exists x\delta$. It follows by (8) that there is $\mathcal{M}' \models T', \exists xPx$ such that $\mathcal{M}'_{P_x}|L = \mathcal{M}|L$. Since $\mathcal{M}|L \models T, \exists x\delta$, we get $\mathcal{M}'_{P_x}|L \models \exists x\delta$, which means that $\mathcal{M}' \models T', \exists x(Px \wedge \delta)$. Moreover, since $\mathcal{M}'_{P_x}|L = \mathcal{M}|L$, we get $\mathcal{M}'_{P_x \wedge \delta}|L = \mathcal{M}_\delta|L$. \square

Together with Theorem 5, the desired result follows immediately:

Theorem 7. *For any finite vocabulary L , recursive L -theory T and L -formula $\delta(x)$ such that $T \vdash \exists x\delta$, there is a recursive theory T' semantically L -equivalent to T over δ such that $T' \vdash \forall x\delta$.*

Proof. Let L be a finite vocabulary, T a recursive L -theory, and $\delta(x)$ an L -formula. By Lemma 6, there is a finite vocabulary $L^+ \supseteq L$, finite L^+ -theory T^+ and L^+ -formula $\delta^+(x)$ such that $\{\mathcal{M}_{\delta^+}|L : \mathcal{M} \models T^+, \exists x\delta^+\} = \{\mathcal{M}_\delta|L : \mathcal{M} \models T, \exists x\delta\}$. By Theorem 5, there is a recursive theory T' in a vocabulary $L' \supseteq L^+$ such that $\{\mathcal{M}|L^+ : \mathcal{M} \models T'\} = \{\mathcal{M}_{\delta^+}|L^+ : \mathcal{M} \models T^+, \exists x\delta^+\}$, and thus $\{\mathcal{M}|L : \mathcal{M} \models T'\} = \{\mathcal{M}_{\delta^+}|L : \mathcal{M} \models T^+, \exists x\delta^+\} = \{\mathcal{M}_\delta|L : \mathcal{M} \models T, \exists x\delta\}$. Since δ is an L -formula, it follows that $T' \vdash \forall x\delta$. \square

At this point, there is good news and bad news for the anti-realist. The good news is that, by inspecting the proofs of Theorem 5 and 6, one can obtain an algorithm for nominalizing any recursive theory formulated in a finite vocabulary.¹⁰ The bad news is that, due to its generality, that algorithm does not yield very attractive theories. In particular, by inspecting the proof of Theorem 5, one can see that, unless the original theory is equivalent to $\exists\bar{x}\forall\bar{y}\varphi(\bar{x},\bar{y})$ for some quantifier-free formula φ not containing any function symbols, the vocabulary of the nominalizing theory will be infinite. And, as a rule of thumb, conceptual parsimony counts towards elegance. At least to me, it is still an open question whether every recursive theory in a finite vocabulary can be nominalized by such a theory.

5 Infinite universes

In this section, we show that theories postulating infinitely many empirical entities are nominalizable in a much stronger sense.

Hodges (1993, p. 208) has a proof of Theorem 4 restricted to classes of infinite models. By filling in the details of that proof, we obtain the following result:

Theorem 8 (Hodges). *Let L be a vocabulary, T an L -theory, and $\delta(x)$ an L -formula, and assume that $\{\mathcal{M}_\delta|L : \mathcal{M} \models T, \exists x\delta\}$ only contains infinite models. For each predicate $P \in L$, let $P^* \notin L$ be a new predicate of the same arity, and define $L' = L \cup \{P^* : P \in L\} \cup \{f\}$, where $f \notin L$ is a new unary function symbol. For each L -formula φ , let φ^* be the result of replacing every predicate P in φ with P^* , and define*

$$\begin{aligned} T' = & \{\varphi^* : \varphi \in T\} \cup \\ & \{\forall x_1 \dots \forall x_n (P x_1 \dots x_n \leftrightarrow P^* f(x_1) \dots f(x_n)) : P \in L\} \cup \\ & \{\forall x \delta^*(f(x)), \forall x (\delta^*(x) \rightarrow \exists y f(y) = x), \forall x \forall y (x \neq y \rightarrow f(x) \neq f(y))\} \end{aligned}$$

Then T and T' are semantically L -equivalent over δ , and $T' \vdash \forall x \delta$.

For theories postulating infinitely many empirical entities, Theorem 8 offers a nominalist paraphrase that is both simple and intuitive. Roughly, the nominalizing theory T' states that the universe contains a map of itself satisfying the translation of T into “map-language”. Let me explain what I mean by that. Observe that a city on a map is not strictly speaking a city, and a road on a map is not strictly speaking a road. Rather, they are a city-on-the-map (city*) and a road-on-the-map (road*), respectively. Likewise, a road on the map does not connect two cities on the map in the sense that a road in the terrain connects two cities in the terrain. Rather, it connects-on-the-map (connects*). The predicates ‘road*’, ‘city*’ and ‘connects*’ belong to the so-called map-language. According to T' , each city x and each road y in the terrain corresponds to a unique city* $f(x)$ and a unique road* $f(y)$ on the map. Moreover, a road x connects cities y and z in the terrain just in case the road* $f(x)$ connects* the city* $f(y)$ with the city* $f(z)$ on the map. The map itself is described by the theory $T^* = \{\varphi^* : \varphi \in T\}$, which is

¹⁰More precisely, one can obtain an algorithm whose input is a description of an algorithm for deciding the axioms of the original theory, and whose output is a description of an algorithm for deciding the axioms of the nominalizing theory.

the translation of T into map-language. To the extent that T postulates non-empirical entities, T^* only postulates entities outside of the map.

