# A Guide to the Bargmann Mass Superselection Rule: Why There Is—and Isn't—Mass Superselection in Non-Relativistic Quantum Mechanics

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What became known as the Bargmann mass superselection rule for non-relativistic quantum mechanics arose from a exercise, set by Arthur Wightman in 1959, to demonstrate that "superposition of two states with different mass gives a state whose existence in Nature would contradict Galilean invariance (Bargmann's Superselection Rule)." A solution to the exercise in the form of a heuristic argument first appeared in journal articles and was then repeated in textbooks. Subsequently, however, attacks on the rule were launched from various directions, and perusing the literature, old and new, leaves one with the impression that the rule enjoys an uncertain status. The goal here is to explain the sense in which there most certainly is a mass superselection rule for ordinary quantum mechanics, as well as a sense in which there isn't a superselection rule for mass. Reaching this goal requires a careful look at the nature of superselection rules and the meaning of Galilean invariance in ordinary quantum mechanics.

## 1 Introduction

The idea of superselection in quantum theory was introduced by Eugene Wigner at a conference talk in 1951 and elaborated the following year in the drei manniche paper of Wick, Wightman, and Wigner (1952) where it was argued that there is a superselection rule for whole-integer/half-integer spin and conjectured that there is also a superselection rule for charge.[1] But

[1]The conference on nuclear physics and the physics of fundamental particles was held at the University of Chicago; for a summary of the Wigner's talk and questions from the audience see Orear et al. (1951). If the summary is accurate, most of the audience failed to grasp Wigner's thesis. The original argument offered for whole-integer/half-integer spin

there is no mention in WWW of a superselection rule for mass in ordinary non-relativistic quantum mechanics (NRQM).

In the same year Inöu and Wigner (1952) had noted an annoying feature of the projective unitary representations of the Galilean group, the presumed symmetry group of NRQM: the unitaries representing a spatial translation by $\mathbf{a}$ and a pure velocity boost by $\mathbf{v}$ do not commute and

> their products taken in a different order differs by a factor $\exp(\frac{i}{\hbar} m\mathbf{v} \cdot \mathbf{a})$. Furthermore, V. Bargmann had shown that these representations are essentially the only 'up to a factor' representations. (p. 706)

(The $m$ is the mass of the particle. I have added back the $\hbar$ which was suppressed in the original.) The reference to Bargmann, Wigner's Princeton colleague, was a "private communication." Bargmann's seminal paper on projective (aka ray) representations of continuous groups was published in 1954 (Bargmann 1954), and the obvious inference is that Bargmann had shown Wigner a draft of this paper or at least had told Wigner of the results for projective representations of the Galilean group.

It is common practice in the physics literature to cite Bargmann (1954) when referring to the Bargmann superselection rule. But there is no mention of superselection rules in Bargmann's paper, much less an argument for mass superselection in NRQM. So who was responsible for proposing this rule? A strong clue is to be found in Inöu and Wigner (1952). The awkward feature of the projective representations of the Galilean group mentioned there led them to investigate "true" (aka proper or vector) representations. But disappointingly, the ones they investigated have even more disturbing features: they are incapable of describing states localized in space or having a definite velocity, and the attempt to identify the infinitesimal spatial translation operator with the momentum operator leads to "absurd results." At this juncture the next logical step would have been for Inöu and Wigner to return to the projective representations of the Galilean group and work out the consequences of the odd feature they had noted at the beginning of their article. Had they done so an argument for the mass superselection would

superselection supposed that time reversal is a symmetry in quantum theory. When this supposition came into question a new argument was given based on rotational invariance; see Hegerfeldt et al.(1968). For charge superselection see Wick et al. (1970) and Strocchi and Wightman (1974).

have quickly suggested itself, for what became the standard argument in the literature starts exactly from this odd feature (see Section 5 below). But they did not take this step, and their article ends with a recitation of the features that rendered physically unacceptable the true representations they investigated.

The first footnote in WWW (1952) mentions that their article is based in part "on a review article which the last two authors [Wightman and Wigner] are preparing with V. Bargmann."[2] Bargmann, Wightman and Wigner were all Princeton colleagues, and the odd feature that Inöu and Wigner had noted about the projective representations of the Galilean group found by Bargmann would have come up in their discussions of superselection rules. Given the ways their minds worked, it is difficult not to believe that this feature served as the spark for them to formulate an argument for mass superselection. Such an argument, or rather an exercise inviting the reader to supply an argument, is found in Wightman (1959). The exercise is to prove that the fact that the multipliers (or factors) for the projective representations of the Galilean group responsible for the odd result noted by Inöu and Wigner (1952)

> cannot be removed by permissible phase changes ... provides us with a superselection rule on the mass for the Galilean group: superposition of two states with different mass gives a state whose existence in Nature would contradict Galilean invariance (Bargmann's Superselection Rule). (p. 86)[3]

The point of the exercise was not only to call attention to a feature of NRQM but also to emphasize the importance of the fact that the Poincaré group admits proper unitary representations and, thus, presumably furnishes no basis for mass superselection (see p. 85). As far as I am aware, Wightman's exercise contains the first mention of "Bargmann's Superselection Rule" in the published literature.[4] It remains undetermined whether Wightman's cho-

[2]As far as I am aware this review article never appeared in print.

[3]It is hard to know whether Wightman himself fully approved of the quoted formulation of the exercise since Wightman (1959) consists of "Notes by A. Barut on Lectures by A. S. Wightman."

[4]Curiously, this paper of Wightman's is rarely cited when authors want to give a reference for the Bargmann rule. Sometimes citations are given to Wightman (1960) where there is no mention of the Bargmann rule (see, for example, Lévy-Leblond 1963 and Bernstein 1967)

sen appellation was due to the fact that his colleague Bargmann supplied a solution to the exercise or whether Wightman decided that the appellation was appropriate because Bargmann's work on representations of the Galilean group supplied the basis of the rule.

What became the standard solution to Wightman's exercise began appearing in the journal literature in the early 1960s (see Lévy-Leblond 1963), and it has been repeated in various forms in textbooks (see Kaempffer 1965, Appendix 7; Jordan 1969, p. 120; Galindo and Pascual 1990, pp. 292-293; Blank et al. 1994, p. 371; de Azcárraga and Izquierdo 1995, p. 157; and Gottfried and Yan 2003, p. 295-296). The origin of this solution remains uncertain, but since the structure of the proof is similar to the proof offered in WWW (1952) for superselection of whole-integer/half-integer spin it is plausible that the standard solution was sketched by Wigner or Wightman in lectures or conversations with colleagues. But whatever its origins the standard argument for Bargmann's mass superselection seems to have convinced the physics community.

Starting in the mid-1990s the history takes a curious turn: the once seemingly well-established mass superselection rule was attacked from various directions. Some of the skepticism was motivated by a combination of a suspicion that formal arguments for superselection based on symmetry principles are hiding physical assumptions and by a desire for a dynamical explanation of superselection, e.g. environmental factors may block some coherent superpositions. More direct attacks claimed that the standard argument for mass superselection is incoherent. And perhaps unkindest of all, Steven Weinberg (1995) charged that the issue of mass superselection in NRQM is a tempest in a tea pot because the Galilean symmetry group of NRQM can be expanded to a central extension which has the same physical consequences as the Galilean group but which admits proper unitary representations and, therefore, obviates the need for a mass superselection rule.[5]

The standard argument for Bargmann's mass superselection rule has a heuristic character, and as such it can be faulted for lack of rigor and, perhaps, also as begging the question. I will offer a proposal for what would count as a mathematically rigorous and non-question-begging solution to Wightman's exercise, and will offer a proof of this version of the solution.

[5] For a sampling of the critical literature see Guilini (1996), Greenberger (2001a, 2000b), Hernandez-Coronado (2012), Weinberg (1995), and Zych and Greenberger (2019).

But whether one uses the standard heuristic argument or the more formal one, there is no reason to doubt that there is a solution to Wightman's exercise that provides a valid basis for *a* sense of mass superselection in NRQM. I will claim, however, that this sense of mass superselection does not satisfy Wigner's and Wightman's own criteria for what counts as a superselection rule. To explain this seeming conundrum requires a slog through mathematical physics that may test the reader's patience; but patience is rewarded by a deeper understanding of the role of symmetries in quantum physics and the nature of superselection.

The present guide to this tangled subject is structured as follows. Section 2 provides a preliminary look at the nature of superselection rules. Section 3 introduces the Galilean group and gives a brief overview of projective representations of groups. Section 4 explains the origin of the projective representation of the Galilean group presupposed in Wightman's exercise. Section 5 reviews the standard solution to Wightman's exercise, and the shortcomings of this argument are laid out in Section 6. Section 7 sketches a way to repair these shortcomings, providing a more rigorous basis for the sense of mass superselection in NRQM demanded by Wightman's exercise. Section 8 reveals how this sense of superselection extends to other, often unnoticed, cases. Also discussed in this section is how the standard argument for mass superselection is extended to the total mass of a composite system. Central extensions of the Galilean group and their representations are reviewed in Section 9. The implications (and non-implications) for mass superselection of the fact that central extensions of the Galilean group admit proper unitary representations are discussed in Section 10. The Poincaré group, the symmetry group that replaces the Galilean group in relativistic quantum mechanics (RQM), does not lead to mass superselection. The emergence of mass superselection in the Newtonian limit when the Poincaré group contracts to the Galilean group is discussed in Section 11. Section 12 presents the case that the sense of mass superselection used in the solution to Wightman's exercise does not meet the standards for mass superselection that Wightman himself set for other cases. Conclusions are presented in Section 13. An Appendix discusses the nomenclature of superselection rules when there is an infinity—even an uncountable infinity—of selection sectors.

## 2 Superselection rules: a first look

I will treat superselection in terms of the algebraic formulation of QM in which a quantum system is characterized by a von Neumman algebra $\mathfrak{N}$ of observables acting on a Hilbert space $\mathcal{H}$. $\mathcal{H}$ is often assumed to be separable, but in order to treat superselection it may prove useful to employ non-separable Hilbert spaces (see the Appendix). In NRQM *sans* superselection the algebra of observables is $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$, the von Neumann algebra of all bounded operators acting on $\mathcal{H}$.[6] An algebraic $\chi$ state on $\mathfrak{N}$ is a normed positive linear functional $\chi : \mathfrak{N} \to \mathbb{C}$. A vector state is is a state for which there is $\psi \in \mathcal{H}$ such that $\chi(A) = \langle \psi, A\psi \rangle$ for all $A \in \mathfrak{N}$. A pure state $\chi$ has the property that there do not exist distinct states $\chi_1$ and $\chi_2$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 + \lambda_2 = 1$ and $0 < \lambda_1, \lambda_2 < 1$ such that $\chi = \lambda_1\chi_1 + \lambda_2\chi_2$.

