# PROBABILITY AND SYMMETRIC LOGIC

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**Abstract**

In this paper we study the interaction between symmetric logic and probability. In particular, we axiomatize the convex hull of the set of evaluations of symmetric logic, yielding the notion of probability in symmetric logic. This answers an open problem of Williams [12] and Paris [8].

**Keywords:** Symmetric logic, Kleene algebra, Belief functions.

## 1 INTRODUCTION

This paper contributes to an ongoing project of axiomatizing convex hulls of evaluations for different logics. In particular, we answer an open problem of Williams [12] and Paris [8] by axiomatizing the convex hull of the set of evaluations of symmetric logic, yielding the notion of probability in symmetric logic.

By a logic we mean a pair $(\mathfrak{F}, \models)$, where $\mathfrak{F}$ is a set of formulas generated by a finite set of atomic propositions $P$ using a finite set of logical connectives, and $\models$ is a consequence relation between formulas (or sets of formulas) defined by means of certain truth assignments $w : P \to [0,1]$. Such truth assignments extend to evaluations $w : \mathfrak{F} \to [0,1]$ by applying the truth tables of the connectives. A belief function is an arbitrary mapping $B : \mathfrak{F} \to [0,1]$. The convex hull of the evaluations consists of belief functions that one might call 'probabilities'; indeed, this is the approach adopted in the field to which this paper aims to contribute, and for reading about which [12] serves as a convenient starting point.[1]

Let us introduce the issues under discussion by recalling a few earlier results.

**Probability and classical propositional logic.** In classical propositional logic the set $\mathfrak{F}$ of formulas is built up using a finite set $P$ of elementary propositions and the logical connectives $\land$ and $\neg$. One defines the consequence relation $\models$ via evaluations and truth tables. Evaluations are mappings $w : P \to \{0,1\}$ from propositional letters into truth values, which extend to mappings $w : \mathfrak{F} \to \{0,1\}$ by applying the Tarski truth conditions. (That is, each $w$ is a homomorphism from $\mathfrak{F}$ into the two-element Boolean algebra $\mathbf{2}$). We denote by $W$ the set of all evaluations.

De Finetti's theorem identifies belief functions that are probability functions with the convex hull of the evaluations.

[1] Note also [11], who writes about "generalized probabilities" (p. 527). In the larger context of norms of rationality, it is usually said that the convex hull of evaluations contains the belief functions which are not "Dutch-bookable"; one then typically applies the classical Dutch Book justification of identifying rational belief functions with probabilities (even in nonclassical settings). This argument and its justification are not topics of the present paper; we refer the Reader to [8, 12, 2].

**1.1 THEOREM** (DE FINETTI [4],[8]). Let $(\mathfrak{F}, \models)$ be a classical propositional logic and $B : \mathfrak{F} \to [0,1]$ a belief function. The following are equivalent.

(A) **Probability:** $B$ satisfies the axioms below.

> (P1) If $\models \varphi$ then $B(\varphi) = 1$, and if $\models \neg\varphi$ then $B(\varphi) = 0$,
> (P2) If $\varphi \models \psi$ then $B(\varphi) \leq B(\psi)$,
> (P3) $B(\varphi \vee \psi) + B(\varphi \wedge \psi) = B(\varphi) + B(\psi)$.

(B) **Convex combination:** $B$ is the convex combination of the functions in $W$. ∎

The non-trivial part of Theorem 1.1 is the direction from (A) to (B). The derivation is based on the observation that evaluations can be identified with atoms of the Lindenbaum–Tarski algebra of $\models$-equivalent formulas, and any probability mapping is determined by its value on the atoms via the rule (P3).

Similar results have been proven in the case when $(\mathfrak{F}, \models)$ is a non-classical logic. Let us recall such results in the next paragraph.

**Probability and non-classical logics.** In what follows we deal with logics where the set $\mathfrak{F}$ of formulas is generated by a finite set of propositional letters $P$ using a finite set of logical connectives $\tau$. Thus, $\mathfrak{F}$ is in fact the absolutely free term algebra generated in the finite similarity type $\tau$.

To recall the next result, we let **2** be a distinguished $\tau$-type algebra with the universe $\{0,1\}$. When the (derived) binary operations $\wedge$ and $\vee$ belong to $\tau$, then they are interpreted in **2** in the usual way, making $(\mathbf{2}, \wedge, \vee)$ a distributive lattice. By a $(\wedge, \vee)$-homomorphism we mean a mapping that is a homomorphism with respect to the operations $\wedge$ and $\vee$. Such mappings might not be homomorphisms with respect to other operations in $\tau$.

**1.2 THEOREM** (PARIS [8]). Suppose $\wedge$ and $\vee$ are (possibly derived) connectives and let $W$ be a set of $(\wedge, \vee)$-homomorphisms $w : \mathfrak{F} \to \mathbf{2}$. Suppose $\models$ is defined such that

$$\varphi \models \psi \quad \text{iff} \quad (\forall w \in W)\, (w(\varphi) = 1 \ \Rightarrow\ w(\psi) = 1). \tag{1}$$

Then the following are equivalent.

(A) $B : \mathfrak{F} \to [0,1]$ is the convex combination of the functions in $W$.

(B) $B : \mathfrak{F} \to [0,1]$ satisfies the axioms below.

> ($\mathcal{L}1$) If $\models \varphi$ then $B(\varphi) = 1$, and if $\varphi \models$ then $B(\varphi) = 0$,
> ($\mathcal{L}2$) If $\varphi \models \psi$ then $B(\psi) \leq B(\psi)$,
> ($\mathcal{L}3$) $B(\varphi \vee \psi) + B(\varphi \wedge \psi) = B(\varphi) + B(\psi)$. ∎

We presented Theorem 1.2 in a more algebraic way than Paris. The condition that each $w \in W$ is a homomorphism into **2** with respect to $\wedge$ and $\vee$ amounts to saying that the connectives $\wedge$ and $\vee$ behave classically. These are precisely the requirements ($T2$) and ($T3$) in [8].

Paris [8] gives an elementary proof of Theorem 1.2, and an alternative proof can be given using Choquet's [3, 41.1] (see also Bradley [2]). A key ingredient in these proofs is that $\models$ generates a congruence on the $(\wedge, \vee)$-reduct of $\mathfrak{F}$ and the quotient of this reduct with respect to the congruence is a distributive lattice. The reason for this is that $W$ consists of $(\wedge, \vee)$-homomorphisms and thus the quotient can be embedded into a suitable power of **2** which itself is a distributive lattice.