Clearly, T' is finite/recursive if T is. More generally, they belong to the same computational complexity class. To see why, observe that a finite set can always be decided in constant time. That means there is an algorithm and a constant $c \in \mathbb{N}$ such that the time it takes for the algorithm to decide whether a string belongs to the set is less than c . Moreover, it is reasonable clear that the second set of axioms,

$$\{\forall x_1 \dots \forall x_n (Px_1 \dots x_n \leftrightarrow P^* f(x_1) \dots f(x_n)) : P \in L\}$$

can be decided in linear time. That means there is an algorithm and a linear function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that, for any string of length n , the time it takes for the algorithm to decide whether the string belongs to the set is less than $f(n)$. Likewise, the translation from T^* to T can also be computed in linear time, bounded by a linear function $g : \mathbb{N} \rightarrow \mathbb{N}$. Assuming that the time it takes to decide¹¹ T is bounded by a function $h : \mathbb{N} \rightarrow \mathbb{N}$, it follows that the time it takes to decide whether a string of length n belongs to T' is less than $f(n) + g(n) + h(n) + c$, for some constant c . Hence, if T can be decided in constant/linear/polynomial/exponential/etc. time, so can T' .

It is interesting to compare this generic nominalization strategy with that of [Field \(1980\)](#), who uses mereology for nominalizing Newtonian physics – especially since Field operates under the assumption that there are infinitely many empirical entities, namely space-time regions. To some extent, [Theorem 8](#) trivializes that project. On the other hand, as we shall see in the next section, some would argue that the generic strategy does not yield attractive enough alternatives. I am not convinced by some of those arguments, and I will try to explain why.

We end this section by noting that, as an immediate consequence of [Theorem 6](#) and [8](#), if the original theory is recursive and has a finite vocabulary, the nominalizing theory can also be finite:

Corollary 2. *For any finite vocabulary L , recursive L -theory T and L -formula $\delta(x)$ such that T entails that there are infinitely many δ :s, there is a finite theory T' semantically L -equivalent to T over δ such that $T' \vdash \forall x \delta$.*

6 A comparison with Burgess and Rosen

Although formulated in the framework of two-sorted first-order logic, [Theorem 8](#) of the previous section was essentially already noted by [Burgess and Rosen \(1999\)](#) in their discussion of general strategies for eliminating mathematical entities from the empirical sciences. The corresponding strategy is there called *Skolemite reduction*. They argue that, unless the original theory has what they call *the representation property* (which essentially means that f can be defined by the original theory), the strategy does not yield attractive enough nominalist alternatives. I am not convinced by their line of argument. In particular, I think their success criteria are far too demanding. Most importantly, as I will show at the end of this section, they fail for very simple empirical theories that

¹¹*Deciding a theory* usually means deciding whether something is a theorem of the theory. Here, it just means deciding whether something is an axiom of the theory.

do not make any substantial claims about mathematical entities whatsoever. Adopting these criteria would thus seem to entail the absurd claim that certain nominalist theories are not nominalizable.

The following is a summary of the central definitions given in [Burgess and Rosen \(1999, pp. 83-92\)](#):

- A two-sorted first-order language has a set x, y, z, \dots of **primary** variables, and a set X, Y, Z, \dots of **secondary** variables. Each predicate is assigned a particular number of argument places for variables of each sort. Predicates only taking primary (secondary) variables as arguments are called **primary (secondary)**, and predicates taking both sorts are called **mixed**. The terminology is extended to formulas and theories in the obvious way.
- Let T be a theory in a two-sorted vocabulary L . The **primary restriction** L° of L is the set of primary predicates of L , and the **primary restriction** T° of T is the set of primary sentences of T . T is **deductively conservative** over T° just in case, for any primary sentence φ , if $T \vdash \varphi$ then $T^\circ \vdash \varphi$. T is **expressively conservative** over T° just in case, for any L -formula φ with only primary free variables x_1, \dots, x_n , there is an L° -formula ψ with the same free variables such that $T \vdash \forall x_1 \dots \forall x_n (\varphi(x_1, \dots, x_n) \leftrightarrow \psi(x_1, \dots, x_n))$.¹² T is a **fully conservative** extension of T° just in case it is both a deductively and expressively conservative extension of T° .
- An extension T^+ of T in a vocabulary $L^+ \supseteq L$ is **definitionally redundant** just in case L^+ is obtained by adding finitely many new predicates, and T^+ is obtained by adding a single axiom for each new predicate that defines it in terms of an L -formula. A further extension T^\dagger of T^+ in the same vocabulary $L^\dagger = L^+$ is **implicationally redundant** just in case it is obtained by adding the instances of finitely many axioms schemes, each of which was already deducible from T^+ . Such an extension T^\dagger is a **merely redundant** extension of T just in case T^+ is a definitionally redundant extension of T . Finally, T has the **elimination property** just in case it has a merely redundant extension T^\dagger that is fully conservative over its primary restriction $T^{\dagger\circ}$.
- T has the **representation property** just in case there is an L -formula $\varphi(x_1, \dots, x_k, X)$ such that

