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Péter on Church's Thesis, Constructivity and Computers*

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Abstract

The aim of this paper is to take a look at Péter's talk *Rekursivität und Konstruktivität* delivered at the *Constructivity in Mathematics* Colloquium in 1957, where she challenged Church's Thesis from a constructive point of view. The discussion of her argument and motivations is then connected to her earlier work on recursion theory as well as her later work on theoretical computer science.

1 Introduction

Rózsa Péter and László Kalmár, lifelong colleagues and friends, were both invited to the famous *Constructivity in Mathematics* Colloquium held in Amsterdam in 1957. In their talks, both of them challenged Church's Thesis, albeit in quite different ways. The aim of this paper is to discuss Péter's less frequently cited contribution *Rekursivität und Konstruktivität* (1959). And to provide more context and background to her argument and to connect it to her earlier and later work in recursion theory and theoretical computer science respectively. Besides her published paper (Péter 1959), Péter and Kalmár's correspondence (Kalmár and Péter 1930-1976) will be used to provide such context. Due to the lack of space, attention will be focused on Péter's work and connections to the ideas of other scholars will merely be indicated.

2 Church's Thesis in Péter's Recursive Functions

Péter's most well known scholarly work, *Recursive Functions* (1951/1967), famously the first monograph in recursion theory, was first published in 1951.

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The phrase 'Church's Thesis' does not appear as a label anywhere in the book, as it was popularized by Kleene's *Introduction to Metamathematics* (1952) that was published the next year.[1] But, of course, the identification of the notion of 'calculable functions' with the general recursive functions "proposed" by Church is discussed. As it is described below in detail, in general Péter displayed a "non-committal" attitude towards the Thesis in (1951/1967) and in later editions she kept including materials that challenged it.

In §20, *Calculable Functions*, Péter "quote[s] some of the arguments used in attempts to make plausible the identification." However, this chapter is entirely devoted to a detailed description of Turing's machines and their capability to compute every general recursive function. This emphasis is due to Péter's view that the finite calculability of a function is strongly tied to the existence of a repeatable and communicable "mechanical procedure," where the "single steps of the calculation" could be carried out even by a machine "in principle" (p. 225). Péter acknowledges that Turing's machines satisfy these requirements while his analysis makes it plausible that "this interpretation correctly reflects real mathematical activity" (p. 240) and that it gives "the impression that a very general concept of calculability has been captured here." (p. 234).

Church's "proposed" identification is only discussed in the less than a page long 7th section of §21, *History and Applications*. As support for the identification, in addition to what was already given in §20, Péter points to the 'confluence' of the multiple attempts to formalize the "vague concepts of 'calculability', 'constructibility', [and] 'effectibility"' (p. 245). More precisely, she states the equivalence of general recursive functions, functions computable by Turing's machines, Hilbert and Bernays' reckonable functions, and the functions calculable by Markov's algorithms.

At the same time, whenever Péter presents some of the arguments for Church's Thesis, she also remarks that the acceptance of it would lead to "a certain demarcation of the concept of calculability" (p. 240). She finds this problematic, as she strongly believed that mathematics and its methods would develop without an end. As she puts it here: "the future evolution of mathematics may bring about methods of calculation completely unexpected nowadays" (p. 240). For this reason, while she considers the concept of general recursivity a "most welcome generalization," nevertheless she sees it "merely [as] a stage – albeit a very high one" (p. 9). Thus the concept of calculability should not be formally confined based on our current mathematical knowledge, as it is expected to develop further in the future. For this reason Péter concludes: "In my opinion, no 'final word' is possible here: the concept of calculability cannot be definitely comprised once and for all" (p. 9).[2]

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[1] Already in his (Kleene 1943, p. 60) Kleene labeled the assertion that "Every effectively calculable function is general recursive" as a 'thesis' that was stated by Church (and implicitly by Turing) but did not use the phrase 'Church's Thesis' anywhere in the paper. The phrase became widely used after it was popularized in Kleene's (1952).

[2] Péter also concludes her popular book, *Playing with Infinity* (1961/1976), with the same thought. The last chapter explains Gödel's incompleteness and Church's undecidability results and raises the question whether we have "come up against final obstacles?" (p. 264). This is

In later editions of *Recursive Functions,* Péter kept adding materials that supported her "non-committal" attitude towards Church's Thesis. In the *Preface of the Second German Edition*, written in 1955, Péter acknowledges that "it has been noted" by the reviewers and readers of the first edition that she was "non-committal on the question of the identification of calculability with general recursivity." And she adds that "It was at that time my intention, however, to set beside one another the possible positions in this question" (1951/1967, p. 9).

Next Péter alludes to her view that based on the endless development of mathematics, the concept of calculability should not be formally confined based on our current knowledge. She claims that this point of view "has lately been very strongly supported by the new results" of Kalmár, which are then discussed in §19.2 and §20.8. The first one is essentially a summary of an early version of Kalmár's *An Argument Against the Plausibility of Church's Thesis* (1959). Here, assuming Church's Thesis while adopting a much less restricted notion of 'effective calculability' than customary, Kalmár constructed a proposition that is intuitively considered to be false, but its falsity cannot be proven by "any correct means." He regarded this "very strange consequence" of the Thesis as an argument against its plausibility (without claiming to have refuted it). Kalmár concluded his paper with his belief that certain mathematical concepts, such as 'effective calculability' or 'provability,' "cannot permit any restriction imposed by an exact mathematical definition" due to the endless development of mathematics, and thus, the possible further development of these concepts in the future.[3] Then Péter discusses Kalmár's (1955). Here, answering a question of Karl Schröter's in the negative, he showed that a system of functional equations (without restrictions on the operations to compute its values) may have a unique solution without the determined function being general recursive.