Theorem 1.2 applies to a number of well-known propositional logics, for example the standard modal logics $K$, $T$, $S_4$, $S_5$, etc.and to certain paraconsistent logics in which conjunction and disjunction retain their classical interpretation. For similar results concerning two-valued logics we refer to the Dempster–Shafer belief functions, see Jaffray [7], Shafer [10] or Paris [8].

So far we have focused on two-valued semantics only. Paris [8] proves the analogous result for Łukasiewicz's many-valued logics $Ł_{k+1}$. Formulas of $Ł_{k+1}$ are built up using the standard connectives $\vee$, $\wedge$, $\neg$ and $\to$. Write $\mathfrak{A} = \mathfrak{A}_{k+1}$ for the algebra with the universe

$$0,\ {}^1/k,\ {}^2/k,\ \ldots,\ 1, \tag{2}$$

and interpret $\neg$, $\vee$, $\wedge$ and $\to$ in $\mathfrak{A}$ by the functions $1-a$, $\min\{1, a+b\}$, $\max\{0, a+b-1\}$ and $\min\{1, 1-a+b\}$ for $a, b \in \mathfrak{A}$. The elements of the universe of $\mathfrak{A}$ are the possible truth values and the algebraic structure of $\mathfrak{A}$ yields the truth tables of the logical connectives. Let $W$ be the set of all homomorphisms from $\mathfrak{F}$ into $\mathfrak{A}$. The consequence $\vDash$ is defined in the standard way using $W$. The next theorem is proven in [8].

**1.3 THEOREM** (PARIS [8]). Let $(\mathfrak{F}, \vDash)$ be Łukasiewicz's $Ł_{k+1}$ and let $W$ be the set of all homomorphisms from $\mathfrak{F}$ into $\mathfrak{A}_{k+1}$. Then the following are equivalent.

(A) $B : \mathfrak{F} \to [0,1]$ is the convex combination of the functions in $W$.

(B) $B : \mathfrak{F} \to [0,1]$ satisfies the axioms below.

(Ł1) If $\vDash \varphi$ then $B(\varphi) = 1$, and if $\varphi \vDash$ then $B(\varphi) = 0$,

(Ł3) $B(\varphi \vee \psi) + B(\varphi \wedge \psi) = B(\varphi) + B(\psi)$. ∎

**Probability and the Kleene truth tables.** To get to the central topic of this paper, let us describe three non-classical three-valued logics: Kleene's "strong logic of indeterminacy" (KL), Priest's "logic of paradox" (LP), and "symmetric logic" (SL); see [9], [12]. In each case the set $\mathfrak{F}$ is generated by a non-empty finite set $P$ of propositional variables using the logical connectives $\wedge$, $\vee$ and $\neg$. An *evaluation* (or truth assignment) $h$ assigns *truth values* to propositional variables. Here, we have three the possible truth statuses: 1, 1/2, and 0. Each evaluation extends to a mapping $h : \mathfrak{F} \to \{1, 1/2, 0\}$ by the rules given by the *Kleene truth tables* as follows:

| $\wedge$ | 1 | 1/2 | 0 |
|---|---|---|---|
| 1 | 1 | 1/2 | 0 |
| 1/2 | 1/2 | 1/2 | 0 |
| 0 | 0 | 0 | 0 |

| $\vee$ | 1 | 1/2 | 0 |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 1/2 | 1 | 1/2 | 1/2 |
| 0 | 1 | 1/2 | 0 |

| $\neg$ | 1 | 1/2 | 0 |
|---|---|---|---|
| | 0 | 1/2 | 1 |

The difference between Kleene logic, LP and Symmetric logic is in the definition of their consequence relations $\vDash$:

**Kleene logic:** $\varphi \vDash_{KL} \psi$ iff for every evaluation $h$ we have

$$\text{if } h(\varphi) = 1, \text{ then } h(\psi) = 1. \tag{3}$$

**LP:** $\varphi \vDash_{LP} \psi$ iff for every evaluation $h$ we have

$$\text{if } h(\varphi) = 1 \text{ or } 1/2, \text{ then } h(\psi) = 1 \text{ or } 1/2. \tag{4}$$

**Symmetric logic:** $\varphi \vDash_{SL} \psi$ iff for every evaluation $h$ we have

$$\text{if } h(\varphi) = 1, \text{ then } h(\psi) = 1; \text{ and} \tag{5}$$

$$\text{if } h(\varphi) = 1/2, \text{ then } h(\psi) = 1 \text{ or } 1/2. \tag{6}$$

In Kleene logic the excluded middle $\varphi \vee \neg\varphi$ is not a tautology (in fact, this logic has no tautologies at all). LP is a paraconsistent logic, where $\varphi \wedge \neg\varphi$ is not contradictory. Symmetric logic has both features[2]. The characterization of probabilities in symmetric logic (and that of KL and LP with the truth values described here)[3] remained an open problem in Williams [12] and (implicitly) in Paris [8]. The main aim of this paper is to provide such a characterization. In particular, our main result is the following theorem (that we prove in section 3).

**THEOREM A.** Let $(\mathfrak{F}, \vDash)$ be a symmetric logic and let $W$ be the set of all Kleene evaluations. The following are equivalent.

(A) $B : \mathfrak{F} \to [0,1]$ is the convex combination of the functions $w$ for $w \in W$.

(B) $B : \mathfrak{F} \to [0,1]$ satisfies the axioms below.

(SL1) If $\varphi \vDash \psi$ then $B(\varphi) \leq B(\psi)$,
(SL2) $B(\neg\varphi) = 1 - B(\varphi)$,
(SL3) $B(\varphi \vee \psi) = B(\varphi) + B(\psi) - B(\varphi \wedge \psi)$,
(SL4) $B(\varphi) = B(\varphi \wedge \psi) + B(\varphi \wedge \neg\psi) - B(\varphi \wedge \neg\varphi \wedge \psi \wedge \neg\psi)$. ∎

In comparison with Paris' Theorem 1.2 it is apparent that axiom ($\mathcal{L}1$) is missing from the characterization of the convex hull of evaluations in SL. The reason is that SL has neither tautologies nor contradictions. (SL1) and (SL3) are respectively the same as ($\mathcal{L}2$) and ($\mathcal{L}3$). (SL2) is a natural requirement about the nature of negation. The only axiom which might seem surprising is (SL4).

(SL4) is related to the fact that in the Lindenbaum–Tarski algebra of SL, join-irreducible elements can exist. These are elements that cannot be expressed as a disjunction of other elements that are smaller. Therefore, unlike in the case of classical logic, not every element can be expressed as the union of atoms and thus a probability mapping in SL is not determined by its values on the atoms. Section 2 contains a particular example and some further discussion.