$$T \vdash \forall X \exists x_1 \dots \exists x_k \varphi(x_1, \dots, x_k, X)$$

and

$$T \vdash \forall x_1 \dots \forall x_k \forall X \forall Y (\varphi(x_1, \dots, x_k, X) \wedge \varphi(x_1, \dots, x_k, Y) \rightarrow X = Y)$$

Although never explicitly stated, the suggestion seems to be that the secondary entities postulated by a theory are dispensable only if the theory has the elimination property. For the sake of comparison, let us extend our own notions of nominalizability to the two-sorted case by saying that a two-sorted L -theory T is

¹²More appropriately, one should instead say that T is *expressively conservative over* L° , since L° rather than T° occurs in the defining expression.

1. **syntactically nominalizable** just in case there is a primary theory T' in a vocabulary $L' \supseteq L^\circ$ such that

- (a) T and T' are syntactically L° -equivalent¹³, and
- (b) T' is no more computationally complex than T .

and

2. **semantically nominalizable** just in case there is a primary theory T' in a vocabulary $L' \supseteq L^\circ$ such that

- (a) T and T' are semantically L° -equivalent¹⁴, and
- (b) T' is no more computationally complex than T .

Theorem 9. *The representation property implies semantic nominalizability.*

Proof. Let T be a two-sorted L -theory, and assume that there is a mixed L -formula $R(\bar{x}, X)$ such that

$$T \vdash \forall X \exists \bar{x} R(\bar{x}, X)$$

and

$$T \vdash \forall \bar{x} \forall X \forall Y (R(\bar{x}, X) \wedge R(\bar{x}, Y) \rightarrow X = Y)$$

For each non-primary L -predicate P taking variables x, y, \dots, X, Y, \dots , introduce a new primary predicate P° taking variables $x, y, \dots, \bar{x}, \bar{y}, \dots$, yielding the extended primary vocabulary $L' \supseteq L^\circ$. For each L -formula φ , define its L' -translation $[\varphi]^\circ$ recursively:

- 1. $[Px_1 \dots x_n]^\circ = Px_1 \dots x_n$
- 2. $[Qxy \dots XY \dots]^\circ = Q^\circ xy \dots \bar{x}\bar{y} \dots$
- 3. $[\neg \varphi]^\circ = \neg [\varphi]^\circ$
- 4. $[\varphi \wedge \psi]^\circ = [\varphi]^\circ \wedge [\psi]^\circ$
- 5. $[\forall x \varphi]^\circ = \forall x [\varphi]^\circ$
- 6. $[\forall X \varphi]^\circ = \forall \bar{x} [\varphi]^\circ$

Finally, define the L' -theory $T' = \{[\varphi]^\circ : \varphi \in T\}$.

To show that T and T' are semantically L° -equivalent, assume that \mathcal{M} is a model of T' with domain D . We construct a two-sorted L -model \mathcal{M}' by adding the set of secondary entities D^k . For each non-primary L -predicate P , let $P^{\mathcal{M}'}(a, b, \dots, A, B, \dots)$ iff $P^{\circ \mathcal{M}}(a, b, \dots, A, B, \dots)$. It is straightforward to show by induction on the complexity of formulas that, for any L -formula $\varphi(x, y, \dots, X, Y, \dots)$, we have $\mathcal{M}' \models \varphi[a, b, \dots, A, B, \dots]$

¹³For any L° -sentence φ , we have $T' \vdash \varphi$ iff $T \vdash \varphi$.

¹⁴Any model of T' can be extended to a model of T by adding a secondary domain, and the primary part of any model of T can be expanded to a model of T' .

iff $\mathcal{M} \models [\varphi]^\circ[a, b, \dots, A, B, \dots]$.¹⁵ Hence, \mathcal{M}' is a model of T . For the other direction, assume that \mathcal{M} is a model of T with primary domain D and secondary domain E . Let $f : D^k \rightarrow E$ be a total surjection extending \mathcal{M} 's interpretation of R . We construct a one-sorted L' -model \mathcal{M}' with the single domain D . For each non-primary L -predicate P , let $P^{\circ\mathcal{M}'}(a, b, \dots, \bar{a}, \bar{b}, \dots)$ iff $P^{\mathcal{M}}(a, b, \dots, f(\bar{a}), f(\bar{b}), \dots)$. As before, it is straightforward to show by induction that, for any L -formula $\varphi(x, y, \dots, X, Y, \dots)$, we have $\mathcal{M}' \models \varphi^\circ[a, b, \dots, \bar{a}, \bar{b}, \dots]$ iff $\mathcal{M} \models \varphi[a, b, \dots, f(\bar{a}), f(\bar{b}), \dots]$.¹⁶ Hence, \mathcal{M}' is a model of T' . \square

