

# Entanglement, Complexity, and Causal Asymmetry in Quantum Theories

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## Abstract

It is often claimed that one cannot locate a notion of causation in fundamental physical theories. The reason most commonly given is that the dynamics of those theories do not support any distinction between the past and the future, and this vitiates any attempt to locate a notion of causal asymmetry — and thus of causation — in fundamental physical theories. I argue that this is incorrect: the ubiquitous generation of entanglement between quantum systems grounds a relevant asymmetry in the dynamical evolution of quantum systems. I show that by exploiting a connection between the amount of entanglement in a quantum state and the algorithmic complexity of that state, one can use recently developed tools for causal inference to identify a causal asymmetry — and a notion of causation — in the dynamical evolution of quantum systems.

## 1 Introduction

Consider the plight of Emma Flake. Dr. Flake runs a lab that works on applications of quantum state tomography, but she has kind of checked out lately. Her grad students, Alice and Bob, have conducted the following experiment in her absence: a source prepares an ensemble of bipartite quantum systems in identical pure states  $|\alpha\rangle$ . Alice knows the state  $|\alpha\rangle$ , but Bob does not. Each system then passes through a region of spacetime governed by a Hamiltonian  $H$ , ending up in some pure state  $|\beta\rangle$ . Bob's task is to reconstruct the pure state  $|\beta\rangle$  by performing a large number of measurements on different observables of the system.

Dr. Flake, feeling guilty about her absenteeism, offers to write up the paper on her own and tells Alice and Bob to take some time off. When she looks at Alice and Bob's notes, she discovers that there is no record of which state was prepared by Alice and which was reconstructed by Bob; all she knows is the two pure states  $|\alpha\rangle$  and  $|\beta\rangle$ . Is there any way for

Dr. Flake to determine which was the initial state and which was the final? In other words, can Dr. Flake determine whether  $|\alpha\rangle$  was caused by time-evolving  $|\beta\rangle$  or vice versa?<sup>1</sup>

Dr. Flake is confronted with a special case of the general problem I will take up in this paper: if one knows (i) the identity of two quantum states and (ii) that they are related by unitary time evolution, does quantum theory provide resources to determine which state was caused by time-evolving the other? It is commonly said that the answer is “no”: the failure of the dynamical equations describing microscopic physical systems to distinguish between past and future leaves Dr. Flake with insufficient resources to solve her problem. This failure is commonly presented as an obstacle to identifying any notion *at all* of causal direction – and thus of causation – using the resources of microscopic physics (see, for example, (Russell, 1912; Albert, 2000; Field, 2003; Woodward, 2007; Loewer, 2012)).

One consequence has been the widespread belief that any notion of causal direction, and thus causation, must be located elsewhere. For many, like (Albert, 2000; Woodward, 2007), that has been in macroscopic systems. For some, it has been in our psychological experience as deliberators (Ismael, 2016; Fernandes, 2017). And for a few, it has been nowhere: causal direction is illusory, tantamount to a preferred choice of coordinates (Price, 2007).

The notion of causation I employ in this paper is a minimal interventionist one: X is a cause of Y if and only if there is an intervention that can be performed on the value of X, while holding all other variables fixed, that produces a change in the value of Y.<sup>2</sup> In this paper I will restrict to the case of two variables and presuming the absence of confounders, so any causal relation will be a *direct* causal relation.

It is well known that without additional conditions, this minimal interventionist notion of causation cannot use observational statistical data alone to identify a direction of causation between two variables X and Y, even when they are related by a known invertible function (see, e.g., (Pearl, 2009, chapter 2)). Unitary time evolution between two quantum states is deterministic and invertible; without information about the actual time ordering of two states  $|\alpha\rangle$  and  $|\beta\rangle$ , familiar causal inference methods are unable to distinguish between  $|\alpha\rangle \longrightarrow |\beta\rangle$  and  $|\beta\rangle \longrightarrow |\alpha\rangle$ .

In this paper, I argue that a generic physical fact about the time evolution of quantum systems identifies a causally relevant asymmetry: interaction between quantum systems almost always entangles those systems but almost never disentangles them. This fact enables

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<sup>1</sup>One might wonder why I describe this as a problem of inferring causal direction rather than temporal direction. I will return to this question in section 6.

<sup>2</sup>This generalizes straightforwardly to probabilistic settings: X causes Y if and only if there is an intervention that can be performed on  $p_X(x)$ , the probability distribution over the values of X, while holding all other variables fixed, that produces a change in  $p_Y(y)$ . See Woodward (2005) for a much richer philosophical development of the interventionist account of causation than I will need here.

one to identify a causally relevant asymmetry between quantum states related by unitary time evolution. To connect this physical fact to formal methods of causal inference, I employ recently developed tools whose applications include the ability to identify the causal direction of deterministic, noiseless processes solely from observational data (Janzing and Schölkopf, 2010; Daniušis et al., 2010; Janzing et al., 2012; Peters et al., 2017), (Mooij et al., 2016, sections 3 and 5.2) to argue that the generic entanglement of quantum systems by unitary time evolution allows one to identify criteria for inferring causal direction even between states related by time-symmetric dynamical laws. The majority of my discussion focuses on paradigmatically microscopic physical systems: bipartite quantum systems undergoing unitary time evolution. That one can identify causally relevant asymmetries in such evolutions is indication that one can locate a notion of causation in microscopic physics after all.

The structure of the paper is as follows. In Section 2, I review the sense(s) in which quantum theories fail to distinguish between past and future and the challenge this poses to any attempt to locate a notion of causation in microscopic physics. In Section 3, I review relevant properties of entanglement in quantum theories: the ubiquity of entangled states, the generic creation of entanglement by interactions between systems, and quantitative measures of how entangled a quantum state is. I argue that already, on physical grounds, these features ground a (fairly weak) causal asymmetry between quantum states related by unitary evolution. Section 4 reviews the causal inference methods I later use to argue that one can identify a direction of causation even between states related by time-symmetric dynamical laws, along with some necessary concepts from algorithmic information theory. Section 5 introduces a generalization of algorithmic information to quantum theories and demonstrates that in the restricted but important setting of a system of  $N$  qubits, one can derive a stronger version of the asymmetry from section 3 as a theorem of the causal inference methods introduced in section 4. In section 6 I conclude with some remarks about the relationship between more familiar temporal asymmetries in physics and the causal asymmetry argued for here, and situate my discussion within a more general understanding of the relationship between the epistemology of causation and its metaphysics.

I want to emphasize three things before proceeding.<sup>3</sup> The first thing is that my primary interest here is the *epistemology* of causal direction.<sup>4</sup> One can view sections 4 and 5 in particular as an exercise in causal epistemology: taking a procedure for inferring causal direction based on observational data that has proved successful in other domains and applying it to quantum theories. I think of this as akin to expanding an experimental technique that has

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<sup>3</sup>Well, four things. The fourth thing is that throughout the paper  $\log = \log_2$  and I adopt units where  $\hbar = c = 1$ .

<sup>4</sup>See (Eberhardt, 2009) for an overview. For a clarifying sketch of several types of projects one might pursue when thinking about causation, see (Woodward, 2014).

proved successful at detecting a certain object or property in one department of nature to a novel domain, where the presence of that object or property is not settled, and seeing what the experiment turns up. In section 6 I return to a discussion of how this primarily epistemological exercise bears on questions about the physical ground of the causal asymmetry I identify.

The second thing is that unless otherwise stated, I will be considering only unitary time evolution. Once departures from unitary evolution are allowed, either through the tracing out of the environment associated with decoherence (Wallace, 2012, chapter 9), spontaneous collapse of GRW or CSL type (Albert, 2000, chapter 7), or old-fashioned measurement-induced collapse, then the time evolution distinguishes between past and future and an asymmetry between cause and effect is introduced.<sup>5</sup>

The third thing concerns the discussion of entanglement. Entanglement of quantum systems by interaction is a generic physical fact. In this paper I employ a particular measure of entanglement, the Schmidt measure, for reasons discussed in section 3. The reader should bear in mind that this is only one of multiple ways to quantify entanglement and the physical fact that interactions generically entangle but almost never disentangle – the physical foundation for the causal inference strategy developed in this paper – does not depend on this particular choice of entanglement measure nor the particular causal inference methods used. Indeed, the idea that the generation of entanglement by dynamical evolution is connected to a closely related asymmetry, the asymmetry of time, has been explored by a number of authors (Popescu et al., 2006; Reimann, 2008; Linden et al., 2009; Jennings and Rudolph, 2010; Short and Farrelly, 2012; Goldstein et al., 2013; Malabarba et al., 2014; Goldstein et al., 2015a).

Finally, this paper contributes to a large and rapidly growing literature on causal structure and causal inference in quantum theories (Leifer and Spekkens, 2013; Barrett et al., 2019; Chiribella et al., 2009; Oreshkov et al., 2012; Rubino et al., 2017; Costa and Shrapnel, 2016; Ried et al., 2015; Allen et al., 2017; Wood and Spekkens, 2015; Chaves et al., 2015, 2014). A number of mathematical and conceptual frameworks for analyzing causal structure have been developed in this literature. For example, (Leifer and Spekkens, 2013) develop a generalization of Bayesian inference and an associated concept of a conditional quantum state, while the process matrix formalism (Oreshkov et al., 2012; Barrett et al., 2019) introduces the “process matrix” which generalizes the standard notions of a quantum state and a quantum channel and has proven especially valuable for analyzing quantum processes

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<sup>5</sup>This doesn’t mean that all causal inference puzzles are solved once one allows departures from unitary evolution. Far from it: it is only after considering such departures that one encounters the most vexing causal inference problem posed by quantum theories: the explanation of EPR-type correlations.

with indeterminate causal order. And some of this work has taken up a similar project as I undertake here: locating causal or temporal asymmetries in the structure of quantum theory (Thompson et al., 2018; Schmid et al., 2020; Hardy, 2021; Di Biagio et al., 2021). In the course of this paper, I introduce an additional set of tools into this literature by taking the initial steps toward generalizing the algorithmic-information-theoretic framework for causal inference developed by (Janzing and Schölkopf, 2010) to quantum theories.

## 2 Time Evolution in Quantum Theories

The time evolution of a quantum system over an interval of duration  $t$  is described by the unitary operator  $U(t) = e^{-iHt}$  where  $H$  is the Hamiltonian governing the system.<sup>6</sup> These unitary operators form a group: Stone’s theorem ensures that as long as the Hamiltonian  $H$  is self-adjoint then it defines a strongly continuous one-parameter group of unitary operators  $U(t)$  (see, e.g. (Hall, 2013, chapter 10.2)). These unitary operators describe time evolution.

Quantum theories can fail to distinguish between past and future in each of the following two senses:<sup>7</sup>

1. Given a state  $|\psi(t)\rangle$ , the unitary dynamics determine both the earlier state  $|\psi(t - \tau)\rangle$  and the later state  $|\psi(t + \tau)\rangle$  for arbitrary  $\tau$ .

Quantum theories *always* fail to distinguish between past and future in this sense. It is true whether or not the theory is time-reversal invariant. It is a consequence of the fundamental fact secured by Stone’s theorem: the time evolution operators  $U(t)$  form a one-parameter group, which ensures that every  $U(t)$  has an inverse  $U(t)^{-1}$ . Given a quantum state  $|\psi(t = 0)\rangle$ , the operator  $U(t) = e^{-iHt}$  uniquely determines the future state  $|\psi(t)\rangle$  and its inverse  $U(t)^{-1} = e^{iHt}$  uniquely determines the past state  $|\psi(-t)\rangle$ . A consequence of this is that given any two quantum states related by unitary time evolution, we have  $|\alpha\rangle = U(t) |\beta\rangle$  and  $|\beta\rangle = U(t)^{-1} |\alpha\rangle$ . If one does not have any additional information about their temporal ordering then one cannot determine whether  $|\alpha\rangle$  causes  $|\beta\rangle$  or vice-versa.<sup>8</sup>

It is worth pausing here to address a natural question: isn’t it obvious that whichever evolution is generated by  $U(t)$  is the true one, with  $U(t)^{-1}$  generating the backward-in-time,

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<sup>6</sup>This presumes time-translation invariance. Time-translation invariance will always be assumed in this paper, a reflection of the fact that I am considering closed quantum systems.

<sup>7</sup>See (Earman, 2002, section 4) or (Farr and Reutlinger, 2013) for a discussion of the importance of distinguishing between the two when considering how the time-symmetry of a theory bears on our ability to distinguish cause and effect in microphysics.

<sup>8</sup>The distinct but related question of whether the two unitaries  $U(t)$  and  $U(t)^{-1}$  are equally implementable in practice is complicated (Janzing et al., 2002; Janzing, 2019; Janzing and Wocjan, 2018).

or acausal, evolution? The reason this does not work is that without antecedent information about the direction of time, one is really considering *two* candidate time variables:  $t$  and  $T$ , related by  $T = -t$ . Whether one considers  $U(t)$  as generating forward-in-time evolution or backward-in-time evolution depends on the time variable one chooses. The notation  $U(t)^{-1}$  is thus a bit misleading; writing the same unitary operator as  $U(T)$  makes the symmetry between  $t$  and  $T$  manifest.