As an interesting additional corollary of Theorem A we obtain an axiomatization of the convex hull of the set of evaluations of Kleene's and Priest's logic:

**THEOREM B.** Let $(\mathfrak{F}, \vDash)$ be a KL or LP and let $W$ be the set of all Kleene evaluations. The following are equivalent.

(A) $B : \mathfrak{F} \to [0,1]$ is the convex combination of the functions in $W$.

(B) $B : \mathfrak{F} \to [0,1]$ satisfies the axioms below.

(KLP1) If $\varphi \vDash \psi$ and $\neg\psi \vDash \neg\varphi$, then $B(\varphi) \leq B(\psi)$,
(KLP2) $B(\neg\varphi) = 1 - B(\varphi)$,
(KLP3) $B(\varphi \vee \psi) = B(\varphi) + B(\psi) - B(\varphi \wedge \psi)$,
(KLP4) $B(\varphi) = B(\psi \wedge \varphi) + B(\neg\psi \wedge \varphi) - B(\varphi \wedge \neg\varphi \wedge \psi \wedge \neg\psi)$. ∎

[2] A word of comment is in order here. Łukasiewicz's $Ł_3$ uses the same Kleene truth tables for $\wedge$, $\vee$ and $\neg$, but it also uses $\to$ as a connective, which is *not* a derived connective defined by $\neg\varphi \vee \psi$.

[3] Williams [12] defines the logics KL, LP and SL using *truth statuses* $T$, $O$ and $F$. Then he assigns truth values to the truth statuses via cognitive loads according to the table below.

| Truth value: | $T$ | $O$ | $F$ |
|---|---|---|---|
| In $KL$ | 1 | 0 | 0 |
| In $LP$ | 1 | 1 | 0 |
| In $SL$ | 1 | 1/2 | 0 |

In the case of SL the truth values given in [12] coincide with our presentation. For further discussion we refer to [6].

The characterization of the convex hull of the evaluations in the case of KL, LP and SL differ only in the axiom (KLP1) and (SL1). This is no surprise as the first axiom is the only one that concerns with the logical consequence relation $\models$, and the three logics differ only in this relation. The proof of Theorem 3.2 below establishes a connection between $\models_{KL}$, $\models_{LP}$ and $\models_{SL}$.

Let us note again that in contrast with Williams [12] we used the truth values 0,1/2, and 1 when defining the (Kleene) evaluations of KL and LP. Williams defines the logics KL and LP using *truth statuses* $T$, $O$ and $F$; and then he assigns the truth values 0 and 1 to the truth statuses via cognitive loads. Therefore, our convex hull of evaluations is different than Williams' convex hull of cognitive evaluations – yielding a different axiomatization.

The rest of the paper is structured as follows. In the next Section we axiomatize the convex hull of evaluations of the free symmetric algebras. Such algebras are the Lindenbaum–Tarski algebras of symmetric logic. This axiomatization is used to prove the main result in Section 3.

## 2 PROBABILITIES ON THE FREE SYMMETRIC ALGEBRA

This section makes use of some universal algebra. We do not recall the standard notions but refer e.g. to the textbook [1] or [5].

**2.1 DEFINITION.** The *three-element Kleene algebra* $\mathfrak{A}$ is the structure $\mathfrak{A} = \left\langle \{0, 1/2, 1\}, \wedge, \vee, - \right\rangle$, where the operations are given by the Kleene truth-tables.

$\mathfrak{A}$ is a distributive lattice that satisfies the De Morgan rules and the Kleene rule: for each $a, b \in \mathfrak{A}$ we have

$$a \wedge b = -(-a \vee -b) \tag{7}$$

$$a \vee b = -(-a \wedge -b) \tag{8}$$

$$a \wedge -a \leq b \vee -b. \tag{9}$$

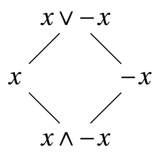
We denote by $\leq$ the ordering of $\mathfrak{A}$ coming from the lattice structure ($0 \leq 1/2 \leq 1$).

**The Lindenbaum–Tarski algebra of Symmetric logic.** The key observation of this section is that symmetric logic is algebraizable in the sense described below. Let $(\mathfrak{F}, \models)$ be a symmetric logic generated by the non-empty finite set $P$ of propositional variables as defined in the previous section. Evaluations $w \in W$ are precisely the homomorphisms $w : \mathfrak{F} \to \mathfrak{A}$. The consequence relation $\models$ induces an equivalence relation on $\mathfrak{F}$ by letting $\varphi$ and $\psi$ be equivalent ($\varphi \sim \psi$ in symbols) if $\varphi \models \psi$ and $\psi \models \varphi$ hold. It is not hard to verify that $\varphi \sim \psi$ if and only if $w(\varphi) = w(\psi)$ for all evaluations $w \in W$.[4] It follows that $\sim$ is not only an equivalence relation but also a congruence[5] of $\mathfrak{F}$. Therefore the quotient structure $\mathfrak{F}/_\sim$ exists. This quotient structure is the Lindenbaum–Tarski algebra of the symmetric logic generated by $P$ and we use the notation $\mathfrak{L} = \mathfrak{F}/_\sim$. Let us remark that $\mathfrak{L}$ is a free algebra in the variety $\mathbf{HSP}(\mathfrak{A})$, hence the name *free symmetric algebra*. Also, $\mathfrak{L}$ can be embedded into $\mathfrak{A}^W$ by the mapping $\varphi/_\sim \mapsto \langle w(\varphi) : w \in W \rangle$. As each homomorphism $w : \mathfrak{F} \to \mathfrak{A}$ is determined by its values $w \restriction P$ on the generator elements, there are exactly $|\mathfrak{A}|^{|P|}$ homomorphisms, hence $\mathfrak{L}$ is finite. Homomorphisms $w : \mathfrak{F} \to \mathfrak{A}$ can be identified with homomorphisms $\bar{w} : \mathfrak{L} \to \mathfrak{A}$ by letting $\bar{w}(\varphi/_\sim) = w(\varphi)$. We employ this identification and write $W = \{ w : \mathfrak{L} \to \mathfrak{A} : w$ is a homomorphism$\}$.