To get a clearer grasp of what the elimination property is about, we offer the following characterizing lemma:

Lemma 7. *An L -theory T has the elimination property just in case there is a finite number of L -formulas $\varphi_1(\bar{x}_1), \dots, \varphi_n(\bar{x}_n)$ whose free variables are all primary such that, with additional primary predicates $P_1, \dots, P_n \notin L$ of corresponding arities,*

$$\Delta = \{\forall \bar{x}_1(P_1 \bar{x}_1 \leftrightarrow \varphi_1(\bar{x}_1)), \dots, \forall \bar{x}_n(P_n \bar{x}_n \leftrightarrow \varphi_n(\bar{x}_n))\}$$

and $L^\dagger = L \cup \{P_1, \dots, P_n\}$, the following obtains:

1. For any L -formula $\varphi(\bar{x})$ with only primary free variables, there is an $L^{\dagger\circ}$ -formula $\psi(\bar{x})$ with the same free variables such that $T \cup \Delta \vdash \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$, and
2. there is an $L^{\dagger\circ}$ -theory Γ such that
 - (a) T and $T^\circ \cup \Gamma$ are syntactically L° -equivalent, and
 - (b) Γ is axiomatizable by finitely many axiom schemes.

Proof. Right to left follows by letting $T^\dagger = T \cup \Delta \cup \Gamma$. For left to right, assume that T^\dagger is a merely redundant extension of T that is fully conservative over $T^{\dagger\circ}$ (i.e. deductively conservative over $T^{\dagger\circ}$ and expressively conservative over $L^{\dagger\circ}$.) Let Δ be the set of definitions extending T . By implicational redundancy of T^\dagger over $T \cup \Delta$, it follows that $T \cup \Delta$ is expressively conservative over $L^{\dagger\circ}$. Let $\Sigma = T^\dagger - T - \Delta$ and $\Gamma = \Sigma^\circ$. Thus, Γ is an $L^{\dagger\circ}$ -theory. Again, by implicational redundancy, Σ is axiomatizable by finitely many axiom schemes, and hence so is Γ . Let φ be an L° -sentence. If $T^\circ \cup \Gamma \vdash \varphi$, it follows that $T \cup \Delta \cup \Sigma \vdash \varphi$. Implicational redundancy yields $T \cup \Delta \vdash \varphi$, and hence $T \vdash \varphi$. If $T \vdash \varphi$, it follows that $T \cup \Delta \cup \Sigma \vdash \varphi$. Deductive conservativeness yields $(T \cup \Delta \cup \Sigma)^\circ \vdash \varphi$, and hence $T^\circ \cup \Gamma \vdash \varphi$. \square

¹⁵In the case of quantifiers, assume as induction hypothesis that the claim holds for $\varphi(x, y, \dots, X, Y, \dots)$. In the primary case, we have $\mathcal{M}' \models \forall x \varphi[b, \dots, A, B, \dots]$ iff $\mathcal{M}' \models \varphi[a, b, \dots, A, B, \dots]$ for all $a \in D$ iff (by induction hypothesis) $\mathcal{M} \models [\varphi]^\circ[a, b, \dots, A, B, \dots]$ for all $a \in D$ iff $\mathcal{M} \models [\forall x \varphi]^\circ[b, \dots, A, B, \dots]$. In the secondary case, we have $\mathcal{M}' \models \forall X \varphi[b, \dots, B, \dots]$ iff $\mathcal{M}' \models \varphi[a, b, \dots, A, B, \dots]$ for all $A \in D^k$ iff (by induction hypothesis) $\mathcal{M} \models [\varphi]^\circ[a, b, \dots, A, B, \dots]$ for all $A \in D^k$ iff $\mathcal{M} \models [\forall \bar{x} \varphi]^\circ[a, b, \dots, B, \dots]$.

¹⁶In the case of secondary quantifiers, assume as induction hypothesis that the claim holds for $\varphi(x, y, \dots, X, Y, \dots)$. We have $\mathcal{M}' \models [\forall X \varphi]^\circ[a, b, \dots, \bar{b}, \dots]$ iff $\mathcal{M}' \models \forall \bar{x} [\varphi]^\circ[a, b, \dots, \bar{b}, \dots]$ iff $\mathcal{M}' \models [\varphi]^\circ[a, b, \dots, \bar{a}, \bar{b}, \dots]$ for all $\bar{a} \in D^k$ iff (by induction hypothesis) $\mathcal{M} \models \varphi[a, b, \dots, f(\bar{a}), f(\bar{b}), \dots]$ for all $\bar{a} \in D^k$ if (by totality of f) and only if (by surjectivity of f) $\mathcal{M} \models \varphi[a, b, \dots, A, f(\bar{b}), \dots]$ for all $A \in E$ iff $\mathcal{M} \models \forall X \varphi[a, b, \dots, f(\bar{b}), \dots]$.

