The Introduction of Topology into Analytic Philosophy: Two Movements and a Coda

Samuel C. Fletcher and Nathan Lackey
scfletch@umn.edu and lacke039@umn.edu
Department of Philosophy
University of Minnesota, Twin Cities
Minneapolis, Minnesota, USA

Abstract: Both early analytic philosophy and the branch of mathematics now known as topology were gestated and born in the early part of the 20th century. It is not well recognized that there was early interaction between the communities practicing and developing these fields. We trace the history of how topological ideas entered into analytic philosophy through two migrations, an earlier one conceiving of topology geometrically and a later one conceiving of topology algebraically. This allows us to reassess the influence and significance of topological methods for philosophy, including the possible fruitfulness of a third conception of topology as a structure determining *similarity*.

1 Introduction and Motivation

In 1935, the MIT mathematician Philip Franklin published an article entitled "What Is Topology?" in the new journal of the fledgling Philosophy of Science Association (Franklin 1935).[1] In sections of at most a few paragraphs each, Franklin contrasted topology with geometry, introduced qualitatively the concepts of a topological manifold and algebraic invariants, and broached, among other topics, fixed-point theorems and the four-color problem in graph theory. He expounded no explicit thesis and cited no philosophical literature. Why did Franklin write this expository article? And why did William Malisoff, the editor of the journal, judge it fit to publish?

To answer these questions, one must understand why topological ideas were seen as important enough to philosophers, especially philosophers of science, to introduce them so unadorned. Towards such answers, our primary aim in this essay is to describe how topological ideas, such as those Franklin describes, were introduced into philosophy in the first half of the twentieth century, at the same time that analytic philosophy was coalescing as a distinct and professionalized philosophical school or program. In sections 2 and 3 we chronicle what motivated those who, like Franklin, introduced topological concepts, delineating their scope of influence, such as it was, but we also expound those concepts' future promise in section 4.

This promise contrasts with topological concepts' influence so far. Despite their early introduction, topological ideas are usually understood to have had a relatively limited effect on

[1] The first volume of *Philosophy of Science* had been published only the year before.

philosophical methodology, especially in comparison with those of formal logic, whose pervasive stamp on that era is well known. For instance, Thomas Mormann (2013, 425) has written that

> the philosophical neglect of topology was just the harbinger of a fundamental sea-change in philosophy of science, namely, the substitution of geometry, and more generally of mathematics, as a core issue of philosophy of science, by logic.[2]

In light of this, he implores historians of philosophy to investigate "why in the beginnings of the last century geometry lost its privileged status in philosophy and couldn't pass it on to topology" (2013, 433).

Our own account of the introduction of topology into analytic philosophy partly answers Mormann's plea, but it also rejects one of its presuppositions. We show how there were two *distinct* initial movements of topological concepts into analytic philosophy, each of which conceived of topology in fundamentally different conceptual terms. These movements are thus distinguished not primarily by which sorts of mathematical objects (e.g., point sets or filters) they took as basic, but by what they took as the subject matter of topology's philosophical application. The first, which we detail extensively in section 2.1 through the early works of Russell and Carnap, took a geometric conception of topology, understanding it as an *abstraction* from the geometry of real space. It is this movement, especially based in early 20th century philosophy of physics, which Mormann has in mind as the heir to philosophical concerns about geometry: according to him, "topology is concerned with the conceptual analysis of spatial notions" (2013, 426) and therefore "as a general theory of space, investigates the structure of these generalized spaces" (2013, 433).[3] Indeed, in considering the few philosophers who have engaged with topology as such, he too considers Russell and Carnap (Mormann 2013, 429–30).[4]

In section 2.2, we then return to resolve the mystery case of Franklin, discussed above. We will show that our evidence for Franklin's motivations and impetus for writing his short expository essay supports the conclusion that his work fits squarely into that first movement, too. Although Mormann is right that this movement never had the influence in analytic philosophy that logical methods did, it was never extinguished, and in section 2.3, we outline one research tradition in which it has survived: mereotopology. Given this movement's geometrical conception of topology, the scope of this continued application is quite understandable even if it is less expansive than that of logic.

Now, in framing the historical question of the (lack of) centrality of topology to philosophy, Mormann presumes that topology should be understood as this first movement did,

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[2] See also Mormann (2013, 423, 426). Although Mormann emphasizes the philosophy of science, he evidently understands this broadly, including within it the philosophy of geometry and space. This is why there is no harm in his unqualified reference to "philosophy," and we shall often follow this practice.

[3] This is a common conception of topology among philosophers. For instance, Callender and Weingard write that "Topology is a kind of abstraction from metrical geometry" (1996, 21).

[4] See also Mormann (2008) and Mormann (2007), respectively, for more on his view of Russell and Carnap's use of topological and geometrical ideas.

purely as a development or outgrowth of geometry in all relevant respects: "from the mathematical point of view there is no essential epistemological, ontological, or methodological difference between geometry and topology" (2013, 425). However, in section 3.1 we show in some detail that there was a second movement of topological concepts into philosophy which, in contrast with the first, took an essentially *algebraic* conception of logic—not in the sense of algebraic topology, which uses algebra to classify topological spaces, but in the sense that topological structures *themselves* are conceived as algebraic structures. Such a conception connects them with the algebraic semantics found in logic, most famously through Stone's representation theorem (Stone 1936; 1937; Johnstone 1982), which exhibits the mathematical duality of Boolean algebra with the algebra of clopen (closed and open) sets of certain topological spaces, now known as Stone spaces. McKinsey and Tarski (McKinsey 1941; McKinsey and Tarski 1944; 1946), among others, extended this idea to interpret intuitionistic logics and certain modal logics as logics of the topological closure operator. Within logic, these ideas have been very influential.

Nevertheless, one might protest that these connections fell largely within formal and mathematical logic, rather than philosophical logic or philosophy more broadly. For instance, Mormann remarks that "among philosophers [Stone's] work has remained virtually unknown up to this day" (2013, 428).[5] By contrast, in section 3.2, we exhibit contemporary research in philosophy of science that builds on this algebraic conception of topology—in particular, that based on Kevin Kelly's *Logic of Reliable Inquiry* (1996). As far as we are aware, this is a novel observation: published work in this research program has not directly recognized this connection heretofore.

In section 4.1, we then review how the previous two sections answer Mormann's question about the peripheral influence of topological concepts in philosophy (albeit an influence wider than Mormann describes). In a word, the narrow scope of influence arose from the equally narrow conceptions of topology's subject matter. Unlike logic, with its ambitions to model descriptively any linguistic activity and formalize the normative dimensions of argumentation itself, topology in its geometric conception concerned just the topic of physical space, while topology in its algebraic conception just provided a different yet fruitful way to understand the algebraic structure of logical systems.

As a coda to our historical discussion, in our concluding section 4.2 we highlight a nascent third movement of topological ideas into philosophy. This movement conceives of topology more like the first, geometric movement than the second, algebraic one, in that it emphasizes topology as a structure on a set. But unlike the geometric movement, that set needn't consist of spatial points, and the structure needn't be an abstraction from geometry in any relevant sense. Instead, the structure encodes relations of *similarity* among the elements of an

[5] See also Mormann (2013, 431): "neither Russell nor any other philosophers of science ever took notice of the path-breaking work of the American mathematician Marshall H. Stone." Russell may not be as culpable as Mormann states, since by the time Stone had announced and published his results in 1934–1937, Russell was no longer actively working on these topics.

arbitrary collection. Because this level of generality encompasses many sorts of problems in philosophy, this movement has the potential to become more far-reaching in its influence, as Mormann suggested would be worthy to enact.

2 First Movement: Geometric Topology and Philosophy

2.1 Russell and Carnap

2.1.1 Russell's *The Analysis of Matter*

In this section, we trace the origins and subsequent development of the relationship between the geometric conception of topology and analytic philosophy, beginning in the early 20th century. We show that philosophical trends during this period primarily relegated discussions of topology (or its conceptual predecessor *analysis situs*)[6] to its application in models of the structure of space and time. Thus, the philosophy of physics figured significantly into the earliest years of these discussions. Both Bertrand Russell (1914; 1927) and Rudolf Carnap ([1922] 2019; [1925] 2019) were aware of significant developments in mathematics that were concurrent to their early writings on the foundations of physical theory. In this section, we also describe the connection between their work and the relevant work of several key mathematicians.

The first points of contact between geometric topology and philosophy were simultaneous with the inception of analytic philosophy itself. Russell's early background in mathematics and his contributions to the philosophical foundations of that field are well known. His specific interests in geometry primed him for engagement with topology. Indeed, some of Russell's earliest published works concerned the foundations of both Euclidean and non-Euclidean geometry.[7] Methodologically, these essays represent an early synthesis of Russell's mathematical training and philosophical inclinations. In a broad sense, the concern with the mathematical foundations of space and its geometry segues into a similar concern with

[6] Although this term finds its origin in the work of Leibiniz, by the time of Russell and Carnap's writings, it had become synonymous with topology. Indeed, Poincare's influential early work in topology bore the title "Analysis Situs" (Poincaré 1895). This work has been noted as an originator of modern topological concepts; mathematician and historian of mathematics Jean Dieudonné has claimed that before this work of Poincaré we can only speak of the pre-history of the discipline (Dieudonné 2009, 15). Oswald Veblen's textbook of topology was also titled *Analysis Situs* (Veblen 1922). Additionally, it is clear from the context in *Analysis of Matter* that Russell took the terms to be synonymous; he first mentions analysis situs immediately prior to introducing Hausdorff's definitions (Russell 1927, 295–96).

[7] Key examples include *An Essay on the Foundations of Geometry* (Russell 1897), which Russell wrote on fellowship shortly after completing his BA in mathematics (Wahl, 2018: x–xiv), and his earlier work "The Logic of Geometry" (Russell 1896), which concerns the nature of the Euclidean axiom of congruence. He argues (against Helmholtz) that this axiom is strictly a priori and as such it cannot be proven through experience of objects in space. As a result, he concludes that rejecting this axiom is absurd from both a logical and philosophical standpoint. In support of these results, Russell provides argumentation that is both philosophical and mathematical.

topological concepts in Russell's works in the 1910s and '20s. The move from geometry proper into geometric topology was a natural one, as both concern the properties of space.