The loose and misleading way of introducing the idea of a superselection rule is to say that it places a limitation on the superposition principle. Taken literally this is nonsense: a Hilbert space is a vector space and as such the superposition of two vectors is again a vector in the space. The more accurate statement is that a superselection rule is a limitation on *coherent* superpositions. When $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$ every vector state is pure, but for other algebras this implication is broken: the vector state corresponding to the superposition of two pure vector states may not be coherent, i.e. it can be an impure (aka mixed) state, signalling a superselection rule at work. Just as importantly, a superselection rule imposes a restriction on what counts as a genuine observable in QM. When von Neumann wrote *Mathematical Foundations of Quantum Mechanics* (1932) he assumed that quantum observables are represented by (essentially) self-adjoint operators; and he also presumed the converse, i.e., any (essentially) self-adjoint operator represents an observable or, in the language of the algebraic formulation, the spectral projections of the unique self-adjoint extension of any essentially self–adjoint operator are all in the von Neumann algebra that characterizes the system, which will certainly be the case if that algebra is $\mathfrak{B}(\mathcal{H})$. When superselection is at work this presumption is undercut because the relevant algebra of observables is

[6]The von Neumann algebra of observables may be thought of as being generated by the set $\mathcal{O}$ of (not necessarily bounded) self-adjoint operators on $\mathcal{H}$ that correspond to genuine physical observables in the intended sense of quantities that can, in principle, be measured. Then $\mathfrak{N}(\mathcal{O}) := \mathcal{O}'' := (\mathcal{O}')'$, where $\mathcal{O}'$ denotes the commutant of $\mathcal{O}'$, i.e. the set of all bounded operators $B \in \mathfrak{B}(\mathcal{H})$ that commute with all $A \in \mathcal{O}$. That $B$ commutes with an unbounded self-adjoint $A$ means that $B$ commutes with all the spectral projections of $A$.

a proper subalgebra of $\mathfrak{B}(\mathcal{H})$.

Superselection rules come in different strengths. According to Strocchi and Wightman (1974), a superselection rule in the broadest sense "can be defined as any restriction on what is observable in the theory" (p. 2198). In the present nomenclature this means that $\mathfrak{N}$ is a proper subalgebra of $\mathfrak{B}(\mathcal{H})$ or, equivalently, that the commutant $\mathfrak{N}'$ of $\mathfrak{N}$ does not consist of multiples of the identity. This may a bit too broad since usually one expects that the superselection operators—operators that, among the other properties they satisfy, must at least commute with everything in $\mathfrak{N}$—are themselves in $\mathfrak{N}$. With this understanding the existence of superselection operators requires that the center $\mathcal{Z}(\mathfrak{N}) := \mathfrak{N}' \cap \mathfrak{N}$ of $\mathfrak{N}$ does not consist solely of multiplies of the identity. A yet stronger requirement is the "commutativity of superselection operators," which means that $\mathfrak{N}'$ is abelian. This is equivalent to requiring that $\mathfrak{N}' \subseteq \mathfrak{N}$ and, thus, that $\mathcal{Z}(\mathfrak{N}) = \mathfrak{N}'$. Further, the commutativity of superselection operators is equivalent to the more familiar notion of the existence of a complete set of commuting observables, which in the present nomenclature means that $\mathfrak{N}$ contains an abelian subalgebra maximal in $\mathfrak{B}(\mathcal{H})$.

When the commutativity of superselection operators holds and these operators have a discrete spectrum the algebra of observables has the form of a direct sum $\oplus_{a\in\mathcal{I}}\mathfrak{N}_a$ acting on a direct sum of Hilbert spaces $\oplus_{a\in\mathcal{I}}\mathcal{H}_a$. In NRQM the $\mathfrak{N}_a$ are typically $\mathfrak{B}(\mathcal{H}_a)$. The projections onto the subspaces $\mathcal{H}_a$ are superselection operators, and the $\mathcal{H}_a$ are referred to as the superselection sectors. The meaning of the direct sum expressions when the index set $\mathcal{I}$ is infinite will be discussed in Section 12. But for present purposes it is sufficient to concentrate on the case of two selection sectors.

The direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2$ of the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ is constructed by forming a vector space consisting of pairs of vectors $\vartheta \in \mathcal{H}_1$ and $\zeta \in \mathcal{H}_2$ and imposing the rules for scalar multiplication and vector addition given by $\alpha(\vartheta \oplus \zeta) = \alpha\vartheta \oplus \alpha\zeta$, $\alpha \in \mathbb{C}$, and $(\vartheta_1 \oplus \zeta_1) + (\vartheta_2 \oplus \zeta_2) = (\vartheta_1 + \vartheta_2) \oplus (\zeta_1 + \zeta_2)$. This direct sum space is complete in the norm derived from the inner product $\langle \oplus_\vartheta, \oplus_\zeta \rangle := \sum_{\alpha\in\{1,2\}} \langle \vartheta_\alpha, \zeta_\alpha \rangle_{\mathcal{H}_\alpha}$ and is, therefore, a Hilbert space. The superselection algebra is $\mathfrak{B}(\mathcal{H}_1) \oplus \mathfrak{B}(\mathcal{H}_2)$, a proper subalgebra of $\mathfrak{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$, illustrating the limitation on what counts as an observable. Consider the vector states $\chi_{\vartheta_1}$ and $\chi_{\zeta_2}$ on $\mathfrak{B}(\mathcal{H}_1) \oplus \mathfrak{B}(\mathcal{H}_2)$ where the vectors $\vartheta_1, \zeta_2 \in \mathcal{H}_1 \oplus \mathcal{H}_2$ have non-zero components only in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Then the vector state $\chi_{\lambda_1\vartheta_1+\lambda_2\zeta_2}$ corresponding to the superposition $\lambda_1\vartheta_1 +$

$\lambda_2\zeta_2$, $0 < \lambda_1, \lambda_2 < 1$, $\lambda_1 + \lambda_2 = 1$, is the mixed state $|\lambda_1|^2\chi_{\vartheta_1} + |\lambda_2|^2\chi_{\zeta_2}$, illustrating the limitation on coherent superpositions. Affirming that the relative phases in the superposition are unobservable, the projection onto a ray spanned $\lambda_1\vartheta_1 + \lambda_2\zeta_2$ is not in the algebra $\mathfrak{B}(\mathcal{H}_1) \oplus \mathfrak{B}(\mathcal{H}_2)$. And for any $A \in \mathfrak{B}(\mathcal{H}_1) \oplus \mathfrak{B}(\mathcal{H}_2)$, the transition probability $\langle\vartheta_1, A\zeta_2\rangle_{\mathcal{H}_1\oplus\mathcal{H}_2} = 0$.

# 3 The Galilean group and projective representations of groups

## 3.1 The Galilean group[7]

The Galilean group $\mathcal{G}$ is a ten-parameter group. An element of $\mathcal{G}$ is denoted by $g = (b, \mathbf{a}, \mathbf{v}, R)$, and the group composition operation is given by $g'g = (b' + b, \mathbf{a} + R'a + \mathbf{v}t, \mathbf{v}' + R\mathbf{v}, R'R)$. The inverse of $g$ is $g^{-1} = (-b, -R^{-1}(\mathbf{a} - \mathbf{v}b), -R^{-1}\mathbf{v}, R^{-1})$. A pure time translation, spatial translation, velocity boost, and rotation will be denoted respectively by $g_b = (b, 0, 0, 1)$, $g_{\mathbf{a}} = (0, \mathbf{a}, 0, 1)$, $g_{\mathbf{v}} = (0, 0, \mathbf{v}, 1)$, and $g_R = (0, 0, 0, R)$; and the identity element of the group is $g_e = (0, 0, 0, 1)$.

The action of $\mathcal{G}$ on spacetime is specified by choosing an inertial coordinate system $(\mathbf{x}, t)$ and stipulating that $g = (b, \mathbf{a}, \mathbf{v}, R) \triangleright (\mathbf{x}, t) \mapsto (\mathbf{x}', t') = (R\mathbf{x} + \mathbf{a} + \mathbf{v}t, t + b)$. These are symmetry maps of a classical spacetime (sometimes called neo-Newtonian spacetime) endowed with an inertial structure, an absolute simultaneity, and a Euclidean spatial structure for the planes of simultaneity. The addition of absolute space in the guise of a distinguished inertial frame would diminish the spacetime symmetries to a subgroup of $\mathcal{G}$ in which the velocity boosts are killed. Eschewing absolute space and removing the inertial structure would produce a spacetime (sometimes called Leibnizian spacetime) with an expanded symmetry group that includes acceleration boosts. The question of what NRQM would look like if based on these modified spacetimes is an intriguing one, but this matter will not be pursued here.

$\mathcal{G}$ is a Lie group, and for future reference its Lie algebra is listed here. Using $H$ for the generator of time translations, $P_j$ for the generator of space translations along the $j$-axis, $K_j$ for the generator of velocity boosts along the $j$-axis, and $J_j$ for the generator of rotation around the $j$-axis

[7] For a comprehensive treatment see Lévy-Leblond (1963, 1971).

$$
\begin{aligned}
[J_j, J_k] &= \epsilon_{jkl} J_l \qquad\qquad (j,k,l = 1,2,3) \\
[J_j, K_k] &= \epsilon_{jkl} J_l \\
[J_j, P_k] &= \epsilon_{jkl} P_l \\
[J_j, H] &= 0 \\
[K_j, K_k] &= 0 \\
[K_j, P_k] &= 0 \\
[K_j, H] &= 0 \\
[P_j, P_k] &= 0 \\
[P_j, H] &= 0
\end{aligned}
\tag{$\mathcal{G}$Lie}
$$

## 3.2 Projective representations of a group

A proper (aka true, linear, vector) unitary representation of a group $G$ on a Hilbert space $\mathcal{H}$ is an assignment to each $g \in G$ of a unitary operator $U(g) : \mathcal{H} \to \mathcal{H}$ such that $U(g')U(g) = U(g'g)$ for all $g', g \in G$. A projective unitary representation is a representation on the projective Hilbert space $\mathbb{P}\mathcal{H} = \mathcal{H}/e^{i\phi}I$, i.e. $\mathbb{P}\mathcal{H}$ consists of equivalence classes of vectors where $\psi \sim \psi'$ iff $\psi = e^{i\phi}\psi'$. The multiplication law for the unitaries takes the more complicated form

$$U(g')U(g) = \omega(g', g)U(g'g) := \exp(i\xi(g', g))U(g'g), \quad g', g \in G \tag{3.1}$$

where the multipliers (aka factors) are functions $\omega : G \times G \to \mathbb{C}$ of modulus one and the exponents are functions $\xi : G \times G \to \mathbb{R}$. The normalization condition $U(e) = 1$ is assumed and this implies a normalization condition $\omega(g_e, g_e) = 1$ on the multipliers.[8]

We can define a different form of the same projective representation by choosing an $h : G \to \mathbb{C}$ such that $|h(g)| = 1$ for all $g \in G$ and setting $\widehat{U}(g) := h(g)U(g)$. The multipliers for these versions are related by $\widehat{\omega}(g', g) = \omega(g', g)[h(g')h(g)/h(g'g)]$.[9] If there is such a redefinition in which the new

[8] And it also follows that $\omega(g_e, g) = \omega(g, g_e) = 1$ and $\omega(g^{-1}, g) = \omega(g, g^{-1}) = 1$.

[9] Mathematicians would call the $\omega_{cob}(g', g) := [h(g')h(g)/h(g'g)]$ term a two-coboundary. Being related by a two-coboundary term is equivalent to being unitarily equivalent as projection representations (see Section 7 below).

multipliers $\widehat{\omega}(g', g) = 1$ for all $g', g \in G$ then projective representation is converted into a proper representation.

In general two types of obstructions stand in the way of such a de-projectivization: topological and algebraic. The former arises when the group is not simply connected, but this obstruction can be bypassed by moving to the universal covering group of $G$. The latter obstruction is more formidable. The associativity of the products of the group elements places an algebraic constraint on the multipliers (and the exponents):

$$\begin{aligned} \omega(g'', g')\omega(g''g', g) &= \omega(g'', g'g)\omega(g', g) \\ \xi(g'', g') + \xi(g''g', g) &= \xi(g'', g'g) + \xi(g', g) \mod 2\pi. \end{aligned} \tag{C}$$

If, as above $\widehat{U}(g) = h(g)U(g)$, then the multipliers $\widehat{\omega}(g', g)$ satisfy (C) iff the multipliers $\omega(g', g)$ do. In some instances the constraint cannot be solved to produce a proper representation.