Although depending a bit on how fine grained a notion of computational complexity one employs, it should be reasonably clear from this lemma that the elimination property entails syntactic nominalizability. I do not know whether it also entails semantic nominalizability, or whether it implies the representation property. As defined, however, the elimination property cannot serve as general criteria of dispensability, for the following simple reason:

Theorem 10. *The elimination property is not preserved under logical equivalence.*

Proof. Take any primary theory T that is not axiomatizable by a finite number of axiom schemes, and let $T' = T \cup \{\forall X\forall Y(X = Y)\}$. Since $T' \vdash \forall X\forall Y(X = Y \leftrightarrow \forall x(x = x))$, it easily follows by induction on the complexity of formulas that T' has the elimination property. To each of its primary axioms, add $\forall X(X = X)$ as a conjunct. The resulting theory is equivalent to T' and deductively conservative over the original theory T , but its primary restriction is empty.¹⁷ Hence, by Lemma 7, it does not have the elimination property. \square

A similar construction also yields the following result, contradicting the claim made by Burgess and Rosen (1999, p. 86):

Theorem 11. *The representation property does not imply the elimination property.*

Proof. Take any primary theory that is not axiomatizable by a finite number of axiom schemes, and extend it with the representation axioms for a mixed binary predicate R , yielding a deductively conservative extension (each model of the original theory can be expanded to a model of the extended theory by adding an equally large domain of secondary entities and interpreting R as a bijective relation between the two domains). By definition, the extended theory has the representation property. Construct a new theory by adding $\forall X(X = X)$ as a conjunct to each primary axiom. The new theory also has the representation property, but its primary restriction is empty. Hence, by Lemma 7, it does not have the elimination property. \square

More charitably, we may of course understand their claim as being restricted to theories that are axiomatizable by finitely many axiom schemes. Indeed:

Theorem 12. *For theories that are axiomatizable by finitely many axiom schemes, the representation property does imply the elimination property.*

Proof. Importing the assumptions from the proof of Theorem 9 concerning T , L , $R(\bar{x}, X)$, L' and T' , let $L^\dagger = L \cup L'$ and let Δ be the set of definitions

$$\forall x\forall y\dots\forall\bar{x}\forall\bar{y}\dots[P^\circ xy\dots\bar{x}\bar{y}\dots \leftrightarrow \exists X\exists Y\dots(R(\bar{x}, X) \wedge R(\bar{y}, Y) \wedge \dots \wedge Pxy\dots XY\dots)]$$

for each non-primary predicate $P \in L$. It can be established that, for any L -formula $\varphi(x, y, \dots, X, Y, \dots)$,

$$\forall x\forall y\dots\forall X\forall Y\dots\forall\bar{x}\forall\bar{y}\dots[R(\bar{x}, X) \wedge R(\bar{y}, Y) \wedge \dots \rightarrow ([\varphi]^\circ(x, y, \dots, \bar{x}, \bar{y}, \dots) \leftrightarrow \varphi(x, y, \dots, X, Y, \dots))]$$

¹⁷As mentioned earlier, we do not identify theories with their deductive closure, as neither do Burgess and Rosen.

is a theorem of $T \cup \Delta$, by induction on the complexity of φ . By considering the case where $\varphi(\bar{x})$ contains no free secondary variables, it follows that there is always an L° -formula $\psi(\bar{x})$ with the same free primary variables such that $T \cup \Delta \vdash \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$. Moreover, assuming that T is axiomatizable by finitely many axiom schemes, it follows that its L' -translation T' is as well. As we saw in the proof of Theorem 9, T and T' are semantically (and thus syntactically) L° -equivalent. By Theorem 7, T has the elimination property. \square

Even so, as witnessed by T in the proof of the following result, there are simple theories not making any substantial claims whatsoever about secondary entities that still lack the elimination property:

Theorem 13. *Even for theories that are axiomatizable by finitely many axiom schemes, semantic nominalizability does not imply the elimination property.*

Proof. Let L be a finite two-sorted vocabulary containing no mixed predicates, and consider the simple L -theory

$$T = \{\forall x \forall y(x = y)\} \cup \{\forall \bar{x} P \bar{x} : P \in L^{\circ}\}$$

We will show that this theory does not have the elimination property.