To illustrate this by the example above, if one chooses  $t$  as the time variable then given an initial state  $|\psi(t=0)\rangle$ , the operator  $U(t) = e^{-iHt}$  uniquely determines the future state  $|\psi(t)\rangle$  and its inverse  $U(t)^{-1} = e^{iHt}$  uniquely determines the past state  $|\psi(-t)\rangle$ . Using  $T$ , the temporal ordering is inverted:  $e^{-iHT}$  uniquely determines the *past* state  $|\psi(-T)\rangle$  and its inverse  $U(T)^{-1} = e^{iHT}$  uniquely determines the *future* state  $|\psi(T)\rangle$ . Any attempt to determine the causal direction that requires first choosing between  $t$  or  $T$  would beg the question. (See (Donoghue and Menezes, 2019, 2020) for a pedagogical presentation and additional discussion of the facts above. In particular, see their interesting discussion of the fact that a choice of the time variable is fixed by the choice of a sign convention for the canonical commutation relations  $[x, p] = \pm i\hbar$ .)

In addition to the invertibility of the dynamics, a quantum theory may also be time-reversal invariant:<sup>9</sup>

2. There exists a time-reversal operator  $\mathcal{T}$  such that

- (a)  $\mathcal{T}$  commutes with the Hamiltonian of the theory,  $[\mathcal{T}, H] = 0$ , and
- (b) If the state  $|\psi_0(t=0)\rangle$  evolves under  $H$  as  $|\psi_0(t=0)\rangle, |\psi_1(t_1)\rangle, \dots, |\psi_n(t_n)\rangle$ , then the state  $\mathcal{T}|\psi_n(-t_n)\rangle$  evolves under  $\mathcal{T}H\mathcal{T}^{-1} = H$  as  $\mathcal{T}|\psi_n(-t_n)\rangle, \mathcal{T}|\psi_{n-1}(-t_{n-1})\rangle, \dots, \mathcal{T}|\psi_1(-t_1)\rangle, \mathcal{T}|\psi_0(t=0)\rangle$ .

It is important to note that the state  $|\psi_0(t=0)\rangle$  and the time-reversed state  $\mathcal{T}|\psi_0(t=0)\rangle$  will not, in general, be the same state. Momentum eigenstates (with  $p \neq 0$ ) provide a simple example: the time-reverse of the state  $|p\rangle$  is  $\mathcal{T}|p\rangle = |-p\rangle$  (up to a phase), so the state and its time-reverse are orthogonal:  $\langle p|\mathcal{T}|p\rangle = 0$ . Eigenstates of angular momentum observables for systems with non-integer spin exhibit the same behavior – both the spin observables  $\sigma_x, \sigma_y, \sigma_z$  and orbital angular momentum observables  $L_x, L_y, L_z$ .<sup>10</sup> One can

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<sup>9</sup>The appropriate understanding of time-reversal in quantum mechanics has received a fair amount of philosophical attention in recent years (Albert, 2000; Callender, 2000; Earman, 2002; Roberts, 2017; Allori, 2019; Farr, 2020; Donoghue and Menezes, 2019, 2020; Callender, 2020; Struyve, 2020). The initial stimulation for much of this work were the arguments for a non-standard definition of time reversal by (Albert, 2000) and (Callender, 2000). For reasons compactly summarized in (Roberts, 2019), I remain partial to the traditional account and will adopt it throughout this paper.

<sup>10</sup>This is a special case of the fact that the state of any degree of freedom that is *odd* under time-reversal – i.e.  $\mathcal{T}^2|\psi\rangle = -|\psi\rangle$  – is orthogonal to its time-reversed state (Sachs, 1987, section 3.2).

specify conditions that ensure that a quantum state and its time-reverse are the same state (for example (Earman, 2002, section 4) or (Sakurai and Napolitano, 2011, theorem 4.12)), but such cases are the exception, not the rule.

Time-reversal invariance of the laws of microphysics is often identified as an obstacle to interpreting microphysics causally, but I think it is a red herring.<sup>11</sup> Time-reversal invariance secures the following: if a sequence of states  $|\psi_a\rangle, |\psi_b\rangle, \dots, |\psi_z\rangle$  is dynamically allowed by unitary evolution under a Hamiltonian  $H$ , then a sequence of generally distinct states  $\mathcal{T}|\psi_z\rangle, \dots, \mathcal{T}|\psi_b\rangle, \mathcal{T}|\psi_a\rangle$  is also allowed by unitary evolution governed by the time-reversed Hamiltonian  $\mathcal{T}H\mathcal{T}^{-1} = H$ .

For this to be an obstacle to interpreting microphysics causally it would have to be the case that given two states  $|\psi_a\rangle$  and  $|\psi_z\rangle$  related by some unitary evolution  $U(t)$ , the time-reversal invariance makes it impossible to determine whether  $|\psi_a\rangle$  is the cause of  $|\psi_z\rangle$  or vice-versa. Time-reversal invariance is unnecessary for this; as I discussed above, this obstacle arises from the invertibility of the time-evolution operators  $U(t)$  and that follows from Stone's theorem, whether the theory is time-reversal invariant or not. Time-reversal invariance is also in general insufficient for this, except in the trivial sense that if quantum theory is time-reversal invariant, that entails that one was able to define a Hamiltonian for the system and that, in turn, gets us back to the real obstacle to determining whether  $|\psi_a\rangle$  is the cause of  $|\psi_z\rangle$  or vice-versa: the invertibility of  $U(t)$  ensured by Stone's theorem.<sup>12</sup> If, on whatever basis, I were to claim to have good reason to believe that  $|\psi_a\rangle$  is the cause of  $|\psi_z\rangle$ , it would do nothing to undermine my belief to tell me that the theory also allows for a physically distinct state  $\mathcal{T}|\psi_z\rangle$  to be the cause of another physically distinct state  $\mathcal{T}|\psi_a\rangle$ .

This is especially true for the situation I am considering in this paper: I am imagining that one *knows* the two states  $|\psi_a\rangle$  and  $|\psi_z\rangle$ . It is difficult to understand the belief that the difficulty with identifying whether  $|\psi_a\rangle$  causes  $|\psi_z\rangle$  or vice-versa stems from a fact about dynamically allowed sequences of states that are, in general, physically distinct and observationally distinguishable from  $|\psi_a\rangle$  and  $|\psi_z\rangle$ . The real difficulty with identifying cause and effect in quantum theories stems from the invertibility of the dynamics: one cannot distinguish between the two possibilities  $|\psi_a\rangle = U(t)|\psi_z\rangle$  and  $|\psi_z\rangle = U(t)^{-1}|\psi_a\rangle$  in the absence of information about the temporal ordering of  $|\psi_a\rangle$  and  $|\psi_z\rangle$ .

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<sup>11</sup>For two arguments for the same conclusion in classical and quantum statistical mechanics that have points of contact with the argument I offer here, but which are ultimately distinct, see (Maudlin, 2007, chapter 4) or (Myrvold, 2020).

<sup>12</sup>For a related point about the relationship between time-reversal invariance and determinism to the past and future and additional discussion, see (Earman, 2002, section 4).

### 3 Interactions Generate Entanglement

Use of the Schmidt decomposition of quantum states is ubiquitous and useful when discussing entanglement. I will rely on it throughout the paper so what follows is a brief reminder about some of its relevant properties.<sup>13</sup>

Any pure state  $|\Psi\rangle$  of a bipartite quantum system can be written as its Schmidt decomposition

$$|\Psi\rangle = \sum_i \lambda_i |a_i\rangle_A \otimes |b_i\rangle_B$$

where  $\{|a_i\rangle_A\}$  and  $\{|b_i\rangle_B\}$  form orthonormal bases for the Hilbert spaces  $\mathcal{H}_A$  and  $\mathcal{H}_B$ , respectively. The coefficients  $\lambda_i$  are called Schmidt coefficients and are non-negative real numbers that satisfy  $\sum \lambda_i^2 = 1$ . The *Schmidt rank* of  $|\Psi\rangle$  is the number of non-zero Schmidt coefficients  $\lambda_i$ ; a state is entangled iff it has a Schmidt rank greater than 1 and it is “fully” entangled – in a sense I will elaborate on in a moment – if its Schmidt rank is equal to  $\min\{\dim(\mathcal{H}_A), \dim(\mathcal{H}_B)\}$ . For simplicity I will consider systems where  $\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B)$ , but nothing in the paper depends on this.

If the Schmidt coefficients are non-degenerate, then the basis vectors  $|a_i\rangle_A$  and  $|b_i\rangle_B$  are unique (up to a phase) so the Schmidt decomposition of a state  $|\Psi\rangle$  is itself unique (up to a phase). If any of the Schmidt coefficients are degenerate then there is more freedom in the choice of basis vectors, but both the Schmidt rank and the values of the Schmidt coefficients are the same for any allowed choice of basis. A familiar example of this latter case is the EPR state. It can be Schmidt decomposed as

$$|\text{EPR}\rangle = \frac{1}{\sqrt{2}} |0\rangle_A |0\rangle_B + \frac{1}{\sqrt{2}} |1\rangle_A |1\rangle_B$$

where the Schmidt coefficients are  $\lambda_1 = \lambda_2 = \frac{1}{\sqrt{2}}$ . One could equally well have written the state in the Schmidt decomposition

$$|\text{EPR}\rangle = \frac{1}{\sqrt{2}} |+\rangle_A |+\rangle_B + \frac{1}{\sqrt{2}} |-\rangle_A |-\rangle_B$$

where  $|\pm\rangle = \frac{1}{\sqrt{2}} |0\rangle \pm \frac{1}{\sqrt{2}} |1\rangle$ . The values of the Schmidt coefficients are unchanged, while the non-uniqueness of the basis reflects the degeneracy of those coefficients.

This highlights a useful feature of the Schmidt decomposition: the Schmidt coefficients are invariant under unitary operations  $U_A \otimes U_B$  that act on Alice and Bob’s subsystems alone and, as an obvious corollary, the Schmidt rank is invariant too. This reflects the physical

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<sup>13</sup>See (Nielsen and Chuang, 2010, section 2.5) for a pedagogical presentation.

fact that entanglement between separated systems cannot be created by unitary operations performed locally on each system.<sup>14</sup> As a result, the set of Schmidt coefficients  $\lambda_i$  completely characterizes the entanglement of a bipartite system in a pure state and is sometimes referred to as the *entanglement spectrum*.

This brings me to the sense in which a bipartite pure state with full Schmidt rank can be considered “fully” entangled. Consider an entangled bipartite system  $|\Psi\rangle \in \mathcal{H}_{AB}$  with  $\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B) = N$ . If  $|\Psi\rangle$  has Schmidt rank 1 then it is separable and Alice and Bob’s measurement results will be uncorrelated for any measurements they perform. If  $|\Psi\rangle$  has Schmidt rank  $N$ , however, Alice and Bob’s results will be correlated for *every* measurement they perform in the same Schmidt basis on  $|\Psi\rangle$ : if Alice finds her system in state  $|a_k\rangle$  then Bob will find his in state  $|b_k\rangle$ , and so on. This is the sense in which the two subsystems are as entangled as they could be.

The Schmidt rank may seem like a rather coarse entanglement measure. It does not distinguish states that an apparently more fine-grained measure of entanglement, like the commonly adopted entanglement entropy, would distinguish. Consider the density operators  $\rho_1$  and  $\rho_2$  corresponding to the entangled states

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}}|11\rangle + \frac{1}{\sqrt{2}}|00\rangle$$

and

$$|\Psi_2\rangle = \sqrt{1-\varepsilon}|11\rangle + \sqrt{\varepsilon}|00\rangle$$

defined on  $\mathcal{H}_{AB}$  with  $\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B) = 2$ . Both  $\rho_1$  and  $\rho_2$  have the full Schmidt rank 2. However, the entanglement entropy of a density operator

$$EE(\sigma) = -\text{Tr}(\sigma \log \sigma) = -\sum_i \lambda_i^2 \log \lambda_i^2$$

distinguishes the two:  $EE(\rho_1) = 1$  while  $EE(\rho_2)$  is  $\mathcal{O}(\varepsilon)$ .<sup>15</sup> Why not use the apparently more fine-grained measure of entanglement?

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<sup>14</sup>It also reflects the mathematical fact that the amount of entanglement between two systems is independent of a unitary change of basis.

<sup>15</sup>The squares of the Schmidt coefficients  $\lambda_i^2$  in the Schmidt decomposition of a bipartite pure state  $|\Psi\rangle$  are the eigenvalues of the density operator  $\sigma = |\Psi\rangle\langle\Psi|$ . A convenient fact about the Schmidt decomposition is that the  $\lambda_i^2$  are *also* the eigenvalues of each of the reduced density operators  $\rho_A$  and  $\rho_B$  representing each entangled subsystem. Since these eigenvalues fully determine the entanglement entropy, the Schmidt decomposition reveals that one can calculate the entanglement entropy of the bipartite system in two equivalent ways: by computing the Shannon entropy of the probability distribution generated by the square of the coefficients of  $\sigma$  for the full bipartite system or, as is more common, computing the von Neumann entropy of the reduced density operators  $\rho_A$  or  $\rho_B$ .

My reasons are partially pragmatic. The first such reason is that such a measure is more fine-grained than I need for present purposes since the topological, measure-theoretic, and dynamical facts about quantum states that I invoke in the remainder of this section can be proved using only information about the Schmidt rank. The second is that there is virtue in adopting a measure of entanglement can be extended to multipartite systems and to mixed states. As a measure of entanglement, the Schmidt rank generalizes naturally to such cases: the Schmidt measure (Eisert and Briegel, 2001), (Hein et al., 2004, section II.c) or the Schmidt number (Terhal and Horodecki, 2000; Bruß, 2002; Sperling and Vogel, 2011) are generalizations of the Schmidt rank to multipartite pure and mixed states. In fact, the availability of the Schmidt measure will be important in section 5 when I consider a multipartite system of  $N$  qubits. A third reason, closely related to the second, is that the Schmidt measure has been used to give a definition of the algorithmic information content (also called the algorithmic complexity or the Kolmogorov complexity) of quantum states (Mora and Briegel, 2005, 2006). I will make use of this connection when I turn to causal inference in section 5.