The development and confirmation of the general theory of relativity in the 1910s brought new attention to the theory's innovative conception of space and time (viz., space-time). Russell sought to engage philosophically with this conception, according to which it was no longer possible to consider space in exclusively Euclidean terms. Thus, the emphasis in the foundations of modern physics moved from Euclidean geometry to non-Euclidean geometry and the abstract core of concepts they share, as described by *analysis situs*. Russell had already established scholarship in the former as evidenced by his previous essays on the foundations of geometry. His engagement with geometric topology proper would become explicit in *The Analysis of Matter* (Russell 1927). But, the problems that Russell addresses in his application of topological concepts find their origin in Russell's earlier work, *Our Knowledge of the External World* (OKEW) (Russell 1914).

OKEW is a collection of lectures in philosophical methodology and epistemology articulating what can be known and how this knowledge can be acquired. At that time, Russell was still sensitive to the influence of British idealism. By attempting to overcome the idealist's skepticism regarding sense data and reality, Russell applied the new logic-centric methodology to mathematical physics. In the chapter "The World of Physics and the World of Sense," Russell was concerned with reconciling everyday sense experiences of the world with contemporary developments in modern physics. Therein, he treated three aspects of the physical world as described by physics: matter, space, and time.

Since the aim of the lecture was to derive a link between the conceptual foundations of physics and sense data, Russell provided a construction of geometric points that obviates a notion of the infinitesimal. Because points are ideal objects, they cannot be perceived, even indirectly. Russell noted Alfred North Whitehead's method of using a relation of *enclosure* to derive points as spatial objects (Russell 1914, 114).[8] Russell contends that with the satisfaction of some specific conditions, this relation and an everyday notion of containment allow one to replace the concept of an infinitesimal point which is found in geometry with a logical construction from spatial objects. He is content to follow Whitehead on the matter without modification at this juncture. These comments on points differ from those found in his *The Analysis of Matter* (TAM) (Russell 1927) where the shift into geometric topology is explicit.

[8] Here, Russell credits Whitehead with this approach to points without citing a specific work. But as noted by Thoman Mormann (2013, 430), Russell elsewhere credits a specific work by Whitehead in reference to his construction of points: the fourth volume of the *Principia*. This work never saw publication. As a result it is not possible to consult Whitehead's point construction in his own words, at least not in that particular work. However, in his "On the Mathematical Concepts of the Material World" (Whitehead 1906), we do find a strikingly similar construction to the one recited in OKEW. Whitehead's construction of points (Whitehead 1906, 488–92) builds points out of a theory of interpoints, which are similar to the enclosures described by Russell. Whitehead's techniques did not involve topological concepts. Incidentally, that work by Whitehead cites Veblen, who would later become well known for his work in topology. However, the citations are to Veblen's dissertation which concerned geometry and predated his work as a topologist. There does not seem to be evidence that the topological notions employed by Russell later in *The Analysis of Matter* had their origin in Whitehead's point construction.

TAM is a long-form work in the philosophy of modern physics. The first section concerns the relationship between logic and physics and the second pertains to the relationship between physics and human perception. Topological concepts arise in the third and final section of TAM, which is titled “The Structure of the Physical World,” in particular chapters 28 and 29. A noteworthy detail comes from a footnote on the first page of chapter 28, “The Construction of Points,” in which Russell thanks Max Newman for his criticisms (Russell 1927, 290). Newman was a Cambridge mathematician who would later become known for his book *Elements of the Topology of the Plane Sets of Points* (Newman 1939). We will return to Russell’s engagement with topologists, Newman in particular, after describing his construction of points.

In this revised construction of points, Russell employs the notion of events rather than the enclosure series of OKEW and he explicitly objects to constructing points in terms of enclosures. He argues that the previous method fails to correspond with the actual world when events are treated as points (Russell 1927, 292). Part of Whitehead’s construction involves the assumption that each event is enclosed by other events and vice versa. The questionable implication of this assumption is that on this account events have no maximum or minimum. With respect to an event of a minimum size, Russell argues that neither direct nor indirect observations can be made of the smallest event possible. However, he states, without elaboration, that quantum theory may (in principle) be able to demonstrate that events do indeed have a minimum (Russell 1927, 292). In any case, he also goes on to argue that the idea of a maximum for events is also problematic. But it is in his positive account of events that the conceptual aspects of topology are invoked.

In order to define points in terms of events, Russell introduces the property of being “compresent,” which describes the overlap between events in space-time. His definition of a point comes out of two conditions being satisfied for a group of events: any two members are compresent and no event not in the group is compresent with every member of the group. Given these conditions, a point might be defined as the location (common overlap) in space-time that is occupied by all the events in a group of events, as described by the concepts above. This is a promising start, but Russell noted that additional rigor is required. He described the possibility that, depending on the shape of the events involved, a single event could both: 1) share regions with each of the other events in the group and 2) not share any region simultaneously with all of the other events in the group. The required further analysis of points in terms of events is a matter concerning “such properties of figures as are unaffected by continuous deformation” (Russell 1927, 295). These properties, as Russell points out, are precisely the topological properties of those figures.

Russell reproduces definitions of relations between points and sets of points relevant to his topological investigations from Leopold Vietoris (1921). The concepts of neighborhood and continuity are central. After having introduced Vietoris on the matter of points, Russell then draws from Felix Hausdorff (1914) on topological space. The significance of Hausdorff’s textbook, the first on set theory and the first on topology, is profound with respect to our argument in this section, for it is the earliest common source of topological ideas for philosophy. The full extent of its influence will unfold in our discussion of Carnap later in this section and our suggestions for a nascent third conception of topology relevant for philosophy in section 4. What's important for present purposes is that Russell draws from Hausdorff a geometrical definition of a topological space in terms of neighborhood systems:

> A "topological" space is a manifold [set] whose elements *x* are associated with sub-classes [neighborhoods] U_x of the manifold [set] such that:
>
> (A) To every *x* corresponds at least one U_x, and every U_x contains *x*;
> (B) If U_x, V_x are both neighbourhoods of *x*, there is a neighbourhood of *x*, say W_x, which is contained in the common part of U_x and V_x;
> (C) If *y* is a member of U_x, there is a neighbourhood of *y* which is contained in U_x;
> (D) Given any two distinct points, there is a neighbourhood of the one and there is a neighbourhood of the other such that the two have no common point. (Russell 1927, 296)[9]

In addition to these definitions from Hausdorff and Vietoris, Russell describes a few of Hausdorff's axioms that allow for the notion of a limit in the context of topology to have more intuitive properties. These properties enable him to deploy a theorem by Pavel Urysohn that permits the description of topological spaces locally—i.e., in some neighborhood of every point—with a distance function. Equipped with these definitions, axioms, and their consequences, Russell is poised to address the problem mentioned above: an event in a given group might overlap with each member individually but not with all of them together.

Russell's contention with Whitehead's point constructions focused on the assumption that events do not have a determinate maximum or minimum. This assumption does not correspond to physical reality, according to Russell. By using the concept of a neighborhood, Russell argues that it is possible to overcome this problem and construct the four-dimensional space-time of general relativity out of events (Russell 1927, 298). By characterizing the relationship between events and particular neighborhoods, Russell proposed to associate events with neighborhoods that meet specific requirements with respect to maximum and minimum: "We have to assign to our events such properties as will enable us to define the points of a topological space as classes of events, and the neighborhoods of the points as classes of points" (Russell 1927, 299).

Russell introduces a conception of events as spherical objects that take up a region of space-time. The dimensions in space are given by familiar Euclidean coordinates while the dimension of time is bounded by a specific maximum and minimum. Classes of points make up the region of space taken up by a given event. Points themselves are defined through what Russell describes as a "co-punctual" group: a five-term relation between five events that share some region. A point is defined by such a group where a specific condition is satisfied: any increase in size of a member of the group would make the group no longer co-punctual (Russell

[9] Technically speaking, this definition captures what we would now call a Hausdorff neighborhood basis. Condition (D) is the Hausdorff condition, while a neighborhood system is generated by closing the basis under the superset relation.

1927, 299). Russell appears to have thereby arrived at a conceptual understanding of points in terms of events that retains the notion that events have some set maximum and minimum. The relationship between this construction and the topological notions introduced earlier in his chapter are relevant because the continuous distortions of the spheres preserve the relevant property of co-punctuality. Of course, the topological notion of a neighborhood is also crucial for Russell's construction.

Clearly, in this chapter, Russell explicitly draws on crucial work by mathematicians whose research was in topology: Felix Hausdorff, Max Newman, Pavel Urysohn, and Leopold Vietoris. Among these four, Russell interacted personally with Newman, who, as we mentioned above, made comments to Russell regarding philosophy of science, logic, and the application of topology in physics, providing him references to contemporary literature. Ivor Gratan-Guinness has suggested that his influence on Russell's comments related to topology "may have risen in places to ghost authorship" (Grattan-Guinness 2012, 20) because of the contrast between the treatments in OKEW and TAM and the fact that Russell never returned to question with such detail. Newman's own interest in topology can be traced back to his time spent at Vienna University from 1922–23. In the above list of topologists cited by Russell, Newman is the only one from the United Kingdom. Indeed, topology was not especially known or popular in Britain at the time and it was in Vienna that Newman's attention was turned to that newly forming area of mathematics. The aforementioned Vietoris was a junior staff member at Vienna University during Newman's years there (Grattan-Guinness 2012, 8)

Hans Hahn was likely the most influential figure for Newman in Vienna. Hahn was a specialist in topology. He was also interested in the foundations of mathematics and formal logic.[10] This combination of topology and foundational inclinations undoubtedly impressed young Newman, who would devote his own academic endeavors to them. It was with these sensibilities that Newman, as a junior research fellow of St John's College, witnessed Russell's 1926 Tarner lectures at Trinity College, Cambridge, which would go on to become chapters of *The Analysis of Matter*.

