A No-Go Theorem for ψ-Ontic Models

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In this note we give a simple no-go theorem that explains why ψ-ontic models cannot be expected to reproduce quantum mechanics. We will see that, using information theoretic considerations, the lack of overlap of epistemic states requires all states to be orthogonal, which openly contradicts quantum theory. This argument is a simplified version of the one we presented in a previous paper.

I. INTRODUCTION

In 2010 Harrigan and Spekkens (HS) proposed a formal classification to categorize the nature of the quantum state, i.e. to establish whether in a certain model ψ corresponds to a real property instantiated by a quantum object, in which case the model is called ψ-ontic, or to some observer information, making it ψ-epistemic [1]. Although the original aim of this classification was to clarify Einstein's view about quantum mechanics (QM), the HS framework has been widely employed not only to categorize different interpretations of QM, but also to argue what formulations of the theory are admissible ([2–10]; cf. [11–13] for critical discussions).

Given the influence of the HS framework in quantum foundations, and in particular the prominent role played by ψ-ontic models in several arguments supporting the reality of the quantum state—many of them built upon [2]—we re-examine the definitions and features of such models in order to understand whether they constitute a sound basis from which to draw conclusions on the nature of ψ. Our analysis suggests that they are not.

Referring to this, in this note we provide a no-go theorem for ψ-ontic models which is a simplified version of the main result contained in our previous paper [14]. More precisely, here we show that, using information theoretic considerations, ψ-ontic models cannot reproduce QM since the lack of overlap of epistemic states requires all states to be orthogonal, which openly contradicts quantum theory. Therefore, we conclude, they should not be employed in order to classify quantum models.[1]

II. THE ARGUMENT

We aim to show that a set of non-overlapping probability distributions cannot reproduce the entropy given by quantum mechanics. In order to show this claim, let us review a property of the information entropy, both in the classical setting (i.e. the Shannon/Gibbs entropy) and in the quantum setting (i.e. the von Neumann entropy).[2]

Entropy of mixed non-overlapping distributions. Let ρ_1 and ρ_2 be two classical distributions over a space Λ with measure λ. The entropy for each distribution is given by the usual formula[3], for example

$$H(\rho_1) = -\int_\Lambda \rho_1(\lambda) \log \rho_1(\lambda) d\lambda. \tag{1}$$

Suppose the two distributions are disjoint, and let $\rho = \frac{1}{2}\rho_1 + \frac{1}{2}\rho_2$ be a uniform mixture of the two distributions. The entropy of ρ is given by

$$H(\rho) = 1 + \frac{1}{2}H(\rho_1) + \frac{1}{2}H(\rho_2). \tag{2}$$

Note how the non-overlapping assumption fixes the entropy of the mixed state.

Entropy of quantum mixed states. Now suppose ψ and ϕ are two pure quantum states and let $p = |\langle\psi|\phi\rangle|^2$ be the probability of transition from one to the other. Consider the mixed state $\rho = \frac{1}{2}|\psi\rangle\langle\psi| + \frac{1}{2}|\phi\rangle\langle\phi|$. Its entropy is given by

$$H(\rho) = H\left(\frac{1+\sqrt{p}}{2}, \frac{1-\sqrt{p}}{2}\right). \tag{3}$$

Theorem: Since non-overlapping distributions can only represent orthogonal states, ψ-ontic models cannot be consistent with quantum theory.

Proof: Suppose we have a ψ-ontic model defined according to [1]. The epistemic states $p(\lambda|P_\psi)$ and $p(\lambda|P_\phi)$ consist of non-overlapping distributions over a space Λ, therefore eq. 2 applies. Given that ψ and ϕ are pure states, and the entropy for pure states is zero, we must have

$$H(p(\lambda|P_\psi)) = H(p(\lambda|P_\phi)) = 0, \tag{4}$$

and therefore

$$\begin{aligned} &H\left(p(\lambda|\frac{1}{2}P_\psi + \frac{1}{2}P_\phi)\right) = \\ &H\left(\frac{1}{2}p(\lambda|P_\psi) + \frac{1}{2}p(\lambda|P_\phi)\right) = 1. \end{aligned} \tag{5}$$

[1] We assume that the reader is familiar with HS definitions.