In his JSL review of the second edition of (1951/1967) Robinson focused only on these new additions and felt compelled to proclaim that "the reviewer is still convinced that the concept of general recursive function does provide the proper formalization of the intuitive concept of computable function" (1958, p. 363).

3 Recursion and Constructivity

In the summer of 1956, Péter received an invitation from Arend Heyting to the *Constructivity in Mathematics* Colloquium to be held in August of 1957 in Amsterdam (see the *Appendix*). Péter accepted the invitation and her contribution

then answered quite forcefully by the very last paragraph of the book: "Future development is sure to enlarge the framework, even if we cannot as yet see how. The eternal lesson is that Mathematics is not something static, closed, but living and developing. Try as we may to constrain it into a closed form, it finds an outlet somewhere and escapes alive" (p. 265).

[3]For a detailed description and analysis of Kalmár's (1959) visit Szabó's (2018). In addition, Gosztonyi's (2016) discusses Péter and Kalmár's shared views on mathematics and its education within the Hungarian mathematical culture as their broader context, while Máté's (2005) examines their philosophical views on mathematics.

became known as *Rekursivität und Konstruktivität* (1959).

The correspondence of Péter and Kalmár (Kalmár and Péter 1930-1976) reveals that they discussed what they should present at the meeting. In a letter from the 18th of February, 1957, Kalmár recommended to Péter that she either discuss any current work of hers on open problems of recursion theory or, "Since [the Colloquium] is about constructivity, you should discuss why you do not find the concept of general recursive function satisfactory from the standpoint of constructivity (as you usually say, something is fishy about it), or at least why you find the concept of recursion present in the theory of special recursive functions more satisfactory".[4] Péter ended up exploring this latter suggestion.

Before turning to Péter's paper, it is important to note that she did not hold constructivist views. Her views on the foundations of mathematics, like Kalmár's, were most closely aligned with that of the Hilbert school. However, when Péter began research in recursion theory at the very beginning of the 1930s the notions of 'effective' and 'constructive' were used essentially interchangeably in the community, including by Church and Kleene.[5] Thus Péter was not opposed to classical or non-constructivist approaches in mathematics and logic in general, but considered the concept of recursive function to be an attempt to precisely characterize the concept of constructivity.

In the beginning of the paper, Péter asserts that functions defined via special types of recursion (such as primitive recursion, course-of-values recursion, simultaneous and nested recursions, etc.) are obviously finitely calculable, and thus, constructive functions. Then the "Herbrand-Gödel-Kleene" notion of general recursive function is considered. On Péter's account, the main reason to introduce this notion was to precisely formulate the concept of constructivity, and Church's Thesis identifies this very notion with the concept of calculable functions. Then the question is raised whether every 'effectively calculable' function can justifiably be called 'constructive'. Thus Péter challenges the direction of Church's Thesis that is usually considered obvious or unproblematic.

Péter then goes on to argue that Church's Thesis is either non-constructive or it contains a vicious circle. By definition, a function is considered to be general recursive if *there exists* a finite system of equations from which all of its values can be calculated in a finite manner. The existential quantifier in the definition can be interpreted both classically and constructively, and the former is (possibly) not satisfactory from a constructivist point of view.[6] Péter was

[4] The correspondence of Péter and Kalmár is in Hungarian; quotes are translated by the present author.

[5] For examples, see Sieg's (2009, pp. 558-559) quoting Church, and Kleene's (1936b) where he states that "The notion of a recursive function of natural numbers, which is familiar in the special cases associated with primitive recursions, Ackermann-Péter multiple recursions, and others, has received a general formulation from Herbrand and Gödel. The resulting notion is of especial interest, since the intuitive notion of a 'constructive' or 'effectively calculable' function of natural numbers can be identified with it very satisfactorily" (p. 544).

[6] Kleene brings up a related issue in his (1936a): "The definition of general recursive function offers no constructive process for determining when a recursive function is defined. This must be the case, if the definition is to be adequate, since otherwise still more general 'recursive' functions could be obtained by the diagonal process" (p. 738).

not the only one to point out this issue. Not only did Heyting and Skolem raise this exact issue around the time,[7] but Church (1936, p. 351, fn 10) and Kleene (1952, p. 319) were evidently aware of it too. Church's recommendation for his concerned readers was to "take the existential quantifier [...] in a constructive sense", and added that "[w]hat the criterion of constructiveness shall be is left to the reader".[8] But, according to Péter, this leads to a vicious circle since general recursion was offered as a precise formulation of constructivity,[9] yet the notion of constructivity is alluded to in the interpretation of the existential quantifier in its definition.[10] She then remarks that the same vicious circle appears however one tries to get around it.

Péter's paper was reviewed jointly with articles that challenged or criticized Church's Thesis (Mendelson 1963, Moschovakis 1968) and it is usually cited in that context as well. While this assessment is appropriate, especially with Péter's known non-committal attitude towards the Thesis, it misses an aspect of her undertaking, namely that Péter's primary goal in this work was not to undermine the Thesis, but to give a precise account of constructive functions.

Indeed, after the short discussion of Church's Thesis, Péter considers multiple possible definitions to characterize the constructive functions in the second half of the paper. After the presentation of each possible formalization, she points out the "vicious circle" in them. However, these examples are not merely there to strengthen her challenge to Church's Thesis by pointing out the circularity over and over again. These are her genuine (and failed) attempts to give a positive, non-circular characterization of the constructive functions.