There is a connection here that Grattan-Guinness does not explicitly mention, although he provides the necessary details to support it. Not long after returning to Cambridge, Newman wrote an essay entitled "The Foundations of Mathematics from the Standpoint of Physics" (Newman 1923) to support his (successful) application to a junior fellowship at St John's College. This essay, though never published, concerns the differences between the ideal abstractions of applied mathematics and the physical objects of the world to which they are applied. Mathematics concerns abstract and ideal entities, whose existence may be inferred by logic alone, whereas the existence of physical objects, when inferred, must be constructed from what is given. We see here a confluence with Newman's concerns and Russell's own concerning the logical construction of point events. It appears to be reasonable to infer that Newman's influence was present in both the motivations for improving upon Whitehead's construction of points and its topological solution.

[10] Indeed, his most famous student was Kurt Gödel.

An examination of the record of citations of Russell's topological point constructions in TAM reveals that they were not influential among his contemporaries. However, for our purposes, TAM serves as a very early and *topically* typical example of the manner in which topological concepts, interpreted geometrically, began to find application in the analytic tradition. Those few philosophers who did initially engage with the topological aspects of TAM did so critically,[11] but they were nonetheless inspired to employ topological tools in more sophisticated accounts or interpretations of space (Grünbaum 1951; Abramov 1973; Winnie 1977). In Grünbaum's 1951 dissertation, he laid the foundations for some of his later influential work in the philosophy of space and time (e.g., Grünbaum 1973), and Winnie's 1977 essay introduced philosophical audiences to more sophisticated treatments of the topological features of space-time, such as that deriving from the work of mathematician A. A. Robb (1914; 1921; 1936). In the following section we turn to another early analytic philosopher whose work involving topology has been more widely of direct influence.

2.1.2 Carnap's *Grundlegung Der Geometrie*, *Der Raum*, and "On the Dependence of the Properties of Space on those of Time"

Russell was not the only major figure in the early history of analytic philosophy whose work on the foundations of physics led to the application of concepts from topology. Carnap is another case.[12] There are several examples from Rudolf Carnap's early work on the philosophy of space that make use of topological concepts (Carnap [1922] 2019; [1925] 2019). The earliest of these was his doctoral dissertation: *Der Raum: Ein Beitrag zur Wissenschaftslehre*[13]. Like Russell, Carnap's earliest post-undergraduate work was focused on the philosophy of geometry. His MA thesis was titled *Grundlegung Der Geometrie*[14] (Carnap 1920) and a thorough revision of that paper went on to become *Der Raum* (Carus and Friedman 2019, xxxi). Significantly, his discussions of space in the MA thesis did not include topological space, so he must have started to see the significance of topological concepts in 1920–1922.

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[11] For the most focused critical discussion of Russell's scholarship regarding the construction of points, see Mormann (2008). Note also that Russell's attempted construction of points bears resemblance to later approaches to mereotopology using (ultra)filters, for which see footnote 22.

[12] It is worthwhile to acknowledge that one of Carnap's early philosophical influences came not from the analytic tradition as he explicitly invoked the Husserlian concept of *Wesenserschauung* in his characterization of intuitive space in "Der Raum" (T. Ryckman 2007, 103; Friedman 2019, 204). Nevertheless, historians also give good reason to understand the early Carnap as operating squarely in the early analytic tradition. Gottfried Gabriel points out that Carnap himself gave "greatest tribute" to Frege out of his influential teachers during his youthful time in Jena and Frieberg from 1910 to 1914 (Gabriel 2007, 65–66), among other nuances of the relationship between Carnap and the "grandfather of analytic philosophy." Additionally, in his editorial notes that accompany a new translation of "Der Raum," Michael Friedman cites the shadow cast by Russell and Whitehead's *Principia* (Friedman 2019, 174) in Carnap's own pointers to relevant literature in his account of formal space. If we take Russell to be a progenitor of analytic philosophy, then his influence on Carnap cited here is pertinent when attempting to categorize Carnap's early work.

[13] Space: A Contribution to the Theory of Science

[14] Foundations of Geometry

In *Der Raum*, Carnap's goal was to identify and clearly articulate the different meanings of the term "space," defending one sense in which the Kantian conception of Euclidean space could be saved from apparent empirical conflict with the newly confirmed general theory of relativity, which describes space(-time) as having in general a non-Euclidean geometry. The source of confusion and disagreements in conversations between philosophers and scientists about these matters, Carnap held, is that each respective discipline uses the same term referring to a different concept of space. Thus, these interdisciplinary problems could be eliminated by making the distinctions that Carnap proposes in *Der Raum* (Carnap [1922] 2019, 27). The first three sections of this dissertation are each devoted to different conceptions of space: formal space, intuitive space, and physical space.

Formal space is a logical construction from an axiomatic theory of objects and relations (Carnap [1922] 2019, 31). After introducing some familiar logical and mathematical concepts (truth, falsity, series, continuity) and relations (transitivity, one-one, many-valued etc), Carnap uses them to define an n-dimensional ("nth level") formal space R_{nt} as locally an n-fold product of the real number line with the number line's standard topological structure (Carnap [1922] 2019, 33–41).[15] This is the structure that is shared by the manifolds described by Riemann ([1864] 2016). By adding axioms concerning direction and distance, one determines projective and metrical spaces.

> Projective and metrical space are thus related to R_{nt} as species and subspecies to a genus (not as individuals to a species). Similarly, from topological space with three dimensions R_{3t} we get projective space R_{3p} and metrical space R_{3m}, as well as further subspecies. (Carnap [1922] 2019, 41)

The significance of R_{nt} for Carnap is that it is also the most comprehensive or general form of intuitive space (Carnap [1922] 2019, 69), which is the a priori form of perceptual space. The axioms determining this are derived from phenomenological insight. In contrast with Kant, this space does not have Euclidean projective and metrical structure, so is compatible with the implications of the general theory of relativity concerning physical space. The subject of a priori determination of the form of perceptual experience is just of a different space than the subject of the physics of space and time.[16]

Carnap here draws from work focused on topology and the foundations of metrical and projective geometry (Carnap [1922] 2019, 149, 155). A notable point of overlap with Russell in this respect is Hausdorff's textbook (1914). As noted above, Russell drew his definition of a topological space in terms of neighborhood systems from Hausdorff. Unlike Russell, Carnap was

[15] Carnap, like his predecessors, does not clearly distinguish between a manifold's topological and differentiable structure, the distinction between which would only come about a decade later (Veblen and Whitehead 1931; 1932). Nevertheless, even though formal space has the standard differentiable structure associated with products of the real line, Carnap does focus his attention on its topological properties.

[16] It is worth noting that Friedman argues that Carnap's approach is substantially flawed. But, for our present purposes we need not weigh in on the efficacy of the structure of intuitive space in *Der Raum* (Friedman 2019, 187–88).

not concerned with the construction of points in his use of topology. But he also drew from Hausdorff the general notion of topological space (represented in the quote above as R_{nt}). The significance of the fact that both Carnap and Russell had knowledge of Hausdorff's presentation of topological ideas will be further developed in a later section.

There are other important connections between Russell's chapters and Carnap's dissertation. Firstly, Carnap cites multiple works by Russell; the works notable for our discussion here are his essay on the foundations of geometry (Russell 1897) and OKEW. The former may have been more influential on Carnap's earlier MA thesis than it was on *Der Raum*. As Michael Friedman notes, in his MA thesis Carnap defended a view similar to that espoused in (Russell 1897) in which the projective form of space is the most general. In *Der Raum*, the topological form of space takes the place of projective space as being the a priori representation of space (Friedman 2019, 177). As we pointed out earlier, OKEW did not contain references to topological concepts. This came later in the point construction found in (Russell 1927) which was published several years after *Der Raum*. Thus, the introduction of topology that occurred between Carnap's 1920 MA thesis and *Der Raum* did not come directly from the works of Russell cited in the latter.

Friedman suggests that the shift towards topology that occurred between the MA thesis and *Der Raum* should be understood as being due to the centrality of general relativity in the latter. This is because general relativity applies a metric space with variable curvature which therefore does not fall under the most general projective space mentioned in Carnap's MA thesis. This prompted the move to topological space as being the most general form of space in *Der Raum* (Friedman 2019, 177).

Topological considerations are central in another of Carnap's early publications, "On the Dependence of the Properties of Space on those of Time." Carnap defends the thesis that, broadly construed, there is such a dependence. This thesis is interesting, in part, because the mathematical description of space and time coordinates in classical mechanics are characterized as each being independent of one another. But, the relationships between each of these four variables were altered by the introduction of special relativity (Carnap [1925] 2019, 301). Carnap's project to demonstrate this dependence consists in constructing a reduction of the topological structure of physical space to that of temporal order in the context of special relativity (Malament 2019, 328).

Carnap introduces topology in the article early on to demarcate metrical properties from topological properties. He broadly states that "The non-metric properties of the temporal and spatial order are called *topological*. They concern only the neighborhood and connection relations" (Carnap [1925] 2019, 303 emph. orig.). The detailed construction of temporal topology concerns a substantial portion of the article. In the process, Carnap cites Hausdorff's neighborhood system axioms for topology, the same that Russell quoted. Carnap defines what he calls "spatial point neighborhoods" and sketches how they satisfy Hausdorff's axioms (Carnap [1925] 2019, 317).

Carnap cites two sources related to topology, in addition to Hausdorff. Both are works specifically in the topology of time. The first was Kurt Lewin's "Die zeitliche Geneseordnung" (Lewin 1923) and the second was Hans Reichenbach's *Axiomatik der relativistischen Raum-Zeit-Lehre* (Reichenbach 1924).[17] An illuminating account of the relationship between Lewin and Reichenbach's ideas on time topology can be found in Flavia Padovani's "Genidentity and Topology of Time: Kurt Lewin and Hans Reichenbach" (Padovani 2013). There, she argues that the differences between their respective accounts of topology of time can be understood through their characterizations of genidentity, or identity over time. She points out that the application of genidentity diverges because Lewin was interested in developing order with respect to actual events whereas Reichenbach was focused on physically possible events (Padovani 2013, 98).