The following section makes the case that the physically relevant representations of the Galilean group are projective, and for such representations de-projectivization is not possible. This in turn leads to mass superselection—or so Wightman's exercise presupposes.

# 4 Physically relevant representations of the Galilean group

It is easy to define proper unitary representations of the Galilean group $\mathcal{G}$ on a Hilbert space. For instance, take the Hilbert space for a spinless particle moving in $\mathbb{R}^3$ to be to be $L^2_{\mathbb{C}}(\mathbb{R}^4, d^4x)$ (rather than the more conventional $L^2_{\mathbb{C}}(\mathbb{R}^3, d^3x)$) and set

$$(U(g)\psi)(\mathbf{x}, t) := \psi(g(\mathbf{x}, t)), \quad \psi \in \mathcal{H}, \ g \in \mathcal{G}. \tag{4.1}$$

But we are only interested in representations that, in some appropriate sense, make $\mathcal{G}$ the (or at least *a*) symmetry group of the Schrödinger equation; and, arguably, these turn out to be projective representations.

To make $\mathcal{G}$ the symmetry group of the Schrödinger equation we adapt the Wigner doctrine for RQM (see Section 10) to NRQM: solutions of the Schrödinger equation correspond to unitary representations of the Galilean group. To implement this doctrine we need to know how the Galilean group, which acts as the symmetry group of (neo-)Newtonian spacetime, is implemented on Hilbert space. To find the Hilbert space implementation of $\mathcal{G}$ we traverse a virtuous circle and require that the implementation $\mathcal{G}$ on Hilbert space carries solutions to the Schrödinger equation onto solutions. There is no a priori guarantee that the results of traversing the circle will be pleasing either theoretically or experimentally. In fact, some commentators find the message so displeasing that, in time honored fashion, they want to shoot the messenger.

For sake of simplicity, consider a spinless free particle of mass $m$ and its associated Schrödinger Hamiltonian $H_m = \dfrac{-\hbar^2}{2m}\nabla^2$. There is a bijection between elements of the familiar one-particle Hilbert space $L^2_{\mathbb{C}}(\mathbb{R}^3, d^3x)$ and solutions $\psi_m(\mathbf{x}, t)$ to the Schrödinger equation for a free particle of mass $m$

$$\frac{-\hbar^2}{2m}\nabla^2\psi_m(\mathbf{x}, t) = i\hbar\frac{\partial\psi_m(\mathbf{x}, t)}{\partial t} \tag{4.2}$$

given by

$$L^2_{\mathbb{C}}(\mathbb{R}^3, d^3x) \ni \psi(\mathbf{x}) \leftrightarrow \psi_m(\mathbf{x}, t) := \exp(-\frac{i}{\hbar}H_m t)\psi(\mathbf{x}) \tag{4.3}$$

Denote the Hilbert space under this correspondence by $\mathcal{H}_m$. We seek a unitary representation $U_m(\mathcal{G})$ of $\mathcal{G}$ acting on $\mathcal{H}_m$, and in analogy with (4.1) we might set

$$(U_m(g)\psi_m)(\mathbf{x}, t) = \psi_m(g(\mathbf{x}, t)), \quad g \in \mathcal{G} \tag{4.4}$$

Unfortunately, it is not the case that $\psi'(\mathbf{x}', t') := (U_m(g)\psi_m)(\mathbf{x}, t)$ also satisfies the Schrödinger equation in the Galilean transformed coordinates, i.e.

$$\frac{-\hbar^2}{2m}\nabla'^2\psi'_m(\mathbf{x}', t') = i\hbar\frac{\partial\psi'_m(\mathbf{x}', t')}{\partial t'} \tag{4.2$'$}$$

But since pure states correspond to rays rather than vectors, we can modify (4.4) by inserting a phase factor on the rhs without changing the physical state:

$$(U_m(g)\psi_m)(\mathbf{x}, t) = \exp(\frac{i}{\hbar}f_g(\mathbf{x}, t))\psi_m(g(\mathbf{x}, t)), \quad g \in \mathcal{G} \tag{4.4$'$}$$

where the subscript on the phase factor indicates that it may depend on $g$. Now we can hope that $f_g$ can be be chosen so that this modified action of $U_m(g)$ guarantees that $\psi'(\mathbf{x}', t')$ satisfies (4.4′). Leaving aside rotation which will play no role in what follows, i.e. for Galilean transformations of the form $g = (b, \mathbf{v}, \mathbf{a}, 1)$, our hopes are fulfilled by the choice $f_g(\mathbf{x}, t) = m(\mathbf{v} \cdot \mathbf{a} + \frac{1}{2}\mathbf{v}^2 b)$. This implies that $U_m(\mathcal{G})$ is a projective representation of the form $U_m(g')U_m(g) = \exp(\frac{i}{\hbar}\xi_m(g', g))U(g'g), \quad g', g \in G$, where $\xi_m(g', g) = m(\mathbf{v}' \cdot \mathbf{a} + \frac{1}{2}\mathbf{v}'^2 b)$ (see de Aczcárrage and Izquierdo 1995, Secs. 3.1 and 3.2).[10] This projective unitary representation under which $\mathcal{G}$ can be viewed as a symmetry group of the Schrödinger equation cannot be de-projectivized, as shown by the results of Bargmann (1954). For an accessible demonstration the reader is referred to Guilini (1996) and Annigioni and Moretti (2013).

For what follows, the important consequence of this implementation of the action of the Galilean group on Hilbert space concerns the case involving a pure velocity boost $g_{\mathbf{v}} = (0, \mathbf{v}, 0, 1)$, a pure spatial translation $g_{\mathbf{a}} = (0, 0, \mathbf{a}, 1)$, and their combination $g_{\mathbf{v}}g_{\mathbf{a}} = g_{\mathbf{a}}g_{\mathbf{v}} = (0, \mathbf{v}, \mathbf{a}, 1)$. We can recapture the feature flagged by Inöu and Wigner (1952) by noting that $U_m(g_{\mathbf{v}})U_m(g_{\mathbf{a}}) = \exp(\frac{i}{\hbar}m\mathbf{v} \cdot \mathbf{a})U_m(g_{\mathbf{v}}g_{\mathbf{a}})$ while $U_m(g_{\mathbf{a}})U_m(g_{\mathbf{v}}) = U_m(g_{\mathbf{a}}g_{\mathbf{v}}) = U_m(g_{\mathbf{v}}g_{\mathbf{a}})$ and, thus,

$$U_m(g_{\mathbf{v}})U_m(g_{\mathbf{a}}) = \exp(\frac{i}{\hbar}m\mathbf{v} \cdot \mathbf{a})U_m(g_{\mathbf{a}})U_m(g_{\mathbf{v}}) \tag{4.5}$$

# 5 The standard solution to Wightman's exercise

Start from the feature noted by Inöu and Wigner (1952) and recapitulated in the preceding section. Multiplying each side of (4.5) by $U_m(g_{-\mathbf{v}})U_m(g_{-\mathbf{a}})$ gives

[10]With the introduction of $\hbar$ a different symbol should be used for the exponent $\xi_m$. But in the interest of simplicity I hope the reader will tolerate the notational solecism of continuing to use the same symbol.

$$
\begin{aligned}
U_m(g_{-\mathbf{v}})U_m(g_{-\mathbf{a}})U_m(g_{\mathbf{v}})U_m(g_{\mathbf{a}}) &= \exp(\frac{i}{\hbar}m\mathbf{v}\cdot\mathbf{a})U_m(g_{-\mathbf{v}})U_m(g_{-\mathbf{a}})U_m(g_{\mathbf{a}})U_m(g_{\mathbf{v}}) \\
&= \exp(\frac{i}{\hbar}m\mathbf{v}\cdot\mathbf{a})I_m \qquad (5.1)
\end{aligned}
$$

The lhs of (5.1) represents a succession of Galilean transformations whose combination is the identity of $\mathcal{G}$. So the result of acting on a state with the lhs should give the same state, i.e. acting on a state vector with the lhs should give the same state vector up to an overall phase. But (the argument goes), this condition fails if states corresponding to different masses could be coherently superposed. Consider the superposition $\psi_{m_1}+\psi_{m_2}$ for $m_1 \neq m_2$. The action on Hilbert space of the succession of Galilean transformations of $g_{\mathbf{a}}$, followed by $g_{\mathbf{v}}$, followed by $g_{-\mathbf{a}}$, followed by $g_{-\mathbf{v}}$ results in

$$
\begin{aligned}
\psi_{m_1}+\psi_{m_2} &\longmapsto \exp(\frac{i}{\hbar}m_1\mathbf{v}\cdot\mathbf{a})\psi_{m_1}+\exp(\frac{i}{\hbar}m_2\mathbf{v}\cdot\mathbf{a})\psi_{m_2} \qquad (*) \\
&= \exp(\frac{i}{\hbar}m_1\mathbf{v}\cdot\mathbf{a})[\psi_{m_1}+\exp(\frac{i}{\hbar}(m_2-m_1)\mathbf{v}\cdot\mathbf{a})\psi_{m_2}]
\end{aligned}
$$

which, because of the change in the relative phase factor, is not of the form $\exp(i\phi)(\psi_{m_1}+\psi_{m_2})$ when the masses are different. The way out (the argument concludes) is to recognize that $\psi_{m_1}+\psi_{m_2}$ is not a coherent superposition; that is, it is a mixed state so that relative phase factors do not matter.

An analogy/disanalogy with the twin paradox of SRT suggests itself. In this apparent paradox the traveling twin rejoins her identical twin only to discover that she (the traveling twin) is chronologically younger than her stay-at-home sibling. In the NRQM case the traveling twin who undergoes a succession of Galilean spatial translations and velocity boosts discovers that she and her stay-at-home sibling assign to the system different states consisting of superpositions that differ in relative phase. In the SRT case the apparent paradox is resolved by noting that there is a real asymmetry between the twins due to their different worldlines through Minkowski spacetime and the fact that chronological aging is proportional to elapsed proper time along the twins' worldlines. In the NRQM case the resolution of the apparent paradox is (supposedly) that, due to the superselection rule for mass, the result of the traveling twin's journey through (neo-)Newtonian spacetime doesn't produce any objective difference between the state she assigns and the state her stay-at-home sibling assigns.

## 6 Critique of the standard solution

Even taken at face value the standard argument needs buttressing. It implicitly assumes that the projective representation of the Galilean group it uses is *the* correct representation and that there is no other physically acceptable representation—in particular, no other physically acceptable proper unitary representation—that doesn't lead to mass superselection. How one could establish this implicit assumption and, indeed, what counts as a "physically acceptable" representation of the Galilean group, is far from obvious. But this seems only a minor quibble compared with the charge that the standard solution is incoherent.