Péter's main aim was to provide a formal characterization of constructive functions that includes all the special recursive functions[11] but does not exhaust the general recursive ones. When she recognized the circularity in all of her attempts, she was discouraged. In a letter to Kalmár, written on the 2nd of July, 1957, not long before the Colloquium, she reported that "sadly it appears to me that the whole idea is bankrupt." Péter ended her paper on the same note with the following, resigned last sentence: "It seems that the concept of constructivity cannot be captured in a non-circular way at all."[12]

Nevertheless Péter did not entirely give up on the idea to provide a precise formal characterization of constructive functions. The next section describes her subsequent attempt after the Colloquium.

[7] See Coquand's (2014) for a discussion of their relevant writings.

[8] Here Péter remarks that since no "real" general recursive function is known, i.e. one that is general recursive but does not belong to any of the special types of recursive functions, it is not clear what the difference between the two interpretations could actually amount to.

[9] See Sundholm's (2014, pp. 13-14) on Church's view on the Thesis and its relation to constructivism.

[10] Heyting raises a rather similar concern about the circularity involved in the definition of recursive functions from a constructivist point of view in his (1962) without referring to Péter's (1959).

[11] The term "special recursive functions" is used here loosely, to refer to the collection of those recursive functions that are defined via a specific type of recursion (see above) and are seen as obviously finitely calculable and constructive functions.

[12] Translated by Tamás Lénárt.

4 Péter's Road to Computer Science

Another aspect of Péter's *Rekursivität und Konstruktivität* worth mentioning is its connection to her later work on recursive functions in the field of theoretical computer science. Péter became involved with the field through the active encouragement of Kalmár. As the interests of Kalmár turned towards the applications of logic in cybernetics, automata theory, computer design and even in engineering in general (Szabó 2019, Makay 2007), he started a research seminar devoted to these topics in the spring of 1956 at the University of Szeged.[13] To involve Péter, who was in Budapest, with the work by the members of the seminar, he sent the papers they were reading and open questions to Péter by mail. The specific research problems he posed to Péter were connected both to theoretical interests, such as Victor Shestakov's (1941) and Claude Shannon's (1938) work on representing relay and switching circuits with Boolean algebras, as well as their use in actual computer design questions that Kalmár planned to undertake in the near future. By the end of the summer, Péter was sending her results frequently and via express mail to Kalmár in order to be part of the "delightful work" of the group.

Péter's recent engagement with the field that we would today call theoretical computer science provides the background for her remark below. While discussing administrative details about their trip with Kalmár to the Colloquium in Amsterdam in a letter on the 14th of January 1957, Péter casually inserted the following comment mid-sentence: "by the way, and I mention this only to you, it seems to me that the notion of definability by calculators[14] is very closely tied to constructivity." This cryptic comment turned out to be rather prescient, as her first paper published after the Colloquium (even before its proceedings appeared), *Graphschemata und Rekursive Funktionen* (1958), engaged with programming in yet another attempt to characterize constructive functions.

In the first paragraph of (1958) Péter directly refers to her talk at the Colloquium and her "failed attempts" to characterize constructive functions through non-circular definitions. This paper presents another attempt to provide such a characterization that includes the special recursive functions but does not ex-

[13] Kalmár's travel report (1956-1957) on the *Constructivity in Mathematics* Colloquium also stands witness to these interests of his. In the second page of the report he mentions that:

> [After the Colloquium I had] the opportunity to visit the "Mathematisch Centrum" and take a look at their operating ARMAC electronic calculator. In addition I had scientific discussion with [Jurjen Ferdinand] Koksma, a mathematics professor, and his colleagues; and with [Adriaan] van Wijngaarden, an engineering professor, about the practical applications of the calculator, as well as about the logical machine under construction in Szeged. (Translated by the present author)

In the report, Kalmár mistakenly refers to "A. Koksma." On Kalmár's description of the ARMAC computer as an electronic calculator, see footnote 14. For short descriptions of the ARMAC computer and the logical machine in Szeged, visit http://www-set.win.tue.nl/UnsungHeroes/machines/armac.html and (Szabó 2016), respectively.

[14] At the time computers were referred to as (high speed) electronic or digital calculators in Hungary.

haust the general recursive ones. Péter introduces "graphschemata," a graphical representation of flow-charts used in programming, which later became known as "Kalužnin-Péter diagrams."[15] Then she considers a rather restricted type of diagram, "normalschemata," as a possible candidate for the characterization of constructive functions. However, at the end it turns out that normalschemata are unsuited for her purposes as she shows that every partial recursive function is computable by a "normalschema." Hence Péter considered this attempt to be yet another failed one.

5 Computers and Church's Thesis

From the end of the 1950s until her passing in 1977, Péter published several papers on the use and applications of recursion theory in computer science. Her late, lesser known book, *Recursive Functions in Computer Theory* (1976/1981), first published in German in 1976 and translated to English in 1981, provides a great introduction and summary of more than a dozen of her papers from this period and deserves greater attention than it has received so far. This section examines how Church's Thesis is discussed in the book in the context of computing machines and programming.

Surprisingly, while Péter showed a non-committal attitude towards Church's Thesis in the earlier works mentioned above, here in (1976/1981), she seemingly commits to what we would today call a (particular version of the) physical version of the thesis in the Preface of:

> The action of a computer can always be thought of as a process such that in response to given input data, the machine produces certain outputs. Since both the input data and the sequential output of the results can be encoded into natural numbers, it follows that the functioning of the computer can always be considered as the computation value of a numeric function. With the idealization that the contents of the computer store are unlimited,[16] it can be shown that the functions computable by a computer are identical with the class of "**partial recursive** functions." [emphasis in the original] (p. 9)

The reason to take such a strong stance already in the 'first' page is to justify the approach taken in the book. From this statement, Péter draws the conclusion that "if we study how the computation of partial recursive number-theoretic functions can be programmed, essentially all questions concerning the problems solvable by a computer will be studied" (p. 9). This statement is repeated

[15]To learn more about Péter's and Kalužnin's work on diagrams in the historical context of automata theory, visit (Mosconi 2014); for a short biography of Kalužnin see (Sushchanskii et al. 1998).