All of this is to say that Carnap ([1925] 2019) appears to have arrived at his topological characterization of time through Lewin via Reichenbach. Lewin (1923) provides a topology of time from a mereological perspective which was rooted in actual events. Reichenbach borrowed this model and modified it (Padovani 2013, 106–7). The resulting version shares the most explicit similarities with Carnap's own, as he acknowledges:

> Reichenbach constructs, before the introduction of metrical concepts, a time topology that is close to ours. For our point of view, he provides very valuable discussions about the co-ordination of certain system concepts to the synonymous physical concepts (e.g. coincidence, simultaneity, and so on). (Carnap [1925] 2019, 324–25)

From this we can see that although Lewin was the originator of the sort of time topology utilized by Carnap, Reichenbach's iteration was more influential. This is likely to be true also because of Reichenbach's application of the concept in the philosophy of modern physics.

2.1.3 Comparisons

There is a trend (in the broad sense) shared between the succession of each of Russell and Carnap's early publications involving topology. The earliest works completed by each concerned the foundations of geometry: Russell's fellowship essay (Russell 1897) and Carnap's MA thesis (Carnap 1920).[18] Then, once they were coming to grips with the theory of relativity, they introduced topological concepts. So, there are two connections that were apparently shared in the development of both Russell and Carnap's thought with respect to their employment of topology. The first is between geometry and topology and the second is between the theory of relativity and topology.

The former may be seen as a result of developments in mathematics that were internalized by these mathematically inclined early figures in analytic philosophy. Mormann has described topology as "geometry's most promising offspring" (Mormann 2013, 425).[19] It seems

[17] This work was later translated as *The Axiomatization of the Theory of Relativity* (Reichenbach 1965).
[18] For more on geometrical themes in Carnap's early work, see Mormann (2007).
[19] We agree with Mormann's comments insofar as they apply to *geometric* topology specifically. In this sense, his assertions apply in the context of the historical figures of interest in this section (Russell and Carnap). However, we

reasonable to conclude that this shared connection in Russell and Carnap's scholarship between geometry and topology can be understood in part through their sustained contact with developments in mathematics. In this they had Hausdorff's classic textbook (1914) in common.

The second shared connection is between topology and relativity in Carnap and Russell's philosophy of science. In the case of Russell, the general theory of relativity was introduced between the publication of OKEW and TAM. In the first chapter of the latter, Russell describes the aim of the book as being concerned with finding "an interpretation of physics which gives a due place to perceptions; if not, we have no right to appeal to the empirical evidence" (Russell 1927, 7). In the chapters that come, Russell grapples with the theory of relativity and the phenomena of sense experience. As noted in the discussion of this work above, it is here that we find explicit application of topological concepts via Newman as a means for reconciling the use of point events in relativity theory with the lack of sense experience of them. There is a similar correlation in the trajectory of Carnap's work. In between the writing of his MA thesis (where topology does not feature) and his dissertation, his focus shifts to the philosophical implications of general relativity for our a priori synthetic knowledge, such as of geometry. The connection between relativity and topology is not a coincidence. Firstly, there is evidence that early interpreters of relativity such as Lorentz, Ehrenfest, and Kretschmann were already gesturing towards its topological features (Giovanelli 2021). Indeed, Einstein himself noted that under the coordinate transformations in his theory the only relations which are preserved are topological relations (T. A. Ryckman 2018, sec. 2.3). Secondly, there is a common source of influence among these fields in Poincaré's work, which was known to both Russell and Carnap.
For Russell, Poincaré's influence came from his contributions to the field of topology in his capacity as a mathematician.[20] In the closing portion of his chapter on space-time order in *Analysis of Matter*, Russell compares his space-time construction with topological manifolds. By providing a definition of neighborhoods and assuming that the total number of events is $\aleph_0$, Russell asserts that the theorems of topology apply to space-time manifolds. It is here that Russell inserts Poincaré's "purely" topological definition of dimensions (Russell 1927, 311–12).

Interestingly, however, it is not in this capacity that his work was most influential for Carnap. Rather, Poincaré's conventionalism arguably contributed more to the application of topology in *Der Raum*. In *Science and Hypothesis*, Poincaré famously argued that the geometric structure of space was purely conventional by a thought experiment involving a world with negative spatial curvature empirically equivalent to a Euclidean world (Poincaré [1902] 2017). In *Der Raum*, Carnap went beyond Poincaré by attempting to demonstrate that worlds with either negative *or* positive spatial curvature were empirically equivalent to a Euclidean world. This led Carnap to the point mentioned above: topological relations are those that are a priori, while metrical relations are completely matters of convention (Mormann 2007, 50).

will see in section 3 that this view of the relationship between geometry and topology is not conceptually required by the features of topology. Starting in the 1930s and more so in the 1940s, the connections between topology and algebra became pronounced.

[20] Indeed, Poincaré's famous namesake conjecture is a noteworthy early problem in topology.

In sum, the introduction of topology to early analytic philosophy of science via Russell and Carnap can be understood through a confluence of elements. Both of these philosophers were attentive to the vanguard developments in mathematics and physics. Their contact with topology came through foundationally interested mathematicians and their own early interests in topology's older relative: geometry. Their interest came through their need to address the novel implications of relativity that had just emerged. Their applications were facilitated by Poincaré, Lewis, Reichenbach, and especially Hausdorff's work on the definition of a topological space. This is because Hausdorff's formulation in terms of neighborhood systems is a perspicuous abstraction from geometrical spaces with a distance function, thus applying to a larger variety of models of space.

2.2 Reprise: Franklin

In the previous subsection, we read how topological ideas were first introduced into philosophy just as they were being developed and codified within mathematics. Understood as abstractions from geometrical concepts, they played a role in the expression of space-time in the general theory of relativity, thus were taken up by philosophers interested in the philosophy of space. In this context, it becomes easier to understand *Philosophy of Science* editor Malisoff's probable motivations in publishing Franklin's short 1935 expository essay: topology, coming into its own as a distinct branch of mathematics, was largely unknown among philosophers of science, even those initially trained as scientists. Malisoff, himself trained as a biochemist, "had a very encompassing and engaged view of philosophy of science… [yet] struggled with quality issues" as an editor (Malaterre, Chartier, and Pulizzotto 2019, 228). Malisoff would have thus likely appreciated the gentle and accessible review of its basic concepts as they might have appeared relevant to philosophers interested in the nature of space, and the application of topology within physics more generally, and not worried so much about the lack of an explicit thesis.

Franklin's own probable motivations are similar, as might be gleaned from the only direct motivation he gives:

> Some elementary topological ideas have made their appearance in modern physics, e.g. the closed spaces of relativity and the phase spaces of statistical mechanics. This makes it desirable to attempt an answer to the question "what is topology?", comprehensible to the mythical educated layman. (1935, 40)

Franklin's own work touched on a variety of topics, including mathematical relativity theory (1922b) and coloring problems in graph theory (1922a; 1934), which he saw as topological (as is evident from his inclusion of this topic in his 1935 paper). The latter was in fact his 1921 dissertation topic under the supervision of one of the first American pioneers in topology, Oswald Veblen (O'Connor and Robertson 2010).

Although we do not presently have any direct evidence about why exactly Franklin was compelled to write his note to philosophers, we do have ample indirect evidence that it was through the personal and professional influence of his MIT mathematics colleague, Norbert Wiener. Wiener had studied philosophy and mathematics, taking a PhD in mathematical logic in 1914 (O'Connor and Robertson 2003). In 1918, he met and befriended Franklin as a co-worker under Veblen's supervision at the U.S. Army Proving Grounds in Aberdeen, Maryland, working on mathematical ballistics for the war effort. Franklin and Wiener's sister, Constance, would go on to marry in 1924, the same year in which both Franklin and Wiener were together again at the MIT mathematics department (O'Connor and Robertson 2010).[21] Soon after they published a paper together in approximation theory and topology (Franklin and Wiener 1926). Wiener himself continued to be interested in modern physics and philosophy during his time at MIT, about which he would engage in far-ranging conversations with colleagues (Struik 1989, 172)

In any case, Franklin's note in *Philosophy of Science* continues the theme of the previous subsection: direct interactions between philosophers and mathematicians developing the nascent subfield of topology were responsible for its introduction into philosophical methodology, especially that pertaining to the nature of space. Although Franklin's own essay would not have any substantial influence—as of the end of 2020 it had only about 10 citations—there are other ways in which the geometrical conception of topology has had a lasting impact on philosophical methodology. In the next subsection, we turn to one prominent example of this: mereotopology.

2.3 Modern Developments: Mereotopology

Since the time of Russell's writing on the nature of events, contemporary philosophers have continued to apply topology to address longstanding problems with respect to spatial and temporal parthood. *Mereology*, the field concerned with the analysis of parts and wholes (and parthood relations), has a long history in philosophy dating back to ancient and medieval ontologists. Since our interest in this paper is with topology, our focus will be on philosophical works that integrate topological concepts into mereology, resulting in the field of study identified by a portmanteau: mereotopology.

Fabio Pianesi and Achille Varzi have argued that one can describe temporal (Pianesi and Varzi 1996) and spatial (Varzi 2007) relations adequately only by combining mereological and topological framework. To do so, they point out several shortcomings of mereology that prompt the integration of topological notions. First, they argue that mereological concepts themselves have no way to differentiate between objects that are one-piece and self connected (like a uniform ball of clay) and objects that are made up of disconnected parts (like a baseball game). Furthermore, they assert that there are some spatio-temporal relations that cannot be adequately defined using the apparatus of mereology alone. In their view, these relations include those between a physical object and the surface on which it is resting, one object physically

[21] It was also in that year that Whitehead moved to the philosophy department at Harvard, and as noted in the previous subsection, Whitehead influenced Russell's engagement with topological ideas. It would be worth investigating further whether Whitehead had any influence on Franklin or Wiener.

surrounding another, and the relation of continuity between two successive events (Pianesi and Varzi 1996, 91).