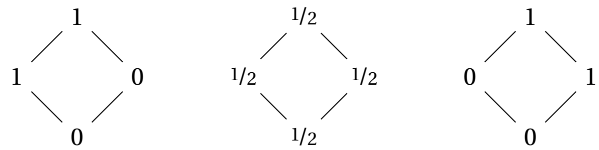
[4] Let us note that KL and LP do not have this feature. Write $\varphi \sim_{KL} \psi$ if $\varphi \models_{KL} \psi$ and $\psi \models_{KL} \varphi$ hold (and similarly with LP). There are formulas $\vartheta_1$ and $\vartheta_2$ such that $\vartheta_1 \sim_{KL} \vartheta_2$ (similarly, $\vartheta_1 \sim_{LP} \vartheta_2$) but $w(\vartheta_1) \neq w(\vartheta_2)$ for some evaluation $w$. Such formulas are, for example, $\vartheta_1 = p \wedge \neg p \wedge q$ and $\vartheta_2 = p \wedge \neg p \wedge \neg q$.

[5] For each homomorphism $w$ the kernel $\ker(w) = \{(\varphi, \psi) : w(\varphi) = w(\psi)\}$ is a congruence; $\sim$ equals $\bigcap_{w \in W} \ker(w)$; and the intersection of congruences is a congruence.

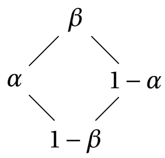
Before stating the main result let us give two examples. Consider first the one-generated free symmetric algebra. The set of propositional variables is $P = \{x\}$ and there are exactly three homomorphisms: $W = \{w_1, w_{1/2}, w_0\}$, where $w_i(x) = i$. The free symmetric algebra, considered as a subalgebra of $\mathfrak{A}^W$, is generated by the element $\langle w_1(x),\ w_{1/2}(x),\ w_0(x)\rangle = \langle 1, 1/2, 0\rangle$. The generated algebra $\mathfrak{L}$ is isomorphic to the four-element (Boolean) algebra illustrated below.




The homomorphisms induce three probabilities $w_i$ as follows:

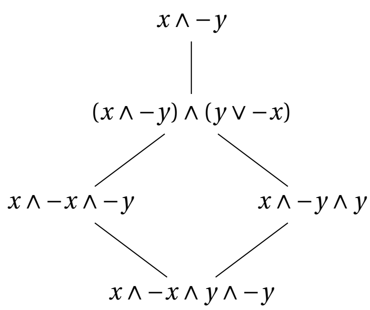



It is easy to verify that the convex combinations of the $w_i$'s are the mappings of the form:




with $\beta \geq \alpha, 1-\alpha$, and $0 \leq \alpha, \beta \leq 1$. We would like to stress an important difference between the symmetric and the classical case. $\mathfrak{L}$ here is isomorphic to the 4-element Boolean algebra, which features also as a Lindenbaum–Tarski algebra in classical propositional logic. Nevertheless, the probabilities are different, as $w_{1/2}$ is not allowed in the classical case. Further, observe that in the symmetric case the values $\alpha$, $1-\alpha$ on the atoms do not determine the whole function, as $\alpha$ and $\beta$ can be set almost independently. This is a striking difference between the two logics, as in the classical case the proof of De Finetti's characterization result (Theorem 1.1) depended exactly on this feature.

Next, consider the two-generated free symmetric algebra $\mathfrak{L}$. Here the set of propositional variables is $P = \{x, y\}$. There are nine homomorphisms $w_{\langle i,j\rangle} : \mathfrak{L} \to \mathfrak{A}$, where $w_{\langle i,j\rangle}(x) = i$, and $w_{\langle i,j\rangle}(y) = j$, $(i, j \in \{0, 1/2, 1\})$. Calculations show that the algebra $\mathfrak{L}$ has 82 elements[6], thus, instead of displaying the entire algebra we only draw a small part of it, namely, the principal ideal generated by $x \wedge -y$:




[6]Ralph Freese, Emil Kiss and Matthew Valeriote: Universal Algebra Calculator, available at: `www.uacalc.org`, 2011.

As one can see, $x \wedge -y$ is not a join of any smaller elements. Such elements are called *join-irreducible*. This is a general phenomenon which appears once at least two propositional variables are admitted in the language: except for the one-generated case above, the free symmetric algebras contain elements that cannot be expressed by the disjunctions of atoms.

The new axiom (SL4) is related to the presence in the Lindenbaum-Tarski algebra for symmetric logics of join-irreducible elements. As join-irreducibles are not the disjunction of smaller elements, it wouldn't be possible to use the additivity axiom (SL3) to calculate their probability based on the probability of smaller elements. It is in such cases in which (SL4) is employed.

Our main result is the following theorem.

**2.2 THEOREM.** Let $B\colon \mathfrak{L} \to [0,1]$ be a function. There are $\lambda_w \geq 0$ with $\sum_{w\in W} \lambda_w = 1$ such that

$$B(x) = \sum_{w \in W} \lambda_w w(x) \tag{10}$$

if and only if the stipulations below hold for $B$:

$$x \leq y \quad \Rightarrow \quad B(x) \leq B(y) \tag{11}$$
$$B(-x) = 1 - B(x) \tag{12}$$
$$B(x \vee y) = B(x) + B(y) - B(x \wedge y) \tag{13}$$
$$B(y) = B(y \wedge x) + B(y \wedge -x) - B(y \wedge -y \wedge x \wedge -x) \tag{14}$$

PROOF. That equations (11)–(14) hold for every $w$ in place of $B$ ($w \in W$) and thus for the convex combinations of the $w$'s is routine to check. We only give justification for (14) by the truth table below. Each column between the double bars corresponds to an evaluation (of the variables $x$ and $y$).

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $x$ | 0 | 0 | 0 | 1/2 | 1/2 | 1/2 | 1 | 1 | 1 |
| $y$ | 0 | 1/2 | 1 | 0 | 1/2 | 1 | 0 | 1/2 | 1 |
| $y \wedge x$ | 0 | 0 | 0 | 0 | 1/2 | 1/2 | 0 | 1/2 | 1 |
| $y \wedge -x$ | 0 | 1/2 | 1 | 0 | 1/2 | 1/2 | 0 | 0 | 0 |
| $y \wedge -y \wedge x \wedge -x$ | 0 | 0 | 0 | 0 | 1/2 | 0 | 0 | 0 | 0 |

In each column the value of $y$ is equal to the value of $y \wedge x$ plus the value of $y \wedge -x$ minus the value of $y \wedge -y \wedge x \wedge -x$, yielding equation (14) for evaluations $w \in W$ in place of $B$.