[16]Later on the page Péter adds the following remark: "The above idealization (which will be assumed throughout in what follows) always arises if a general mathematical theory is applied to practical problems. This is often expressed by saying 'the infinite is a useful approximation to the large but finite"' (p. 9).

again later: "Thus if we study the programming problems of the computation of partial recursive functions, this means, in principle, the study of programming of all the machine solvable problems" (p. 63).

There are two noteworthy qualifications in the above statement: the emphasis on the *partiality* of the functions involved and the claim that the identification in this form "can be shown." The latter is especially surprising, as later in the book Péter describes (the classical) Church's Thesis as neither provable nor disprovable mathematically, as it is, "of course, [...] not an exact mathematical proposition" (p. 142).

Péter devotes Chapter 4, *The Recursivity of Everything Computable*, to *showing* that "for every partial recursive function there is a program such that computation with this program yields the value of the function, if it is defined, and goes on forever, without calculating anything if it is not" (p. 63). Péter then uses an idealized assembly language with a simple system of statements. First she shows that every primitive recursive function is machine computable and then claims that "[m]achine computability is also preserved in the application of a μ-operation, not only in the bounded case [...], but in the unbounded case as well" (p. 57). Here a cycle is used for the μ-operation which exits only if the smallest number that was searched for is found and goes on forever otherwise. Then she points to her construction of a universal program for the calculation of partial recursive functions in (Péter 1969). Finally, to establish the reverse direction, Péter shows through examples how to encode programs written in assembly language via partial recursive functions.

Thus it seems that the possibility of "showing" the "recursivity of everything [machine] computable" rests on the availability of concrete computers and their known operations and (assembly) languages. On the other hand, Péter does not provide any argument or even raise as a question whether current computers are capable of computing everything that is machine computable in principle (for example, as Robin Gandy did in his (1980)). The inclusion of an argument of this sort most likely could not be considered as "an exact mathematical proposition" either.

Interestingly, Péter later eases up on both qualifications in the remarkably short 10th Chapter, *Does Recursivity Mean Restriction?* First she returns to the question of partiality and states that: "Actually, a really *partial* recursive function might not be obtained at all." Here, surprisingly, Péter alludes to programming practices. The problem with the computation of the arguments of a proper partial recursive function is that "one can never know whether the computer has failed to stop because the computation is lengthy, or if it will work on forever, without computing anything." Hence "[o]ne always strives to feed 'reasonable' programs into the computer, whereby [...] the calculation will come to a halt after a (large) finite number of computing steps." After this practical restriction, Péter arrives at the conclusion that "whatever can really be obtained by the use of a computer is general recursive" (p. 141).

According to Péter this conclusion raises the question in the title of the chapter, namely, whether the recursivity of what is computable means "an essential restriction on the abilities of the computer" (p. 141). This question, then, leads her to Church's Thesis, i.e. the statement that "every numeric function is general recursive if its values are computable in a finite number of steps for all arguments" (p. 142). Under the assumption of the Thesis, recursivity not only does not pose any restriction, but it means that "computers, which in principle are capable of computing every general recursive function, yield the most that can be expected according to the present state of our knowledge" (p. 142).

Of course when discussing Church's Thesis, while admitting that "there are many arguments for it," Péter has to mention that there are "some" against it as well. Here she mentions the informal character of the Thesis and points to Kalmár's argument against its plausibility (1959). More importantly, she re-emphasizes the conviction she shared with Kalmár in the endless development of mathematics, namely that "effective calculability is one of those notions the definition of which can never be considered complete in the course of the development of mathematics" (p. 142).

This allusion to possible future developments leads to the last and most interesting remark of the short chapter:

> Let us hope, provided a counter-example to Church's thesis is made known [...that...] the technological means will develop to modify computers to enable them to compute such functions. (p. 142)

Hence, Péter not only believes in the endless development of mathematics and mathematical methods, but in the possibility of advancements in computing technologies as well.[17] This latter belief might explain why Péter did not even attempt to characterize in any way what actual computing machines are capable of in principle. At the same time, it seems to undermine the possibility of "showing" that the machine computable functions are identical with the (partial) general recursive functions, or at least casts into doubt what it could amount to in general.

Thus in the end, it seems that Péter was not philosophically committed to what we would call a physical version of Church's Thesis either. However, it appears that she truly endorsed it under "the present state of our knowledge" and saw it as appropriate justification of the approach taken in the book.

6 Conclusion

This short overview of Péter's main discussions of Church's Thesis show a constant and consistent non-committal attitude towards it. She never questioned the usefulness of the concept of general recursive functions within recursion theory and the importance of undecidability results based on Church's Thesis. Nevertheless she persisted in her belief in the endless development of mathematics and that such a development is an argument against the exact formalization and confinement of the concept of calculability once and for all.

[17] This is in stark contrast with Gödel's view that the human mind infinitely surpasses any finite machine for which the "inexhaustibility" of mathematics or the possible future discovery of humanly effective but non-mechanical processes would provide an argument. See Sieg's (2013) for a detailed analysis of Gödel's writings on this issue.

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APPENDIX: The Amsterdam Colloquium

In the summer of 1956, Heyting sent Péter the following invitation (found in (Kalmár and Péter 1930-1976)) to the Amsterdam Colloquium, which was then in the early planning phase.