Consider a finite number of L -formulas $\varphi_1(\bar{x}_1), \dots, \varphi_n(\bar{x}_n)$ whose free variables are all primary, and new primary predicates $P_1, \dots, P_n \notin L$ of corresponding arities. Let

$$\Delta = \{\forall \bar{x}_1(P_1 \bar{x}_1 \leftrightarrow \varphi_1(\bar{x}_1)), \dots, \forall \bar{x}_n(P_n \bar{x}_n \leftrightarrow \varphi_n(\bar{x}_n))\}$$

$$L^+ = L \cup \{P_1, \dots, P_n\}$$

and

$$T^+ = T \cup \Delta$$

Assume, towards contradiction, that for any L^+ -formula $\varphi(\bar{x})$ with only primary free variables, there is an L° -formula $\psi(\bar{x})$ with the same free variables such that

$$T^+ \vdash \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$$

In particular, we assume that

(9) For any L^+ -sentence φ , there is an L° -sentence ψ such that $T^+ \vdash \varphi \leftrightarrow \psi$.

By construction of T , we have for any atomic L° -formula $\varphi(\bar{x})$ that $T \vdash \forall \bar{x} \varphi(\bar{x})$, and thus

(10) For any atomic L° -formula $\varphi(\bar{x})$: $T \vdash \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \forall X(X = X))$.

Let $\varphi'_1, \dots, \varphi'_n$ be the secondary L -sentences gotten by first replacing each atomic L° -formula in $\varphi_1(\bar{x}_1), \dots, \varphi_n(\bar{x}_n)$ with $\forall X(X = X)$, and then removing all primary quantifiers (which are now redundant). It follows that

$$T \vdash \forall \bar{x}(\varphi_1(\bar{x}) \leftrightarrow \varphi'_1), \dots, T \vdash \forall \bar{x}(\varphi_n(\bar{x}) \leftrightarrow \varphi'_n)$$

and thus

$$(11) \quad T^+ \vdash \forall \bar{x}(P_1 \bar{x} \leftrightarrow \varphi'_1), \dots, T^+ \vdash \forall \bar{x}(P_n \bar{x} \leftrightarrow \varphi'_n).$$

By (10) and (11), it now follows that every $L^{+\circ}$ -sentence is equivalent (under T^+) to a Boolean combination of $\varphi'_1, \dots, \varphi'_n, \forall X(X = X)$. Hence, by assumption (9), it follows that for any secondary L -sentence φ , there is a secondary L -sentence ψ that can be written as a Boolean combination of $\varphi'_1, \dots, \varphi'_n, \forall X(X = X)$ such that

$$T^+ \vdash \varphi \leftrightarrow \psi$$

But since $\varphi \leftrightarrow \psi$ contains no primary predicates (not even the primary identity predicate), its truth depends only on the secondary domain, about which T^+ says nothing (since the latter is merely a definitional extension of T). Hence, it must be true in all models, and thus

$$\vdash \varphi \leftrightarrow \psi$$

Let k be a natural number such that $\varphi'_1, \dots, \varphi'_n, \forall X(X = X)$ all have a lower quantifier rank than k . It follows that any secondary L -sentence is logically equivalent to a secondary L -sentence whose quantifier rank is less than k . This, however, is not the case. It is a well known fact that, since L is finite, there are at most finitely many pairwise non-equivalent secondary L -sentences of quantifier rank less than k , but infinitely many pairwise non-equivalent secondary L -sentences, e.g. $\exists_{\geq 1} X(X = X), \exists_{\geq 2} X(X = X), \exists_{\geq 3} X(X = X), \dots$ \square

7 Finite theories

Suppose we have a finite L -theory T and an L -formula $\delta(x)$ such that $T \vdash \exists x \delta \wedge \exists x \neg \delta$. It follows from the argument in the beginning of section 4 that, if T has a model with finitely many δ 's, it is a trivial matter to find a finite L -theory T' semantically L -equivalent to T over δ such that $T' \not\vdash \exists x \neg \delta$. It is not, however, a trivial matter to nominalize T with a theory T' such that $T' \vdash \forall x \delta$. In some cases, it is impossible. This is due to the fact, pointed out by Hodges (1993, p. 211, Exercise 4), that some finitely relativized pseudo-elementary class is not finitely pseudo-elementary. We offer the following example:

Example 3. Let P be a unary predicate, $L = \{P\}$, and let Q be the finite set of axioms of Robinson arithmetic (sufficient for representing all recursive relations) in the vocabulary $L_Q = \{0, s, +, \cdot, <\}$. Let c be a new constant, let $L' = L \cup L_Q \cup \{c\}$, and let $\varphi(x)$ be an L_Q -formula representing a recursive but not primitive recursive set $A \subseteq \mathbb{N}$ in Q . Finally, let

$$T = Q \cup \{\varphi(c), \forall x(Px \leftrightarrow x < c), \exists x Px\}$$

and consider the finitely relativized pseudo-elementary class

$$\mathcal{C} = \{\mathcal{M}_{Px} | L : \mathcal{M} \models T\}$$

Assume, towards contradiction, that there is a finite theory T' in some vocabulary extending L such that

$$\{\mathcal{M} | L : \mathcal{M} \models T'\} = \mathcal{C}$$

Observe that $T' \vdash \forall xPx$. Consider the set of natural numbers

$$A' = \{|P^{\mathcal{M}}| : \mathcal{M} \models T' \text{ and } P^{\mathcal{M}} \text{ is finite}\}$$