My reasons are not entirely pragmatic, however, and there are two conceptual issues worth highlighting before moving on. There is a sense in which the Schmidt rank and its generalizations are not measures of precisely the same property as the entanglement entropy. Consider the following remarks from (Preskill, 1998) and (Bruß, 2002):

So a number used to quantify entanglement ought to have the property that local operations do not increase it. An obvious candidate is the Schmidt [rank], but on reflection it does not seem very satisfactory. Consider

$$|\Psi_\varepsilon\rangle = \sqrt{1 - 2|\varepsilon|^2} |00\rangle + \varepsilon |11\rangle + \varepsilon |22\rangle$$

which has Schmidt [rank] 3 for any  $|\varepsilon| > 0$ . Should we really say that  $|\Psi_\varepsilon\rangle$  is “more entangled” than  $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|11\rangle + |00\rangle)$ ? Entanglement, after all, can be regarded as a resource – we might plan to use it for teleportation, for example. It seems clear that  $|\Psi_\varepsilon\rangle$  (for  $|\varepsilon| \ll 1$ ) is a less valuable resource than  $|\phi^+\rangle$  (Preskill, 1998, chapter 5.5).<sup>16</sup>

Bruß draws a similar distinction as Preskill while introducing a method for determining the Schmidt number of an arbitrary mixed state:

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<sup>16</sup>The Schmidt *rank* of a bipartite pure state is sometimes referred to interchangeably as its Schmidt *number*. This is unfortunate since I will occasionally mention a quantity introduced by (Terhal and Horodecki, 2000) that they call the Schmidt number, which is an extension of the Schmidt rank to mixed states. I’ve altered Preskill’s terminology to cohere with mine in this paper.

A slightly different question from “how much entangled is a state  $\rho$ ?” can be addressed via the generalization of entanglement witnesses to so-called Schmidt witnesses. They give an answer to the question “how many degrees of freedom are entangled in  $\rho$ ?” This corresponds to a *finer classification* of entangled states (Bruß, 2002, section V.C).

It is not hard to find similar remarks in the literature on entanglement which suggest that rather than the entanglement entropy being a finer measure than the Schmidt rank or vice versa, the two instead capture subtly different aspects of entanglement.<sup>17</sup> Heuristically, the entanglement entropy quantifies how *strongly* subsystems are entangled while the Schmidt rank captures how *broadly* the subsystems are entangled. This distinction can track important practical differences in certain contexts: for example, it was proved in (Van den Nest, 2013) that the Schmidt rank is an informative measure of entanglement for determining the potential speedup of a quantum computation over a classical one, but any entanglement measure that is continuous and vanishes on product states – like the entanglement entropy – is not.

The second conceptual issue concerns a connection between measures of entanglement and measures of information that I will exploit when discussing causal inference.<sup>18</sup> The von Neumann entropy – what I have thus far been calling the entanglement entropy – can also be given an information-theoretic interpretation as a quantum analogue of Shannon information. The Shannon information does not quantify the information content of an individual message, but rather characterizes an *ensemble* of messages produced by a specified source that produces each bit  $x_i$  with some probability  $p_i$ . The Shannon information quantifies the minimum number of bits required to encode an arbitrary message selected from an ensemble of typical messages produced by the source. This is also true of the von Neumann entropy; a quantum source prepares qubits in pure states  $|x_i\rangle$  with probabilities  $p_i$  and the entanglement entropy quantifies the minimum number of qubits required to transmit a typical message selected from the ensemble of messages produced by the specified quantum source.

The causal inference methods I will use in the subsequent sections rely on the notion of *algorithmic* information.<sup>19</sup> This is not a measure of the amount of information required to describe a typical object in a specified ensemble but of the “intrinsic” amount of information

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<sup>17</sup>For example, see the remarks in (Sanpera et al., 2001) or, more substantively, the proof in (Van den Nest, 2013) mentioned below. This is not to say that there is never any relationship between the entanglement entropy and the Schmidt rank; for example, see (Sperling and Vogel, 2011, section III). For philosophical discussion of some of the multiple notions of entanglement see (Earman, 2015).

<sup>18</sup>See (Timpson, 2013, chapters 2.2 & 3.6) for a clear presentation and philosophical discussion of the interpretation of the Shannon and von Neumann entropies as measures of information.

<sup>19</sup>See (Grunwald and Vitányi, 2010) for a helpful comparison of Shannon information and algorithmic information.

contained in the object considered on its own. I will have more to say about algorithmic information in section 4 but what matters presently is that the algorithmic information content of an object is *not* a statistical property of an ensemble containing that object. This suggests that the entanglement entropy cannot provide a satisfactory measure of the algorithmic information content of an individual pure quantum state. The Schmidt rank of a quantum state is a property of that state alone and is thus a more appropriate measure for drawing a connection between entanglement and the algorithmic information content of a quantum state. (They do not motivate their choice in this way but this does make the use of the Schmidt measure by (Mora and Briegel, 2005, 2006) a conceptually natural choice for explicitly connecting entanglement and the algorithmic information content of a quantum state.)

With those justifications provided, I will now make use of the Schmidt rank to introduce some topological, measure-theoretic, and dynamical facts about the ubiquity of entangled pure states in the state space  $\mathcal{H}_{AB}$  of a bipartite quantum system. All of these facts are well-known; my aim in rehearsing them is to lay out the physical grounds of the causal asymmetry for which I argue at the end of this section, as well as to make maximally plausible the similar conclusions I draw using causal inference methods in section 5. If this discussion results in those conclusions seeming inevitable then it will have been successful.

First consider the case where  $\mathcal{H}_{AB}$  is finite-dimensional. Recall that the rank of a matrix is the number of linearly independent rows (or columns) it contains and that a matrix  $M$  of less-than-full rank has  $\det(M) = 0$ . This means that the density matrix corresponding to any bipartite pure state of less-than-full Schmidt rank has determinant equal to zero. One can use this fact to show that the set of pure states in  $\mathcal{H}_{AB}$  with less-than-full Schmidt rank is nowhere-dense in the set of all pure states in the topology induced by any norm on the space.<sup>20</sup> This means that each pure state in  $\mathcal{H}_{AB}$  of less-than-full Schmidt rank is enveloped by a ball of fully entangled pure states; equivalently, each pure state of less-than-full Schmidt rank is a limit point of a sequence of fully entangled pure states.

The set of pure states of less-than-full Schmidt rank is also sparse measure-theoretically: it has measure zero in the set of all pure states in  $\mathcal{H}_{AB}$ .<sup>21</sup> This is well-known but I couldn't find a proof to cite, so here is a proof sketch. Since  $\mathcal{H}_{AB} \cong \mathbb{C}^{N \times N}$ , the determinant  $\det: \mathbb{C}^{N \times N} \rightarrow \mathbb{C}$  is a polynomial in the complex entries of an  $N \times N$  matrix and

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<sup>20</sup>See (Brock, 2005) for an elementary proof. The fact that all norms on a finite-dimensional vector space are equivalent justifies the statement that this is true for "any norm" on  $\mathcal{H}_{AB}$ .

<sup>21</sup>In the finite-dimensional case, the restriction to pure states is important: for *mixed* states of a bipartite system, the separable states are no longer as sparse, and the set of separable mixed states always contains an open ball around the maximally mixed state (Życzkowski et al., 1998). For an infinite-dimensional Hilbert space, this is no longer true: as (Clifton and Halvorson, 1999) showed, the set of mixed states is nowhere dense in the set of all states, in the topology induced by the trace-norm.

any complex polynomial is a holomorphic function. The zero set  $Z(f)$  of any holomorphic function – the set of points in the domain of  $f$  on which that function equals zero – is at most countable (Rudin, 1987, theorem 10.18) and so the determinant equals zero on at most countably many points in  $\mathbb{C}^{N \times N}$ . The determinant of a density matrix vanishes if and only if it has less-than-full Schmidt rank so there can be at most countably many such density matrices. Using  $\mathbb{C}^{N \times N} \cong \mathbb{R}^{2N \times 2N}$  we can conclude that the set of density matrices with less-than-full Schmidt rank has Borel measure zero.

In light of the sparseness of separable pure states it may seem obvious that non-trivial time evolution on  $\mathcal{H}_{AB}$  – any time evolution that includes interactions between Alice and Bob’s subsystems – has to take any pure state  $|\psi\rangle_{AB}$  that is not fully entangled into a fully entangled state, and that the curve traced out by  $U(t)|\psi\rangle_{AB}$  will consist almost entirely of fully entangled states.<sup>22</sup> This is true.<sup>23</sup> In fact, the previous discussion ensures that any initial bipartite state with less-than-full Schmidt rank will, after an infinitesimal time interval, develop into a state with full Schmidt rank and will then remain in a state of full Schmidt rank for all subsequent times except for a set of measure zero.

This should not be surprising at this point. The topological sparseness of pure states of less-than-full Schmidt rank means that there is nowhere else for those states to evolve except into states with full Schmidt rank, at least for infinitesimal time evolution, and their measure-theoretic sparseness means that any curve through  $\mathcal{H}_{AB}$  generated by time evolution for *any finite time interval* will pass through a state of less-than-full Schmidt rank in at most countably many instants. The upshot of this is that if one chooses an arbitrary state  $|k\rangle$  of less-than-full Schmidt rank  $k$  and evolves it for any finite time  $t$ , except for set of instants of measure zero the state  $U(t)|k\rangle$  will be a state of full Schmidt rank.

Now suppose instead that one is interested in the time evolution of an arbitrary state  $|N\rangle$  of full Schmidt rank  $N$ . Are there any time evolution operators  $U(t)$  such that  $U(t)|N\rangle$  has less-than-full Schmidt rank? Since specifying a Hilbert space representation for a quantum system requires fixing a Hamiltonian  $H$ , the only freedom in choosing  $U(t)$  comes from fixing  $t$ .<sup>24</sup> Thus this question is equivalent to asking how many instants  $t_i$  there are such

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<sup>22</sup>I will drop the “non-trivial” qualifier for the remainder of the paper. Considering time evolutions that are non-trivial in this sense is equivalent to requiring that the time evolution operator  $U(t)$  does not factorize into a product  $U(t)_A \otimes U(t)_B$  of operators evolving Alice and Bob’s subsystems independently. Any unitary operator that factorizes in this way cannot change the Schmidt coefficients of a quantum state, as I mentioned previously, and so cannot create entanglement between subsystems.

<sup>23</sup>A general argument is sketched in (Binney and Skinner, 2013, section 6.1). There are an immense number of concrete examples of dynamically generated entanglement; see, for instance, the discussions in many-body physics (Calabrese and Cardy, 2006; Eisert and Osborne, 2006; Amico et al., 2008), quantum field theory (Peschanski and Seki, 2016; Cervera-Lierta et al., 2017; Kharzeev and Levin, 2017), and non-relativistic quantum mechanics (Mishima et al., 2004; Schroeder, 2017).

<sup>24</sup>This is clearest in the algebraic approach to quantum theories, where a quantum system is defined by

that  $U(t_i)|N\rangle$  has less-than-full Schmidt rank.

One such operator can obviously be reverse-engineered. Find a state of less-than-full Schmidt rank  $|k\rangle$  such that  $U(t)|k\rangle = |N\rangle$ . Applying  $U(t)^{-1}$  to  $|N\rangle$  will then produce  $|k\rangle$ , a state of less-than-full Schmidt rank. Are there more such operators?

There can be at most countably many. Let  $\tau_R$  be the recurrence time for the system and consider the operator  $U(\tau_R)$ .<sup>25</sup> Suppose that one starts with the state  $|N\rangle$  and that the dynamics are such that  $|N\rangle$  visits *every* state of less-than-full Schmidt rank in  $\mathcal{H}_{AB}$  between  $t = 0$  and  $t = \tau_R$ . Then the curve  $U(\tau_R)|N\rangle$  will visit countably many states of less-than-full Schmidt rank on its tour through  $\mathcal{H}_{AB}$ . Any finite time interval  $t$  can be split up into at most  $n$  intervals of length  $\tau_R$  (plus some remainder  $q$ ) so we can write  $U(t) = U(n\tau_R + q)$  and, since a finite union of countable sets is itself countable, the state  $|N\rangle$  evolves into a state of less-than-full Schmidt rank for at most countably many instants. This means that the set of times  $t_i$  for which  $U(t_i)|N\rangle$  has less-than-full Schmidt rank has measure zero.

Recall that I began with a simple problem: if one is given two bipartite pure states  $|\alpha\rangle$  and  $|\beta\rangle$  and knows that they are related by unitary time evolution, can they determine whether  $|\alpha\rangle$  was the cause of  $|\beta\rangle$  or vice versa? My discussion of the sparseness of pure states of less-than-full Schmidt rank in  $\mathcal{H}_{AB}$  lays the groundwork for an answer. Suppose that either of the following is true:<sup>26</sup>

1. *The state  $|\alpha\rangle$  is separable (has Schmidt rank 1) and  $|\beta\rangle$  is entangled (has Schmidt rank  $> 1$ );*

or

2. *Both of the states  $|\alpha\rangle$  and  $|\beta\rangle$  are entangled but  $|\alpha\rangle$  has less-than-full Schmidt rank and  $|\beta\rangle$  has full Schmidt rank.*

It is then overwhelmingly likely that  $|\beta\rangle$  was caused by time-evolving  $|\alpha\rangle$  rather than vice versa.