Elsewhere, Varzi has noted the Whiteheadian origin of event construction in mereology (Varzi 1996, 270), which we noted in section 2.1 was also an important influence for Russell's engagement with topology. On this note, Varzi highlights conundrums that arise within attempts to reconcile theories of parts and theories of wholes. Parthood is conceptually constituted as a relation while wholeness is a global notion (Varzi 1996, 269). There, Varzi also argues that a mereological approach alone cannot accommodate the intuitive distinctions between a whole and sums of parts. He further argues that "the question of what constitutes a natural whole cannot even be formulated in mereological terms" (Varzi 1996, 270). In the illustration of this, Varzi draws out the history of this subset of philosophy in reference to Whitehead's early scholarship. Here, Varzi argues that Whitehead's definition of the necessary condition for the sum of two events fails to capture the "connectedness" relation that is required.

At this point, the need for topological concepts is already apparent. As Varzi points out, this failed attempt is grasping at the notion of connectedness that we find in topological reasoning (Varzi 1996, 270). Lastly, we can see the historical link to Whitehead and topology as evidenced in Whitehead's own revisions in *Process in Reality* (Whitehead 1929). Varzi interprets Whitehead as adopting the view that topology subsumes mereology by his use of the "C" predicate that can be traced back to *Process and Reality*. This predicate is taken to represent the topological concept of "connected" (Varzi 1996, 271). Varzi argues that in the final version of his mereological theory, Whitehead defined all other mereological terms with reference to "C". This indicates a view that has mereology being subsumed by topology (Varzi 1996, 281). By contrast, Pianesi and Varzi consider mereology and topology as conceptually independent. In the framework that they develop, there is a primitive specific to each domain. The notion of *part* comes solely from mereology and the notion of *boundary* is exclusively derived from topology (Pianesi and Varzi 1996, 91). From this perspective, they argue in favor of a unique event structure that constructs temporal ordering from restrictions on the relevant mereotopological structure (Pianesi and Varzi 1996, 98). (For more on the variety of and technical detail on (first-order) mereotopology, see Pratt-Hartmann (2007).)

Whereas Pianesi and Varzi were primarily concerned with event construction and temporal order, Peter Forrest has taken a topological approach to the mereology of regions in space (or spacetime) (Forrest 1996).[22] The thesis that Forrest supports is reminiscent of Russell's early account of events (in which he followed Whitehead) described in section 2.1. He argues in favor of an ontology of space that includes regions only, where points are constructed from regions. In support of this view, Forrest (1996, 34) uses topology to respond to two questions. First, which assumptions about regions are needed in order to achieve point constructions? Secondly, how can regions be described?

Instead of starting with points as primitives and describing regions in terms of them, Forrest begins with providing topological properties of regions. By thinking of sets of regions

[22] For another account of topological regions that includes a study of maps between regions, see Roeper (1997).

that have arbitrarily small diameters—effectively, neighborhood systems in the sense of Hausdorff described in section 2.1—it is possible to generate topological concepts for regions. For example, Forrest defines two regions as being connected if and only if the set of regions overlapping both of them is the zero norm (Forrest 1996, 42). He further invokes a special kind of set of regions called a filter as follows: “filter F is a non-empty set of regions such that: (i) if U and V belong to F then F contains a lower bound of U and V (ii) if U belongs to F and U is part of V then V belongs to F” (Forrest 1996, 42). With this topological machinery in place, points are defined in terms of regions in terms of equivalence classes of zero norm filters that hold under a particular distance relation (Forrest 1996, 43).

More recent work in mereotopology concerns space as constructed out of regions. Forrest has contributed to this literature (Forrest 2010), but an earlier influential example is Peter Roeper’s “Region-Based Topology” (Roeper 1997). In this work, Roeper takes as given that points are not parts of space, but are rather locations in space. Regions are parts of space, and as a result they, not points, are the subject of spatial properties and relationships. So, Roeper aims to describe the structure of space in terms of regions and describe points only in terms of this structure (Roeper 1997, 251). The description of regions and their relationships is topological in nature, as the title of the article would suggest.

Roeper’s approach has been influential and has a direct connection to Whitehead’s aforementioned approach to the construction of points from regions.[23] But the conception of topology it uses is distinct from the point-set topology we have discussed so far. Roeper uses ultrafilters and the Boolean algebra of regions (Roeper 1997, 258), which begins from an *algebraic* conception of the relationships that regions bear to one another. (We discuss ultrafilters shortly in section 3.1.) Indeed, the set of ultrafilters of a Boolean algebra are the *stone spaces* of the algebra (Roeper 1997, 308). This connects his approach with another conception of topology that has been fruitfully introduced into philosophy. It is that movement to which we turn in the next section.

3 Second Movement: Logic and Topology through Algebra

3.1 Stone, Tarski, and McKinsey

In the previous section, we saw how a geometric conception of topology informed philosophical discussions of space by Russell, Carnap, Franklin, and others, a conception which continues to have lasting influence in contemporary mereotopology. Now, we turn to a second movement of topological methods into philosophy, one focused less on the foundations of space-time and more on logic. Additionally, this thread of scholarship reflects an algebraic conceptualization of topology. This emphasis on logic and algebra in the application of topology has lent itself to

[23] Roeper also cites an earlier predecessor, by Arnold Johanson, that takes a similar approach using category theory (Johanson 1981). Johanson cites Whitehead and Russell’s TAM, among others, as inspiration.

somewhat varied fields. In this subsection, we focus on the genesis of this idea in logic, and in the next subsection, we show how it has been applied in computer science and formal epistemology.

The link between logic and topology has an origin through the work of Marshall H. Stone over the course of several publications. He first announced his results in “Boolean Algebras and their Application to Topology” (Stone 1934). The proofs for the theorems listed in that brief article appeared in two much lengthier articles: “The Theory of Representation for Boolean Algebras” (Stone 1936) and “Applications of the Theory of Boolean Rings to General Topology” (Stone 1937). As this latter title indicates, this 1937 article is relevant to our present topological considerations. That article provides the proof of a mathematical duality between Boolean algebras and a specific type of topological space, now known as a Stone space.[24]

To explain the significance of Stone’s result, in what follows we first review the close connection between classical logic and Boolean algebra. Then we identify the Boolean algebraic structure found within any topology—namely, the algebra of clopen sets of a topological space—and describe, conversely, how to construct a topology from any Boolean algebra’s ultrafilters.[25] We explain how these constructions are dual for Stone spaces, in that applying one construction procedure after the other always brings one back to the structure or space (or one isomorphic with it) with which one started. Lastly, we elaborate how this idea was influential and extended to connect it with other types of logic, especially modal logic.

A Boolean algebra is a non-empty set A equipped with two two-argument functions, $\wedge$ and $\vee$, one one-argument function $\neg$, and two designated elements, 0 and 1, all satisfying the following axioms for all $a, b, c \in A$:

Associativity	$a \wedge (b \wedge c) = (a \wedge b) \wedge c$	$a \vee (b \vee c) = (a \vee b) \vee c$
Commutativity	$a \wedge b = b \wedge a$	$a \vee b = b \vee a$
Absorption	$a \wedge (a \vee b) = a$	$a \vee (a \wedge b) = a$
Identity	$a \wedge 1 = a$	$a \vee 0 = a$
Distributivity	$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$	$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
Complement	$a \wedge \neg a = 0$	$a \vee \neg a = 1$

By inspection, one can see that Boolean algebras provide structures for the semantics of classical propositional logic. We have already chosen the two two-place function symbols $\wedge$ and $\vee$ to evoke the conjunction and disjunction, the one one-place function symbol $\neg$ to evoke the negation, the equality symbol = to evoke the biconditional, and the symbols for the two designated elements 0 and 1 to evoke a contradiction and tautology, respectively. One can then

[24] Stone himself had called them Boolean spaces for obvious reasons (Johnstone 1982, xvi).

[25] Alternatively, one can proceed from the Boolean algebra’s prime ideals, which are dual to the ultrafilters and more faithful to Stone’s original presentation (Johnstone 1982, xv–xvi).

easily extend this to first-order logic by defining the infimum and supremum of non-empty subsets of A, which correspond roughly to infinite applications $\wedge$ and $\vee$, respectively, hence universal and existential quantifiers (Button and Walsh 2018, 296–300). Algebras for which the infimum and supremum are always defined are said to be *complete*.

Given what we have developed so far, we can already identify a complete Boolean algebra within a topological structure. To do so, we just need to introduce a couple more concepts from topology. Recall from section 2 that the original formulation of a topological space by Hausdorff assigned a neighborhood system to each point of a set. A subset of this set is said to be *open* when it is a neighborhood of every element it contains. The open sets of a topological space with underlying set *S* satisfy the following conditions:

1. Both the empty set and *S* are open.
2. If $\{O_\alpha\}$ is a collection of open sets, then $\bigcup_\alpha \{O_\alpha\}$ is open.
3. If $\{O_\alpha\}$ is a finite collection of open sets, then $\bigcap_\alpha \{O_\alpha\}$ is open.

Condition 1 is satisfied because (i) the empty set contains no elements and thus it is vacuously true that it is a neighborhood of each element it contains (that is to say, none); and (ii) the underlying set *S* is always a neighborhood of each of its points. Condition 2 is satisfied because each element in the union is in some set in the collection, each set in the collection is a neighborhood of each element it contains, and the union is a superset of each set in the collection. Condition 3 is satisfied because any point in the intersection is in each set in the finite collection, each such set is a neighborhood of that point, and neighborhood systems are closed under finite intersections.