Baldly stated, the complaint is that mass in NRQM is simply not a candidate for superselection. This is because (the objection goes) in NRQM mass is not an observable, and the mass subscripts on the component vectors in superposition in ($*$) do not label different eigenvalues of a mass operator; rather, mass is a parameter whose values labels different systems. Two examples of the complaint: "In a theory with mass being a parameter there are no states for a single particle that are associated with different masses" (Zych and Greenberger 2019, p. 1); "[I]n order "to make sense of a mass superselection rule one should regard mass as a dynamical variable" (Guilini 1996, p. 229).[11]

The polemical situations for both the proponents and the critics of the standard solution are murky. The proponents aim to show that in NRQM a superselection rule "prevents the existence ... of states with a mass spectrum, and therefore of unstable particles" (Lévy-Leblond 1963, p. 785). Towards this goal they offer a reductio argument where the reductio assumption is that a particle is in a coherent superposition of states of different mass. The critics charge that the reductio assumption is incoherent since there are no states for a single particle that are associated with different masses. But this charge seems to grant the proponents what they seek to prove!

[11] I will not be discussing the project of finding a dynamical explanation of mass superselection and other superselection rules. But it is an interesting project that offers new insights into the nature of superselection. For some progress in this project see Annigioni and Moretti (2013).

On the other hand the proponents of the standard solution appear to be somewhat better off, for at least they offer an appealing heuristic argument that fits the pattern of argumentation used by Wick, Wightman and Wigner (1952) and Hegerfeldt, Kraus, and Wigner (1968) to prove the superselection rule for whole-integer/half-integer angular momentum. In the present case, however, the heuristic argument leaves something to be desired. The $\psi_{m_1}$ and the $\psi_{m_2}$ on the lhs of $(*)$ belong to different unitary projective representations $U_{m_1}, \mathcal{H}_{m_1}$ and $U_{m_2}, \mathcal{H}_{m_2}$ of $\mathcal{G}$, so the meaning of the superposition of $\psi_{m_1}$ and $\psi_{m_2}$ is unclear. Consequently, the action of $\mathcal{G}$ on the Hilbert space $\mathcal{H}_{m_1,m_2}$ to which $\psi_{m_1}$ and $\psi_{m_2}$ supposedly both belong and which gives the meaning to the superposition $\psi_{m_1} + \psi_{m_2}$ is unclear. *If* $\mathcal{H}_{m_1,m_2}$ were $\mathcal{H}_{m_1} \oplus \mathcal{H}_{m_2}$ and *if* $\mathcal{G}$ acts by $U_{m_1}(\mathcal{G}) \oplus U_{m_2}(\mathcal{G})$ on this space then all would be well, for then $(*)$ becomes

$$\begin{aligned} \psi_{m_1} \oplus \psi_{m_2} \quad &\longmapsto \quad \exp(\frac{i}{\hbar}m_1\mathbf{v}\cdot\mathbf{a})\psi_{m_1} \oplus \exp(\frac{i}{\hbar}m_2\mathbf{v}\cdot\mathbf{a})\psi_{m_2} \qquad (**) \\ &= \quad \exp(\frac{i}{\hbar}m_1\mathbf{v}\cdot\mathbf{a})(\psi_{m_1} \oplus \exp(\frac{i}{\hbar}(m_2 - m_1)\mathbf{v}\cdot\mathbf{a}))\psi_{m_2}) \end{aligned}$$

So *if* the direct sum structure lies behind $(*)$ then the action of the sequence of Galilean spatial translations and velocity boosts does produce the change of relative phase of the components of the superposition corresponding to different particle masses, as claimed in $(*)$. But assuming the *ifs* is assuming part of what needs to be proved since the direct sum structure for the Hilbert space is half way to proving that the rhs of $(**)$ represents a mixed state.

Before turning to the issue of what would constitute a rigorous and satisfying solution to Wightman's exercise, a final remark about the standard argument for mass superselection. In some minds it encourages the idea that mass superselection can be escaped by moving from the Galilean group to a central extension of this group since such extensions admit proper unitary representations, apparently short-circuiting the standard argument which relies on special features of the projective representation of $\mathcal{G}$. We will see in Section 9 that this idea is mistaken.

# 7 Solving Wightman's exercise

A fully satisfying solution to Wightman's exercise should demonstrate that, as a consequence of the features of the projective unitary representations of the Galilean group found by Bargmann, the Hilbert space hosting superpositions of states of different masses for a particle is a direct sum space. More precisely, consider the projective representations $\mathcal{H}_{m_i}, U_{m_i}(\mathcal{G})$, $i = 1, 2$ and $m_1 \neq m_2$. Let $\mathcal{K}$ be a Hilbert space $\mathcal{K}$ such that there are subspaces $\mathcal{H}'_{m_i} \subset \mathcal{K}$, $i = 1, 2$, and unitary maps $V_i : \mathcal{H}_{m_i} \rightarrow \mathcal{H}'_{m_i}$. Let $\mathbf{U}(\mathcal{G})$ be a representation of $\mathcal{G}$ on $\mathcal{K}$ such that the subrepresentations $\mathbf{U}(\mathcal{G})|\mathcal{H}'_{m_i}$ are identified with the $U_{m_i}(\mathcal{G})$, i.e. $\mathbf{U}(g)|\mathcal{H}'_{m_i} = V_i U_{m_i}(g) V_i^{-1}$ for all $g \in \mathcal{G}$. The Hilbert space $(\mathcal{H}'_{m_1} + \mathcal{H}'_{m_2})_{lin}$ is the candidate space for forming superpositions of mass states.[12] The goal is to prove that $(\mathcal{H}'_{m_1} + \mathcal{H}'_{m_2})_{lin} = \mathcal{H}'_{m_1} \oplus \mathcal{H}'_{m_2}$ and $\mathbf{U}(\mathcal{G}) = U'_{m_1}(\mathcal{G}) \oplus U'_{m_2}(\mathcal{G})$, i.e. $\mathbf{U}(g) = (V_1 U_{m_1}(g) V_1^{-1}) \oplus (V_2 U_{m_2}(g) V_2^{-1})$, for all $g \in \mathcal{G}$. The goal would be obtained by showing that $\mathcal{H}'_{m_1} \perp \mathcal{H}'_{m_2}$, for then the natural isomorphism from $\vee_i \mathcal{H}'_{m_i}$, $i = 1, 2$, to $\mathcal{H}'_{m_1} \oplus \mathcal{H}'_{m_2}$ implements the equivalence between $\mathbf{U}(\mathcal{G})$ and the direct sum $U'_{m_1}(\mathcal{G}) \oplus U'_{m_2}(\mathcal{G})$ of its subrepresentations $U'_{m_1}(\mathcal{G})$ and $U'_{m_2}(\mathcal{G})$.

The strategy for proof is first to show that $\mathcal{H}_{m_1}, U_{m_1}(\mathcal{G})$ and $\mathcal{H}_{m_2}, U_{m_2}(\mathcal{G})$ (and, thus, $\mathcal{H}'_{m_1}, U'_{m_1}(\mathcal{G})$ and $\mathcal{H}'_{m_2}, U'_{m_2}(\mathcal{G})$) are unitarily inequivalent as projective representations of $\mathcal{G}$ when $m_1 \neq m_2$, and then to show that this inequivalence entails the orthogonality of the representations. For projective representations the definition of unitary equivalence is a little more complicated than for proper representations.

> *Def.* (a) Proper unitary representations $\mathcal{H}, U$ and $\overline{\mathcal{H}}, \overline{U}$ of a group $G$ are unitarily equivalent iff there a unitary map $V : \mathcal{H} \rightarrow \overline{\mathcal{H}}$ such that $\overline{U}(g) = V U(g) V^{-1}$ for all $g \in G$.
>
> (b) (Lévy-Leblond 1963, fn. 7, p. 781) Projective unitary representations $\mathcal{H}, U$ and $\overline{\mathcal{H}}, \overline{U}$ of a group $G$ are unitarily equivalent iff there is a function $h : G \rightarrow \mathbb{C}$ of modulus 1 and a unitary $V : \mathcal{H} \rightarrow \overline{\mathcal{H}}$ such that $\overline{U}(g) = h(g) V U(g) V^{-1}$ for all $g \in G$.

Now suppose for reductio that $\mathcal{H}_{m_1}, U_{m_1}(\mathcal{G})$ and $\mathcal{H}_{m_2}, U_{m_2}(\mathcal{G})$ are unitarily equivalent as projective representations of the Galilean group $\mathcal{G}$ when $m_1 \neq m_2$. So there is a $h : \mathcal{G} \rightarrow \mathbb{C}$ and a unitary $V : \mathcal{H}_{m_1} \rightarrow \mathcal{H}_{m_2}$ such that $U_{m_2}(g) = h(g) V U_{m_1}(g) V^{-1}$ for all $g \in \mathcal{G}$. From the lead-up to the standard argument we know that

[12] The linear hull $(\mathcal{H}'_{m_1} + \mathcal{H}'_{m_2})_{lin}$ is the minimal closed subspace of $\mathcal{K}$ containing vectors $\alpha\psi'_{m_1} + \beta\psi'_{m_2}$, with $\alpha, \beta \in \mathbb{C}$ and $\psi'_{m_1} \in \mathcal{H}'_{m_1}$ and $\psi'_{m_2} \in \mathcal{H}'_{m_2}$.

$$U_{m_1}(g_{-\mathbf{v}})U_{m_1}(g_{-\mathbf{a}})U_{m_1}(g_{\mathbf{v}})U_{m_1}(g_{\mathbf{a}}) = \exp(\frac{i}{\hbar}m_1\mathbf{v}\cdot\mathbf{a})I_{m_1} \qquad (7.1a)$$
$$U_{m_2}(g_{-\mathbf{v}})U_{m_2}(g_{-\mathbf{a}})U_{m_2}(g_{\mathbf{v}})U_{m_2}(g_{\mathbf{a}}) = \exp(\frac{i}{\hbar}m_2\mathbf{v}\cdot\mathbf{a})I_{m_2}. \qquad (7.1b)$$

Multiplying the lhs of (7.1a) from the left by $V$ and from the right by $V^{-1}$ and setting $h^{-1}(g) := \overline{h}(g)$ results in

$$\begin{aligned}
& VU_{m_1}(g_{-\mathbf{v}})V^{-1})(VU_{m_1}(g_{-\mathbf{a}})V^{-1})(VU_{m_1}(g_{\mathbf{v}})V^{-1})(VU_{m_1}(g_{\mathbf{a}})V^{-1}) \\
= \; & \overline{h}(g_{-\mathbf{v}})U_{m_2}(g_{-\mathbf{v}})\overline{h}(g_{-\mathbf{a}})U_{m_2}(g_{-\mathbf{a}})\overline{h}(g_{\mathbf{v}})U_{m_2}(g_{\mathbf{v}})\overline{h}(g_{\mathbf{a}})U_{m_2}(g_{\mathbf{a}}) \\
= \; & \overline{h}(g_{-\mathbf{v}})U_{m_2}(g_{-\mathbf{v}})\overline{h}(g_{-\mathbf{a}})U_{m_2}(g_{-\mathbf{a}})\overline{h}(g_{\mathbf{v}})U_{m_2}(g_{\mathbf{v}})\overline{h}(g_{\mathbf{a}})U_{m_2}(g_{\mathbf{a}}) \qquad (7.2) \\
= \; & \overline{h}(g_{-\mathbf{v}})\overline{h}(g_{-\mathbf{a}})\overline{h}(g_{\mathbf{v}})\overline{h}(g_{\mathbf{a}})[U_{m_2}(g_{-\mathbf{v}})U_{m_2}(g_{-\mathbf{a}})U_{m_2}(g_{\mathbf{v}})U_{m_2}(g_{\mathbf{a}})] \\
= \; & \overline{h}(g_{-\mathbf{v}})\overline{h}(g_{-\mathbf{a}})\overline{h}(g_{\mathbf{v}})\overline{h}(g_{\mathbf{a}})\exp(\frac{i}{\hbar}m_2\mathbf{v}\cdot\mathbf{a})I_{m_2}
\end{aligned}$$

where the last line follows by (7.1b). And multiplying the rhs of (7.1a) from the left by $V$ and from the right by $V^{-1}$ results in

$$\exp(\frac{i}{\hbar}m_1\mathbf{v}\cdot\mathbf{a})I_{m_2} \qquad (7.3)$$

But $\overline{h}(g_{-\mathbf{v}})\overline{h}(g_{-\mathbf{a}})\overline{h}(g_{\mathbf{v}})\overline{h}(g_{\mathbf{a}}) = \overline{h}(g_{-\mathbf{v}})\overline{h}(g_{-\mathbf{v}})\overline{h}(g_{-\mathbf{a}})\overline{h}(g_{\mathbf{a}})$ and $\overline{h}(g_{-\mathbf{v}})\overline{h}(g_{\mathbf{v}}) = 1 = \overline{h}(g_{-\mathbf{a}})\overline{h}(g_{\mathbf{a}})$, so (7.2) reduces to $\exp(\frac{i}{\hbar}m_2\mathbf{v}\cdot\mathbf{a})I_{m_2}$, and the equality of (7.2) and (7.3) now implies $\exp(\frac{i}{\hbar}m_1\mathbf{v}\cdot\mathbf{a}) = \exp(\frac{i}{\hbar}m_2\mathbf{v}\cdot\mathbf{a})$, which is a contradiction if $m_1 \neq m_2$.[13]