Dear[18] Madam Péter,

The International Union for Logic, Methodology and Philosophy of the Sciences has charged the Netherlands Society for Logic and Philosophy of Sciences to organize in 1957 at Amsterdam a colloquium on "The different notions of constructivity in mathematics". I hope to organize this colloquium in the summer of 1957, presumably in August. Subventions have been asked from UNESCO and from the Dutch government, by which probably we shall be able to pay a considerable part of the expenses of the participants.

The object of the colloquium will be to study the different notions of constructivity which have been proposed and the relations between them. I shall be please very much if you will participate in it. I join to this letter a list of persons whom I intend to invite.[19] However, this list, as the plan in general is in a preliminary state. I shall be thankful for suggestions w[h]ich you can give me. If the funds are sufficient, I should like to invite a number of young and promising mathematicians as auditors. You will also oblige me by mentioning names which can be considered in this respect, but I beg you to remember that it is by no means sure that such invitations will be possible.

Yours sincerely
A. Heyting

The working title of the Colloquium appears to be more pluralistic at this stage than the final *Constructivity in Mathematics*. Nevertheless, in the *Preface* of the Proceedings, Heyting states in a similar vein that "Several different notions of constructivity were discussed in these lectures" (p. 9).

Based on Péter and Kalmár's correspondence (Kalmár and Péter 1930-1976) and Kalmár's travel report (1956-1957) we know that Péter was among the first 18 logicians invited to the Colloquium, and Kalmár was later invited on her strong recommendation. Altogether 24 logicians received invitations to the event, 21 attended and 19 gave talks, while about the same number of "promising young mathematicians" were among the audience.

Originally the organizers offered to cover 60% of the costs of the attendees. However, since even the remaining costs were essentially insurmountable for the Hungarian scholars, the costs of Péter, Kalmár, and their student, the set theorist and combinatorist András Hajnal (later frequent Paul Erdős co-author) were almost entirely covered by the organizers.

[18] The greeting is stapled over, thus 'dear' is merely an educated guess here.
[19] Sadly this list was not kept in (Kalmár and Péter 1930-1976).

APPENDIX TO THE PREPRINT:

Péter - Rekursivität und Konstruktivität (1959)

This Appendix appears only in this preprint version of my article and is not included in the printed version!

The text of Péter's *Rekursivität und Konstruktivität* article is included below, as it is not trivial to get a hold of, especially not a searchable and copy-pastable digital version. Since this text was generated via a freely available optical character recognition software from low resolution digital images, it may contain typos even after it was inspected and corrected by hand. If the reader finds such a typo, please send me an email to www.mate@gmail.com.

The text below follows the original typesetting and marks the page breaks, the only change is in the few quotation marks, where the „X" style quotation was changed to the "X" style.

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REKURSIVITÄT UND KONSTRUKTIVITÄT

RÓZSA PÉTER
Professor der Eötvös Loránd-Universität, Budapest

Bereits bei der Entstehung des Begriffes der rekursiven Funktion war die Forderung der Konstruktivität entscheidend. Die erfolgreiche Entwicklung der Mengenlehre hat zu seiner Zeit die Tendenz hervorgebracht, die verschiedenen Zweige der Mathematik, auch die Zahlentheorie, als Teile der Mengenlehre au behandeln. Das Auftauchen der mengentheoretischen Antinomien hat dann eben die entgegengesetzte Tendenz hervorgerufen. Skolem [11] hat gezeigt, dass sich die Zahlentheorie mit Hilfe der rekurrierenden Denkweise ohne Berufung auf unendliche Mengen begründen lässt. Die dabei verwendeten Rekursionen erhielten ihre wichtige Rolle eben als Definitionen, welche die Werte der definierten Funktionen an jeder Stelle in endlich vielen Schritten zu berechnen gestatten. (Im Folgenden verstehe ich unter "Zahl" immer eine nichtnegative ganze Zahl und unter "Funktion", wenn nicht etwas anderes gesagt wird, eine zahlentheoretische Funktion.)

Die Endlich-Berechenbarkeit gilt zweifellos für die primitive Rekursion, wo sich der Funktionswert an der Stelle $n+1$ auf den Funktionswert an der vorangehenden Stelle n zurückführen lässt; und für Rekursionsarten, die sich auf primitive Rekursionen zurückführen lassen, wie z.B. die Wertverlaufsrekursion [5], wobei zur Definition von $\varphi(n+1)$ beliebig viele an früheren Stellen angenommenen Werte von φ verwendet werden können; die einfache eingeschachtelte Rekursion [5], wobei für die in der Rekursion nicht teilnehmenden Variablen, für die sogenannten Parameter, Ersetzungen erfolgen; die uneingeschachtelte mehrfache Rekursion [6], wobei die Rekursion zugleich nach mehreren Variablen erfolgt; und die beschränkte mehrfache Rekursion [9], wobei die eben zu definierende Funktion durch eine bereits definierte Funktion majorisiert wird. Aber auch die allgemeine mehrfache Rekursion ist eine konstruktive Definition, und dasselbe gilt auch für eine | Erweiterung dieser Rekursionsart: Die Anordnung der Stellen 227
ist nämlich bei einer k-fachen Rekursion vom Typus ω^k, und daher lassen sich die k-fachen Rekursionen auf transfinite Rekursionen vom Typus ω^k zurückführen [7]. Eine transfinite Rekursion vom Typus α (α eine Ordnungszahl) bedeutet nach ACKERMANN [1] eine Definition, wobei die Funktionswerte mit Hilfe früherer Funktionswerte angegeben werden, wo als "frühere" Stellen die in einer geeigneten Wohlordnung vom Typus α frühere Stellen gelten. Da nun in einer Wohlordnung von einer beliebigen Stelle ausgehend nur endlich viele Schritte rückwärts getan werden können, so lässt sich der Wert einer durch transfinite Rekursion definierten Funktion an jeder Stelle in endlich vielen Schritten berechnen. So ist eine transfinite Rekursion immer konstruktiv - vorausgesetzt natürlich, dass die betreffende Wohlordnung konstruktiv angegeben ist, wie das z.B. für die Ordnung vom Typus ω^k in der zitierten Arbeit angegeben wurde. Man sieht leicht, dass auch die bisher untersuchten Rekursionen der HILBERTschen höheren Stufen [2], [8] konstruktiv sind. Dabei werden zur Definition einer