Conversely, a subset of *S* is said to be *closed* when it contains all of its *boundary points*, i.e., the points all of whose neighborhoods contain at least one point in the set and one point outside of the set. It is then elementary to prove that every closed set *C* is the complement of an open set *O*—i.e., $C = S - O$— and that the collection of closed sets satisfy the following properties:

1. Both the empty set and *S* are closed.
2. If $\{O_\alpha\}$ is a finite collection of closed sets, then $\bigcup_\alpha \{O_\alpha\}$ is closed.
3. If $\{O_\alpha\}$ is a collection of closed sets, then $\bigcap_\alpha \{O_\alpha\}$ is closed.

These properties follow from the analogous properties for open sets and the set algebra for complements.

To summarize, the collection of open sets of any topological space is closed (in the sense from the theory of relations) under arbitrary unions and finite intersections, while the collection of closed sets is closed (in the sense from the theory of relations) under finite unions and arbitrary intersections. Moreover, closed subsets are complements of open sets and vice versa. These conditions suffice to characterize a topological space: a set of subsets satisfying the three listed properties for open sets determines a neighborhood system for each point as the collection of sets containing an open set itself containing that point. In fact, this algebraically elegant characterization has become the most common *definition* of a topological space in modern textbooks.

The connection between sets of open and closed sets to Boolean algebras follows from the following identifications: take the empty set as 0 and the entire set S as 1, complementation as ¬, and intersection and union as ∧ and ∨. Then each of the collections of open and closed sets of a topological space is a Boolean algebra. However, they are not necessarily complete Boolean algebras, because the open sets are not closed under arbitrary intersections and the closed sets are not closed under arbitrary unions. This means that their corresponding Boolean algebras may not have some infima or suprema, respectively. But if one considers the sets which are both closed and open—the *clopen* sets—then the result is a complete Boolean algebra.

So, every topological space characterizes a Boolean algebra through its clopen sets. Conversely, every Boolean algebra characterizes a topological space. To see how, we introduce the concept of an *ultrafilter* for a Boolean algebra A (Button and Walsh 2018, 297–98). It is any $F \subseteq A$ such that

1. $0 \notin F$,
2. if $a \in F$ and $b \in F$, then $a \wedge b \in F$,
3. if $a \in F$ and $a \wedge b = a$, then $b \in F$, and
4. for any $a \in A$, either $a \in F$ or $\neg a \in F$.

The first three conditions make F a filter—cf. the discussion of filters as sets of regions in section 2.3—so an ultrafilter is a maximal filter in the sense that it contains, for every element, either that element or its complement. One can interpret these conditions in terms of classical logic. Condition 1 states that an ultrafilter does not contain any contradictions. Condition 2 states that it is closed under conjunction. The significance of the formula $a \wedge b = a$ in condition 3 comes from the fact that in classical logic, $(a \rightarrow b) \leftrightarrow ((a \wedge b) \leftrightarrow a)$. So, this condition states that ultrafilters are closed under material implication. Condition 4 requires ultrafilters to be, in a sense, syntactically complete.

Now let $U(A)$ be the set of ultrafilters of the underlying set of the Boolean algebra A. For each $a \in A$, define $u(a) = \{F \in U(A) : a \in F\}$. These sets determine a base for a topology on $U(A)$, i.e., a collection of sets unions of which form (an algebra of) all the open sets of the topology. Stone showed two things about such topological spaces (the Stone spaces). First, they satisfy a number of equivalent, simple characterizations as topological spaces.[26] Second, he proved that if one followed this construction by the construction of a Boolean algebra by clopen sets, the resulting algebra would be isomorphic with the algebra one started with, and mutatis mutandis for starting with a topological space. Moreover, these constructions preserve morphisms between pairs of algebras or Stone spaces (Button and Walsh 2018, 341–42).

Stone's connection of topology to the field of abstract algebra was quite innovative. In fact, prior to Stone's work, Garrett Birkhoff resisted the suggestion of potential importance of

[26] For instance, they are exactly the spaces that are totally separated and compact. See Johnstone (1982, 69–70) for a number of other, equivalent characterizations.

topology for the field of algebra (Johnstone 1982, xiv–xv). Stone's work demonstrates an equivalence in the study of aspects of Boolean algebra and topological spaces so that one can study algebraic structures through topology and certain topological structures—see footnote 26—through algebra, as was immediately recognized (MacLane 1939, 89). Slightly later, Stone (1938) emphasized an extension of his results involving order and lattice theory that allows study of a wider class of topological spaces. The basic idea is very simple. Define a partial order $\leq$ on the elements of a bounded lattice so that for every pair of elements a and b, $a \leq b$ iff $b = a \vee b$. (A bounded lattice is like a Boolean algebra except without a complement and its associated axioms.) Logically, the order represents entailment. So, we can define for elements a and b an element $a \rightarrow b$ as the greatest element (in the partial order) c such that $a \wedge c \leq b$, and then for any element a the element $\neg a$ as $a \rightarrow 0$. When such elements always exist, the resulting structure is called a Heyting algebra (although Stone himself called it a "Brouwerian lattice"). Stone showed that the elements of any complete Heyting algebra can be put into one-to-one correspondence with the open sets of a topological space by correlating the partial order of the algebra with the subset relation on the open sets. One of the reasons this is especially philosophically interesting is that Heyting algebras provide semantical models for intuitionistic logic.

In the decades after its publication, Stone's work received much attention in the field of mathematics. Of the copious citations to his work, a few noteworthy examples are Jonsson and Tarski's "Boolean Algebras with Operators" (Jonsson and Tarski 1951) and Wallman's "Lattices and Topological Spaces" (Wallman 1938).[27] Wallman extended Stone's ideas to study geometry, namely the homology theory of topological spaces, where they remain important. Jonsson and Tarski extended it to Boolean algebras with operators, such as those for closure or projection. The study of the Stone representation theorem remained a topic of graduate texts in mathematics departments (Johnstone 1982), and a part of subsequent textbooks on topology more generally (Vickers 1989, 128–29). Indeed, Johnstone argues that Stone's work had some influence on every branch of mathematics except for finite group theory, combinatorics, classical analysis, and number theory.

In the case of mathematical logic, the usefulness of Stone's theorem came later, being chiefly recognized in the 1970s and early 1980s (Johnstone 1982, xix–xx). However, Dana Scott's "Extending the Topological Interpretation to Intuitionistic Analysis" (Scott 1968) was influenced by Stone and it has garnered some modest attention among philosophers working in intuitionistic logic. For example, the *Stanford Encyclopedia of Philosophy* article "Intuitionism in the Philosophy of Mathematics" makes reference to Scott's topological model in theory analysis for intuitionistic logics (Lemhoff 2020). Also, in their recent book *Varieties of Continua*, Geoffrey Hellman and Stewart Shapiro cite Scott's techniques in reference to an intuitionistic construction of real numbers on an Aristotelian continuum (Hellman and Shapiro 2018, 96).

[27] According to Google Scholar, Stone's article (Stone 1937) has over 1800 citations, and each of these subsequent works (Jonsson and Tarski 1951; Wallman 1938) has hundreds of citations each.

Stone's theorem is perhaps the most famous demonstration of the algebraic properties of topological structures. But, like the trunks of the Pando tree which share their roots, investigations depending on an algebraic conception of topology arose elsewhere. Tarski discovered his own results on the connections between algebra and topology around 1935.[28] These findings were not published until 1937 (Tarski 1956). In this work, Tarski interprets sentential calculus in the following manner: "With every sentence μ of the sentential calculus we correlate, in one-one fashion, a sentence μ_1 of topology in such a way that μ is provable in the two-valued calculus if and only if μ_1 holds in every topological space" (Tarski 1956, 421). But the resulting topological structures to which these results apply end up being only the discrete topologies, which are uninteresting because they simply represent each point of a space as being isolated from each other point (Grosholz 1985, 149; 2007, 271). Nevertheless, Tarski was encouraged by Stone's progress (Stone 1936; 1937). In the reprinting of this article, Tarski explicitly notes that there is a strong connection between this work of his and that of Stone (1938).[29]

In 1944 Tarski co-authored a paper with J.C.C. McKinsey titled "The Algebra of Topology" (McKinsey and Tarski 1944). As the title would suggest, the motivation for the paper is to contribute to the development of an algebra that relates to point-set topology in very specific ways. The ultimate goal of the paper is to shed some light on the axiomatic foundations of topology (McKinsey and Tarski 1944, 141). To accomplish this, they construct the foundation of a novel algebraic calculus and investigate its relationship to topology. The postulates of this algebra are provided by including the properties of closure (as found in topology) in addition to the standard Boolean algebra postulates. They call this a "closure algebra."

This paper paved the way for future work applying topology in conjunction with modal logic. Prior to 1944, McKinsey had published work on the relationship between topology and Lewis systems of modal logic (McKinsey 1941). This earlier work includes a proof of a theorem that is very similar to theorem 2.5 in their (McKinsey and Tarski 1944). But Tarski recognized broader applications of topology in logic that extended beyond the case of modality in McKinsey's earlier work. This included modeling the axiomatization of the consequence relation on the topological notion of closure and the finiteness property of satisfiability for sets of sentences on the compactness property of a topological space (Feferman 2004, 8).

Although Tarski recognized these general connections, one of most significance was the topological semantics for modal logic which arose from his work with McKinsey. These ideas have been applied in intellectual circles at the intersection of logic and epistemology.[30] For

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[28] Grosholz (1985, sec. 2; 2007, chap. 10.3) provides a comparison of Tarski and Stone's work in this area and offers an account of the resulting significance of topology for logic as a branch of mathematics. In particular, she comments on the initial limitations of the analogy between topology and logic, which was initially restricted to quite limited logics and topological spaces, that had to be overcome for it to become widely fruitful.

[29] Specifically, theorem 4.11 in Tarski (1956) and theorem 7 in Stone (1938).