The intuitive reason for this is that it is overwhelmingly likely that a state of low Schmidt rank will evolve into a state of higher Schmidt rank but the converse is extremely unlikely. If

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a  $C^*$  algebra. The GNS reconstruction theorem enables one to move from the abstract  $C^*$  algebra to a Hilbert space representation for the system, but this requires data about the full  $C^*$  algebra, including the Hamiltonian.

<sup>25</sup>See (Wallace, 2015) for proofs that in any finite-dimensional quantum system, and any infinite-dimensional quantum system that satisfies modest constraints on the Hamiltonian, there exists a single time  $\tau_R$  such that after  $\tau_R$  *every* state in  $\mathcal{H}_{AB}$  will have returned to a state arbitrarily close to itself.

<sup>26</sup>Note that this excludes the case where both  $|\alpha\rangle$  and  $|\beta\rangle$  are entangled but neither has full Schmidt rank. I'm not aware of any demonstration that within the set of bipartite pure states of less-than-full Schmidt rank, the states of lower rank are less prevalent than those of higher rank. The causal inference methods used in section 5 will do better for this case.

the state  $|\alpha\rangle$  is the cause then it is almost certain that the state  $U(t)|\alpha\rangle$  would have higher Schmidt rank, for an arbitrarily chosen  $t$ . One does not have to know anything about the dynamics – about how  $|\alpha\rangle$  would evolve over different intervals  $t$  – to be essentially certain that the resultant state will have full Schmidt rank. If one evolved the state  $|\alpha\rangle$  for  $t + dt$  or  $t + 10^{23}$ , or  $t + \pi t$ , or . . . the resultant state would not be  $|\beta\rangle$ , but one can be essentially certain that it would be a fully entangled state. That it happens to be  $|\beta\rangle$  in particular reflects the unmysterious fact that one evolved the state  $|\alpha\rangle$  for  $t$  rather than some other interval of time. In short, identifying  $|\alpha\rangle$  as the cause renders the effect  $|\beta\rangle$  utterly unsurprising.

If instead one adopts  $|\beta\rangle$  as the cause, then the effect  $|\alpha\rangle$  is rendered *fantastically* surprising. It is impossibly unlikely that  $|\beta\rangle$  just happened to evolve for precisely an interval  $t$  lying in the measure-zero set of times that would result in  $U(t)|\beta\rangle$  having less-than-full Schmidt rank. It suggests an incredible degree of fine-tuning; indeed, it is nearly impossible to imagine a human experimenter who was *trying* to accomplish such a precise fine-tuning doing so successfully. Even with complete knowledge of both the state  $|\beta\rangle$  and its behavior under the dynamics – how it would evolve under  $U(t)$  for different  $t$  – and the explicit goal of producing a state of lower-Schmidt rank, they would have to control the experiment with a level of precision typically reserved for deities.<sup>27</sup> In fact, insofar as one thinks that a cause ought to explain its effects then identifying  $|\beta\rangle$  as the cause is unacceptable: it not only fails to explain  $|\alpha\rangle$  in any meaningful sense but also raises additional explanatory concerns more urgent than those with which one began.

Some readers may have noticed a parallel with classical statistical mechanics: precisely this form of reasoning has been used in that context to argue that there is an identifiable explanatory asymmetry between two microstates despite the time-symmetry of the dynamical laws describing their time evolution. For example, (Maudlin, 2007, p. 132) introduces the problem as follows:

. . . postulate a macroscopically atypical [low entropy] but microscopically typical [chosen at random from microstates compatible with the atypical macrostate] state, plus the laws, and one can explain the macroscopically typical [high entropy] but microscopically atypical [evolves to a lower entropy macrostate in one

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<sup>27</sup>An observation about the impracticality of arranging a similar evolution was made in a different context by (Englert et al., 1988). They consider sending a spin-1/2 particle in a  $\sigma_x$  eigenstate through a Stern-Gerlach device set to measure  $\sigma_z$ , which will split the incoming beam into a superposition of the two eigenstates of  $\sigma_z$ . They investigate the degree of precision with which an experimenter would need to control the magnetic field in the Stern-Gerlach device to ensure that the spin-1/2 particle returns to its original  $\sigma_x$  eigenstate after passing through the Stern-Gerlach device. They show that an *exact* return to the original  $\sigma_x$  eigenstate is unattainable in practice and that even to reproduce the original state with 99% accuracy would require the ability to control the gradient of the macroscopic magnetic field in the Stern-Gerlach device to *at least* 5 decimal places.

temporal direction] state from them: the latter was generated from the former by means of the operation of the laws. But equally: postulate a macroscopically typical but microscopically atypical state at one end, plus the laws, and one can ‘generate’ a macroscopically atypical but microscopically typical state from them. Pick one end, add the laws, and you can explain the other end: which end you pick as *explanans* and which as *explanandum* is up to you.

Maudlin argues that this apparent explanatory symmetry is illusory, invoking a principle that lies at the foundation of the causal inference methods I discuss in section 4. He points out that the asymmetry is hiding in the way that one specifies the *explanans* microstate and the *explanandum* microstate in the two candidate explanations. Consider explaining an atypical microstate in a high-entropy macrostate by demonstrating that it resulted from the time evolution of a typical microstate in a low-entropy macrostate. The meaning of “typical” used to describe the *explanans* is measure-theoretic: the low-entropy macrostate is atypical because it occupies a small volume of the system’s phase space and the microstate is typical because it was chosen at random from that low-entropy macrostate. Importantly, one can entirely specify an *explanans* that will produce the desired *explanandum* while knowing basically nothing about the dynamics governing the system.

This is no longer true if one tries to invert the order of explanation. If one chooses as the *explanans* an atypical microstate in a high-entropy macrostate, the meaning of “atypical” becomes crucially different. “Atypical” now means *dynamically* atypical: the microstate is atypical in the new sense if its time evolution takes it into a low-entropy macrostate. Such atypical microstates are scattered throughout the volume of the high-entropy macrostate and share no common property that could be used to identify them except their dynamical behavior. Specifying an atypical microstate in this sense requires thorough knowledge of the dynamics governing its evolution. Not only can such microstates only be specified by describing their dynamical behavior, but their identity *as* atypical microstates depends very sensitively on the specific form of the dynamics governing the system. As Maudlin points out,

... a slight modification of the dynamical laws would lead to essentially no change in which initial states are macroscopically atypical, in that they have low entropy, *but would completely alter the set of atypical high-entropy states whose time evolution in either direction leads to low entropy* (Maudlin, 2007, p. 133) [my emphasis].

The explanatory asymmetry comes from this asymmetry in how we have to specify the *explanans* and the *explanandum* in the two cases. In one direction, one can randomly

pick a microstate from a low-entropy macrostate, knowing essentially nothing about the specific dynamics governing the system, and then show that those dynamics will evolve that *explanans* microstate into a microstate lying in a high-entropy macrostate. In the other direction, one needs detailed knowledge about the specific form of the dynamics to even know what counts as an atypical microstate, let alone to select as their *explanans* an atypical microstate that the dynamics will evolve into a low-entropy macrostate. Maudlin argues that this is no explanation at all: it amounts to explaining the system’s evolution into a low-entropy macrostate by saying that the initial conditions were such that they would evolve into a low-entropy macrostate, given the dynamics governing the system. One might as well ascribe such microstates a *virtus entropia*.<sup>28</sup>

The parallel strategy for using entanglement to identify explanatory asymmetries between quantum states is clear. The set of all states in  $\mathcal{H}_{AB}$  can be partitioned into disjoint subsets containing states with Schmidt rank 1, 2,  $\dots$ ,  $\dim(\mathcal{H}_{AB})$ . We know from the above discussion that almost all states in  $\mathcal{H}_{AB}$  will be in the subset of states with full Schmidt rank; call a quantum state in  $\mathcal{H}_{AB}$  “macroscopically atypical” if it lies in any set of states with less-than-full Schmidt rank. A state *within* that set is “microscopically typical” if it is chosen from that set at random. A state is “macroscopically typical” in  $\mathcal{H}_{AB}$  if it is in the set of states with full Schmidt rank; a state within that set is “microscopically atypical”, given a specific time-evolution operator  $U(t_i)$ , if it evolves into a state of less-than-full Schmidt rank.

The discussion above establishes that time evolution from full Schmidt rank to less-than-full Schmidt rank is highly non-generic; the only way to specify an *explanans* state  $|\phi\rangle$  that would do so under  $U(t_i)$  requires extremely detailed knowledge of both the details of the Hamiltonian  $H$  appearing in  $U(t_i)$  and the duration of time evolution  $t_i$ . Just like in the classical statistical mechanical case, “explaining” a state  $|\phi_1\rangle$  of less-than-full Schmidt rank as resulting from the time evolution of a state  $|N\rangle$  of full Schmidt rank amounts to saying that  $|\phi_1\rangle$  occurred because the initial condition  $|N\rangle$  was such that it would evolve to a state of lower Schmidt rank, given the dynamics  $U(t_i)$ .

Maudlin extracts from his discussion a principle for evaluating explanations, though he doesn’t dwell on it:<sup>29</sup>

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<sup>28</sup>Sklar expresses a similar dissatisfaction with such candidate explanations of why subsystems of a larger system obey the second law: “. . . we would simply posit an initial state that gives rise to parallel entropic increase in branch systems with each other and with the main system. But to characterise the state in that way would, of course, not be offering us an explanation of the sort we expected. It would be one thing to be able to characterize the initial state in some simple way. . . and be able to derive the Second Law from that. But to derive the Second Law from a bald assertion that “initial conditions were such that they would lead to Second Law behavior” hardly seems of much interest” (Sklar, 1993, p. 330).

<sup>29</sup>Although Maudlin doesn’t dwell on it, (Woodward, 2020) has given an extended and enlightening examination of the very similar principle that forms the foundation of the causal inference methods I describe in section 4.

The problem is this: in order to account for the universe as we see it, we need more than the laws: we need a constraint on one of the boundaries. That constraint, together with the operations of the laws, then suffices to account for the nature of the other boundary. But in order for this to work [be explanatory] *the constraint must itself be specifiable independently of what will result from the operation of the laws* (Maudlin, 2007, p. 132).

I think there is an argument to be made at this point that as long as two quantum states satisfy one of the two conditions outlined above, my initial goal of identifying a causal asymmetry between quantum states related by time-symmetric dynamics has been accomplished. Recall that I have adopted a minimal interventionist notion of causation in this paper: X is a cause of Y iff there is an intervention that can be performed on the value of X, while holding all other variables fixed, that produces a change in the value of Y. Two quantum states  $|\alpha\rangle$  and  $|\beta\rangle$  related by a unitary time evolution  $U(t)$  satisfy this condition, but they satisfy it for both candidate causal orderings: intervening to set  $|\alpha\rangle$  to  $|\alpha'\rangle$  will produce a change in  $|\beta\rangle$ , since  $U(t)|\alpha'\rangle \neq |\beta\rangle$ . But this is also true if the candidate causal order is inverted: intervening to set  $|\beta\rangle$  to  $|\beta'\rangle$  will produce a change in  $|\alpha\rangle$  since  $U(t)^{-1}|\beta'\rangle \neq |\alpha\rangle$ .

Some additional information is required to break the symmetry. The above discussion suggests that, under conditions widely satisfied in quantum theories, one has such information: one should identify the “cause” state as the state that can produce the effect state *and* can be specified without any reference to its behavior under the dynamics  $U(t)$ . In the case under consideration, this amounts to identifying the “cause” state as either (1) the state with Schmidt rank 1, if the other state has Schmidt rank  $> 1$  or (2) the state with less-than-full Schmidt rank, if the other state has full Schmidt rank. Many people have claimed that the time-symmetry of the dynamics governing microscopic systems makes any such asymmetry impossible; I have argued that this is not correct.

That said, I want to emphasize some limitations of my discussion thus far. First, I have discussed only microscopic systems *par excellence*: pure states of bipartite systems related by unitary evolution. Second, conditions (1) or (2) are satisfied for a pair of states only when one of those states comes from a set of measure zero: the set of states of less-than-full Schmidt rank. It is true that such states, particularly product states, play a more significant role in foundational and mathematical discussions of quantum mechanics than their sparseness in  $\mathcal{H}_{AB}$  alone might suggest, but they are sparse nonetheless. I invoked an analogy with the case of classical statistical mechanics above, but this is an important disanalogy: the measure of the set of low-entropy microstates is exponentially small and vanishes as the size of the system  $N \rightarrow 0$ , but it is non-zero for any finite system; for estimates and discussion of classical and quantum multipartite systems (including exceptions to this estimate), see (Goldstein et al.,

2015b) and (Goldstein et al., 2017, section 7). I will return to this disanalogy in section 5.

The final limitation is that while conditions (1) and (2) show that one can, in principle, identify a causal asymmetry in quantum theories, they are of little practical value for the real-world epistemology of causation. The argument presented in this section does identify an asymmetry present in the *mathematical foundations* of quantum theories, but it relies on a standard of precision that is unsatisfiable in practice. I have been considering states that are *exactly* product states or, more generally, states whose Schmidt decomposition has *exactly*  $k$  non-zero coefficients. This presumption of precision is often made without comment in foundational discussions but is never achievable – or at least never verifiably achievable – in any real world situation. At best, one can verify that a state is *approximately* a product state or that a state has  $k$  Schmidt coefficients larger than some  $\varepsilon > 0$ . However, the topological sparseness of the states of less-than-full Schmidt rank means that if  $|\psi\rangle$  has less-than-full Schmidt rank, any state  $|\phi\rangle$  that is within  $\varepsilon$  of  $|\psi\rangle$  in the Hilbert space norm will have full Schmidt rank. Conditions (1) and (2) no longer identify a causal asymmetry if modified to

1 $_{\varepsilon}$ : The state  $|\alpha\rangle$  is *within  $\varepsilon$  of a state that is* separable (has Schmidt rank 1) and  $|\beta\rangle$  is *within  $\varepsilon$  of a state that is* entangled (has Schmidt rank  $>1$ ), or;

2 $_{\varepsilon}$ : Both of the states  $|\alpha\rangle$  and  $|\beta\rangle$  are entangled but  $|\alpha\rangle$  is *within  $\varepsilon$  of a state that has* less-than-full Schmidt rank and  $|\beta\rangle$  is *within  $\varepsilon$  of a state that has* full Schmidt rank.