[2] We show the explicit calculations in the appendix.
[3] The logarithm is assumed to be in base 2.

If we compare the above with eq. 3, it follows that p must be zero. That is, we must have that

$$\langle\psi|\phi\rangle = 0 \tag{6}$$

no matter what ψ and ϕ are.

Hence, the non-overlapping assumption built into the ψ-ontic model necessarily implies that all pure states are orthogonal. Since this is not true in quantum mechanics, any ψ-ontic model will fail to reproduce the results of quantum information, quantum statistical mechanics and, therefore, quantum theory in general. $\square$

In retrospect this should not be surprising for two reasons. First of all, the case where all states are orthogonal corresponds exactly to the case in which all observables commute, i.e. the classical case. Therefore, we are simply finding that the implicit use of classical probability in ψ-ontic models pushes us to classical interpretation of quantum models. Secondly, we already know that quantum information theory has features that are not reproducible in classical information theory, and that is exactly what makes it such an exciting new field.

III. CONCLUSION

We have presented a simpler and more compact version of the main result presented in [14]. Both arguments reach the same conclusion: ψ-ontic models are not compatible with quantum information theory, and therefore with quantum mechanics itself.

One may think the issue could be circumvented by substituting the Shannon entropy formula. That is, the problem is not with the non-overlapping assumption but rather with how the entropy is calculated. However, the link between probability theory, information theory and measure theory is so tight that it makes this approach unfeasible. If we take a set of states U with a measure $\mu(U)$ that quantifies the states in the set, the probability of each state for a uniform distribution is given by $\rho = 1/\mu(U)$ and the entropy by $\log \mu(U)$. The measure used to count states, the probabilities assigned to states and the entropy assigned to distributions are not independent mathematical structures: failure of one means failure of all. We will look deeper into this issue in a later work.

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APPENDIX: CALCULATION

Entropy of mixed non-overlapping distributions. We want to show that, given two non-overlapping probability distributions ρ_1 and ρ_2, the entropy of $\rho = \frac{1}{2}\rho_1 + \frac{1}{2}\rho_2$ is given by

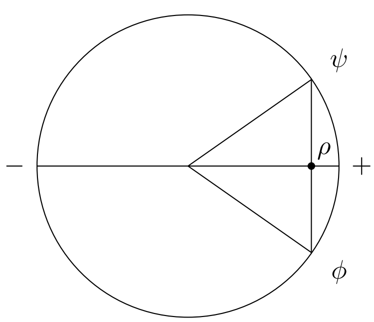
$$H(\rho) = 1 + \frac{1}{2}H(\rho_1) + \frac{1}{2}H(\rho_2). \quad (2)$$

Let $U_1, U_2 \subset \Lambda$ be the respective supports of the distributions. Since the distributions are non-overlapping, we have $U_1 \cap U_2 = \emptyset$. We have

$$\begin{aligned} H(\rho) &= -\int_\Lambda \rho \log \rho d\lambda \\ &= -\int_{U_1} \rho \log \rho d\lambda - \int_{U_2} \rho \log \rho d\lambda \\ &= -\int_{U_1} \frac{1}{2}\rho_1 \log \frac{1}{2}\rho_1 d\lambda - \int_{U_2} \frac{1}{2}\rho_2 \log \frac{1}{2}\rho_2 d\lambda \\ &= -\frac{1}{2}\int_{U_1} \rho_1 \log \frac{1}{2} d\lambda - \frac{1}{2}\int_{U_1} \rho_1 \log \rho_1 d\lambda \\ &- \frac{1}{2}\int_{U_2} \rho_2 \log \frac{1}{2} d\lambda - \frac{1}{2}\int_{U_2} \rho_2 \log \rho_2 d\lambda \\ &= -\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2} + \frac{1}{2}H(\rho_1) + \frac{1}{2}H(\rho_2) \\ &= 1 + \frac{1}{2}H(\rho_1) + \frac{1}{2}H(\rho_2). \end{aligned}$$