[30] What follows is not a comprehensive statement of the extensive publications by these authors and other scholars working on topological formal learning theory and epistemic logic. Rather, we intend for the reader to be merely introduced to one fruitful branch of intellectual development that has arisen from the introduction of topology into philosophy.

example, Alexandru Baltag and colleagues have generated numerous results in modal variants of epistemic logics that have elements originating from Tarski and McKinsey's seminal work (Baltag, Bezhanishvili, and Fernández González 2019; Baltag et al. 2017; 2016). In their work, evidence for beliefs is modeled in modal terms as non-empty sets of possible worlds. Aggregates of evidence are further modeled as non-empty intersections of finite sets of evidence. The "evidential topology" is formed by the family of combined evidence that constitutes a topological basis. In this context, the authors examine the operator "having (combined) factive evidence for P." They demonstrate that this operator matches McKinsey and Tarski's own modal topological semantics (Baltag et al. 2016, 83–84).

McKinsey and Tarski's work has inspired much work in mathematical logic, especially modal logic, spatial logic and intuitionistic logic. The details of these developments fall outside the scope of our present project because we are chiefly concerned with the application of topology in philosophy. Nonetheless, it is worthwhile to note the extent of the influence of these concepts in mathematical logic if only because of their vast propagation in that consanguineal field. The McKinsey-Tarski topological interpretation of modal logic has been essentially foundational in multiple research programs concerning logics for spatial structures (van Benthem and Bezhanishvili 2007, 217–18). These include developments in modal logics of geometrical structures as applied in physical space (Shehtman 1983) and the application of topology for duality within closure algebras and various extensions thereof (Goldblatt 2003, 361). The latter of these developed into a long-term research program, especially from the 1970s up to the present, on extending Stone's duality to other types of logics, as Jonsson and Tarski had done in one direction (Esakia 1974; 2004; Bezhanishvili, Esakia, and Gabelaia 2010; Beklemishev and Gabelaia 2014).[31]

3.2 Modern Developments: Logics of Reliable Inquiry

Conspicuously, citations to Stone's theorem outside of mathematics are harder to come by and virtually non-existent in philosophy and philosophy of science. This was not due to the irrelevance of Stone's work to philosophy. Rather, the interdisciplinary interactions were gradual and the avenues into philosophy were multiple. We have already discussed in the last subsection the interaction with logic. Another entry point was through developments in computer science and formal learning theory in the 1980s that Kevin Kelly and collaborators introduced into the philosophy of science.

Central to these developments was the algebraic and logical conception of topology, which Stone inaugurated, concerned with algebras of open, closed, and clopen sets. Researchers working with this conception explicitly contrast their point of view with the geometric one,

[31] Again, this just scratches the surface of the application of topology in mathematical logic. See also, for instance, Grosholz (2007, chap. 10.5) for an accessible treatment of some connections between topology and model theory, and Aiello et al. (2007) for a variety of surveys of spatial logics that employ topology.

whose influence we discussed in section 2. For instance, Vickers writes in the preface to his textbook, *Topology via Logic* (1989), that

> The traditional—spatial—motivation for general topology and its axioms relies on abstracting first from Euclidean space to metric spaces, and then abstracting out, for no obvious reason, certain properties of their open sets. I believe that the localic [algebraic] view helps to clarify these axioms, by interpreting them not as set theory…, but as logic….

The logic Vickers refers to has its basis in domain theory as a computation-theoretic semantics for programming languages. Smyth (1983) first articulated the view that "open sets are [i.e., represent] semidecidable properties," which are the properties that some effective method can verify (but not necessarily refute). Just a few years later, Abramsky (1987) re-interpreted the algebra of open sets as a "logic of finite observations," connecting observations with verifications of empirical properties.

In formal learning theory, one often considers sequences of such observations and the circumstances under which there is an effective method for a learner to acquire the rule governing the outcomes of the observations, at least approximately and with high probability. For example, such a rule within evolutionary biology might take the form: "trait *x* is usually present in individuals surviving to maturity." So, an observation of an individual that possesses trait *x* and survives to maturity might be encoded as a 1. Conversely, an observation of an individual surviving to maturity without trait *x* is assigned a 0 (Oliver Schulte and Juhl 1996, 142).

One can then consider how sequences of observations of this sort confirm or disconfirm the rule, which one can represent by the sequences of observations that it predicts. In line with the interpretation from Abramsky, verifiable hypotheses or rules can then be represented by open sets in the space of sequences, and refutable hypotheses or rules by closed sets (for each of their negations, represented by their complements in the sequence space, is verifiable) (Oliver Schulte and Juhl 1996, 142–43).

Kelly develops this point of view and its applications in formal epistemology and philosophy of science[32] in great detail in *The Logic of Reliable Inquiry* (LRI) (Kelly 1996), where he investigates the question of finding effective methods that are guaranteed to arrive at true rules eventually. Kelly partitions these questions into different categories according to levels of abstraction. For example, the zeroth level refers to questions about a method's bearing on hypotheses for some finite data sequence, and the first level questions pertain to a method's implications for the kinds of potential inputs (finite data streams, hypotheses) which are possible. Topological ideas appear in the fourth chapter of Kelly's book: "Topology and Ideal Hypothesis Assessment." This chapter addresses questions in the fourth (and highest) level of abstraction in the book, which involve questions about the class of inductive problems themselves (Kelly 1996, 35–36).

[32] For instance, one of Kelly's main theses is that computability should play as important a role in formal epistemology and induction in the philosophy of science as probability enjoys (Kelly 1996, 8; Suppes 1998, 351).

Kelly is interested in characterizing which sorts of hypotheses, characterized by data sequences, are verifiable, refutable, and decidable verifiability, refutability, and decidability (Kelly 1996, 83–88). (The decidable hypotheses are the ones that are both verifiable and refutable.) Arbitrary disjunctions of certainly verifiable hypotheses are certainly verifiable, since verifying one verifies the disjunction. Analogously, arbitrary conjunctions of certainly refutable hypotheses are certainly refutable, since refuting one refutes the conjunction. Just as for Abramsky, data sequences characterizing such hypotheses are exactly the open and closed sets, respectively. The data sequences characterizing decidable hypotheses are clopen. Kelly goes beyond these original identifications to characterize the sorts of hypotheses represented by different members of the Borel hierarchy, a way of describing the complexity of a set in a topological space by reference to its open sets (Kelly 1996, 89–90). He applies this specifically to the Baire space, which is the space of infinite sequences of natural numbers, hence represents data sequences. The first level of this hierarchy begins with the clopen sets and generates new levels by complementation (negations of hypotheses) and countable union (disjunctions of hypotheses). The first application results in the open and closed sets, respectively. Kelly shows why higher levels correspond with hypotheses that may not be certainly verifiable (refutable, decidable) but only so in the infinite limit, of those that are only gradually verifiable (refutable, decidable), such as those concerning limiting relative frequencies. (Kelly 1996, 116).[33]

The fruitfulness of topology in fields such as formal learning theory, formal epistemology, logic, and philosophy of science has been recognized in literature that propagated after the publication of LRI. For example, Oliver Schulte (1999; 1999) has taken a similar approach to long-standing problems of induction. Subsequent works by Kelly and colleagues have explored statistical applications of the topological ideas in LRI (Genin and Kelly 2017). Additionally, Alexandru Baltag, Sonja Smets, and colleagues (Baltag et al. 2013; 2016; Baltag, Gierasimczuk, and Smets 2016) draw on the topological work of LRI to combine it with their work in epistemic logic, which, as we discussed at the end of section 3.1, itself draws directly from ideas stemming from Stone's representation theorem. Additionally, recent debates concerning convergence of opinion in Bayesian epistemology have employed topological structure on the space of outcomes related to that in LRI (Belot 2013; Elga 2016; Norton 2011). In his survey of the relationships between knowledge and logic, Johan van Benthem notes that Kelly's formal model of learning has potential to contribute to fruitful developments in logical dynamics (van Benthem 2006a, 69). Similarly, in "Open Problems in Logical Dynamics," van Benthem cites the "event trees" found in LRI as possessing an underexplored link with dynamic logics of information and modal logic (van Benthem 2006b, 183). Elsewhere, in a description of a broad view of information dynamics, van Benthem and Maricarmen Martinez point out that their account has connections with the formal learning theory of LRI (van Benthem and Martinez

[33] In this chapter on topology, Kelly also draws on theoretical resources found in the literature related to recursion and decidability (Putnam 1965; Gold 1965; Kugel 1977). Although Kelly's application of these ideas is interesting, our focus in this section remains on the use of topology as facilitated by the conceptual connections derived from Stone's theorem. For more on the connections between recursion theory, topology, and the Borel hierarchy, see Grosholz (1985, sec. 3; 2007, chap. 10.4).

2008, 238).[34] Much of this relevance to epistemic logic is centered around modeling the acquisition of knowledge. Christopher Steinsvold has exhibited the topological properties of logic that models an agent's *lack* of knowledge (Steinsvold 2008). Here, Steinsvold acknowledges that through his application of topology in epistemic logics he is operating in the same vein as Kelly's work in LRI (Steinsvold 2008, 386).

4 Coda: Conclusions and Prospects

4.1 Conclusions Regarding the Scope of Topology in Philosophy

In section 2, we showed how topological concepts entered into philosophical discussion of the nature of space and time. Largely through the significant attention that Einstein's general theory of relativity garnered starting in the 1920s, philosophers such as Russell and Carnap came to deploy topology in their early work on the subject. We argued that this explains the presence of Franklin's 1935 expository essay in Philosophy of Science, and showed the continued influence and centrality of topological concepts in describing the nature and structure of space within mereotopology.

In section 3, we introduced and reviewed a different conception of topology that Stone inaugurated through his representation theorem for Boolean algebras. In contrast with the geometric conception of topology as an abstraction from geometry—the conception that featured in section 2—Stone's work revealed an algebraic conception of topology through the algebraic structure of its open, closed, and clopen sets. The connection with Boolean algebras then provides a bridge between topology and semantical structures for logic. Besides its impact across many parts of mathematics, Stone's ideas have thus also been extended to modal and intuitionistic logics. In computer science in the 1980s, the algebraic conception yielded an interpretation of open and closed sets as representing verifiable and refutable propositions, which in the 1990s Kelly and collaborators applied in the philosophy of science.