The goal is now to show that this inequivalence implies that $\mathcal{H}'_{m_1} \perp \mathcal{H}'_{m_2}$. Suppose to the contrary that that there is a non-null $\psi_0 \in \mathcal{H}'_{m_1} \cap \mathcal{H}'_{m_2}$. For any $g \in \mathcal{G}$ the $U'_{m_i}(g)$ are bounded operators and, thus, their respective domains are of all of $\mathcal{H}'_{m_i}$ and, in particular, $U'_{m_i}(g)\psi_0 \in \mathcal{H}'_{m_i}$. Further, the subspace $(U'_{m_i}(\mathcal{G})\psi_0)_{lin}$ of $\mathcal{H}'_{m_i}$ is invariant under $U'_{m_i}(\mathcal{G})$. Since the $\mathcal{H}'_{m_i}, U'_{m_i}$ are irreducible representations the only non-null invariant subspace of $\mathcal{H}'_{m_i}$ is

[13]The assumption that there is unitary $V$ and an $h(p)$ such that $U_{m_2}(g) = h(g)VU_{m_1}(g)V^{-1}$ for all $g \in \mathcal{G}$ implies that $\omega_{m_2}(g', g) = \omega_{m_1}(g', g)[h(g')h(g)/h(g'g)]$. But $\omega_{m_2,m_1}(g, g^{-1}) = 1$. So $\omega_{m_2}(g_{\mathbf{v}}, g_{-\mathbf{v}}) = 1 = \omega_{m_1}(g_{\mathbf{v}}, g_{-\mathbf{v}})[h(g_{\mathbf{v}})h(g_{-\mathbf{v}})/h(g_{\mathbf{v}}g_{-\mathbf{v}})] = h(g_{\mathbf{v}})h(g_{-\mathbf{v}})$. Similarly, $h(g_{\mathbf{a}})h(g_{-\mathbf{a}}) = 1$.

$\mathcal{H}'_{m_i}$ itself, and $(U'_{m_i}(\mathcal{G})\psi_0)_{lin} = \mathcal{H}'_{m_i}$. The map that sends $U'_{m_1}(g)\psi_0 \in \mathcal{H}'_{m_1}$ to $U'_{m_2}(g)\psi_0 \in \mathcal{H}'_{m_2}$ is norm preserving and has an inverse, and extending it by linearity produces a unitary map between $\mathcal{H}'_{m_1}$ and $\mathcal{H}'_{m_2}$. Under this identification we can set $\mathcal{H}'_{m_1} = \mathcal{H}'_{m_2}$ and can conclude that $\mathbf{U}(\mathcal{G})|\mathcal{H}'_{m_1} = \mathbf{U}(\mathcal{G})|\mathcal{H}'_{m_2}$, i.e. $V_1U_{m_1}(g)V_1^{-1} = V_2U_{m_2}(g)V_2^{-1}$ for all $g \in \mathcal{G}$. So for all $g \in \mathcal{G}$, and $U_{m_2}(g) = (V_2^{-1} \circ V_1)U_{m_1}(g)(V_1^{-1} \circ V_2)$ and, thus, $W : \mathcal{H}_{m_1} \to \mathcal{H}_{m_2}$, with $W := V_2^{-1} \circ V_1$ makes $U_{m_1}(\mathcal{G})$ and $U_{m_2}(\mathcal{G})$ unitarily equivalent, which we have seen is not the case. Hence, $\mathcal{H}'_{m_1} \perp \mathcal{H}'_{m_2}$ and $\mathbf{U}(\mathcal{G}) = U'_{m_1}(\mathcal{G}) \oplus U'_{m_2}(\mathcal{G})$ acting on $\mathcal{H}'_{m_1} \oplus \mathcal{H}'_{m_2}$.

Now that we have a solid motivation for $(**)$ we can return to the logic of the standard argument and assert that the physical (= algebraic) states corresponding to the vectors on the lhs and rhs of the $\mapsto$ in $(**)$ are the same state and, thus, the state corresponding to the vector on the rhs is a mixed state. This will be the case if and only if the relevant algebra of observables is not $\mathfrak{B}(\mathcal{H}'_{m_1} \oplus \mathcal{H}'_{m_2})$ but rather $\mathfrak{B}(\mathcal{H}'_{m_1}) \oplus \mathfrak{B}(\mathcal{H}'_{m_2})$ or some subalgebra thereof. It would be preferable to have a more direct argument for this limitation on the observables. Towards this end, recalling that a von Neumann algebra is generated by its unitaries, one could argue that for $\mathbf{U}(\mathcal{G})$ the relevant algebra of observables is the von Neumann algebra $\mathbf{U}(\mathcal{G})''$ generated by $\mathbf{U}(\mathcal{G})$, and this is $((U'_{m_1}(\mathcal{G}) \oplus U'_{m_2}(\mathcal{G}))'' = U'_{m_1}(\mathcal{G})'' \oplus U'_{m_2}(\mathcal{G})'' \subseteq \mathfrak{B}(\mathcal{H}'_{m_1}) \oplus \mathfrak{B}(\mathcal{H}'_{m_2})$. Since the only bounded operators on $\mathcal{H}'_i$ that commute with $U'_{m_i}(g)$ for all $g \in \mathcal{G}$ are multiples of the identity, $U'_{m_i}(\mathcal{G})'' = \mathfrak{B}(\mathcal{H}'_{m_i})$ and, thus, $\mathbf{U}(\mathcal{G})'' = \mathfrak{B}(\mathcal{H}'_{m_1}) \oplus \mathfrak{B}(\mathcal{H}'_{m_2})$. Either way of fleshing out of the standard argument provides a more solid basis for the solution Wightman wanted for his exercise.

This formal solution to Wightman's exercise may strike one as so pedantic and tedious as not to be worth the candle, especially since the standard argument for mass superselection gives the "right" answer. But while the standard argument may be retained as a useful heuristic, it is reassuring to be able to check that the heuristic can be backed up by a more rigorous argument.

Having established that the correct way to think about superposition of states of a given particle of masses $m_1$ and $m_2$ is by using a direct sum structure with Hilbert space $\mathcal{H}_{m_1} \oplus \mathcal{H}_{m_2}$, with $\mathcal{G}$ acting on this space as $U_{m_1}(\mathcal{G}) \oplus U_{m_2}(\mathcal{G})$ and the algebra of observables being $\mathfrak{B}(\mathcal{H}_{m_1}) \oplus \mathfrak{B}(\mathcal{H}_{m_2})$, there is what might seem to seem to be an adequate response to the complaint that the standard argument for mass superselection is incoherent because mass is not an observable but a parameter in NRQM. For in the direct sum structure mass plays both roles: there is a self-adjoint mass operator $M := m_1 I_{m_1} \oplus m_2 I_{m_2}$ which belongs to the algebra of observables $\mathfrak{B}(\mathcal{H}_{m_1}) \oplus \mathfrak{B}(\mathcal{H}_{m_2})$. The selection sectors are the eigenspaces of the observable $M$, and the different values $m_1$ and $m_2$ of the mass parameter label these sectors. However, Section 12 will advance serious reservations about this response and, correspondingly, reservations about whether the sense of mass superselection demonstrated by the solution to Wightman's exercise should count as superselection by the criteria advanced in the works of Wightman and Wigner. But since it is the sense of superselection demonstrated by the solution to Wightman's exercise that is at issue in the vast majority of discussions of the Bargmann superselection rule, I will continue to operate with it until further notice. And the standard argument for mass superselection will be retained as a useful heuristic.

# 8 More superselection

## 8.1 Descriptions of different systems vs. descriptions of different behaviors of the same system

Intuitions can vary about when different representations describe the behaviors of different systems vs. different behaviors of the same system. But one commonality is the idea that representations using different values of a universal constant describe different systems. But what is a universal constant other than a dimensional number that does not vary among physically possible descriptions of the same system? QM offers a way to break out of this circle: a universal constant can be identified by the fact that coherent superpositions of states belonging to representations using different values of the constant are not physically realizable.

This idea is exemplified in representations of the canonical commutation relations (CCR). A unitary representation of the Weyl form of the CCR for one degree of freedom is given by unitaries $U(a)$ and $V(b)$, $a, b \in \mathbb{R}$, acting on a separable $\mathcal{H}$ with

$$\begin{aligned} U(a_1)U(a_2) &= U(a_1 + a_2), \quad V(b_1)V(b_2) = V(b_1 + b_2) \\ U(a)V(b) &= \exp(i\hbar ab)V(b)U(a). \end{aligned} \tag{8.1}$$

Setting $W(a, b) := U(a)V(b)$ we have

$$W(a,b)W(c,d) = \exp(-i\hbar bc)W(a+c, c+d) \tag{8.2}$$

which gives a unitary projective representation of the additive group $\mathbb{R}^2$. This representation cannot be de-projectivized (see Hall 2013, Ex. 16.56, p. 362-363).

One way to satisfy the Weyl CCR is the Schrödinger representation:

$$\begin{aligned}(U(a)\psi)(x) &= \psi(x-a), \quad \psi \in L^2_{\mathbb{C}}(\mathbb{R}, dx) \\ (V(b)\psi)(x) &= \exp(i\hbar bx)\psi(x).\end{aligned} \tag{8.3}$$

Von Neumann (1931) proved the essential uniqueness of the Schrödinger representation: any irreducible and strongly continuous representation of the Weyl CCR on a separable Hilbert space is unitarily equivalent to the Schrödinger representation.[14]

Or rather, he proved the uniqueness for any fixed value of $\hbar \neq 0$. Representations for different values of $\hbar \neq 0$ are unitarily inequivalent and, thus, form different superselection sectors, confirming that $\hbar$ behaves as a universal constant as regards superselection rules. To see this note that the identity element $(0,0)$ of the additive group $\mathbb{R}^2$ equals $(-c,-d)(c,d)$. The projective representation $W_\hbar(a,b)$ of $\mathbb{R}^2$ acting on a Hilbert space $\mathcal{H}_\hbar$ maps $(-c,-d)(c,d)$ not to the identity operator $I_\hbar$ but rather to $I_\hbar$ times a phase factor that depends on $\hbar$:

$$W_\hbar(-c,-d)W_\hbar(c,d) = \exp(i\hbar cd)W_\hbar(0,0) = \exp(-i\hbar cd)I_\hbar. \tag{8.4}$$

The standard argument for mass superselection can be adapted here to give a heuristic explanation for why $\hbar$ is a superselected quantity. Or one can proceed more rigorously by first showing that $W_{\hbar_1}$ and $W_{\hbar_2}$ for different $\hbar_1, \hbar_2 \neq 0$ are unitarily inequivalent as projective representations of the additive group $\mathbb{R}^2$, and then that this inequivalence entails a direct sum structure. Insofar as it is persuasive, the line of reasoning in this section provides grounds to think of mass in NRQM as analogous to a universal constant.