Entropy of quantum mixed states. We want to show that, given two states ψ and ϕ, the entropy of the mixed state $\rho = \frac{1}{2}|\psi\rangle\langle\psi| + \frac{1}{2}|\phi\rangle\langle\phi|$ is

$$H(\rho) = H\left(\frac{1 + |\langle\psi|\phi\rangle|}{2}, \frac{1 - |\langle\psi|\phi\rangle|}{2}\right). \quad (7)$$



Note that ψ and ϕ will identify a two-dimensional subspace which can be thought, without loss of generality, as a qubit and therefore can be represented by a Bloch sphere. The picture represents the intersection of the Bloch sphere with the plane identified by ψ and ϕ. As ρ is an equal mixture of the two states, it will be represent by the midpoint between the two. Taking the line that goes through ρ and the center of the sphere, we can see that ρ can also be seen as the mixture of the states $+$ and $-$ which, since they represent equal and opposite directions, form a basis. To diagonalize ρ, then, means to express it in terms of $+$ and $-$.

If $\theta_{\psi\phi}$ is the angle between ψ and ϕ, we have

$$|\langle\psi|\phi\rangle|^2 = \cos^2\frac{\theta_{\psi\phi}}{2}. \quad (8)$$

The angle is divided by two because the angle on the Bloch sphere (i.e. in physical space) is double the angle in the Hilbert space. For example, for z^+ and z^- the angle on the Bloch sphere would be π and the inner product is zero (i.e. opposite directions in physical space correspond to orthogonal states).

Now we express ψ and ϕ in terms of $+$ and $-$, remembering that they form a basis. Given that ρ is at the midpoint, the figure is vertically symmetric. The angle between ψ and $+$, then, is half of $\theta_{\psi\phi}$. The inner product between ψ and $+$ is

$$\begin{aligned} |\langle\psi|+\rangle|^2 &= \cos^2\frac{\theta_{\psi+}}{2} \\ &= \cos^2\frac{\theta_{\psi\phi}}{4}. \end{aligned} \quad (9)$$

Keeping in mind that we are composing vectors in the Hilbert space (and not in the geometry of the physical space) we have

$$\begin{aligned} |\psi\rangle &= \cos\frac{\theta_{\psi\phi}}{4}|+\rangle + \sin\frac{\theta_{\psi\phi}}{4}|-\rangle \\ |\phi\rangle &= \cos\frac{\theta_{\psi\phi}}{4}|+\rangle - \sin\frac{\theta_{\psi\phi}}{4}|-\rangle. \end{aligned}$$

The density matrices corresponding to the pure states are

$$\begin{aligned} |\psi\rangle\langle\psi| = \cos^2\frac{\theta_{\psi\phi}}{4}|+\rangle\langle+| \\ + \cos\frac{\theta_{\psi\phi}}{4}\sin\frac{\theta_{\psi\phi}}{4}(|+\rangle\langle-| + |-\rangle\langle+|) \\ + \sin^2\frac{\theta_{\psi\phi}}{4}|-\rangle\langle-| \\ |\phi\rangle\langle\phi| = \cos^2\frac{\theta_{\psi\phi}}{4}|+\rangle\langle+| \\ - \cos\frac{\theta_{\psi\phi}}{4}\sin\frac{\theta_{\psi\phi}}{4}(|+\rangle\langle-| + |-\rangle\langle+|) \\ + \sin^2\frac{\theta_{\psi\phi}}{4}|-\rangle\langle-|. \end{aligned}$$

We can now calculate the mixture

$$\begin{aligned} &\frac{1}{2}(|\psi\rangle\langle\psi| + |\phi\rangle\langle\phi|) \\ &= \cos^2\frac{\theta_{\psi\phi}}{4}|+\rangle\langle+| + \sin^2\frac{\theta_{\psi\phi}}{4}|-\rangle\langle-| \\ &= \frac{1 + \cos\frac{\theta_{\psi\phi}}{2}}{2}|+\rangle\langle+| + \frac{1 - \cos\frac{\theta_{\psi\phi}}{2}}{2}|-\rangle\langle-| \\ &= \frac{1 + |\langle\psi|\phi\rangle|}{2}|+\rangle\langle+| + \frac{1 - |\langle\psi|\phi\rangle|}{2}|-\rangle\langle-|. \end{aligned}$$

As ρ is in a diagonal form, the entropy is given by eq. 7.