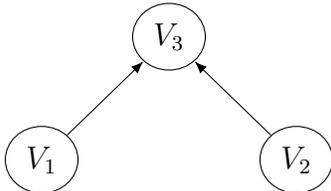
In section 5 I will show that one can do better than this while still making use of a principle much like the one Maudlin employed to identify an explanatory asymmetry in classical statistical mechanics and which I have used in quantum theories. Such a principle provides part of the foundation of a set of mathematical and conceptual methods for causal inference that have proven empirically reliable in a number of disparate domains. My aim is to make use of those methods to give a formal demonstration that the presence of entanglement can be used to infer causal direction in quantum theories under conditions that are less restrictive than those invoked in this section.

## 4 Causal Inference Methods

I began with the problem of determining whether, and under what conditions, one could identify a causally relevant asymmetry between quantum states related by unitary evolution. Faced with such a problem, it would be natural to turn to the tools of causal inference. Unfortunately, if one takes up commonly used tools to infer causal relationships from observational data that rely solely on conditional statistical independence, such as those in (Pearl,

2009, chapter 2), they will find those tools inadequate for the problem at hand. The reason for this is fairly straightforward.

Any causal graph relating classical statistical variables  $V_1, V_2, \dots, V_N$  defines a joint probability distribution  $P(V_1, V_2, \dots, V_N)$  over those variables.<sup>30</sup> Consider a simple graph relating  $V_1, V_2$ , and  $V_3$ :



This graph defines a set of possible joint probability distributions  $P(V_1, V_2, V_3)$ . Suppose that all causal relations are deterministic; then possible joint distributions could be  $P(V_1=1, V_2=0, V_3=1)=1$ ,  $P(V_1=0, V_2=0, V_3=1)=0$ , and so on. Causal inference problems begin with a joint probability distribution over variables  $V_1, V_2, \dots, V_N$  and attempt to reconstruct the causal graph representing the true causal structure that generates the given probability distribution.

For each candidate causal graph, one typically requires that a probability distribution satisfies two conditions relative to that graph: the Causal Markov Condition and a faithfulness condition.<sup>31</sup> The Causal Markov Condition is a constraint on how the joint probability distribution  $P(V_1, V_2, \dots, V_N)$  factorizes into a product of conditional dependence relations according to a candidate causal graph. Speaking loosely, one requires that if all of the direct causes of a variable  $O$  in the causal graph are specified, then one cannot learn any additional information about the value of  $O$  from any variable that is not itself a descendent of  $O$ . Formally, one requires that any candidate causal graph satisfy:

$$P(V_1, \dots, V_N) = \prod_{k=1}^n P(V_k \mid PA(V_k))$$

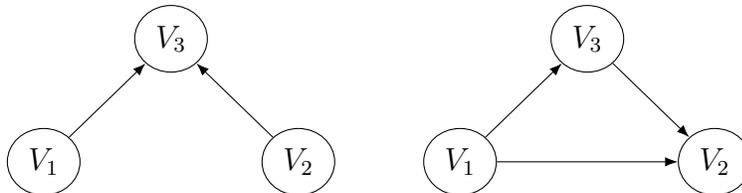
where  $PA(V_k)$  represents the direct causes (or “parents”) of the variable  $V_k$ . This captures the requirement that all of the conditional dependence relations in the joint distribution  $P(V_1, V_2, \dots, V_N)$  be accounted for by causal relations between  $V_1, V_2, \dots, V_N$  in the causal graph.

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<sup>30</sup>The restriction to *classical* statistical variables is important because a joint probability distribution for non-commuting variables, like conjugate observables in quantum mechanics, is generally not well-defined. If one restricts to quantum observables that commute then one can define a joint probability distribution over their possible values. See (Fine, 1982) for a review and connection to hidden variable theories.

<sup>31</sup>See, for example, (Peters et al., 2017, chapter 6.5), (Spirtes et al., 2000, chapter 3), or (Pearl, 2009, chapter 2.4), where the faithfulness condition is called “stability”.

Faithfulness requires that *only* the conditional dependence relations in the joint probability distribution are reflected in the causal relations between  $V_1, V_2, \dots, V_N$  in the causal graph. The idea is easiest to illustrate with an example. Suppose one knows the joint distribution  $P(V_1, V_2, V_3)$  and suppose that  $V_1 \perp\!\!\!\perp V_2$ , but  $V_1 \not\perp\!\!\!\perp V_2 \mid V_3$ . (Notation:  $\perp\!\!\!\perp$  indicates statistical independence and  $\not\perp\!\!\!\perp$  indicates statistical dependence.) This is consistent with both of the following causal graphs:



Further suppose that, in the graph on the right, the causal influence of  $V_1$  on  $V_2$  is precisely canceled by the causal influence of  $V_3$  on  $V_2$ . Then both of these graphs entail  $V_1 \perp\!\!\!\perp V_2$  and  $V_1 \not\perp\!\!\!\perp V_2 \mid V_3$ , but there is something unsatisfying about the graph on the right: the required conditional dependence relations have been recovered only by fine-tuning causal influences to cancel precisely. This is why (Pearl, 2009, chapter 2.4) calls this a stability condition: holding fixed the strength of the causal influence  $V_1$  on  $V_2$  while perturbing the strength of the causal influence of  $V_3$  on  $V_2$  by any  $\delta > 0$  will destroy the conditional independence relations. It is these kind of finely-tuned graphs that are ruled out by faithfulness.<sup>32</sup>

In short, if a probability distribution satisfies the Causal Markov and Faithfulness conditions relative to a candidate causal graph that entails all, and only, the conditional dependence relations present in the joint distribution  $P(V_1, V_2, \dots, V_N)$  are reflected in the causal structure of the graph. If a joint probability distribution satisfies these two properties relative to multiple graphs, those graphs are said to form a *Markov equivalent set*.

One can now see why these conditions are insufficient for causal inference with two variables  $V_1$  and  $V_2$ . Suppose one already knows that any statistical dependence between the two variables is due to a direct causal relationship between them, allowing them to rule out confounders. Even then, the above conditions are insufficient to identify whether  $V_1$  causes  $V_2$  or vice versa. One is stuck with the following Markov equivalence class of graphs:

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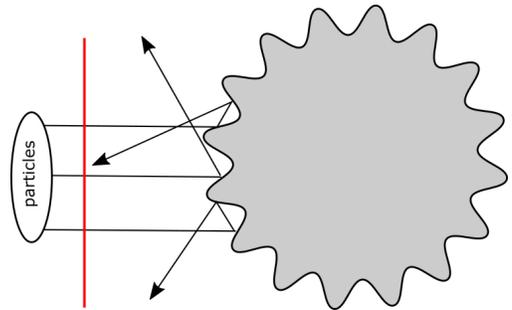
<sup>32</sup>Faithfulness is often motivated by the fact that the set of parameter values quantifying causal influence that produce this type of cancellation are measure zero in the set of all parameter values Meek (1995), (Spirtes et al., 2000, theorem 3.2). For a clarifying discussion of alternative justifications for imposing faithfulness, see (Weinberger, 2018).



The reason is simple: any causal graph relating the two variables  $V_1$  and  $V_2$  will be fully connected, and fully connected graphs do not predict any conditional independence relations. This is why causal inference from conditional independences in observational statistical data, using only the Causal Markov Property and Faithfulness, can successfully identify the causal relationship between two variables only if they are embedded in a larger set of at least three variables.

Happily, it turns out that one can do better than this if they are willing to introduce additional conditions on a satisfactory causal graph. A set of causal inference strategies has recently been developed that aims to address the problem of causal inference from observational data for only two variables (Janzing and Schölkopf, 2010; Daniušis et al., 2010; Janzing et al., 2012; Peters et al., 2017). The central condition they introduce is simple: given two candidate causal graphs that generate the joint distribution  $P(V_1, V_2)$ , the true causal graph is the one that entails that the distribution for the “cause” variable  $P(V_1)$  does not contain any information about the conditional distribution  $P(V_2 | V_1)$  for the “effect” variable, and vice versa. A perhaps more intuitive way to describe this condition is as follows: the mechanism that determines the probability distribution over the cause variable operates independently of the mechanism that determines the conditional distribution over the effect variable, given the distribution over the cause; (Peters et al., 2017, chapter 2) label this “The Principle of Independent Mechanisms”.

The range of contexts in which such a principle is justified has been given an enlightening examination in (Woodward, 2020), but is quite reasonable in the particular context of physical theory: one can specify an initial “cause” state – or a probability distribution over initial “cause” states – without any detailed knowledge of the dynamical mechanism that will take that initial state as input and output a conditional distribution over final, or “effect”, states. If that is possible according to the way  $P(V_1, V_2)$  factorizes according to one candidate causal graph but impossible according to the second, then it seems quite reasonable that the first causal graph is the correct one.



**Figure 1:** A jagged potential.

One can illustrate the principle with a simple example. Suppose one prepares a beam

of  $N$  classical particles and scatters them off a jagged potential like the one in Figure 1.<sup>33</sup> Draw a line that all  $N$  particles will cross, both before they encounter the potential *and* after they are scattered back off it, some distance away from the region where the potential is non-zero. Repeat the experiment many times and record the position of the particles each time they cross the line on their way in and again on their way out. This will produce a joint probability distribution over the positions of the  $N$  particles  $P(X_{\text{in}}, X_{\text{out}})$ , and the scattering potential is the mechanism that determines the conditional distributions  $P(X_{\text{in}} | X_{\text{out}})$  and  $P(X_{\text{out}} | X_{\text{in}})$ . The causal inference problem is whether one could infer that one distribution is the cause and the other the effect *in the absence of any temporal information*.

The Causal Markov and Faithfulness conditions are insufficient, as discussed above, but it is easy to see how the Principle of Independent Mechanisms can make this possible. The distribution  $P(X_{\text{out}})$  will be highly disordered, with the positions of the  $N$  particles distributed roughly randomly across the line. However  $P(X_{\text{in}})$  will be quite uniform: the beam of  $N$  particles will cross the line at roughly the same positions on each run of the experiment. For  $P(X_{\text{out}})$  to be the cause distribution, it would have to be such that the potential would funnel the random distribution into a highly ordered one. This would be impossibly unlikely given the geometry of the scattering potential unless each state of the incoming beams of  $N$  particles had been *extremely* carefully fine-tuned to be funnelled into a more ordered state by that particular scattering potential. Any such preparation procedure would require extremely detailed information about the geometry of the mechanism determining  $P(X_{\text{in}} | X_{\text{out}})$  – the scattering potential – and so identifying  $P(X_{\text{out}})$  as the cause would violate the Principle of Independent Mechanisms.

A natural precisification of this principle uses algorithmic information theory: the true causal graph should render the *mutual information*  $I(P_{\text{cause}} : P_{\text{effect}|\text{cause}})$  between the probability distribution over the cause and the conditional distribution over the effect equal to zero. This can be stated in an equivalent form that is more transparent in the context of physical theory: the true causal graph should render equal to zero the mutual information  $I(s : M)$  between the “cause” state  $s$  and the dynamical mechanism  $M$  that determines the conditional distribution over the effect, given  $s$ . In fact, Janzing, Schölkopf, and collaborators have constructed a framework for causal inference that is founded on the use of the tools of algorithmic information theory in which The Principle of Independent Mechanisms plays a foundational role.

A valuable feature of this framework is that it is *non-probabilistic*: its foundational concepts are those of algorithmic information theory, not statistics (Janzing and Schölkopf,

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<sup>33</sup>This is roughly a combination of two different examples from (Peters et al., 2017, chapter 2) and (Janzing et al., 2016).

2010; Peters et al., 2017). This is not to say that it somehow doesn't work for the more familiar cases of causal inference from statistical data; in fact, the examination of algorithmic informational dependencies can reveal novel statistical dependencies entailed by a causal graph (Janzing and Schölkopf, 2010, section 3). The point is that it also applies more generally, enabling causal inference between objects even when one does not have statistical data about those objects, like a joint probability distribution over their values. Although I do not rely on it here, this may prove valuable for doing causal inference in quantum theory where joint probability distributions are not guaranteed to be well-defined (see fn. 30).

I will begin with a brief introduction to some relevant concepts of algorithmic information theory.<sup>34</sup> Suppose one has a countable set of objects  $\{\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_n, \dots\}$  and a system for identifying each object  $\mathcal{O}_i$  by a binary description  $s_i$ . Note that “objects” here is quite broad: it can include things like probability distributions, states of a physical system, PDF files, *Blood Meridian*, Gila monsters, surfboards, etc. The algorithmic information content of an object  $\mathcal{O}_i$  is meant to quantify the amount of information required by the *shortest* complete description of the object  $\mathcal{O}_i$ .