For the converse direction, suppose that $B$ satisfies (11)–(14). Recall that $P = \{x_i : i < n\}$ is the finite set of propositional variables. We make use of the following notation

$$x^1 = x, \quad x^{1/2} = x \wedge -x, \quad x^0 = -x. \tag{15}$$

For $S \subseteq P$ and $\varepsilon\colon P \to \{0, 1/2, 1\}$ we introduce the elements

$$t^S_\varepsilon = \bigwedge_{x \in S} x^{\varepsilon(x)} \tag{16}$$

that are conjunctions of literals. For instance,

$$t^P_{\langle 1/2, 1/2, \cdots, 1/2 \rangle} = \bigwedge_{x \in P} (x \wedge -x) \tag{17}$$

is the bottom element of $\mathfrak{L}$ as it is the conjunction of all variables and their negations and thus it must be the smallest element of $\mathfrak{L}$. Atoms of $\mathfrak{L}$ are the minimal elements above the bottom element. Each atom is thus the conjunction of all variables and their negations except for one variable, that is, atoms are of the form

$$x_i \wedge \bigwedge_{x \in P \smallsetminus \{x_i\}} (x \wedge -x), \quad \text{and} \quad -x_i \wedge \bigwedge_{x \in P \smallsetminus \{x_i\}} (x \wedge -x). \tag{18}$$

These are the elements $t_\varepsilon^P$ where $\varepsilon$ is 1/2 everywhere except for one coordinate $x_i$, where $\varepsilon(x_i)$ is 1 or 0. Note that already the two-generated example above shows that not every element of $\mathfrak{L}$ can be expressed as the disjunction of atoms.

We split the proof into several lemmas. The main idea of the proof is that first we show that $B$ can be expressed as a convex combination with respect to the elements of the form $t_\varepsilon^P$. This is the statement in Lemma 2.4. Then, by an inductive argument in Lemma 2.10 we extend this result to the elements $t_\varepsilon^S$ for $S \subseteq P$. In order to carry out this induction, using a sieve formula provided in (19), we need to show that once $B$ is in the form of a convex combination with respect to elements $t_\varepsilon^S$, it remains a convex combination with respect to the disjunctions of such elements (this is Lemma 2.9). Finally, using the De Morgan rules and distributivity, every element of $\mathfrak{L}$ can be written as disjunctions of certain $t_\varepsilon^S$'s, hence the convex combination characterization extends to the entire algebra $\mathfrak{L}$.

**2.3 LEMMA.** For any $T \subseteq \mathfrak{L}$ the sieve formula below holds.

$$B(\bigvee_{t\in T} t) = \sum_{\emptyset \neq S \subseteq T} (-1)^{|S|+1} B(\bigwedge_{t \in S} t) \tag{19}$$

PROOF. Iterated application of property (13) of $B$ and distributivity of $\mathfrak{L}$. ∎

Note that the sieve formula holds for each homomorphism $w \in W$ as well, as every such $w$ satisfies the analogous version of (13).

**2.4 LEMMA.** There are $\lambda_w \geq 0$ such that

$$B(t_\varepsilon^P) = \sum_{w \in W} \lambda_w w(t_\varepsilon^P) \tag{20}$$

holds for all $\varepsilon : P \to \{0, 1/2, 1\}$

PROOF. To simplify notation, we omit the superscript $P$ from $t_\varepsilon^P$ and write $t_\varepsilon$ instead. For $\varepsilon, \delta : P \to \{0, 1/2, 1\}$ we denote by $\#(\varepsilon, \delta)$ the number of places where $\varepsilon$ and $\delta$ differ. Let us identify homomorphisms $w \in W$ with sequences $\varepsilon : P \to \{0, 1/2, 1\}$ such that $w(x) = \varepsilon(x)$ for $x \in P$. Throughout $w_\varepsilon$ will denote the homomorphism corresponding to $\varepsilon$. We are about to define the values $\lambda_\varepsilon$. Put

$$\eta(\varepsilon) = \begin{cases} 1/2 & \text{if } 1/2 \in \operatorname{ran}(\varepsilon) \\ 1 & \text{otherwise.} \end{cases} \tag{21}$$

We define $\lambda_\varepsilon$ to be the value that satisfies:

$$\eta(\varepsilon) \cdot \lambda_\varepsilon = \sum_{t_\delta \leq t_\varepsilon} (-1)^{\#(\varepsilon,\delta)} \cdot B(t_\delta) \tag{22}$$

In the following two claims we give alternative (equivalent) definitions for $\lambda_\varepsilon$. These alternative forms will be used in the course of proving that the $\lambda$'s are non-negative, that they sum up to one and that $B(t_\varepsilon)$ is the appropriate convex combination. Write

$$E_\varepsilon = \left\{ t_\delta : t_\delta < t_\varepsilon,\ \#(\delta, \varepsilon) = 1 \right\}. \tag{23}$$

**2.5 CLAIM.** $\eta(\varepsilon) \cdot \lambda_\varepsilon = B(t_\varepsilon) - B(\bigvee E_\varepsilon)$.

PROOF. The set $\{t_\delta : \delta : P \to \{0, 1/2, 1\}\}$ is closed under $\wedge$. For every $t_\sigma < t_\varepsilon$ such that $\#(\varepsilon, \sigma) = n$ there are $t_{\delta_i} \in E_\varepsilon$ $(i < n)$ such that $t_\sigma = \bigwedge_{i<n} t_{\delta_i}$, and vice-versa. It follows that

$$\eta(\varepsilon) \cdot \lambda_\varepsilon \quad = \quad B(t_\varepsilon) - \sum_{\substack{t_\delta < t_\varepsilon \\ \#(\varepsilon,\delta)=1}} B(t_\delta) + \sum_{\substack{t_\delta < t_\varepsilon \\ \#(\varepsilon,\delta)=2}} B(t_\delta) - \dots \tag{24}$$

$$= \quad B(t_\varepsilon) + \sum_{\emptyset \neq S \subseteq E_\varepsilon} (-1)^{|S|} B(\bigwedge_{t \in S} t) \tag{25}$$

$$= \quad B(t_\varepsilon) - B(\bigvee_{t \in E_\varepsilon} t). \tag{26}$$

The last equation follows from the application of the sieve formula (19). ∎

**2.6 CLAIM.** $\eta(\varepsilon) \cdot \lambda_\varepsilon = B(t_\varepsilon) - \sum_{t_\delta < t_\varepsilon} \eta(\delta)\lambda_\delta$.