Funktion der II-ten, III-ten Stufe, usw. als Hilfsfunktionen nicht nur (wie in den bereits erwähnten Rekursionen, die bei der Hilbertschen Klassifikation zur I-ten Stufe gehören) Funktionen von Zahlenvariablen, sondern Funktionen auch von Funktionsvariablen, bzw. auch von Funktionsfunktionsvariablen, usw. zugelassen. Wird dieses "usw." ins transfinite fortgesetzt, so muss freilich bei den Rekursionen der α-ten Stufe auch die Ordnungszahl α konstruktiv definiert werden, um eine konstruktive Definition zu erhalten.

Als eine Zusammenfassung und Verallgemeinerung der durch diesen speziellen Rekursionsarten definierten Funktionen ist der HERBRAND-GÖDEL-KLEENEsche Begriff der allgemein-rekursiven Funktion entstanden [3]. Das ist ein sehr nützlicher Begriff, da er die einheitliche Behandlung sämtlicher speziellen rekursiven Funktionsarten ermöglicht; bisher ist aber keine allgemeinrekursive Funktion bekannt, die für irgendeine mathematische Untersuchung wichtig ist, und nicht unter eine der bekannten speziellen rekursiven Funktionsarten eingereiht werden könnte. Aber der Hauptziel bei der Einführung dieses Begriffes war eben die exakte Fassung des Konstruktivitätsbegriffes. Die sogenannte Church-sche Thesis identifiziert den Begriff der berechenbaren Funktion mit diesem Begriff. Hier möchte ich nicht dar auf ein|gehen, worüber Kalmár sprechen wird, 228
nämlich ob tatsächlich alle berechenbaren Funktionen allgemein-rekursiv sind; ich möchte gerade die entgegengesetzte Frage aufwerfen: können die allgemeinrekursiven Funktionen sämtlich mit Recht "effektiv-berechenbar", d.h. "konstruktiv" genannt werden?

Eine allgemein-rekursive Funktion wird durch ein Gleichungs system angegeben, wobei vorausgesetzt wird, dass es zu jeder Stelle ein endliches Berechnungsverfahren gibt, welche aus Einsetzungen von Zahlen für Variablen und Ersetzungen von Gleichem durch Gleiches besteht, und den Wert der betrachteten Funktion an der angegebenen Stelle eindeutig liefert. Nun ist aber dieses "es gibt" etwas unsicheres, wie darauf schon der sprachliche Ausdruck hinweist, und zwar in den meisten Sprachen. "Es gibt" - wer denn? "Il y a" d.h. "er hat da" - wer und wo? "There is" d.h. "da ist" - wo denn? Kleene meint, wer das in dieser Allgemeinheit nicht annimmt, mag dieses "es gibt" konstruktiv auffassen. Das ist leicht zu sagen, gerade da bisher keine echt-allgemeinrekursive Funktion bekannt ist, und so kann man nicht wissen, was mit einer solchen Einschränkung verloren geht. So werden eigentlich zwei Begriffe der allgemein-rekursiven Funktion definiert: einer mit klassisch aufgefasstem, und einer mit intuitionistisch aufgefasstem "es gibt". Es wäre interessant durch ein Beispiel zu zeigen, inwiefern der letztere Begriff enger ist, nämlich durch eine Funktion, welche klassisch allgemein-rekursiv ist und intuitionistisch nicht; das ist aber kaum zu hoffen, da in den bisherigen Betrachtungen noch überhaupt kein Beispiel für eine allgemein- und nicht speziell-rekursive Funktion vorgekommen ist. Nun, der klassische Begriff der allgemein-rekursiven Funktion ist nicht konstruktiv, und die intuitionistische enthält ein Circulus vitiosus: hier soll das in der Definition auftretende "es gibt" konstruktiv sein – man wollte aber gerade mit dieser Definition der Allgemein-Rekursivität die Konstruktivitāt exakt definieren.

Derselbe Circulus vitiosus taucht überall auf, wie man ihn auch umgehen mag.

Der Aufbau der Theorie der allgemein-rekursiven Funktionen wird sehr erleichtert durch das schöne Ergebnis von KLEENE [3], wonach sich jede allgemein-rekursive Funktion auf eine explizite Form

$$\varphi(a_1, ..., a_r) = \psi(\mu_\omega[\tau(a_1, ..., a_r, \omega) = 0])$$

| bringen lässt, wobei ψ und τ primitiv-rekursiv sind, und $\mu_\omega B(\omega)$ das kleinste 229
ω bedeutet, für welche die Aussage $B(\omega)$ wahr ist (und etwa 0, falls kein solches ω existiert); denn man kann mit einer derartigen expliziten Form viel leichter umgehen, als mit einem definierenden Gleichungssystem. Aber eine derartige explizite Form definiert nur dann eine allgemein-rekursive Funktion, wenn es zu jeder Stelle $(a_1, .., a_r)$ ein ω mit $\tau(a_1, ..., a_r, \omega) = 0$ gibt. Hier haben wir wieder mit diesem "es gibt" zu tun, das konstruktiv gemeint werden muss, wenn man die dadurch definierte Funktion konstruktiv nennen will. Ist das nicht der Fall, so ist die Allgemein-Rekursivität von φ nur ein Schein: ein Gleichungssystem für φ kann zwar aufgeschrieben werden, aber um daraus den Wert von φ an einer gegebenen Stelle zu berechnen, hat man immer grössere Werte für ω einzusetzen, bis einmal $\tau = 0$ wird; und dafür, dass es ein solches ω gibt, liegt vielleicht nur ein Existenzbeweis vor. So haben wir hier wieder den Zirkel.