[14] The result generalizes to any finite number of degrees of freedom but *not* to an infinite number.

## 8.2 Direct sums and tensor products

### 8.2.1 No interaction across superselection sectors

A Hamiltonian operator $H_{m_1,m_2}$ that acts on $\mathcal{H}_{m_1} \oplus \mathcal{H}_{m_2}$, $m_1 \neq m_2$, must be essentially self-adjoint in order to generate Schrödinger evolution. But if $H_{m_1,m_2}$ is an unbounded operator then it will not belong to $\mathfrak{B}(\mathcal{H}_{m_1} \oplus \mathcal{H}_{m_2})$ much less to the smaller superselection algebra $\mathfrak{B}(\mathcal{H}_{m_1}) \oplus \mathfrak{B}(\mathcal{H}_{m_2})$. However, an essentially self-adjoint Hamiltonian can be an observable in the broader sense that the spectral projections of its unique self-adjoint extension all belong to the algebra of observables $\mathfrak{B}(\mathcal{H}_{m_1}) \oplus \mathfrak{B}(\mathcal{H}_{m_2})$ (in the jargon, such a Hamiltonian is said to be affiliated with $\mathfrak{B}(\mathcal{H}_{m_1}) \oplus \mathfrak{B}(\mathcal{H}_{m_2})$). Such a Hamiltonian $H_{m_1,m_2}$ takes the form $H_{m_1} \oplus H_{m_2}$. The Schrödinger evolution operator is then

$$\exp(-\frac{i}{\hbar}(H_{m_1} \oplus H_{m_2})t) = \exp(-\frac{i}{\hbar}H_{m_1}t) \oplus \exp(-\frac{i}{\hbar}H_{m_2}t). \tag{9.5}$$

A state vector initially lying wholly in one of the selection sectors is evolved so that it forever remains in that sector. And the Schrödinger evolutions of the particles of different masses are independent of one another—as far as NRQM is concerned the different mass states describe different possible worlds with no physical connection between them. It is possible to escape this consequence by rejecting the assumption that the Hamiltonian $H_{m_1,m_2}$ is an observable and adding to $H_{m_1} \oplus H_{m_2}$ an interaction term which, perforce, is not in $\mathfrak{B}(\mathcal{H}_{m_1}) \oplus \mathfrak{B}(\mathcal{H}_{m_2})$. But to posit an interaction which is not an observable is to slide from physics to metaphysics.

### 8.2.2 Tensor products

The argument for mass superselection is supposed to convince us that states from different projective representations $\mathcal{H}_{m_i}, U_{m_i}(\mathcal{G})$, $i = 1, 2$, of the Galilean group with $m_1 \neq m_2$, cannot be regarded as states of the *same* particle. But they can certainly be regarded as states of two *different* particles. How then to describe a bi-partite system consisting of two particles of different masses? The conventional answer is to use a tensor product structure: the Hilbert space is $\mathcal{H}_{m_1} \otimes \mathcal{H}_{m_2}$, the action of $\mathbf{U}(\mathcal{G})$ is $U_{m_1}(\mathcal{G}) \otimes U_{m_2}(\mathcal{G})$, and the algebra of observables is $\mathfrak{B}(\mathcal{H}_{m_1} \otimes \mathcal{H}_{m_2})$.[15] A Hamiltonian affiliated with

[15] Various considerations can be mounted to justify the tensor product construction to describe composite systems; see, for example Blank et al. (1994, Ch. 11) and Aerts and

$\mathfrak{B}(\mathcal{H}_{m_1} \otimes \mathcal{H}_{m_2})$ can certainly describe interactions between the component systems.

If we repeat the series of Galilean transformations $g_{\mathbf{a}}$, $g_{\mathbf{v}}$, $g_{-\mathbf{a}}$, $g_{-\mathbf{v}}$ the effect on a superposition $\psi_{m_1} \otimes \psi_{m_2} + \psi'_{m_1} \otimes \psi'_{m_2}$ of states of the bi-partite system is

$$\begin{aligned} \psi_{m_1} \otimes \psi_{m_2} + \psi'_{m_1} \otimes \psi'_{m_2} \quad &\mapsto \quad \exp(\frac{i}{\hbar} m_1 \mathbf{v} \cdot \mathbf{a})\psi_{m_1} \otimes \exp(\frac{i}{\hbar} m_2 \mathbf{v} \cdot \mathbf{a})\psi_{m_2} \qquad (***) \\ &\qquad + \exp(\frac{i}{\hbar} m_1 \mathbf{v} \cdot \mathbf{a})\psi'_{m_1} \otimes \exp(\frac{i}{\hbar} m_2 \mathbf{v} \cdot \mathbf{a})\psi'_{m_2} \\ &= \quad \exp(\frac{i}{\hbar}(m_1 + m_2)\mathbf{v} \cdot \mathbf{a})[\psi_{m_1} \otimes \psi_{m_2} + \psi'_{m_1} \otimes \psi'_{m_2}]. \end{aligned}$$

The vectors on the lhs and rhs of $\mapsto$ correspond to the same physical state since there is no change of relative phase, and there is apparently no reason to see a need for mass superselection since the standard argument for mass superselection gains no traction here.[16] But don't be too hasty; superselection is lurking.

### 8.2.3 Superselection for total mass

Suppose that our bi-partite system consisting of two particles of masses $m_1$ and $m_2$ is treated as a single system of mass $\mathbf{m} = m_1 + m_2$. Now the standard heuristic argument for mass superselection leads to the conclusion that there is a superselection rule for the *total* mass $\mathbf{m}$ of the bi-partite system, forbidding the coherent superposition of states with different values of $\mathbf{m}$ (see, for example, Blank et al. 1994, Remark 10.3.2). If we try to imagine the effect of the series of Galilean transformations $g_{\mathbf{a}}$, $g_{\mathbf{v}}$, $g_{-\mathbf{a}}$, $g_{-\mathbf{v}}$ on a superposition of states of the bi-partite system with different values $\mathbf{m} = m_1 + m_2$ and $\mathbf{m}' = m'_1 + m'_2$ of the total mass, $(***)$ is replaced by

---

Daubechies (1978).

[16] In contrast to how scalar multiplication works for direct sums, for tensor products the rule is $\alpha(\psi_1 \otimes \psi_2) = \alpha\psi_1 \otimes \psi_2 = \psi_1 \otimes \alpha\psi_2$, $\alpha \in \mathbb{C}$.

$$
\begin{aligned}
\psi_{m_1} \otimes \psi_{m_2} + \psi'_{m'_1} \otimes \psi'_{m'_2} \quad &\mapsto \quad \exp(\frac{i}{\hbar} m_1 \mathbf{v} \cdot \mathbf{a}) \psi_{m_1} \otimes \exp(\frac{i}{\hbar} m_2 \mathbf{v} \cdot \mathbf{a}) \psi_{m_2} \qquad (****) \\
&\qquad + \exp(\frac{i}{\hbar} m'_1 \mathbf{v} \cdot \mathbf{a}) \psi'_{m_1} \otimes \exp(\frac{i}{\hbar} m'_2 \mathbf{v} \cdot \mathbf{a}) \psi'_{m_2} \\
&= \quad \exp(\frac{i}{\hbar} \mathbf{m v} \cdot \mathbf{a}) [\psi_{m_1} \otimes \psi_{m_2} + \exp(\frac{i}{\hbar} (\mathbf{m}' - \mathbf{m}) \mathbf{v} \cdot \mathbf{a}) \psi'_{m_1} \otimes \psi'_{m_2}]
\end{aligned}
$$

which does record a change of relative phase when $\mathbf{m}' \neq \mathbf{m}$, resulting in physically different states on the lhs and rhs of $\mapsto$. The more formal argument backs up the superselection for total mass.

The result was illustrated for a bi-partite system, but it holds for a system consisting of any number $N$ of particles described by an $N$-fold tensor product. This result is the basis of a perceived empirical inadequacy in NRQM (see Section 10.2). The consequences of mass superselection for Galilean quantum field theories are examined in Lévy-Leblond (1967); for instance, to accommodate particle production such theories must incorporate different kinds of particles with masses chosen to be compatible with superselection for total mass.

# 9 Central extensions of the Galilean group

## 9.1 Central extensions of groups

Projective representations of some groups—the Galilean group in particular—cannot be de-projectivized. But in a large class of cases, including connected Lie groups, de-projectivization can be achieved by passing to a central extension of the group. There are two one-dimensional central extensions of the Galilean group, one using the additive group $\mathbb{R}$ and the other the group $U(1)$. Here I focus on the former. In the central extension $\widetilde{G}$ of a group $G$ by $\mathbb{R}$, $\mathbb{R}$ is an invariant subgroup in the center of $\widetilde{G}$. Although $\widetilde{G}$ may be deemed an enlargement of $G$ it is not the case that $G$ is a subgroup of $\widetilde{G}$; rather, $G = \widetilde{G}/\mathbb{R}$.

Given a group $G$ and $\xi : G \times G \to \mathbb{R}$, define a new group $\widetilde{G}$ with elements $\widetilde{g} = (\theta, g)$, $g \in G$ and $\theta \in \mathbb{R}$, by the group multiplication law $(\theta'', g'') = (\theta', g')(\theta, g) = (\theta' + \theta + \xi(g', g), g'g)$. For all $(\theta, g_e)$ and all $(\theta', g')$ in $\widetilde{G}$, $(\theta, g_e)(\theta', g') = (\theta', g')(\theta, g_e)$, confirming that $\mathbb{R}$ is a central subgroup of $\widetilde{G}$. If $U(G)$ is a unitary projective representation of $G$ where $U(g')U(g) = \omega(g', g)U(g'g) := \exp(i\xi(g', g))U(g'g)$ then $\widetilde{U}(\widetilde{g}) := \exp(i\theta)U(g)$ is a proper unitary representation of $\widetilde{G}$:

$$\begin{aligned}
\widetilde{U}(\widetilde{g}')\widetilde{U}(\widetilde{g}) &= \exp(i\theta')U(g')\exp(i\theta)U(g) \\
&= \exp(i(\theta' + \theta))U(g')U(g) \\
&= \exp(i(\theta' + \theta))\exp(i\xi(g', g))U(g'g) \\
&= \exp(i(\theta' + \theta + \xi(g', g)))U(g'g) \\
&= \widetilde{U}(\widetilde{g}'\widetilde{g}).
\end{aligned}$$