PROOF. By induction. The statement holds for $\lambda_{\langle 1/2, \cdots, 1/2 \rangle}$. Suppose, by induction, that

$$\eta(\delta) \cdot \lambda_\delta = B(t_\delta) - \sum_{t_\sigma < t_\delta} \eta(\sigma)\lambda_\sigma. \tag{27}$$

holds for $t_\delta < t_\varepsilon$. Using the definition of $\lambda_\varepsilon$ and the inductive hypothesis we get

$$\begin{aligned} \eta(\varepsilon) \cdot \lambda_\varepsilon &= B(t_\varepsilon) + \sum_{t_\delta < t_\varepsilon} (-1)^{\#(\varepsilon,\delta)} \cdot B(t_\delta) && (28) \\ &= B(t_\varepsilon) + \sum_{t_\delta < t_\varepsilon} (-1)^{\#(\varepsilon,\delta)} \cdot \big(\eta(\delta)\lambda_\delta + \sum_{t_\sigma < t_\delta} \eta(\sigma)\lambda_\sigma\big) && (29) \\ &= B(t_\varepsilon) + \sum_{t_\delta < t_\varepsilon} (-1)^{\#(\varepsilon,\delta)} \cdot \eta(\delta)\lambda_\delta + \sum_{t_\delta < t_\varepsilon} \sum_{t_\sigma < t_\delta} (-1)^{\#(\varepsilon,\delta)} \eta(\sigma)\lambda_\sigma. && (30) \end{aligned}$$

In the last equation we count every term $\eta(\sigma)\lambda_\sigma$ for $t_\sigma < t_\varepsilon$ odd many times with alternating signs. Therefore, the last expression is $B(t_\varepsilon) - \sum_{t_\sigma < t_\varepsilon} \eta(\sigma)\lambda_\sigma$. ∎

**2.7 CLAIM.** $\lambda_\varepsilon \geq 0$ holds for all $\varepsilon : P \to \{0, 1/2, 1\}$.

PROOF. It is immediate that $t_\varepsilon \geq \bigvee E_\varepsilon$. Hence, by property (11), $B(t_\varepsilon) - B(\bigvee E_\varepsilon) \geq 0$. By Claim 2.5, $\lambda_\varepsilon \geq 0$. ∎

**2.8 CLAIM.** $B(t_\varepsilon) = \sum_\delta \lambda_\delta w_\delta(t_\varepsilon)$.

PROOF. Observe first that $w_\delta(t_\varepsilon) \neq 0$ if and only if $t_\delta \leq t_\varepsilon$. We differentiate two cases according to whether $1/2 \in \operatorname{ran}(\varepsilon)$. In the first case, suppose $1/2 \in \operatorname{ran}(\varepsilon)$, thus $\eta(\varepsilon) = 1/2$. Then $w_\delta(t_\varepsilon)$ is either 0 or $1/2$.

$$\sum_\delta \lambda_\delta w_\delta(t_\varepsilon) = \sum_{\delta : w_\delta(t_\varepsilon) > 0} w_\delta(t_\varepsilon) \frac{1}{\eta(\varepsilon)} \sum_{t_\sigma \leq t_\delta} (-1)^{\#(\delta,\sigma)} B(t_\sigma) = \sum_{t_\delta \leq t_\varepsilon} \sum_{t_\sigma \leq t_\delta} (-1)^{\#(\delta,\sigma)} B(t_\sigma). \tag{31}$$

In this last sum every term $B(t_\sigma)$ is counted even many times with opposite signs, except for $B(t_\varepsilon)$, which occurs exactly once. Hence the sum is equal to $B(t_\varepsilon)$.

In the second case $\eta(\varepsilon) = 1$. Observe that for such an $\varepsilon$, $w_\delta(t_\varepsilon) = 1$ if and only if $\delta = \varepsilon$. Also, if $t_\delta < t_\varepsilon$, then $1/2 \in \operatorname{ran}(\delta)$, and thus $\eta(\delta) = 1/2$. It follows that

$$\sum_\delta \lambda_\delta w_\delta(t_\varepsilon) = \sum_{t_\delta \leq t_\varepsilon} \lambda_\delta w_\delta(t_\varepsilon) = \lambda_\varepsilon + \frac{1}{2} \sum_{t_\delta < t_\varepsilon} \lambda_\delta = B(t_\varepsilon) - \sum_{t_\delta < t_\varepsilon} \eta(\delta)\lambda_\delta + \frac{1}{2} \sum_{t_\delta < t_\varepsilon} \lambda_\delta = B(t_\varepsilon). \tag{32}$$

∎

The proof of Lemma 2.4 is completed. ∎

Next, suppose $X \subseteq \mathfrak{L}$ is closed under $\wedge$. Let $X^\vee$ be the set of all disjunctions of elements of $X$. Then $X^\vee$ is closed under $\vee$, and by distributivity it is closed under $\wedge$ as well.

**2.9 LEMMA.** Suppose $X \subseteq \mathfrak{L}$ is closed under $\wedge$ and

$$B(t) = \sum_{w\in W} \lambda_w w(t) \tag{33}$$

holds for all $t \in X$. Then (33) holds for all $t \in X^\vee$ as well.

PROOF. Let $T \subseteq X$ and consider an element $\bigvee_{t\in T} t$ of $X^\vee$. As $X$ is closed under $\wedge$ it follows that for any $S \subseteq T$ we have $\bigwedge_{t\in S} t \in X$, in particular $B(\bigwedge_{t\in S} t)$ is of the form of a linear combination (33). The sieve formula (19) gives

$$\begin{aligned} B(\bigvee_{t\in T} t) &= \sum_{\emptyset\neq S\subseteq T} (-1)^{|S|+1} B(\bigwedge_{t\in S} t) && (34)\\ &= \sum_{\emptyset\neq S\subseteq T} (-1)^{|S|+1} \sum_{w\in W} \lambda_w w(\bigwedge_{t\in S} t) && (35)\\ &= \sum_{w\in W} \lambda_w \sum_{\emptyset\neq S\subseteq T} (-1)^{|S|+1} w(\bigwedge_{t\in S} t) && (36)\\ &= \sum_{w\in W} \lambda_w w(\bigvee_{t\in T} t). && (37) \end{aligned}$$

∎

Next, we extend the result of Lemma 2.4 to elements $t^S_\varepsilon$ for $S \subseteq P$. For $S \subseteq P$ let us write

$$X^S = \{t^S_\varepsilon : \varepsilon : P \to \{0, 1/2, 1\}\}. \tag{38}$$

As elements $t^S_\varepsilon$ are conjunctions of literals from $S$, $X^S$ is closed under $\wedge$.