Wenn nun die speziell-rekursiven Funktionen offenbar verdienen, konstruktiv genannt zu werden, die allgemein-rekursiven Funktionen dagegen nur mit einer Einschränkung, die sich ohne ein Circulus vitiosus garnicht formulieren lässt, so erhebt sich die Frage, ob sich vielleicht durch eine zirkelfreie Definition eine Funktionenklasse, die noch mit Recht konstruktiv genannt werden kann, zwischen diese Funktionenklassen einschalten lässt.

Zunächst hat man dazu zu untersuchen, wie weit man über die bisher untersuchten speziell-rekursiven Funktionenklassen hinaus-gehen kann, um noch immer konstruktive Funktionen zu erhalten. Ich habe gezeigt [7], dass das Diagonalverfahren, auf sämtliche mehrfach-rekursive Funktionen angewandt, eine Funktion liefert, welche durch eine transfinite Rekursion vom Typus ω^ω definiert wird. ACKERMANN [1] hat schon früher durch eine transfinite Rekursion vom Typus der ersten ε-Zahl eine Funktion definiert, und zum Beweis der Widerspruchsfreiheit der Zahlentheorie hat er gerade die Konstruktivität dieser Funktion benutzt. Will man aber in dieser Richtung weitergehen, also als die nächste konstruktive Funktionenklasse die Klasse der durch transfinite Rekursionen definierten Funktionen annehmen, so stösst man wieder an ein Circulus vitiosus. Zu einer transfiniten Rekursion ist ja immer eine Wohlordnung der Zahlen anzugeben, und zwar selbstverständlich konstruktiv, wenn die dadurch
definierte Funk|tion konstruktiv sein soll. ROUTLEDGE [10] hat diese Zirkel- 230
haftigkeit benützend sogar bewiesen, dass die allgemein-rekursiven Funktionen mit den durch transfiniten Rekursionen (sogar vom Typus ω) definierten Funktionen identisch sind - wobei jedocj die nötigen Wohlordnungen wiederum durch allgemein-rekursive Funktionen angegeben werden.

Ähnlich kann vermutet werden, dass die Klasse der allgemein-rekursiven Funktionen mit der Klasse der Funktionen der Hilbertschen Stufen identisch ist, falls die Bildung dieser Stufen transfinit fortgesetzt wird. Offenbar können aber die auf der α-ten Stufe definierten Funktionen wiederum nur dann als konstruktiv betrachtet werden, wenn auch die Ordnungszahl α konstruktiv ist, was nach Church und KLEENE [4] im Wesentlichen ebenfalls auf den Begriff der Allgemein-Rekursivität hinauskommt. Da haben wir wieder einen Zirkel, wie bei der transfiniten Rekursion. – Auf diesem Gebiet gibt es noch viele ungelöste Probleme; z.B. die bereits erwähnte Vermutung der Identität der Klasse der allgemein-rekursiven Funktionen mit der Klasse der Funktionen der Hilbertschen Stufen; ferner ob die Hilbertsche II-te Stufe über die I-te Stufe wirklich hinausführt, wenn auch mehrfache Rekursionen zugelassen werden; und die entsprechende Frage für die höheren Stufen.

Man könnte die Suche nach der zweckmässigen Definition auch von anderer Seite her beginnen: man könnte aus einer offenbar konstruktiven Funktionenklasse ausgehen, und mit Hilfe von geeigneten allgemeinen Erweiterungsmethoden weitergehen, welche noch nicht über die intuitiv konstruktiven Funktionen hinausführen. Eine solche Erweiterungsmethode wäre z.B. das Diagonalverfahren. Man könnte diese schrittweise anwenden; also, nachdem das auf die mehrfach-rekursiven Funktionen angewandte Diagonalverfahren zu einer speziellen transfiniten Rekursion geführt hat, nicht gleich auf die allgemeine transfinite Rekursion übergehen, sondern die Diagonalfunktion zu den bisherigen Funktionen hinzugenommen, wieder das Diagonalverfahren verwenden, usw. In diesem "usw." steckt wieder das bereits erwähnte Circulus, wenn nämlich das Verfahren transfinit fortgesotzt wird. Sogar ein zweiter Zirkel steckt darin, wenn die Art der Abzählung der bereits gewonnenen Funktionen nicht von vornherein in einer offenbar konstruktiven Weise fixiert wird, sondern nur soviel verlangt wird, dass diese Abzählung konstruktiv sein soll.

| Eine andere Möglichkeit zur Erweiterung der offenbar konatruktiven Funk- 231
tionenklassen wäre zu untersuchen, wasfür "Gebrauchsanweisungen" zu den Definitionsgleichungen angegeben werden können, um die Werte der definierten Funktionen tatsächlich zu erhalten. (Es kann ja auch ein am willkürlichsten aussehendes Gleichungssystem als konstruktive Definition einer Funktion angenommen werden, wenn dazu eine annehmbare Gebrauchsanweisung zur Berechnung der Funktionswerte gehört.) Z.B. gehören zu der primitiven Rekursion

$$\begin{array}{ll} G_1 & \varphi(0) = k \\ G_2 & \varphi(x+1) = \alpha(x, \varphi(x)), \end{array}$$

wobei α bereits konstruktiv definiert ist, folgende Gebrauchsanweisungen: soll $\varphi(n)$ berechnet werden, so