In light of our review and commentary on these two movements, we can return to answering Mormann's query, adumbrated in section 1, about "why in the beginnings of the last century geometry lost its privileged status in philosophy and couldn't pass it on to topology" (2013, 433) while logic became the dominant formal method in philosophy. Recall that Mormann argued for a contrast between the influence of geometry and topology:

> traditionally geometry had also served as a source for inspiration and as an arsenal of conceptual tools for philosophy itself. This fruitful exchange did not find a continuation between the 20th century philosophy of science and topology. Ideas from topology hardly

[34] Additionally, van Benthem gestured towards the future directions that might be taken by combining aspects of his project in *Logical Dynamics of Information* and the formal learning theory of LRI (van Benthem 2011, 249).

> found their way in the conceptual tool kit of the philosopher of science. (Mormann 2013, 425)[35]

This failure of a continued influence, on his view, illustrates a change in philosophers' understanding of their aims and methods:

> Since from the mathematical point of view there is no essential epistemological, ontological, or methodological difference between geometry and topology, the negligible amount of attention that philosophy paid to topology in the last century must be attributed to a change in the way philosophers understood the aims and methods of philosophy of science. (Mormann 2013, 425)

Clearly Mormann himself conceives of topology geometrically just as, according to our discussion in section 2, the first movement of topological ideas into philosophy did. Topological structure, as an abstraction from the concrete distances, angles, and curvature of metrical structure, is born from geometrical structure and inherits much of the latter's foundational interest, according to this view.

Insofar as geometry was central to the parts of philosophy in the early 20th century with which Mormann is concerned, however, it was in part so because of attention to the implications of the special and general theories of relativity (T. A. Ryckman 2018). To our knowledge, philosophers did not conceive of mathematical geometry as providing a formal method for addressing topics in general philosophy of science, much less philosophy broadly, but rather as one apt to specific topics concerning the nature of space and time. The early work of Russell and Carnap, discussed in section 3.1, engaged with topological concepts for precisely this purpose, as did subsequent work in mereotopology discussed in section 3.2.

This suggests a different hypothesis to explain the changing fortunes of geometry and, by extension, the meager inheritance of topology. If geometry has lost any centrality in philosophy or philosophy of science, it may just be due to a topical shift of focus rather than a methodological one. On this hypothesis, as the novelty of relativity theory waned and philosophers turned their attention to other topics, philosophers adopted methods that they saw fit to address those topics. Indeed, research concerning space and time *continues* to employ concepts from topology. This is compatible with Mormann's ultimate conclusion that the aims and methods of philosophy may have changed, but suggests an alternative to it when it comes to demand an explanation for why topological concepts have not been as central to philosophy.

This topical (rather than methodological) explanation of topology's import in philosophy also comports with the rising fortunes of logic during the birth and youth of analytic philosophy.

[35] Mormann also draws another contrast, that "20th century philosophy of science showed no interest in topology as an object of philosophical reflection. There has been no 'philosophy of topology' in analogy to disciplines such as 'philosophy of physics', 'philosophy of biology', or 'philosophy of geometry' " (Mormann 2013, 425). However, the analogy is not as strong as it might at first seem, since physics and biology are distinct sciences, while geometry historically has been a partly mathematical, partly empirical discipline. Topology, by contrast, is clearly a subdiscipline of mathematics.

The use of logical methods became widespread, at least in part, because of their promise to clarify a wide range of topics across philosophy. They were not so topically specific. As an example of this sentiment, Michael Dummett remarked that analytical philosophy was post-Fregean philosophy, and that what distinguished analytic philosophy from other schools of philosophy was its embrace of logical methods: "Frege's fundamental achievement was to alter our perspective in philosophy, to replace epistemology, as the starting-point of the subject, by what he called 'logic' " (Dummett 1978, 441).

But as section 3 showed, beginning with Stone's representation theorem, there was another movement of topological ideas, one that considered them from an algebraic perspective, which has had an influence on mathematical logic.[36] Does this not raise Mormann's puzzle about the lack of centrality of topology anew? Why did not the fortunes for topology rise with logic, with Stone's theorem tethering them? Again, our exposition in section 3 suggests an answer based in the topical relevance of topological ideas rather than a methodological change: topology *has* had an influence in mathematical logic, where results like Stone's theorem provide powerful tools to prove results about the semantics of logical systems via topology (and vice versa). But the sorts of problems to which *most* of philosophy has applied methods from logic do not benefit from knowledge of this duality. As discussed in section 3.2, it seems that it was not until the 1980s that computer scientists found a non-mathematical application of Stone's duality and the algebraic interpretation of topology it suggests in the logic of verification and falsification. Since then, Kevin Kelly and collaborators have adopted this interpretation in their own work applying formal learning theory to issues in philosophy of science. So, the algebraic perspective on topology *has*, like the geometric perspective, had some influence in philosophy, but limited to the particular types of philosophical problems to which it naturally relates.

4.2 Prospects for Topology beyond Geometry and Algebra: Similarity

Through two movements, topology has in the 20th century found its way into the conceptual toolkit of philosophers, even though it was sometimes decades in the making. Nevertheless, its application has been and continues to be neither familiar nor routine. In section 4.1, we suggested that the particular conceptions of topology associated with each of those movements explains why. In the first movement, in which topology is understood geometrically, topological ideas apply just to topics concerning the philosophy of space and time, and adjacent areas such as mereology. In the second movement, in which topology is understood algebraically, topological ideas allow one to reformulate the semantical structures for many types of logics. Consequently, they are only topically applicable to questions in philosophy that demand a powerful mathematical duality to prove statements of interest about these logics.

In light of these two conceptions, it is less clear that topology has been neglected as a font for concepts and methods in philosophy instead of receiving the limited attention it is due. What

[36] Mormann (2013, 428, 431) does discuss Stone's theorem, including how it connects logic with topology, but seemingly only in the context of a geometric conception of topology. See also Mormann (2020, sec. 3).

prospects could there be, therefore, for topological ideas to be ascendent as Mormann suggests? He writes:

> Only in recent decades logic has begun to [lose] its monopoly and geometry and topology received a new chance to find a place in philosophy of science, as an object for philosophical reflection and as a conceptual tool for doing philosophy. (Mormann 2013, 423)

Mormann desires "to show that the introduction of topological structures may elucidate the role of the spatial structures (in a broad sense) that [underlie] logic and cognition" (Mormann 2020, 135) and hence their "role in epistemology and metaphysics" (Mormann 2020, 137). If one conceives of topology geometrically, as Mormann does, then the promise of topological structures depends on how convincing a case could be made that the concerns of logic, cognition, epistemology, and metaphysics really are spatial.

One of Mormann's key illustrations of this promise is the case of conceptual spaces (Mormann 1993; 2020, sec. 4; 2021).[37] Peter Gärdenfors (2000; 2014) introduced conceptual spaces as vector spaces, some of whose elements represent conceptual prototypes, such as particular colors in a color space. True concepts are not arbitrary subsets of the space, but convex subsets. Some particular thing, as an element of that space, falls under a concept to the extent that it is close in that space to the concept's prototype. Understanding this closeness topologically, rather than geometrically, proves fruitful in understanding the gradations in (generally non-sharp) boundaries between real concepts (Mormann 2021).

However, there is potential for a *third* conception of topology, neither literally geometric nor algebraic, which may play a role across philosophy and has only recently been introduced (Fletcher forthcoming). That conception is of *similarity*: topological structure on a set (and related types of structure, such as uniform structure) describes how the elements of that set are similar and dissimilar to one another in various degrees and sorts. In the neighborhood formulation of a topological space, the neighborhoods of an element describe multiple qualitative similarity relations between those elements and others in the set one is considering. This conception is more directly related to the geometric conception of topology than the algebraic conception, but it eschews interpreting topology negatively as an abstraction from geometry. Rather, it interprets topology positively as a construction from multiple similarity relations, suggesting even ways that topological structures might be modified to take into account a wider variety of ways that elements might be similar and how those ways might interconnect.[38]

The reason the similarity conception of topology has the potential to play a role across philosophical inquiry broadly is that the concept of similarity is pervasive and pertinent to a wide range of topics, while encompassing the geometric conception (interpreting spatial proximity as similarity of location). Perhaps most famously, relations of comparative similarity among

[37] Mormann (2020, sec. 5) also shows how to formulate different versions of Leibnizian identity principles for objects in topological terms (i.e., using topological separation axioms). We also believe that the topological structure employed in such cases should be interpreted in terms of similarity (of the objects involved) instead of in terms of geometry.

[38] We defer to Fletcher (forthcoming, sec. 2) for the technical details.

possible worlds form the basis of David Lewis's semantics for counterfactual (and more generally, subjunctive) conditionals (Lewis 1973). The semantics for these conditionals in turn has downstream implications for a wide range of central issues in epistemology, metaphysics, philosophy of action, philosophy of mind, and philosophy of science, among others (Starr 2021). Bartha (2010, chap. 5.6) has studied how topology encodes mathematical similarity within analogical reasoning in mathematics, which might plausibly extend to reasoning with mathematical models of concrete phenomena. Within philosophy of science, one of us (Fletcher forthcoming) has argued with concrete examples that this conception bears on questions regarding laws of nature and intertheoretic reduction, in addition to the characterization of emergent properties (Fletcher 2020a; 2021b), symmetries (Fletcher 2021a), and the epistemic connections between modeling and inference (Fletcher 2020b). The similarity conception of topology also better interprets Mormann's own suggestions for the fruitfulness of topological concepts: elements in a conceptual space are related not (literally) spatially, but by degrees of similarity along the dimensions of that space.[39] How fruitful and pervasive it may be will be up to future researchers to assess.

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[39] Mormann (2020, 144) acknowledges this particular interpretation: "Every conceptual space is endowed with a metric that measures the similarity between objects." But he does not seem to extend that interpretation to topology more generally.

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