## 9.2 Central extensions of the Galilean group[17]

For the Galilean group $\mathcal{G}$ there are many $\mathbb{R}$ central extensions $\widetilde{\mathcal{G}}_m$ depending on the mass since the exponents for the projective representations for a particle of mass $m$ depend on $m$. (Leaving aside rotations, i.e. for $g = (b, \mathbf{v}, \mathbf{a}, 1)$ and $g' = (b', \mathbf{v}', \mathbf{a}', 1)$, the exponents of $\widetilde{\mathcal{G}}_m$ are given by $\xi_m(g', g) = m(\mathbf{v}'\cdot\mathbf{a} + \frac{1}{2}\mathbf{v}'^2 b)$.) And we can write the proper unitary representations of $\widetilde{\mathcal{G}}_m$ as $\widetilde{U_m}(\widetilde{g}_m) := \widetilde{U_m}(\theta_m, g) = \exp(imr)U_m(g)$, $r \in \mathbb{R}$ and $g \in \mathcal{G}$. The $\widetilde{\mathcal{G}}_m$, $m \neq 0$, are all isomorphic as groups. Consider $\widetilde{\mathcal{G}}_{m_1}$ and $\widetilde{\mathcal{G}}_{m_2}$, $m_1 \neq 0 \neq m_2$, and define $\lambda := m_2/m_1$ so that $\xi_{m_2} = \lambda\xi_{m_1}$. Then $\iota(\theta_{m_1}, g) = (\lambda\theta_{m_1}, g)$ is a group isomorphism from $\widetilde{\mathcal{G}}_{m_1}$to $\widetilde{\mathcal{G}}_{m_2}$ since $(\theta'_{m_1} + \theta_{m_1} + \xi_{m_1}(g', g), g'g) \mapsto (\lambda\theta'_{m_1} + \lambda\theta_{m_1} + \lambda\xi_{m_1}(g', g), g'g) = (\theta'_{m_2} + \theta_{m_2} + \xi_{m_2}(g', g), g'g)$. Despite being isomorphic, the different $\widetilde{\mathcal{G}}_m$s are different group extensions with different physical consequences (see below).

The Lie algebra ($\widetilde{\mathcal{G}}_m$Lie) of the extension $\widetilde{\mathcal{G}}_m$ is given by modifying the Lie algebra ($\mathcal{G}$Lie) for $\mathcal{G}$ by replacing $[K_j, P_k] = 0$ by $[K_j, P_k] = im1\delta_{jk}$ where 1 is the unit of the Lie algebra. There is then an isomorphism (up to sign) between ($\widetilde{\mathcal{G}}_m$Lie) and the Poisson bracket algebra for a free classical particle of mass $m$, where $\{K_j, P_k\}_{PB} = -m\delta_{jk}$. This provides some motivation for regarding $\widetilde{\mathcal{G}}_m$ as the quantum Galilean group for a particle of mass $m$ and, by extension, for regarding the abstract central extension $\widetilde{\mathcal{G}}$ of $\mathcal{G}$, of which the $\widetilde{\mathcal{G}}_m$ are group isomorphic realizations, as the the "quantum Galilean group" for NRQM (see de Azcárraga and Izquierdo 1995, pp. 190-191).

[17] What follows in this section is taken from Azcárraga and Izquierado (1995, p. 162).

The algebra ($\widetilde{\mathcal{G}}$Lie) for the generic central extension $\widetilde{\mathcal{G}}$ is obtained by adding a new basis element $M$ to ($\mathcal{G}$Lie), replacing $[K_j, P_k] = 0$ with $[K_j, P_k] = iM\delta_{jk}$, and requiring that $M$ has vanishing Lie brackets with itself and all the other basis elements. Since $M$ is a central element of the algebra, in any irreducible representation of $\widetilde{\mathcal{G}}$ the operator representing the basis element $M$ of the Lie algebra must, by Schur's lemma, have the form $mI$, for some fixed $m \in \mathbb{R}$. Thus, any proper irreducible unitary representation of $\widetilde{\mathcal{G}}$ will coincide with the representation of some $\widetilde{\mathcal{G}}_m$.

The use of central extensions of the Galilean group allows escape from the complications of projective unitary representations to the cleaner proper representations, but it also comes with a cost. The ten-dimensional Galilean group has a natural *and* faithful action on (neo-) Newtonian spacetime, which justifies dubbing it *the* (neo-) Newtonian spacetime symmetry group (recall: $\mathcal{G} \ni g = (b, \mathbf{a}, \mathbf{v}, R) \triangleright (\mathbf{x}, t) \mapsto (\mathbf{x}', t') = (R\mathbf{x} + \mathbf{a} + \mathbf{v}t, t + b))$. The most obvious stipulation for the action of the eleven-dimensional central extensions $\widetilde{\mathcal{G}}_m$ on (neo-) Newtonian spacetime is: $\widetilde{\mathcal{G}}_m \ni (\theta_m, g) = (\theta_m, b, \mathbf{a}, \mathbf{v}, R) \triangleright (\mathbf{x}, t) \mapsto (\mathbf{x}', t') = (R\mathbf{x} + \mathbf{a} + \mathbf{v}t, t + b)$ for any $m$, which is massively unfaithful. If one is serious about giving the $\widetilde{\mathcal{G}}_m$ a genuine role to play in NRQM one would be tempted to add a fifth dimension to (neo-) Newtonian spacetime and have $\theta_m$ act on this fifth dimension in such a way that faithfulness of the action of $\widetilde{\mathcal{G}}_m$ can be achieved (see Hernandez-Coronado 2012). Such a move would necessitate new interpretational rules and perhaps new physics as well.

Such intriguing issues lie beyond the scope of the present paper. But what needs to be addressed here is whether the use of central extensions of the Galilean group changes any of the above conclusions about mass superselection in NRQM. Some eminent physicists seem to think it does.

# 10 (Non)-implications of central extensions of $\mathcal{G}$ for mass superselection

Steven Weinberg appears to be claiming that the symmetry argument for mass superselection in NRQM is fragile:

> [T]there is nothing to prevent us from formally enlarging the Galilean group, by adding one more generator to the Lie algebra, which commutes with all the other generators, and whose

> eigenvalues are the masses of the various states. In this case the physical states provide an ordinary rather than a projective representation of the expanded symmetry group. The difference appears to be a mere matter of notation, except that with this reinterpretation of the Galilean group there is no need for a mass superselection rule. (Weinberg 1995, p. 62).

And later:

> In short, the issue of superselection rules is a bit of a red herring; *it may or may not be possible to prepare physical systems in arbitrary superpositions of states, but one cannot settle the question by reference to symmetry principles, because whatever one thinks of the symmetry group of nature may be, there is always another symmetry group whose consequences are identical except for the absence of superselection rules.* (Ibid., pp. 90-91; italics in the original)

This is puzzling. The "expanded symmetry group" in the first quotation seems to refer to the central extension $\widetilde{\mathcal{G}}$ of the Galilean group $\mathcal{G}$, and the claims seem to be that $\widetilde{\mathcal{G}}$ and $\mathcal{G}$ have the same physical consequences and that $\widetilde{\mathcal{G}}$ does not imply mass superselection since it admits proper unitary representations. But as we have seen, a superselection rule for mass in NRQM certainly does have physical consequences. So $\widetilde{\mathcal{G}}$ cannot have the same physical consequences as $\mathcal{G}$ if $\widetilde{\mathcal{G}}$ does not lead to mass superselection.

Prima facie, the standard heuristic argument for mass superselection, which relies on features of the projective representations of $\mathcal{G}$, appears not to apply to $\widetilde{\mathcal{G}}$ because $\widetilde{\mathcal{G}}$ admits proper unitary representations. However, a closer look reveals that not only is the resort to central extensions of $\mathcal{G}$ not a means for suppressing mass superselection in NRQM but it is also a means for revealing why the Galilean group leads to mass superselection whereas the Poincaré group used in RQM does not. The Poincaré group does not admit any non-trivial central extensions whereas the Galilean group does. As noted above, in an irreducible unitary representation of $\widetilde{\mathcal{G}}$ the mass has some fixed value $m$ and, hence, any such representation will be the irreducible unitary representation $\widetilde{U_m}(\theta_m, g) = \exp(imr)U_m(g)$, $g \in \mathcal{G}$ and $r \in \mathbb{R}$, of the concrete realization $\widetilde{\mathcal{G}}_m$ of $\widetilde{\mathcal{G}}$; and all of these representations are proper as well as irreducible. The heuristic reductio argument for mass superselection can proceed as before. Obviously the Hilbert space action of the Galilean identity transformation $g_e$ as represented by $\widetilde{U_m}(\theta_m, g_e) = \exp(imr)I_{\mathcal{H}_m}$ does not change the physical state. But if $\psi_{m_1} + \psi_{m_2}$, $\psi_{m_1} \in \mathcal{H}_{m_1}$, $\psi_{m_2} \in \mathcal{H}_{m_2}$, were a coherent superposition for $m_1 \neq m_2$ the Hilbert space action of the Galilean identity transformation would change the physical state since the relative phase of the two branches of the superposition will be changed by the difference between $\exp(im_1 r)$ and $\exp(im_2 r)$. The more formal argument for mass superselection can proceed by noting that, despite the fact that the $\widetilde{\mathcal{G}}_m$s are all isomorphic groups, their proper unitary representations are unitary inequivalent for different mass values (Def. (a) of Section 7). To see this suppose to the contrary that for $m_1 \neq 0 \neq m_2$ there is a unitary $V : \mathcal{H}_{m_1} \to \mathcal{H}_{m_2}$ such that $\widetilde{U_{m_2}}(\theta_{m_2}, g) = V\widetilde{U_{m_1}}(\theta_{m_1}, g)V^{-1}$ for all $g \in \mathcal{G}$ and $r \in \mathbb{R}$. For $g = g_e$ this implies $\exp(im_2 r)I_{\mathcal{H}_{m_2}} = V \exp(im_1 r)I_{\mathcal{H}_{m_1}} V^{-1} = \exp(im_1 r)I_{\mathcal{H}_{m_2}}$ and, thus, $m_1 = m_2$ contrary to assumption.[18] The rest of the formal counterpart of the standard argument for mass superselection presented in Section 7 can now be applied to again reach the conclusion that the Hilbert space hosting superpositions of states of different masses for a particle is a direct sum space.

A more sympathetic reconstruction of Weinberg's position must await a more able reconstructor.

# 11 The Poincaré group and the Newtonian limit

## 11.1 Poincaré, Galileo, and group contraction

In RQM the Poincaré group $\mathcal{P}$ (= inhomogeneous Lorentz group), the symmetry group of Minkowski spacetime, replaces the Galilean group of NRQM. According to the Wigner doctrine for RQM, solutions to relativistically invariant wave equations correspond to unitary representations of $\mathcal{P}$. (But as Wigner emphasized, the converse may fail because some unitary repre-

[18] There is a fancier way underscoring the difference in the $\widetilde{\mathcal{G}}_m$s. For different values of $m$ the $\widetilde{\mathcal{G}}_m$s correspond to different elements of the second cohomology group $H^2(\mathcal{G}, \mathbb{R})$, the group of group extensions of $\mathcal{G}$ by $\mathbb{R}$. The multipliers for the different $\widetilde{\mathcal{G}}_m$s are not related by a two-coboundary term, which is the case iff the representations $U_m(\mathcal{G})$ for different values of $m$ are unitarily inequivalent. See de Aczcárrage and Izquierdo (1995, Sections 4.2 and 5.2(a)).

sentations of $\mathcal{P}$ may not correspond to physically realizable solutions, e.g. representations with spacelike momenta.) In contrast to the Galilean group, the unitary representations of $\mathcal{P}$ can be de-projectivized so that $\mathcal{P}$ admits proper representations, and in proper irreducible representations the mass operator is non-trivial—it is not a multiple of the identity—and there is no mass superselection rule in RQM. Revisiting the heuristic argument for Bargmann mass superselection, when the series of Galilean transformations $g_{\mathbf{a}}$, $g_{\mathbf{v}}$, $g_{-\mathbf{a}}$, $g_{-\mathbf{v}}$ is replaced by Poincaré transformations the relativity of simultaneity produces to first order in $v/c$ a change in proper time by an amount $\mathbf{a} \cdot \mathbf{v}/c^2$, and so that the resulting action on Hilbert space should not leave unchanged the physical state of a superposition of different mass states (see Greenberger 2001a, 2001b).