(1) a) falls $n = 0$, man kopiere G_1,
b) falls $n \neq 0$, man setze in G_2 $n-1$ für x:

$$\varphi(n) = \alpha(n-1, \varphi(n-1)),$$

(2) falls man eine Gleichung hat, wo auf der linken Seite $\varphi(n)$ steht, sind folgende Fälle zu unterscheiden:

α) steht auf der rechten Seite eine Zahl, so ist die Berechnung fertig,

β) kommt auf der rechten Seite $\varphi(m)$ vor, wo m eine von 0 verschiedene Zahl ist, dann soll in G_2 $m-1$ für x gesetzt werden:

$$\varphi(m) = \alpha(m-1, \varphi(m-1)),$$

und in der vorangehenden Gleichung soll $\varphi(m)$ durch die rechte Seite dieser Gleichung ersetzt werden,

γ) kommt auf der rechten Seite $\varphi(0)$ vor, dann soll diese durch k ersetzt werden,

δ) sonst (nämlich, wenn die rechte Seite durch Einsetzungen aus α und Zahlen aufgebaut wird) soll die rechte Seite auf Grund der konstruktiven Definition von α berechnet, und der erhaltene Wert für sie eingesetzt werden.

Diese Gebrauchsanweisungen können leicht "gödelisiert" werden, und so ergibt sich für sie eine einfache primitive Rekursion. Man weiss nuch, dass (1) einmal und die Schritte (2) $n+1$-mal anzuwenden sind.

| Zunächst würde man glauben, dass zu einer komplizierteren Rekursion- 232
sart auch kompliziertere Gebrauchsanweisungen gehören, und so könnte man durch Zulassung komplizierterer Gebrauchsanweisungen zu weiteren konstruktiven Funktionenklassen kommen. Das ist aber nicht der Fall: damit die Berechnung eines Funktionswertes $\varphi(n_1, ..., n_r)$ überhaupt möglich sei, müssen derartige Definitionsgleichungen vorhanden sein, dessen eine Seite durch geeignete Einsetzung zu $\varphi(n_1, ..., n_r)$ wird, und die andere Seite entweder eine Zahl ist, oder aus bereits konstruktiv definierten Funktionen aufgebaut wird, oder aber auch φ enthält, jedoch an Stellen, die in irgendeinem Sinn als "günstigere" Stellen betrachtet werden können, nämlich am Wege zur Berechnung näher zum Ziel liegen. In den bekannten Rekursionen führt immer zum Ziel, wenn man den nächsten Schritt auf ein innerstes φ anwendet (dessen Argumente also kein φ mehr enthalten), oder, falls die Funktionen $\sigma_1, \sigma_2, ..., \sigma_l = \varphi$ simultan definiert werden, auf ein innerstes σ_i. So erhält man immer ähnliche einfache Gebrauchsanweisungen, wie bei der betrachteten primitiven Rekursion. Aber an den Gebrauchsanweisungen sieht man nicht an, ob sie in endlich vielen Schritten zu Ende führen. Die Anzahl der anzuwendenden Schritte wird bei den komplizierteren Rekursionsarten immer komplizierter: bei einer 2-fachen Rekursion ist auch die Anzahl der Berechnungsschritte eine 2-fach rekursive Funktion der Stelle. So wird die Endlichkeit des durch die Gebrauchsanweisung gegebenen Berechnungsverfahrens für eine 2-fach rekursive Funktion nur dadurch gesichert, dass die 2-fache Rekursion konstruktiv ist.

So kommt man also nicht weiter. Auch für beliebige allgemein-rekursive Funktionen kann man sehr einfache Gebrauchsanweisungen angeben: zur Berechnung des Wertes $\varphi(n_1, ..., n_r)$ verwende man auf die Definitionsgleichungen die erlaubten "elementaren" Berechnungsschritte (statt Einsetzung einer Zahl n für x gelten als elementare Schritte: die Ersetzung von x durch die Nachfolgerfunktion - dies geschieht n-mal - und dann durch 0), und zwar erst nur einmal (dies kann endlichvielemal geschehen), dann auf die erhaltenen Gleichungen noch einmal, usw.; hat man eine Gleichung der Form $\varphi(n_1, ..., n_r) = c$, wo c eine Zahl ist,

so ist die Berechnung beendet. Wieviele Schritte sind dazu nötig? Kann man dafür keine konstruktive Schranke angeben, so ist die Definition nicht konstruktiv. Gibt es eine primitiv-rekursive, oder | etwa mehrfach-rekursive Schranke 233
für die Berechnungsschritte, so ergibt sich aus der Ableitung der Kleeneschen expliziten Form (da nämlich das darin vorkommende kleinste ω als ein Mass der zur Berechnung notwendigen Schritte betrachtet werden kann; ferner dieses kleinste ω, wenn eine primitiv- oder etwa mehrfach-rekursive Schranke dafür vorliegt, auch primitiv- bzw. mehrfach-rekursiv ist), dass auch die definierte Funktion primitiv- bzw. mehrfach-rekursiv ist. Auf diesem Wege gelingt es also auch keine konstruktive Funktionenklasse zwischen die speziell-rekursiven und die allgemein-rekursiven Funktionen einzuschalten.

Es hat den Anschein, dass sich der Konstruktivitätsbegriff überhaupt nicht zirkelfrei erfassen lässt.

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