Understanding Time Reversal in Quantum Mechanics: A Full Derivation

Shan Gao
Research Center for Philosophy of Science and Technology,
Shanxi University, Taiyuan 030006, P. R. China
E-mail: gaoshan2017@sxu.edu.cn.

May 1, 2022

Abstract

Why does time reversal involve two operations, a temporal reflection and the operation of complex conjugation in quantum mechanics? Why is it that time reversal preserves position and reverses momentum and spin? This puzzle of time reversal in quantum mechanics has been with us since Wigner's first presentation. In this paper, I argue that the standard account of time reversal in quantum mechanics can be derived from the natural requirement that time reversal reverses velocities by analyzing the continuity equation.

Why does time reversal involve two operations, a temporal reflection and the operation of complex conjugation in quantum mechanics? Why is it that time reversal preserves position and reverses momentum and spin? This puzzle of time reversal in quantum mechanics has been with us since Wigner's (1931) first presentation, although some progress has been made to solve it recently (Roberts, 2017, 2020; Struyve, 2020; Callender, 2021). According to some authors, time reversal "can involve nothing whatsoever other than reversing the velocities of the particles" (Albert 2000, p.20), and "It does not make sense to time-reverse a truly instantaneous state of a system" (Callender, 2000). I am in sympathy with the arguments of these authors. In this paper, I will argue that the standard account of time reversal in quantum mechanics can be derived from the natural requirement that time reversal reverses velocities by analyzing the continuity equation.

Consider the Schrödinger equation for a spin-0 quantum system in an external scalar potential:

$$i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t}=[-\frac{\hbar^2}{2m}\nabla^2+V(\mathbf{r},t)]\psi(\mathbf{r},t), \tag{1}$$

where $\hbar$ is Planck's constant divided by 2π, $\psi(\mathbf{r},t)$ is the wave function of the system, m is the mass of the system, and $V(\mathbf{r},t)$ is an external scalar potential. From this equation we can derive the continuity equation:

$$\frac{\partial \rho(\mathbf{r},t)}{\partial t} + \nabla \cdot \mathbf{j}(\mathbf{r},t) = 0, \tag{2}$$

where $\rho(\mathbf{r},t) = |\psi(\mathbf{r},t)|^2$ and $\mathbf{j}(\mathbf{r},t) = \frac{\hbar}{2mi}[\psi^*(\mathbf{r},t)\nabla\psi(\mathbf{r},t) - \psi(\mathbf{r},t)\nabla\psi^*(\mathbf{r},t)]$ are probability density and probability current density, respectively. By writing the wave function in the polar form $\psi = Re^{iS/\hbar}$, where R and S are real functions, we can obtain the local velocity for the probability current:

$$\mathbf{v}(\mathbf{r},t) \equiv \frac{\mathbf{j}(\mathbf{r},t)}{\rho(\mathbf{r},t)} = \frac{1}{m}\nabla S(\mathbf{r},t). \tag{3}$$

Now let's see how the wave function $\psi(\mathbf{r},t)$ is transformed by the time reversal operator T. First, since the definition of probability density (via the Born rule) does not depend on the direction of time, we have $T\rho(\mathbf{r},t) = \rho(\mathbf{r},-t)$, which leads to $TR(\mathbf{r},t) = \pm R(\mathbf{r},-t)$ due to $\rho(\mathbf{r},t) = R^2(\mathbf{r},-t)$. Next, since time reversal involves reversing velocities, we have $T\mathbf{v}(\mathbf{r},t) = -\mathbf{v}(\mathbf{r},-t)$, which leads to $TS(\mathbf{r},t) = -S(\mathbf{r},-t) + C_0$, where C_0 is a real constant. Note that the continuous equation is time reversal invariant under these transformations. Then we can obtain the standard antiunitary transformation rule for the wave function: $T\psi(\mathbf{r},t) = \psi^*(\mathbf{r},-t)$ when ignoring an overall constant phase. In addition, by analyzing the probability current acceleration, we can obtain the transformation rule for the scalar potential: $TV(\mathbf{r},t) = V(\mathbf{r},-t)$. Notably this transformation rule applies to the electric scalar potential $T\phi(\mathbf{r},t) = \phi(\mathbf{r},-t)$.

Based on the transformation rule for the wave function, we can derive the transformation rule for every observable from its definition (or its operation on the wave function). For example, for position $\mathbf{r}$, we have $T\mathbf{r}T^{-1} = \mathbf{r}$, and for momentum $\mathbf{p} = -i\hbar\nabla$, we have $T\mathbf{p}T^{-1} = -\mathbf{p}$, and for angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, we have $T\mathbf{L}T^{-1} = -\mathbf{L}$.