**2.10 LEMMA.** $B(t^S_\varepsilon) = \sum_{w\in W} \lambda_w w(t^S_\varepsilon)$ holds for all $S \subseteq P$ and $\varepsilon : P \to \{0, 1/2, 1\}$.

PROOF. By induction on $P \smallsetminus S$. The statement clearly holds for $S = P$ by Lemma 2.4. In the inductive step, assume that we know the statement for $S \cup \{x\}$ (where $x \in P \smallsetminus S$) and we want to provide a proof for $S$.

Pick any $t^S_\varepsilon$. By (14) we have

$$B(t^S_\varepsilon) = B(x \wedge t^S_\varepsilon) + B(-x \wedge t^S_\varepsilon) - B(x \wedge -x \wedge t^S_\varepsilon \wedge -t^S_\varepsilon). \tag{39}$$

Notice that $x \wedge t^S_\varepsilon$ and $-x \wedge t^S_\varepsilon$ both belong to $X^{S\cup\{x\}}$, hence by the inductive hypothesis it follows that

$$\begin{aligned} B(x \wedge t^S_\varepsilon) &= \sum_{w\in W} \lambda_w w(x \wedge t^S_\varepsilon), && (40)\\ B(-x \wedge t^S_\varepsilon) &= \sum_{w\in W} \lambda_w w(-x \wedge t^S_\varepsilon). && (41) \end{aligned}$$

Also, $y = x \wedge -x \wedge t^S_\varepsilon$ belongs to $X^{S\cup\{x\}}$. Using distributivity and the De Morgan rules we obtain

$$y \wedge -t^S_\varepsilon = y \wedge -\bigwedge_{v\in S} v^{\varepsilon(v)} = y \wedge \bigvee_{v\in S} -v^{\varepsilon(v)} = \bigvee_{v\in S} y \wedge -v^{\varepsilon(v)} \tag{42}$$

Here, each $y \wedge -v^{\varepsilon(v)}$ is either $y \wedge v$ or $y \wedge -v$ or $y \wedge (v \vee -v) = (y \wedge v) \vee (y \wedge -v)$. Therefore $y \wedge -t^S_\varepsilon$ is the disjunction of elements from $X^{S\cup\{x\}}$. Applying Lemma 2.9 to the $\wedge$-closed $X^{S\cup\{x\}}$ we get

$$B(x \wedge -x \wedge t^S_\varepsilon \wedge -t^S_\varepsilon) = \sum_{w\in W} \lambda_w w(x \wedge -x \wedge t^S_\varepsilon \wedge -t^S_\varepsilon). \tag{43}$$

It follows that the right-hand side of (39) is equal to

$$\sum_{w\in W} \lambda_w \Big( w(x \wedge t^S_\varepsilon) + w(-x \wedge t^S_\varepsilon) - w(x \wedge -x \wedge t^S_\varepsilon \wedge -t^S_\varepsilon) \Big). \tag{44}$$

But each $w$ satisfies (14), thus the sum in (44) equals $\sum_{w\in W} \lambda_w w(t^S_\varepsilon)$. Therefore $B(t^S_\varepsilon)$ is the desired linear combination. ∎

Let us write $X = \left\{t_\varepsilon^S : S \subseteq P,\ \varepsilon : P \to \{0, 1/2, 1\}\right\}$. It is straightforward to see that $X$ is closed under $\wedge$. Consequently

$$B(t) = \sum_{w \in W} \lambda_w w(t) \qquad \text{for all } t \in X^\vee \tag{45}$$

**2.11 LEMMA.** $X^\vee$ exhausts all elements of $\mathfrak{L}$.

PROOF. Using distributivity of $\mathfrak{L}$, each $t \in \mathfrak{L}$ can be written as a disjunctive normal form

$$\bigvee_k (t_{i_0}^k \wedge \cdots \wedge t_{i_k}^k) \tag{46}$$

where each $t_i^k$ is a propositional variable or its negation. The subterms $t_{i_0}^k \wedge \cdots \wedge t_{i_k}^k$ belong to $X$, thus the disjunctive normal form is an element of $X^\vee$. ∎

Combining the lemmas we obtain that there are $\lambda_w \geq 0$ such that

$$B(t) = \sum_{w \in W} \lambda_w w(t) \tag{47}$$

holds for all $t \in \mathfrak{L}$. It remains only to show that $\sum_{w \in W} \lambda_w = 1$. Let $t_\pi = \bigwedge_{x \in P}(x \wedge -x)$ be the least element of $\mathfrak{L}$. Then

$$B(t_\pi) = \sum_\varepsilon \lambda_\varepsilon w_\varepsilon(t) = \frac{1}{2}\lambda_\pi; \tag{48}$$

$$B(-t_\pi) = \sum_\varepsilon \lambda_\varepsilon w_\varepsilon(-t_\pi) = \sum_{\varepsilon \neq \pi} \lambda_\varepsilon + \frac{1}{2}\lambda_\pi. \tag{49}$$

By property (12), we have

$$B(t_\pi) + B(-t_\pi) = \sum_\varepsilon \lambda_\varepsilon = 1. \tag{50}$$

∎

To close this Section, we illustrate the proof in the two-generated free symmetric algebra. Recall that the set of propositional variables is $P = \{x, y\}$ and that there are nine homomorphisms $w_{\langle i,j \rangle} : \mathfrak{L} \to \mathfrak{A}$, where $w_{\langle i,j \rangle}(x) = i$, and $w_{\langle i,j \rangle}(y) = j$, $(i, j \in \{0, 1/2, 1\})$. In the following table we give the elements $t_\varepsilon = t_\varepsilon^P$ as elements of the algebra $\mathfrak{A}^W$ (empty entries are 0's).

| $x$ | 0 | 0 | 0 | 1/2 | 1/2 | 1/2 | 1 | 1 | 1 | |
|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 0 | 1/2 | 1 | 0 | 1/2 | 1 | 0 | 1/2 | 1 | |
| $x \wedge -x \wedge y \wedge -y$ | | | | | 1/2 | | | | | $t_{\langle 1/2,1/2 \rangle}$ |
| $x \wedge y \wedge -y$ | | | | | 1/2 | | | 1/2 | | $t_{\langle 1,1/2 \rangle}$ |
| $-x \wedge y \wedge -y$ | | 1/2 | | | 1/2 | | | | | $t_{\langle 0,1/2 \rangle}$ |
| $x \wedge -x \wedge y$ | | | | | 1/2 | 1/2 | | | | $t_{\langle 1/2,1 \rangle}$ |
| $x \wedge -x \wedge -y$ | | | | 1/2 | 1/2 | | | | | $t_{\langle 1/2,0 \rangle}$ |
| $x \wedge y$ | | | | | 1/2 | 1/2 | | 1/2 | 1 | $t_{\langle 1,1 \rangle}$ |
| $x \wedge -y$ | | | | 1/2 | 1/2 | | 1 | 1/2 | | $t_{\langle 1,0 \rangle}$ |
| $-x \wedge y$ | | 1/2 | 1 | | 1/2 | 1/2 | | | | $t_{\langle 0,1 \rangle}$ |
| $-x \wedge -y$ | 1 | 1/2 | | 1/2 | 1/2 | | | | | $t_{\langle 0,0 \rangle}$ |