In other words, let A' be the set of natural numbers n such that T' has a model with n elements. Observe that, since T' is finite, A' is primitive recursive. By assumption, we get

$$A' = \{|P^{\mathcal{M}}| : \mathcal{M} \models Q, \varphi(c), \forall x(Px \leftrightarrow x < c), \exists xPx \text{ and } P^{\mathcal{M}} \text{ is finite}\}$$

Now, let $A^- = A - \{0\}$. Since A^- is not primitive recursive, we get $A^- \neq A'$. We will show that $A^- = A'$, yielding a contradiction. Assume that $n \in A^-$. That means $Q \vdash \varphi(\mathbf{n})$, and thus $\mathcal{N} \models \varphi(\mathbf{n})$. Let \mathcal{M} be just like \mathcal{N} , with $c^{\mathcal{M}} = n$ and $P^{\mathcal{M}} = \{m \in \mathbb{N} : m < n\}$. Since $n \neq 0$, we get $n \in A'$. Next, assume that $n \notin A^-$. If $n = 0$, we get $n \notin A'$. Otherwise, we get $Q \vdash \neg\varphi(\mathbf{n})$. Since $Q, \exists_{=n}x(x < c) \vdash c = \mathbf{n}$, that means $n \notin A'$. Hence, \mathcal{C} is not finitely pseudo-elementary.

Observe that \mathcal{C} is still *recursively* pseudo-elementary (even recursively *elementary*), with

$$T' = \{\neg\exists_{=n}xPx : n \in \mathbb{N} - A^-\} \cup \{\forall xPx\}$$

To verify that $\mathcal{C} = \{\mathcal{M}|L : \mathcal{M} \models T'\}$, assume that \mathcal{M} is a model of T . If $P^{\mathcal{M}}$ is finite, it has $n > 0$ elements. Since $Q, \exists_{=n}x(x < c) \vdash c = \mathbf{n}$, it follows that $\mathcal{M} \models \varphi(\mathbf{n})$. Hence, $Q \not\vdash \neg\varphi(\mathbf{n})$, which means that $n \in A^-$. Hence, $\mathcal{M}_{Px}|L$ is a model of T' . If $P^{\mathcal{M}}$ is infinite, we only need to observe that T' has a model of every infinite cardinality. For the other direction, assume that \mathcal{M} is a model of T' . If \mathcal{M} is finite, it has $n \in A^-$ elements, in which case it can be extended to a model of T . If \mathcal{M} is infinite, we only need to observe that $T \cup \{\forall x\exists y(x < y \wedge \varphi(y))\}$ has a model of every infinite cardinality, with c interpreted as an arbitrarily large non-standard element.

Moreover, no finite theory T^* with $T^* \vdash \forall xPx$ is even *syntactically* L -equivalent to T over Px . To see why, assume, towards contradiction, that T^* is such a theory. By Lemma 3, T^* is syntactically L -equivalent to T' over Px . Moreover, since both entail $\forall xPx$, it follows that T^* and T' are syntactically L -equivalent simpliciter. We have already established that $A^- = \{|P^{\mathcal{M}}| : \mathcal{M} \models T' \text{ and } P^{\mathcal{M}} \text{ is finite}\}$. Define

$$A^* = \{|P^{\mathcal{M}}| : \mathcal{M} \models T^* \text{ and } P^{\mathcal{M}} \text{ is finite}\}$$

Since A^* but not A^- is primitive recursive, we get $A^- \neq A^*$. Hence, we get two cases:

1. There is $n \in A^-$ such that $n \notin A^*$. That means $T^* \vdash \neg\exists_{=n}xPx$ but $T' \not\vdash \neg\exists_{=n}xPx$, contradicting our assumption.
2. There is $n \in A^*$ such that $n \notin A^-$. That means $T' \vdash \neg\exists_{=n}xPx$ but $T^* \not\vdash \neg\exists_{=n}xPx$, also contradicting our assumption.

Hence, there is no such theory T^* .

The example shows that there are statements that cannot be paraphrased along the lines of [Goodman and Quine \(1947\)](#). For any recursive but not primitive recursive set A of natural numbers, the statement

- (12) The number of concrete objects is an element of A ,

which, as we just saw, can be expressed by a sentence implying that there are infinitely many abstract objects, cannot be expressed by a sentence implying that all objects are concrete.

Since T is finite, it can be decided in constant time. It is clear that any theory syntactically L -equivalent to T' cannot. Intuitively, this is because the algorithm deciding it would have to be able to discriminate between sentences of arbitrary length. Hence, any theory nominalizing T would have to be more computationally complex. How much more is hard to say. Although it is obvious that the computational complexity of T' *itself* will be the same as that of A , it is not clear what this means for the computational complexity of theories *equivalent* to T' .