By analyzing the continous equation for a charged system in an electromagnetic field, we can also obtain the transformation rules for the electromagnetic potentials and fields. The probability current velocity for a spin-0 system with mass m and charge Q in an external electromagnetic field is

$$\mathbf{v}(\mathbf{r},t) = \frac{1}{m}[\nabla S(\mathbf{r},t) - Q\mathbf{A}(\mathbf{r},t)], \tag{4}$$

where $\mathbf{A}(\mathbf{r},t)$ is the magnetic vector potential. Then $T\mathbf{v}(\mathbf{r},t) = -\mathbf{v}(\mathbf{r},-t)$ leads to $T\mathbf{A}(\mathbf{r},t) = -\mathbf{A}(\mathbf{r},-t)$. Using the definition $\mathbf{B} = \nabla \times \mathbf{A}$, we can obtain the transformation rule for the magnetic field $T\mathbf{B}(\mathbf{r},t) = -\mathbf{B}(\mathbf{r},-t)$. By combining with $T\phi(\mathbf{r},t) = \phi(\mathbf{r},-t)$, we can also obtain the transformation rule for the electric field $T\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},-t)$. Note that both $\rho(\mathbf{r},t)$

and $\mathbf{v}(\mathbf{r},t)$ or $\mathbf{j}(\mathbf{r},t)$ are gauge invariant quantities which are physically measurable (for an ensemble of identically prepared systems).

Finally, we can also obtain the time reversal transformation rule for spin in a similar way. The probability current velocity for a spin-s system with mass m and charge Q and magnetic moment μ_s in an external electromagnetic field is

$$\mathbf{v}(\mathbf{r},t) = \frac{1}{m}[\nabla S(\mathbf{r},t) - Q\mathbf{A}(\mathbf{r},t)] + \frac{\mu_s}{s}\frac{\nabla \times (\psi^*(\mathbf{r},t)\mathbf{S}\psi(\mathbf{r},t))}{\psi^*(\mathbf{r},t)\psi(\mathbf{r},t)}, \tag{5}$$

where $\mathbf{S}$ is the spin operator. Then $T\mathbf{v}(\mathbf{r},t) = -\mathbf{v}(\mathbf{r},-t)$ leads to $T\mathbf{S}(\mathbf{r},t) = -\mathbf{S}(\mathbf{r},-t)$. Based on the transformation rules for spin and the wave function, we can also derive the famous result $T^2 = -I$ for spin-1/2 systems.

The above analysis provides a full derivation of the standard time reversal transformation rules in quantum mechanics. Based on this analysis, we can confirm that the Schrödinger equation is time reversal invariant as usually thought. This analysis can be extended to relativistic quantum mechanics and quantum field theory.

The derivation of the transformation rules also provides an intelligible way to understanding time reversal in quantum mechanics. Why time reversal involves complex conjugation is because the phase of the wave function is the spatial derivative of certain velocity and reversing the velocity as required by time reversal amounts to taking the complex conjugation of the wave function. Some authors have given a similar account (Earman, 2002; Sebens, 2015; Callender, 2021). Moreover, why time reversal reverses momentum, spin, and magnetic fields (which are not the rates of change of anything) is because these quantities are closely linked with the velocity in a certain way. In this sense, the above derivation satisfies Albert's stringent requirement that time reversal involves only reversing velocities.

To sum up, I have argued that by analyzing the continuity equation, the standard account of time reversal in quantum mechanics can be derived from the natural requirement that time reversal reverses velocities. This provides an intelligible way to understanding the time reversal invariance of the theory.

References

[1] Albert, D. Z. (2000). Time and Chance. Cambridge, MA: Harvard University Press.

[2] Callender, C. (2000). Is Time 'Handed' in a Quantum World?, Proceedings of the Aristotelian Society, 100 (3): 247-269.

[3] Callender, C. (2021). Quantum Mechanics: Keeping It Real? The British Journal for the Philosophy of Science. https://doi.org/10.1086/715032

[4] Earman, J. (2002). What time reversal is and why it matters. International Studies in the Philosophy of Science 16(3): 245-264.

[5] Roberts, B. (2017). Three myths about time reversal in quantum theory. Philosophy of Science 84, 315-334.

[6] Roberts, B. (2020). Time Reversal. The Routledge Companion to the Philosophy of Physics, Eleanor Knox and Alistair Wilson (eds).

[7] Sebens, C. (2015). Quantum Mechanics as Classical Physics. Philosophy of Science, 82(2): 266-291.

[8] Struyve, W. (2020). Time-reversal invariance and ontology. https://philsci-archive.pitt.edu/17682/.

[9] Wigner, E. P. (1931). Group Theory and its Application to the Quantum Mechanics of Atomic Spectra, New York: Academic Press (1959).