There will always be some shortest binary string that uniquely identifies an object  $\mathcal{O}$ , denoted  $s^*$ . The algorithmic information of  $\mathcal{O}$  (also called the algorithmic complexity or Kolmogorov complexity) is the length  $\ell(p)$  of the shortest program  $p$  that, if run on a (prefix-free) universal Turing machine  $U$ , would output the string  $s^*$  and halt. Formally:<sup>35</sup>

$$K(s) = \min_p \{\ell(p) \mid U(p) = s\}$$

Note that the upper bound on  $K(s)$  for an  $n$ -bit string is  $n$  since one can always write a program of the form “Print  $s_1, \dots, s_n$ ” that reproduces the string  $K(s)$  bit-by-bit. This illustrates one sense in which  $K(s)$  is a measure of the information contained in the object  $\mathcal{O}$ : it captures how much information is required to algorithmically reconstruct its complete binary description.<sup>36</sup>

One can similarly define the conditional algorithmic information of one object, given a second. Let  $t$  and  $s$  be the shortest binary descriptions of objects  $\mathcal{O}_t$  and  $\mathcal{O}_s$ . Then the

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<sup>34</sup>I very loosely follow (Janzing and Schölkopf, 2010, section 2.1) here. See (Li and Vitányi, 2019) for a textbook introduction.

<sup>35</sup>In algorithmic information theory, equalities are generally only equalities up to a constant that is independent of the object itself, but may depend on the alphabet or programming language chosen for the encoding or the particular universal Turing machine being considered. For example, a program to output *Hamlet* may be shorter when written in Python than in FORTRAN. This tells us something about Python and FORTRAN, but nothing about the algorithmic information content of *Hamlet* itself. This “equality up to a constant” is denoted by  $\pm$

<sup>36</sup>Note that this has the somewhat counterintuitive consequence that a binary sequence that is completely random has maximal algorithmic information.

conditional algorithmic information of  $\mathcal{O}_s$  given  $\mathcal{O}_t$ , denoted  $K(s | t)$ , is defined as the length of the shortest program that takes as input  $t$  then generates  $s$  as output and halts. This measures how much information about  $\mathcal{O}_s$  one obtains if given  $\mathcal{O}_t$ , and thus how much computational work is saved by knowing  $t$  when computing  $s$ . If  $t$  contains no information about  $s$  then  $K(s | t) = K(s)$ .

The conditional algorithmic information allows a precise definition of the natural intuition that a cause does not contain information about the mechanism that maps it to the effect. The *mutual algorithmic information* between two objects  $\mathcal{O}_t$  and  $\mathcal{O}_s$  is:

$$I(s : t) = K(t) - K(t | s^*)$$

One can also use the *joint algorithmic information*  $K(s, t) \stackrel{\pm}{=} K(s) + K(t | s^*)$  to give a symmetric formulation of the mutual algorithmic information:

$$I(s : t) \stackrel{\pm}{=} K(s) + K(t) - K(s, t)$$

This gives the intuitive result that if knowing  $s^*$  does not enable a shorter computation of  $t$  (or vice-versa) then  $s$  and  $t$  do not share any mutual information.

In (Janzing and Schölkopf, 2010, section 2) they used a related concept, the conditional mutual information

$$I(s : t | r) \stackrel{\pm}{=} K(s | r) + K(t | r) - K(s, t | r)$$

to define an algorithmic information theoretic formulation of the Causal Markov Property. If  $x_1, x_2, \dots, x_n$  are binary strings describing observations related by a graph, then each string  $x_j$  shares no mutual information with the strings  $nd_j$  associated with its non-descendents in the graph, conditional on the *shortest* binary string associated with the parents  $pa_j^*$  of  $x_j$  in the graph:

$$I(x_j : nd_j | pa_j^*) \stackrel{\pm}{=} 0$$

In general, Janzing, Schölkopf, and collaborators have made use of algorithmic information theory for a multitude of causal inference tasks; see (Mooij et al., 2016) for a review of how some of these methods perform on a variety of empirical data sets.

My interest is in the application of some of these tools to inferring causal direction in physical theory. In particular, I will focus on a connection drawn in (Janzing et al., 2016) between dynamical evolution and the increase of algorithmic complexity in a toy model of statistical mechanics, with the aim of extending it to quantum theories.

They begin by assuming the Principle of Independent Mechanisms: if  $s$  is the initial state of an  $N$ -particle system and the operator  $D_t$  represents applying the dynamics governing the system for some time interval  $t$ , then  $I(s : D_t) \stackrel{\pm}{=} 0$ . (More precisely,  $s$  and  $D_t$  are the shortest binary encodings of the initial state and the dynamics.) This asserts that not only does  $s$  contain no information about the dynamics  $D$  (the geometry and strength of a potential, for example),  $s$  also contains no information about how much time it will be subjected to those dynamics (i.e. it contains no information about the interval  $t$ ).

Although it seems so weak as to be nearly tautological, Janzing, Chaves, and Schölkopf immediately derive from it that the *algorithmic entropy*  $K(s)$  of a toy  $N$ -particle system *must* increase under dynamical evolution.<sup>37</sup> The proof is sufficiently simple that I will include it here before flagging its main limitation:

**No entropy decrease:** *If the dynamics of a system is an invertible mapping  $D_t$  of a finite set  $\mathcal{S}$  of states then  $I(s : D_t) \stackrel{\pm}{=} 0$  implies that the algorithmic information can never decrease when applying  $D_t$  to the initial state, i.e.*

$$K(D_t(s)) \stackrel{+}{\geq} K(s)$$

for all  $s \in \mathcal{S}$ .

Proof: Imposing  $I(s : D_t) \stackrel{\pm}{=} 0$  entails that  $K(s) \stackrel{\pm}{=} K(s | D_t)$ . Since  $D_t$  is invertible,  $s$  can be computed from  $D_t(s)$  and vice versa, which implies  $K(s | D_t) \stackrel{\pm}{=} K(D_t(s) | D_t)$ . From this one has  $K(s) \stackrel{\pm}{=} K(s | D_t) \stackrel{\pm}{=} K(D_t(s) | D_t) \stackrel{+}{\leq} K(D_t(s))$ .  $\square$

The intuitive idea is simple: if  $D_t(s)$  had a shorter description than  $s$  one could obtain a shorter binary description of the initial state by encoding  $D_t(s)$  and adding “then apply  $D_t^{-1}$ ”. But that is impossible since we’ve assumed  $s$  is the shortest binary description of the initial state. Upon establishing this theorem they illustrate that it holds for a toy system of  $N$  particles modeled as a cellular automaton (Janzing et al., 2016, section 2).

The theorem is suggestive but, as it stands, ultimately insufficient for inferring causal direction for time evolution in quantum systems. The reason for this is simple: the set of possible states of any quantum system is infinite. That said, I showed by construction at the end of section 3 that something much like **No entropy decrease** should be true for quantum

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<sup>37</sup>The equation of algorithmic entropy with  $K(s)$  by (Janzing et al., 2016) assumes that the microstate  $s$  is perfectly known to the observer. More generally, one defines the algorithmic entropy of a microstate  $s$  as the sum of the algorithmic information and the thermodynamic entropy  $\mathcal{S}(s) = K(s) + H(s)$ ; see (Zurek, 1989) or (Li and Vitányi, 2019, chapter 8).

systems: the Schmidt rank of a bipartite quantum system in a pure state is guaranteed to not decrease under time evolution, except for a set of time intervals  $t_i$  of measure zero, because time evolution generically creates entanglement.

To connect my discussion at the end of section 3 to the methods of causal inference using algorithmic information theory, I need a measure of the algorithmic information of a quantum state. In the next section, I will show that one can use such a measure and the causal inference methods described above to derive a more general, and more useful, version of the conclusion I reached at the end of section 3.

## 5 Entanglement, Algorithmic Information, and Causal Direction

To embed the inference of causal direction made on physical grounds in section 3 into the framework of causal inference using algorithmic information theory, one needs a measure of the algorithmic information of a quantum state. It would be preferable for the measure to be natural, either in the sense that it shares many or all of the conceptual virtues of classical algorithmic complexity or because it seems to appropriately generalize those properties to the novel physical and mathematical setting of quantum theories. Ideally, there would be a unique such measure.

There are multiple proposals for quantitative measures of quantum algorithmic information, each with some legitimate claim to being a natural generalization of classical algorithmic information to quantum states (see (Vitányi, 2001) for an early overview and (Mora et al., 2007) for a more recent one). We do not live in the best of all possible worlds, however: the measures are demonstrably inequivalent and one thus has to make a choice. I will focus on a measure of algorithmic information that tracks the Schmidt rank of a quantum state (or the Schmidt measure, for multipartite systems). Ultimately I will show that one can reproduce a less restrictive version of the criteria for identifying causal direction offered at the end of section 3 as a theorem that follows from imposing the Principle of Independent Mechanisms. I will first say a bit about the problems faced by any extension of algorithmic information to quantum states.

The question of how much a measure of quantum algorithmic information needs to have in common with its classical precursor to deserve the name is somewhat subjective, but one can identify at least three properties that seem non-negotiable. First, it should be definable from the quantum state alone rather than, say, only for a member of a specified ensemble of quantum states. This was why I deemed the von Neumann entropy unsatisfactory in section

3. Second, there should be some recognizable sense in which it measures the amount of information required to compute, or reconstruct, the state in question using some algorithmic procedure. Finally, just as there is an upper bound on the algorithmic information required to specify any classical bit string of specified length  $n$ , there should be some analogous upper bound on the algorithmic information of a quantum state that scales with the size of the system.

A number of other seemingly essential properties of classical algorithmic information are up for grabs, however. Must quantum algorithmic information be defined in terms of a computation carried out by a quantum generalization of a Turing machine or is a different notion of “algorithmic procedure” appropriate? Should the algorithmic information be measured in classical bits or qubits? Does quantum algorithmic information need to reduce smoothly to classical information in some context? *How* should the upper bound on the algorithmic information of a quantum system scale with the size of the system? Almost all classical bit strings of length  $n$  maximize the classical algorithmic information; should the same be true of the quantum algorithmic information content of quantum states in, say, an  $N$ -dimensional Hilbert space? Different proposals for measures of quantum algorithmic complexity have adopted different answers to these questions (Berthiaume et al., 2001; Vitányi, 2001; Mueller, 2007).

The measure I will focus on defines the amount of algorithmic information in a quantum state  $|\phi\rangle$  as the *classical* algorithmic information required by the shortest description of an algorithmic procedure for preparing it from some reference state  $|0\rangle$  (Mora and Briegel, 2005, 2006). One models the algorithmic preparation procedure as a quantum circuit  $\mathcal{C}_\phi$ : a finite sequence of unitary operations – quantum gates – chosen from a finite set  $\mathcal{G} = \{G_1, \dots, G_n\}$  that, when performed on the reference state  $|0\rangle$ , produce the desired state  $\mathcal{C}_\phi|0\rangle = |\phi\rangle$  up to some fidelity  $\varepsilon$ .<sup>38</sup> The set  $\mathcal{G}$  is finite, so each unitary operation – and thus each circuit  $\mathcal{C}_\phi$  – can be given a finite binary encoding which has a finite quantity of classical algorithmic information. The algorithmic information of a quantum state  $|\phi\rangle$  is identified with the classical algorithmic information of the binary encoding of the simplest circuit  $\mathcal{C}_\phi$ :

$$K_Q^\varepsilon(|\phi\rangle) = \min_{\mathcal{C}_\phi} K(\mathcal{C}_\phi)$$

where the notation  $\mathcal{C}_\phi$  is doing double-duty as the quantum circuit or the binary encoding of that circuit, depending on context.

The value of  $K_Q^\varepsilon$  appears to depend on four quantities not obviously related to the state itself: (i) the choice of the set  $\mathcal{G}$  of quantum gates, (ii) the alphabet  $\Omega$  used to encode the

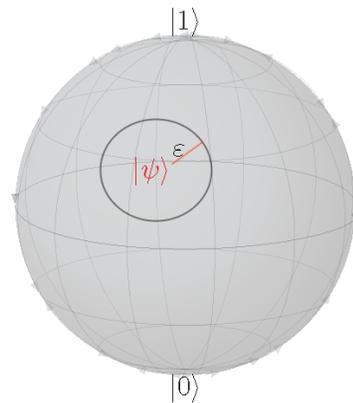
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<sup>38</sup>This means that  $|\langle\phi|\mathcal{C}|0\rangle|^2 \geq 1 - \varepsilon$ .

circuit, (iii) the degree of fidelity  $\varepsilon$ , and (iv) the particular circuit  $\mathcal{C}_\phi$ . As (Mora and Briegel, 2005, 2006) show, (ii) and (iv) are unproblematic: the values of  $K_Q^\varepsilon(|\phi\rangle)$  for two different alphabets  $\Omega_1$  and  $\Omega_2$  differ by at most a constant, and minimizing over all circuits  $\mathcal{C}_\phi$  that prepare the state  $|\phi\rangle$  removes any dependence on an arbitrary choice of circuit. On reflection, it is physically quite reasonable that  $K_Q^\varepsilon$  depend on the degree of fidelity  $\varepsilon$ : changing the precision with which an algorithm has to prepare a state will, and should, change the amount of information the algorithm requires. The arbitrariness in  $K_Q^\varepsilon$  associated with (i), the chosen set of quantum gates  $\mathcal{G}$ , is genuine, although (Mora and Briegel, 2005, 2006) offer several palliative remarks on that front.