Given any $B : \mathfrak{L} \to [0,1]$, the $\lambda_{\langle i,j \rangle}$'s can be read off from the table, for example,

$$1/2 \cdot \lambda_{\langle 1/2,1/2 \rangle} = B(t_{\langle 1/2,1/2 \rangle}); \tag{51}$$

$$1/2 \cdot \lambda_{\langle 1,1/2 \rangle} = B(t_{\langle 1,1/2 \rangle}) - B(t_{\langle 1/2,1/2 \rangle}); \tag{52}$$

$$1/2 \cdot \lambda_{\langle 0,1/2 \rangle} = B(t_{\langle 0,1/2 \rangle}) - B(t_{\langle 1/2,1/2 \rangle}); \tag{53}$$

$$\lambda_{\langle 1,1 \rangle} = B(t_{\langle 1,1 \rangle}) - B(t_{\langle 1,1/2 \rangle}) - B(t_{\langle 1/2,1 \rangle}) + B(t_{\langle 1/2,1/2 \rangle}). \tag{54}$$

## 3 PROBABILITIES AND SYMMETRIC LOGIC

Theorem 2.2 gives a characterization of the convex hull of evaluations on free symmetric algebras. However, belief functions are defined on the set of formulas and not on the free algebras. Hence, we have to "pull back" the characterization from the Lindenbaum–Tarski algebra to the algebra of formulas. This is done in the next theorem.

**3.1 THEOREM.** Let $(\mathfrak{F}, \vDash)$ be a symmetric logic and $W$ be the set of all evaluations. The following are equivalent.

(A) **Probability in Symmetric logic:** $B : \mathfrak{F} \to [0,1]$ satisfies the axioms below.

- (SL1) If $\varphi \vDash \psi$ then $B(\varphi) \le B(\psi)$,
- (SL2) $B(\neg\varphi) = 1 - B(\varphi)$,
- (SL3) $B(\varphi \vee \psi) = B(\varphi) + B(\psi) - B(\varphi \wedge \psi)$,
- (SL4) $B(\varphi) = B(\varphi \wedge \psi) + B(\varphi \wedge \neg\psi) - B(\varphi \wedge \neg\varphi \wedge \psi \wedge \neg\psi)$.

(B) **Convex combination:** $B : \mathfrak{F} \to [0,1]$ is the convex combination of the functions in $W$.

PROOF. That convex combinations of the $w$'s satisfy (SL1)–(SL4) is routine to check. As for the converse direction, suppose $B$ satisfies (SL1)–(SL4). Recall that semantic consequence $\vDash$ induces a congruence relation on $\mathfrak{F}$ by letting $\varphi$ and $\psi$ equivalent ($\varphi \sim \psi$ in symbols) if $\varphi \vDash \psi$ and $\psi \vDash \varphi$. As before, $\mathfrak{L}$ denotes the quotient algebra $\mathfrak{F}/_\sim$. By property (SL1), the mapping $\bar{B} : \mathfrak{L} \to [0,1]$ given by $\bar{B}(\vartheta/_\sim) = B(\vartheta)$ is well-defined and $\bar{B}$ satisfies (11)–(14) of Theorem 2.2. Thus, there are $\lambda_w \ge 0$ with $\sum_{w\in W} \lambda_w = 1$ such that $\bar{B}(\vartheta/_\sim) = \sum_{w\in W} \lambda_w w(\vartheta/_\sim)$ holds for all $\vartheta \in \mathfrak{F}$. As $\varphi \sim \psi$ if and only if for $w(\varphi) = w(\psi)$ for all $w \in W$, it follows that $B(\vartheta) = \sum_{w\in W} \lambda_w w(\vartheta)$, as desired. ∎

Recall that the set of evaluations in SL, KL and LP are exactly the same. The axiomatization of the convex hull of the evaluations is possible in KL and LP too:

**3.2 THEOREM.** Let $(\mathfrak{F}, \vDash)$ be KL or LP and let $W$ be the set of all evaluations. The following are equivalent.

(A) **Probability in KL or LP:** $B : \mathfrak{F} \to [0,1]$ satisfies the axioms below.

- (KLP1) If $\varphi \vDash \psi$ and $\neg\psi \vDash \neg\varphi$, then $B(\varphi) \le B(\psi)$,
- (KLP2) $B(\neg\varphi) = 1 - B(\varphi)$,
- (KLP3) $B(\varphi \vee \psi) = B(\varphi) + B(\psi) - B(\varphi \wedge \psi)$,
- (KLP4) $B(\varphi) = B(\psi \wedge \varphi) + B(\neg\psi \wedge \varphi) - B(\varphi \wedge \neg\varphi \wedge \psi \wedge \neg\psi)$.

(B) **Convex combination:** $B : \mathfrak{F} \to [0,1]$ is the convex combination of the functions in $W$.

PROOF. Since the evaluations in KL, LP and SL are exactly the same, they have one and the same convex hull. The characterization of this convex hull in SL has been done in Theorem 3.1. Therefore, to characterize this convex hull in KL or LP, the only thing we need to do is to formulate the condition (SL1) in the terminology of KL or LP. This can be done as follows:

(i) $\varphi \vDash_{SL} \psi$ if and only if ($\varphi \vDash_{KL} \psi$ and $\neg\psi \vDash_{KL} \neg\varphi$).
(ii) $\varphi \vDash_{LP} \psi$ if and only if ($\neg\psi \vDash_{KL} \neg\varphi$).
(iii) $\varphi \vDash_{SL} \psi$ if and only if ($\varphi \vDash_{LP} \psi$ and $\neg\psi \vDash_{LP} \neg\varphi$).

∎

To sum up, in Theorem 3.1 we provided an axiomatization of the convex hull of the set of evaluations of symmetric logic, yielding the notion of probability in symmetric logic. This axiomatization differs from the standard one (i.e. that of classical logic, cf. Theorem 1.2) in the axiom (SL4).

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