8 Conclusion

Some realists claim that theoretical entities like numbers and electrons are indispensable for describing the empirical world. Motivated by the meta-ontology of Quine, I take this claim to imply that, for some first-order theory T and formula $\delta(x)$ such that $T \vdash \exists x\delta \wedge \exists x\neg\delta$, there is no first-order theory T' such that

- (a) T and T' describe the δ :s in the same way,
- (b) $T' \vdash \forall x\delta$, and
- (c) T' is at least as attractive as T in terms of other theoretical virtues.

In an attempt to refute the realist claim, I try to solve the general problem of *nominalizing* T (with respect to δ), namely to find a theory T' satisfying conditions (a)–(c) under various precisifications thereof. In particular, I note that condition (a) can be understood either in terms of syntactic or semantic equivalence (Definition 3), with the latter strictly stronger than the former (Lemma 3).

The results are somewhat mixed. On the positive side, even under the stronger precisification of (a), I use results by [Craig and Vaught \(1958\)](#), [Makkai \(1964\)](#) and [Hodges \(1993\)](#) to establish that (1) if the vocabulary of T is finite, then a nominalizing theory can always be found that is recursive if T is (Theorem 7), and (2) if T postulates infinitely many δ :s, a nominalizing theory can always be found that is no more computationally complex than T (Theorem 8). On the negative side, even under the weaker precisification of (a), I establish that (3) certain finite theories cannot be nominalized by a finite theory (Example 3). Thus, as far as I can see, the prospects for nominalization look the same from the point of view of both semantic and syntactic equivalence. In either case, postulating non-empirical entities is never necessary for obtaining a recursive theory of the empirical world, but sometimes necessary for obtaining a finite one. Moreover, for theories postulating infinitely many empirical entities, nominalization is cheap.

Lemma 5 (and its proof) shows that, if the empirical world can be described by a recursive theory, then it can be described by a finite theory postulating (infinitely many) non-empirical entities. Insofar as the aim of science is to provide a finite theory of the empirical world, Example 3 shows that the postulated entities may be indispensable. Unlike recursive axiomatizability, however, finite axiomatizability is not invariant under alternative notions of logicity: it depends on where the line is drawn between the

logical and non-logical parts of a system. In a Hilbert-style system, for instance, there are infinitely many logical axioms. In a system of natural deduction, on the other hand, there are no logical axioms, but infinitely many rules (or, if you like, finitely many rules with infinitely many instances). A classical Hilbert system will typically include every instance of the scheme $(\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi)$, which are not considered logical from an intuitionistic point of view. A theory that is finitely axiomatizable in a classical system may therefore not be so in an intuitionistic system. Hence, the value of providing a finite theory may be contested.

References

- Burgess, J. P. and Rosen, G. (1999). *A Subject With No Object*. Oxford University Press.
- Colyvan, M. (2019). Indispensability Arguments in the Philosophy of Mathematics. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, spring 2019 edition.
- Craig, W. (1953). On axiomatizability within a system. *The Journal of Symbolic Logic*, 18(1):30–32.
- Craig, W. and Vaught, R. L. (1958). Finite axiomatizability using additional predicates. *The Journal of Symbolic Logic*, 23(3):289–308.
- Field, H. (1980). *Science Without Numbers*. Princeton University Press.
- Goodman, N. and Quine, W. V. (1947). Steps toward a constructive nominalism. *Journal of Symbolic Logic*, 12(4):105–122.
- Hodges, W. (1993). *Model Theory*. Encyclopedia of Mathematics and its Applications. Cambridge University Press.
- Inwagen, P. V. (1998). Meta-ontology. *Erkenntnis (1975-)*, 48(2/3):233–250.
- Johannesson, E. (2020). Realism and empirical equivalence. *Journal of Philosophical Logic*, 49(3):475–495.
- Ketland, J. (2004). Empirical adequacy and ramsification. *British Journal for the Philosophy of Science*, 55(2):287–300.
- Ketland, J. (2005). Some more curious inferences. *Analysis*, 65(1):18 – 24.
- Kleene, S. C. (1952). Finite axiomatizability of theories in the predicate calculus using additional predicate symbols. two papers on the predicate calculus. *Memoirs of the American Mathematical Society*, 10:27.
- Makkai, M. (1964). On PC_{Δ} -classes in the theory of models. *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, 9:159–194.
- Melia, J. (2000). Weaseling away the indispensability argument. *Mind*, 109(435):455–479.

- Putnam, H. (1965). Craig's theorem. *Journal of Philosophy*, 62(10):251–260.
- Putnam, H. (1979). *Philosophical papers. Vol. 1, Mathematics, matter and method*. Cambridge Univ. Press, Cambridge, 2 edition.
- Quine, W. V. (1951). Ontology and ideology. *Philosophical Studies: An International Journal for Philosophy in the Analytic Tradition*, 2(1):11 – 15.
- Tarski, A. (1954). Contributions to the theory of models. I. In *Indagationes Mathematicae (Proceedings)*, volume 57, pages 572–581. Elsevier.
- van Benthem, J. F. A. K. (1978). Ramsey eliminability. *Studia Logica*, 37(4):321–336.