To see the connection between this notion of algorithmic information and entanglement, consider a quantum system consisting of  $N$  qubits with Hilbert space  $\mathcal{H}_N$ . In this case, one can derive the specific dependence of  $K_Q^\varepsilon$  on the fidelity  $\varepsilon$  by invoking the fact that beginning with the reference state  $|0\rangle = |0\rangle_1 \cdots |0\rangle_N \in \mathcal{H}_N$ , one can prepare any state  $|\phi\rangle \in \mathcal{H}_N$  with *at most*  $\mathcal{O}(2^N)$  quantum gates  $G_i$ . Taking into account the dependence on the fidelity  $\varepsilon$ , (Mora and Briegel, 2006, section 5) show that the number of quantum gates required to compose a circuit  $\mathcal{C}_\phi$  that prepares an arbitrary state  $|\phi\rangle$  from the reference state  $|0\rangle$  is  $\mathcal{O}(2^N \log \frac{1}{\varepsilon})$ . This is an intuitive result: the bigger the system or the greater fidelity demanded, the more unitary operations required.

Note that by specifying a degree of fidelity  $\varepsilon$  one identifies a *patch* around  $|\phi\rangle$ . More precisely, it specifies a patch of radius  $\varepsilon$  on the unit sphere corresponding to all pure states in  $\mathcal{H}_N$ ; see Figure 2. The quantity  $K_Q^\varepsilon(|\phi\rangle)$  captures the amount of information required to specify a circuit  $\mathcal{C}_\phi$  that will put  $|0\rangle$  in that patch.<sup>39</sup> Choosing a degree of fidelity  $\varepsilon$  thus defines an equivalence class of states close to  $|\phi\rangle$  by assigning each of them the same algorithmic information content  $K_Q^\varepsilon(|\phi\rangle)$ . For  $\varepsilon > 0$  such “patches” of states have small but finite measure in the set of pure states in  $\mathcal{H}_N$ ; indeed, one can cover the unit sphere of pure



**Figure 2:** An  $\varepsilon$ -radius patch around  $|\psi\rangle$ .

<sup>39</sup>The two limiting cases may be clarifying. Let  $\varepsilon \rightarrow 1$ ; then the “patch” is the entire sphere of pure states and  $K_Q^{\varepsilon=1}(|\phi\rangle)$  quantifies the information required to construct a circuit  $\mathcal{C}$  that will act on  $|0\rangle$  to put it into *any* pure state inside  $\mathcal{H}_N$ . In this limit  $K_Q^{\varepsilon=1}$  is zero for all states, as it obviously should be:  $|0\rangle$  is already such a state, so just leave it alone. Now let  $\varepsilon \rightarrow 0$ ; then the “patch” approaches the single state  $|\phi\rangle$  and  $K_Q^{\varepsilon=0}(|\phi\rangle)$  quantifies the information required to prepare *exactly*  $|\phi\rangle$  from  $|0\rangle$ . This obviously diverges at the limit; for example, it would require preparing an amplitude to be *exactly*  $\frac{1}{\sqrt{2}}$  rather than the closest rational approximation to  $\frac{1}{\sqrt{2}}$ .

states with finitely many of them<sup>40</sup> Topologically, this means that every state  $|\phi\rangle$  is surrounded by a ball of radius  $\varepsilon$  of states of equivalent algorithmic information.

The information required to describe the circuit  $\mathcal{C}_\phi$  is just the information required to describe the quantum gates that compose it. From the fact that  $\mathcal{C}_\phi$  consists of, at most, a sequence of  $\sim 2^N \log \frac{1}{\varepsilon}$  unitary operations, (Mora and Briegel, 2006) conclude that its binary encoding – and thus the quantum algorithmic information of the state  $|\phi\rangle$  – has the upper bound

$$K_Q^\varepsilon(|\phi\rangle) = \min_{\mathcal{C}_\phi} K(\mathcal{C}_\phi) \lesssim 2^N \log \frac{1}{\varepsilon}$$

It is significant that while the classical algorithmic information of an  $n$ -bit string grows at most linearly with  $n$ , the algorithmic information of a quantum state can grow *exponentially* with the size of the system  $N$ . Given that the dimension of the space of states of a composite classical system in physics generally grows linearly in the dimension of the subsystems, while the dimension of the space of states of a composite quantum system grows exponentially due to the possibility of entanglement, one might think that the upper bound of  $K_Q^\varepsilon(|\phi\rangle)$  reflects the presence of entangled states in  $\mathcal{H}_N$ . As (Mora and Briegel, 2005, 2006) show, that is correct.

At this point the Schmidt rank of a quantum state once again becomes relevant. They adopt as their measure of entanglement the Schmidt measure (Eisert and Briegel, 2001), a generalization of the Schmidt rank to multipartite quantum systems. For an  $N$ -particle system with Hilbert space  $\mathcal{H} = \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_N$ , any state  $|\Psi\rangle$  in  $\mathcal{H}$  can be written as<sup>41</sup>

$$|\Psi\rangle = \sum_i^R a_i |\psi_i^{(1)}\rangle \otimes \dots \otimes |\psi_i^{(N)}\rangle$$

Let  $r$  be the minimum number of product states  $|\psi^{(1)}\rangle \otimes \dots \otimes |\psi^{(N)}\rangle$  needed to write  $|\Psi\rangle$  in the above form. The *Schmidt measure* of the state  $|\Psi\rangle$  is defined as  $S(|\Psi\rangle) = \log r$ . A separable state will have Schmidt measure  $\log 1 = 0$  and a fully entangled state will have Schmidt measure  $\log \dim(\mathcal{H})$ , which will be  $\log 2^N = N \log 2 = N$  for the system of  $N$  qubits.

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<sup>40</sup>See (Nielsen and Chuang, 2010, section 4.5.4) for how the number of patches required scales with the fidelity  $\varepsilon$  and system size  $N$ .

<sup>41</sup>I am presuming a preferred factorization of the big Hilbert space  $\mathcal{H}$  into tensor factors. In general, entanglement measures for multipartite systems are sensitive to different choices of partition; this is part of the difficulty of extending such measures beyond bipartite systems. Where a preferred partition isn't available, one could more generally define the Schmidt measure as the minimum value of  $r$  over all possible partitions. See (Zanardi et al., 2004), (Cotler et al., 2019), or (Carroll and Singh, 2020) for different proposals for identifying preferred factorizations in certain contexts.

The connection between algorithmic information and entanglement follows from the fact one can use the Schmidt measure to bound the algorithmic information  $K_Q^\varepsilon$  of a quantum state. As (Mora and Briegel, 2006) show, knowing the Schmidt measure of a quantum state makes possible a more informative bound on its complexity  $K_Q^\varepsilon$ :

$$K_Q^\varepsilon(|\phi\rangle) = \min_{\mathcal{C}_\phi} K(\mathcal{C}_\phi) \lesssim 3N2^{S(|\phi\rangle)} \log \frac{1}{\varepsilon}$$

The algorithmic complexity of a separable state  $|1\rangle$  can grow at most linearly with  $N$ , while the upper bound for all states in  $\mathcal{H}_N$  is saturated by a state  $|N\rangle$  only if it has full Schmidt measure. Given a state  $|k\rangle$  that is neither separable nor fully entangled – i.e. a state which has Schmidt measure  $0 < k < N$  – the upper bound on the algorithmic complexity of  $|k\rangle$  will thus satisfy  $K_Q^\varepsilon(|\text{SEP}\rangle) < K_Q^\varepsilon(|k\rangle) < K_Q^\varepsilon(|N\rangle)$ .

There are several comforting features of this definition of quantum algorithmic information. The first is that it lays the groundwork for addressing two major limitations of the causal inference criteria of section 3: that they applied only if one state from the pair comes from a set of states that has measure zero in the set of all pure states in  $\mathcal{H}_{AB}$ , and that they no longer held if one replaced states  $|\alpha\rangle$  that were *exactly* of less-than-full Schmidt rank with states that were *within*  $\varepsilon$  of a state of less-than-full Schmidt rank. In algorithmic information theoretic terms, states  $|k\rangle$  of less-than-full Schmidt rank now correspond to  $\varepsilon$ -patches of non-maximal algorithmic information  $K_Q^\varepsilon(|k\rangle)$ . As mentioned above, those patches have non-zero measure on the sphere of pure states in  $\mathcal{H}_N$ . Furthermore, these patches include all states that *approximate*  $|k\rangle$  with fidelity  $\varepsilon$ . According to Schmidt rank alone, the two states

$$|\Psi\rangle = |00\rangle \quad |\Psi_\varepsilon\rangle = \sqrt{1-\varepsilon}|00\rangle + \varepsilon|11\rangle$$

are no less different than are

$$|\Psi\rangle = |00\rangle \quad |\Psi_{\text{EPR}}\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

The algorithmic information  $K_Q^\varepsilon$  is less sensitive: it treats  $|\Psi\rangle$  and  $|\Psi_\varepsilon\rangle$  as equivalent, but distinguishes  $|\Psi\rangle$  from  $|\Psi_{\text{EPR}}\rangle$ . It represents an improvement over the Schmidt rank in this regard.

Nevertheless it can still reproduce an algorithmic information-theoretic version of the topological and measure-theoretic facts relied upon in section 3. In (Mora and Briegel, 2006) they show that for a system of  $N$  qubits, the relative frequency of pure states in  $\mathcal{H}_N$  with non-maximal algorithmic information is exponentially small: it scales as  $\sim 2^{2^N \log \varepsilon}$ .<sup>42</sup>

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<sup>42</sup>Since the algorithmic information of a quantum state is the classical algorithmic information of the circuit

Recall that (i) a state in  $\mathcal{H}_N$  is maximally complex only if it has full Schmidt measure and (ii) the Schmidt measure reduces to the Schmidt rank for pure states of a bipartite system. With that in mind, consideration of the (divergent) limit  $\varepsilon \rightarrow 0$  shows that the relative frequency of pure states of non-maximal algorithmic information goes to zero in the set of all pure states as one demands perfect fidelity. One can thus reproduce in algorithmic information-theoretic terms the measure-theoretic sparseness of states that are not fully entangled, not only for bipartite pure states but also for pure states of *multipartite* systems of  $N$  qubits.

With all this machinery in place, an analogue of the result of (Janzing et al., 2016) for quantum algorithmic information now follows by essentially identical reasoning.

**No entanglement decrease:** *If the dynamics of a quantum system are given by a unitary operator  $U_t$  on a finite-dimensional Hilbert space  $\mathcal{H}$  of states  $|\phi\rangle$ , then  $I(|\phi\rangle : U_t) \stackrel{\pm}{=} 0$  implies that the quantum algorithmic information can never decrease when applying  $U_t$  to the initial state, i.e.*

$$K_Q^\varepsilon(U_t|\phi) \stackrel{+}{\geq} K_Q^\varepsilon(|\phi\rangle)$$

for all  $|\phi\rangle \in \mathcal{H}$ .

Proof: The assumption that  $I(|\phi\rangle : U_t) \stackrel{\pm}{=} 0$  entails that  $K_Q^\varepsilon(|\phi\rangle) \stackrel{\pm}{=} K_Q^\varepsilon(|\phi\rangle | U_t)$ . Unitary time evolution is invertible, so  $|\phi\rangle$  can be computed from  $U_t|\phi\rangle$  and vice versa. This implies that  $K_Q^\varepsilon(|\phi\rangle | U_t) \stackrel{\pm}{=} K_Q^\varepsilon(U_t|\phi) | U_t)$ . From this, one obtains  $K_Q^\varepsilon(|\phi\rangle) \stackrel{\pm}{=} K_Q^\varepsilon(|\phi\rangle | U_t) \stackrel{\pm}{=} K_Q^\varepsilon(U_t|\phi) | U_t) \stackrel{+}{\leq} K_Q^\varepsilon(U_t|\phi)$ .  $\square$

This is unsurprising in light of the facts reviewed above concerning the relative (in)frequency of states of non-maximal complexity in the set of all pure states.

One can also recognize the same intuitive reasoning at work as in the result of (Janzing et al., 2016): Suppose that  $U_t|\phi\rangle$  had less quantum algorithmic information than  $|\phi\rangle$ . By definition there would be a circuit  $\mathcal{C}_{U_t\phi}$ , the concatenation of  $\mathcal{C}_{U_t}$  and  $\mathcal{C}_\phi$ , with a simpler binary encoding than  $\mathcal{C}_\phi$ .<sup>43</sup> That would entail one could obtain a shorter description of the initial circuit  $\mathcal{C}_\phi$  by applying  $\mathcal{C}_{U_t\phi}$  to the reference state  $|0\rangle$  and appending  $\mathcal{C}_{U_t^{-1}}$  to prepare  $|\phi\rangle$ :  $\mathcal{C}_{U_t^{-1}}\mathcal{C}_{U_t\phi}|0\rangle = |\phi\rangle$ . But by definition  $\mathcal{C}_\phi$  is the shortest binary encoding of any circuit that will prepare  $|\phi\rangle$  from  $|0\rangle$ , so this is impossible.

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$\mathcal{C}$  that prepares it, this is a straightforward adaptation of the standard proof that the relative frequency of compressible strings in the set of all  $n$ -bit strings goes to zero as  $n \rightarrow \infty$ .

<sup>43</sup>Time evolution for a large class of Hamiltonians can be well-approximated as a quantum circuit  $\mathcal{C}_{U_t}$ ; see (Nielsen and Chuang, 2010, section 4.7).

One may wonder how this is consistent with the measure-zero set of pure states whose Schmidt rank decreases for a given dynamical evolution  $U(t_i)$ . The answer is simple: they are excluded by imposing the Principle of Independent Mechanisms  $I(|\phi\rangle : U_t) = 0$ . Specifying a particular time interval  $t_i$  of dynamical evolution tells one quite a lot about the candidate “cause” states for an entanglement-decreasing evolution. Indeed, it is only conditional on knowing  $U(t_i)$  that one can even *specify* such states. Even then they can only be picked out from the set of all pure states in  $\mathcal{H}_N$  by including their dynamical behavior in their description. They are thus ruled out as candidate “cause” states because they fail to satisfy  $I(|\phi\rangle : U_t) = 0$ .

From a physical perspective **No entanglement decrease** is quite reasonable. It would be baffling if a quantum system contained information about the dynamics, like the geometry and strength of the potential, *prior* to having experienced them. It is obvious that the system will have acquired some information about the dynamics *after* it is subjected to them. This informational asymmetry will be encoded in the quantum state – in multiple ways, surely, but I have focused on the extent to which this information about the dynamics is captured by the degree of entanglement in the quantum state. This gives us a new criterion for causal inference:

*Given two states  $|\alpha\rangle$  and  $|\beta\rangle$  related by unitary evolution, if  $K_Q^\varepsilon(|\alpha\rangle) < K_Q^\varepsilon(|\beta\rangle)$  then  $|\beta\rangle$  was caused by time-evolving  $|\alpha\rangle$ .*

This criterion represents an improvement in several respects on the criteria identified at the end of section 3. First, the introduction of a fidelity  $\varepsilon$  means that it no longer depends on an unsatisfiable standard of precision, as noted above.

Second, the set of states for which it holds has finite measure in the set of all pure states in  $\mathcal{H}_N$ , as discussed above. The algorithmic information  $K_Q^\varepsilon$  is defined on *patches* of pure states of small, but finite, volume and so states of non-maximal algorithmic information will occupy finite volume in the set of all pure states in  $\mathcal{H}_N$ . A similar conclusion follows from recalling that in the set of all pure states, the relative frequency of pure states of maximal algorithmic complexity is  $\approx 1 - 2^{2^N \log \varepsilon}$ .

It is illuminating to contrast this with the case of classical statistical mechanics; for example, Goldstein et al. (2017) estimate that for a system of  $N \sim 10^{20}$  particles, the relative volume of phase space occupied by microstates in the equilibrium macrostate is  $\approx 1 - e^{-10^{-15}N}$ .<sup>44</sup> The relative frequency of pure states of maximal algorithmic complexity thus represents a difference in degree from the classical statistical mechanical case – possibly

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<sup>44</sup>More precisely, this the volume occupied by the equilibrium macrostate for any energy shell in phase space containing microstates with energies in the range  $E + \Delta E$ .

a dramatic degree, depending on the system size  $N$  and chosen fidelity  $\varepsilon$  – but not a difference in kind.

Third, the criteria proposed in section 3 were too weak to identify a causal direction between two pure states each of less-than-full Schmidt rank, as noted in fn. 26. The algorithmic information-theoretic criterion can do better: given two pure states of *any* Schmidt measure, **No entanglement decrease** entails that the state with strictly lower algorithmic information is the cause state. Finally, the criteria in section 3 were restricted to bipartite systems, while this algorithmic information-theoretic criterion extends also to multipartite systems of qubits. Note, however, that it does *not* straightforwardly extend to infinite-dimensional systems, while the criteria in section 3 applied to both finite and infinite-dimensional systems.

One may naturally wonder if **No entanglement decrease** is an artifact of the chosen definition of quantum algorithmic information. There is good reason to be optimistic on this score. Many proposals for extending algorithmic information to quantum states assign higher algorithmic information to entangled states (see (Mora et al., 2007) for a review) so one can reasonably expect that some version of the conclusion here concerning  $K_Q^\varepsilon$  could be reproduced for other proposed measures of quantum algorithmic information. Additionally, the measure here is defined using quantum gates – the unitary operations composing the circuit  $\mathcal{C}_\phi$  – and one may be concerned that this limits its application to systems of qubits. That concern can be alleviated: there is no principled obstacle to extending the use of quantum gates beyond two-level systems to more interesting multi-level systems, or even to quantum systems with continuous degrees of freedom Braunstein and Van Loock (2005), and thus to extending the notion of algorithmic information employed in this paper to those systems.

There is a significant limitation to this result, however, which is a well-known drawback of any use of algorithmic information theory: the algorithmic complexity of an object is not computable. Making use of this causal inference criterion in any practical setting would thus require identifying contextually appropriate, computable substitutes for the algorithmic complexity. Certain computable substitutes have already been proposed in different contexts; for example, the “information-geometric approach” of (Daniušis et al., 2010; Janzing et al., 2012). (The practical applicability of this method is itself limited by the fact that it requires deterministic relations between cause and effect variables.) It is also the case that the particular scenario here – inferring the causal ordering of two known pure states – is itself of limited practical application in the laboratory; after all, one doesn’t typically find pure states lying around in the world (to borrow a phrase from a referee). As a philosophical result, it succeed in demonstrating that contrary to what Russell and modern-day Russellians say about causation in “fundamental” physics, the time-symmetric nature of dynamical evolution

does not make it impossible for that dynamical evolution to ground relations of cause and effect. However, as a procedure intended for practical use in the laboratory, this is only a small first step in applying the methods of Janzing, Schölkopf, and collaborators to a quantum context.

That said, focusing solely on the strength of the causal inference criterion underwritten by **No entanglement decrease** may be too parochial. Arguably the primary interest of the criterion proposed here lies as much in the method used to reach it as its content. By attributing to quantum states a measure of algorithmic information and incorporating them into an empirically successful framework for causal inference, one can identify causal asymmetries between quantum states related by unitary time evolution. All that is required is a very weak assumption: the Principle of Independent Mechanisms. The extension of the methods of (Janzing and Schölkopf, 2010; Peters et al., 2017) to quantum theories introduces a novel set of mathematical and conceptual tools into the landscape for quantum causal inference. The fact that those methods confirm a conclusion that can also be reached on physical grounds provides a comforting verification of their reliability in this context, while their ability to extend that conclusion in select ways encourages their broader exploration.

## 6 Conclusion

In this paper I have argued that one can give a general and principled identification of a causal asymmetry between pure states of certain quantum systems related by unitary time evolution. The physical fact underlying this identification is simple: interaction between quantum systems entangles those systems but almost never disentangles them. Although there are many ways to make that physical fact precise, I have focused on one in particular for identifying causal asymmetry: the Schmidt rank (more generally, the Schmidt measure) of the two quantum states in question. I argued for the presence of this asymmetry on physical grounds in section 3, while in section 5 I did so by applying algorithmic information-theoretic concepts and methods from causal inference. In both cases, the Principle of Independent Mechanisms played a central role: the requirement that one ought to be able to specify the “cause” state without any information about the dynamics identified the causal asymmetry. Also in both cases, the asymmetry arose not in the laws governing dynamical evolution themselves, but between the states related by that dynamical evolution. In this respect, the asymmetry was identified by abandoning what (Woodward, 2020, section 11) calls the “cause-in-laws” attitude: the belief that if the laws of dynamical evolution themselves do not reflect an asymmetry, then there is no asymmetry to be found. Instead, my discussion here is a reflection of the more general fact that “the directional features of causation are

closely bound up with facts about the initial and boundary conditions of the systems we are analyzing and the way in which these are related to or interact with the [laws] governing those systems” (Woodward, 2020, section 11). That one can identify causal asymmetries in the dynamical evolution of quantum theories is indication that, *pace* Russell and others, one can locate a meaningful notion of causation in microscopic physics after all.

I want to conclude by highlighting a couple of philosophical questions that the previous sections raise. The first is the relationship between causal asymmetry and temporal asymmetry. Couldn’t the conclusions here about causal asymmetry be simply recast as conclusions about inferring temporal asymmetry? That is, couldn’t this have been described as an exercise in determining which of  $|\alpha\rangle$  or  $|\beta\rangle$  was the *initial* or *earlier* state, rather than whether  $|\beta\rangle$  was caused by time-evolving  $|\alpha\rangle$  or vice versa? This is certainly how Janzing, Schölkopf, and collaborators interpret much of their work; when applied to a time series or within the context of physical theory, for instance, they frequently describe their aim interchangeably as distinguishing between cause and effect or between *past* and *future*.

I think that it is quite reasonable to accept that *in the setup I have considered*, a causal asymmetry entails a temporal asymmetry. Indeed, one might think that when applied to states and dynamics in physical theories, some pre-theoretic understanding that causal asymmetries are almost always asymmetries *in time* is what lends the Principle of Independent Mechanisms its plausibility. However, I ultimately do think that what is being inferred from these methods is, first and foremost, a causal relationship. One reason for thinking this is that the methods of Janzing et al. that I use in this paper have been shown to perform well at distinguishing cause from effect even for data sets that are not time series data (e.g. the relationship between altitude and temperature, or longitude and precipitation in (Mooij et al., 2016)). That suggests that the structure being identified by these methods is causal structure; it just so happens that in the restricted setup being considered in this paper, that causal asymmetry coincides with the temporal asymmetry.

Second, one might wonder how this relates to other recognized temporal asymmetries in quantum theories. The clearest relationship is with decoherence.<sup>45</sup> As is well known, in quantum systems of many particles the entanglement between any two subsystems rapidly decoheres as each subsystem interacts, and becomes increasingly entangled, with other particles in the system.<sup>46</sup> It essentially never happens that a multiparticle system that is initially, or evolves into, a highly entangled state is later *disentangled* through some series of interactions between the particles. This is a manifestation of the general physical fact at the

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<sup>45</sup>See (Wallace, 2012, chapter 9) for a discussion of decoherence and temporal asymmetry.

<sup>46</sup>In language that may be more familiar: typically one splits the multiparticle system into two and calls the subsystem(s) of interest “the system” and the rest of the particles “the environment”. The entanglement between the subsystems of interest is then decohered by their respective entanglements with the environment.

foundation of this paper: interactions between quantum systems generically entangle those systems but almost never disentangle them. The non-unitarity of time evolution associated with decoherence that arises when one traces out all but the subsystem(s) of interest is the sharpest formal sense in which decoherence is asymmetric in time. There is an asymmetry that arises prior to the tracing out, however: the increasing entanglement between subsystems.<sup>47</sup>

With this in mind, one could focus on the physical fact underlying my formal discussion of causal inference: the generic creation of entanglement between quantum systems by interactions. One could then view my discussion as merely one way to formalize that fact in a restricted context, in a way chosen specifically to be amenable to a particular set of formal methods of causal inference. In that case, one could see that physical fact as identifying an asymmetry that is conceptually prior both to the asymmetry I identified using causal inference methods *and* to the asymmetry introduced when, after two entangled subsystems are decohered through interactions with some “environment”, that environment is traced out to produce a non-unitary evolution for the subsystems of interest. With that attitude, the causal asymmetry I have identified for bipartite (and multipartite) pure states is just the maximally microscopic manifestation of the very same physical conditions that produce the asymmetry manifest in decoherence.

Finally, I have spoken consistently of *inferring* or *identifying* causal asymmetry. As this epistemic language suggests, I have not primarily been concerned with providing an analysis of what causation or causal asymmetry *is*, in any strong metaphysical sense, but rather with certain circumstances in quantum theories under which one can identify it. That said, empirically accessible manifestations of an object in, or property of, the natural world – and thus successful methods for detecting that object or property – are not independent of the way that object, or that property, *actually is*. By focusing on how one can learn about causation under particular circumstances, one can hope to learn something about the worldly properties that provide the physical basis for, or ground, our reasoning about causal relations in various contexts. I am particularly influenced on this score by (Woodward, 2020):<sup>48</sup>

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<sup>47</sup>As I mentioned in section 1, the relationship between entanglement and the direction of time has received extensive exploration.

<sup>48</sup>See also (Eberhardt, 2009, p. 914):

“... metaphysical accounts have provided essentially no guidance for methods of discovery because it remains unclear how they could be operationalized into discovery procedures that do not depend on the availability of causal knowledge in the first place. Epistemological headway was made by a completely different strategy that largely ignored metaphysical considerations.”

I would only emphasize that I think that once one has made this epistemological headway, it can be valuable to use it as a basis for circling back to address some of the metaphysical considerations that one initially set aside to make epistemic progress.

This is the project, alluded to earlier, of elucidating the worldly infrastructure that underlies and grounds assessments of causal direction. I see this project as connecting epistemological concerns having to do with how we find out about causal direction with the “what is out there” concerns of metaphysicians, although my answer to the what is out there question does not involve any kind of elaborate metaphysics. My general picture is that causal thinking “works” to the extent that it does because it picks up on or is supported by certain generic features of our world ... (Woodward, 2020, p. 5).

The causal inference methods of Janzing, Schölkopf, and collaborators, or of (Pearl, 2009) or (Spirtes et al., 2000), represent relatively unified *methodological* strategies for identifying causal relations in the world without presuming any unified *metaphysical* ground of those relations. These methodological frameworks allow that the physical basis, or metaphysical, ground of any particular causal relationship identified by those methods will likely vary with the worldly situations to which those methods are applied. Aside from the satisfaction of certain statistical or algorithmic information-theoretic properties, there may be little in their respective worldly grounds to link them together *as causal relationships*; the relationships may be metaphysically necessary or contingent, may be fundamental or emergent, and so on. The ability of these methodological frameworks to act as reliable diagnostics for such a diversely instantiated class of causal relations is part of their immense epistemic value. In the case I have considered, the worldly structure that grounds the identification of a causal asymmetry is a differential degree of entanglement in two states related by unitary time evolution. This asymmetry, in turn, is grounded in the physical fact that interactions between subsystems in quantum theories generically produce entanglement. The fact that one can identify this causal asymmetry using methods of causal inference reflects not merely a fact about causal epistemology, but a fact about the nature of the physical world.

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