In the Newtonian limit $c \to \infty$ the Poincaré group goes over to the Galilean group (group contraction) and, correspondingly, the Poincaré Lie algebra goes over to the Galilean Lie algebra (Lie algebra contraction). But the Poincaré group does not cease to be the relevant symmetry group for describing phenomena at velocities $v << c$. Thus, the mass superselection rule in NRQM is, in a sense, an emergent property: 'you can't see it coming' since it takes effect only when the limit $c \to \infty$ is reached and group/Lie algebra contraction is complete. This is not the place to engage in a discussion of the meaning and metaphysics of emergence. Rather I want to emphasize a consequence of the type of emergence (or whatever you choose to label it) just noted.

There are various respects in which, judged from the perspective of RQM, NRQM is approximately true in the $v/c << 1$ regime and in which NRQM has proved to be experimentally accurate. But there is also a respect in which, again judged from the perspective of RQM, NRQM is *not* approximately true in the $v/c << 1$ regime and in which NRQM has proved experimentally inadequate. In particular, because NRQM implies mass superselection it is unable to account for the data observed in low velocity inelastic collisions of elementary particles.

> In nuclear physics (where the [Bargmann mass superselection] rule has been largely ignored), the momenta of the reaction products would be wrong unless the change in rest mass energy is accounted for in energy conservation even when the velocities of the reaction products are small compared to $c$. This is illustrated by the model ... of a nonrelativistic inelastic reaction involv-

> ing particles of different mass, which requires that mass changes be permitted, and which uses superpositions of states of various masses.[19] (Gottfried and Yan 2003, p. 296)

The claim here must be that accounting for the momenta of the reaction products requires a change in the *total* rest mass and not simply a change in the individual masses that leaves the total mass the same (recall Section 8.2.3 above).

## 11.2 Terminology

Terminology is arbitrary, but ill-chosen terminology can lead to confusion. It is not uncommon to hear the Newtonian or non-relativistic limit characterized as the limit in which $v/c << 1$. For example:

> Formally, in the nonrelativistic (NR) limit, the Lorentz transformation (LT) passes over into the Galilean transformation (GT) ... By NR physics, we mean that $v$ lies within the low velocity limit [$v/c << 1$] of relativistic physics.[20] (Greenberger 2001a, 100405-1)

But then a puzzle ensues:

> There is a superselection rule in GT that one cannot coherently superpose particles of different masses ... But there is a very puzzling feature to this result. Relativistically one *can* coherently combine wave functions of different mass states. In the NR limit the LT reduces to the GT, but a phase shift survives in this limit that is independent of $c$. (Ibid., 100405-2)

No such puzzle ensues from the terminology used here: non-relativistic QM in the sense used here—NRQM—is quantum theory with the Galilean group as its symmetry group, and relativistic QM—RQM—is the quantum theory having the Poincaré group as its symmetry group. The former is obtained

[19] The claim must be that accounting for the momenta of the reaction products requires a change in the *total* rest mass and not simply a change in the individual masses that leaves the total mass the same (recall Section 8.2.3 above).

[20] I have substituted $v$ for the authors' $u$.

as the Newtonian limit of the latter, not in the sense of $v/c << 1$ but in the sense of $c \to \infty$. This leaves space for residues of relativistic effects to appear in the $v/c << 1$ limit that do not show up when the $c \to \infty$ limit is reached. If there is any lingering mystery it is general one about emergent properties that announce themselves only at the limit.

## 12 Castillos en el aire?

The foregoing sections have been concerned with one sense in which there is indisputably mass superselection in NRQM. It is time to acknowledge a sense in which there is no mass superselection in NRQM. Recall the Strocchi and Wightman (1974) dictum that a superselection rule in the broadest sense "can be defined as any restriction on what is observable in the theory" (p. 2198). Superselection as restriction on what is observable is a basic theme that runs through all of Wigner's and Wightman's work on superselection. But whether or not one sees this theme played out depends on where one starts. The treatment of mass superselection above concentrated on the simplest case of spinless particles. To illustrate a case where the Wigner-Wightman theme is played out consider now particles with spin. Superselection for whole-integer/half-integer spin splits the initially presumed Hilbert space $\mathcal{H}$ of the system into the direct sum $\mathcal{H}^{whole} \oplus \mathcal{H}^{half}$, where $\mathcal{H}^{whole}$ and $\mathcal{H}^{half}$ consist respectively of states of whole-integer and half-integer spin, and the algebra of observables is diminished from the initially presumed $\mathfrak{B}(\mathcal{H})$ to the proper subalgebra $\mathfrak{B}(\mathcal{H}^{whole}) \oplus \mathfrak{B}(\mathcal{H}^{half})$. Here there is a genuine restriction on what is an observable, and here there are superselection operators that commute with each other and with all elements of the algebra—in particular, the projections onto the subspaces $\mathcal{H}^{whole}$ and $\mathcal{H}^{half}$. Coherent superpositions of states belonging to $\mathcal{H}^{whole}$ and $\mathcal{H}^{half}$ are not possible, and this goes hand in hand with the restriction on what counts as an observable.

The argument for superselection for whole-integer/half-integer angular momentum offered in WWW (1952) and Hegerfeldt, Kraus, and Wigner (1968) resembles the standard heuristic argument for mass superselection in NRQM. A series of spatial or spatiotemporal transformations that compose to the identity transformation should not change the physical state of the system. But the Hilbert space action of these transformations results in a change in the relative phase of the two branches of a superposition of states from $\mathcal{H}^{whole}$ and $\mathcal{H}^{half}$ so that if there is to be no change in the physical state then (the argument goes) the Hilbert space that describes the physical states is $\mathcal{H}^{whole} \oplus \mathcal{H}^{half}$ and the algebra of genuine observables is $\mathfrak{B}(\mathcal{H}^{whole}) \oplus \mathfrak{B}(\mathcal{H}^{half})$. The resemblance between this argument and the standard heuristic argument for mass superselection encourages the mistaken idea that mass superselection has the same meaning as superselection for whole-integer/half-integer spin.

Consider again the examples of a single spinless particle of mass $m$ and a bi-partite system whose subsystems consist of two particles of different masses $m_1$ and $m_2$. The presumed Hilbert spaces and algebras of observables for the single particle system and the bi-partite system are respectively $\mathcal{H}_m$, $\mathfrak{B}(\mathcal{H}_m)$ (where $\mathcal{H}_m$ is the Hilbert space that carries the representation $U_m(\mathcal{G})$), and $\mathcal{H}_{m_1} \otimes \mathcal{H}_{m_2}$, $\mathfrak{B}(\mathcal{H}_{m_1}) \otimes \mathfrak{B}(\mathcal{H}_{m_2})$ (where the $\mathcal{H}_{m_i}$, $i = 1, 2$, are the Hilbert spaces that carry the representations $U_{m_i}(\mathcal{G})$). In neither case does mass superselection discover lurking superselection operators that split the presumed Hilbert space into selection sectors and reduce the presumed algebra of observables to a proper subalgebra.

Mass superselection operators arise in an attempt to represent the superposition of states of different masses, which perforce come from different representations of the Galilean group. For the single particle system the standard heuristic argument for mass superselection and its formal counterpart imply that the representation of a superposition of states of different masses $m_1$ and $m_2$ is given by the Hilbert space $\mathcal{H}_{m_1} \oplus \mathcal{H}_{m_2}$ and the algebra of observables $\mathfrak{B}(\mathcal{H}_{m_1}) \oplus \mathfrak{B}(\mathcal{H}_{m_2})$. In the case of a bi-partite system consisting of two particles of different masses $m_1$ and $m_2$ the representation of a superposition of states of different total mass $\mathbf{m} = m_1 + m_2$ and $\mathbf{m}' = m'_1 + m'_2$ is given by the Hilbert space $(\mathcal{H}_{m_1} \otimes \mathcal{H}_{m_2}) \oplus (\mathcal{H}_{m'_1} \otimes \mathcal{H}_{m'_2})$ and the algebra $\mathfrak{B}(\mathcal{H}_{m_1} \otimes \mathcal{H}_{m_2}) \oplus \mathfrak{B}(\mathcal{H}_{m'_1} \otimes \mathcal{H}_{m'_2})$. In the first case there is a mass superselection operator $M := m_1 I_{m_1} \oplus m_2 I_{m_2}$ which belongs to the algebra $\mathfrak{B}(\mathcal{H}_{m_1}) \oplus \mathfrak{B}(\mathcal{H}_{m_2})$ and whose eigenvalues $m_1$ and $m_2$ label the sectors of the direct sum of Hilbert spaces; and in the second case there is the mass superselection operator $\mathbf{M} : = \mathbf{m} I_{\mathbf{m}} \oplus \mathbf{m}' I_{\mathbf{m}'}$ which belongs to the algebra $\mathfrak{B}(\mathcal{H}_{m_1} \otimes \mathcal{H}_{m_2}) \oplus \mathfrak{B}(\mathcal{H}_{m'_1} \otimes \mathcal{H}_{m'_2})$ and whose eigenvalues $\mathbf{m}$ and $\mathbf{m}'$ label sectors of the direct sum of Hilbert spaces. This seems to involve an expansion rather than a restriction on what counts as an observable! But these mass operators are observables only in the formal sense that they belong to their respective algebras of observables in the direct sum structures. The direct sum structures that support these algebras are castles built in the air; they do not describe the states and observables of the intended target systems or, for that matter, of any particular systems. The projection operators onto rays crossing the selection sectors are not observables, but this is a limitation on castles-in-the-air observables that no one thought to be observables in the first instance.

Where does this leave mass superselection in NRQM? When authors refer to the Bargmann mass superselection rule what they often have in mind is not superselection in the sense of a restriction on what counts as an observable in a target system but rather a rule that "prevents the existence ... of states with a mass spectrum, and therefore of unstable particles" (Lévy-Leblond 1963, p. 785). What is responsible for the latter and, in particular, what role is played by the standard argument for mass superselection? Three factors are involved. First, in any irreducible projective unitary representation of the Galilean group (or any irreducible proper representation of the central extension of the Galilean group) the mass of a particle has a single fixed value. Second, the superposition of states of different mass—that is, states from different irreducible representations of the Galilean group—must be interpreted as the sum of states from the direct sum of the irreducible representations to which the states belong. Thus, such a superposition is not coherent, and the relative phase of the components of such a superposition is not an observable. This is the legitimate moral to be drawn from the standard heuristic argument or, better, its more formal counterpart sketched in Section 7. Third, assuming that the Hamiltonian for the dynamics on the direct sum of irreducible representations is an observable, the states of the different mass sectors evolve independently, and the evolution does allow states to cross from one sector to another. Putting these three factors together leaves no room for states with a mass spectrum or unstable particles.

# 13 Conclusion

The history of the Bargmann mass superselection rule for NRQM is a curious one. The rule arose from a exercise, set by Wightman (1959), to demonstrate that "superposition of two states with different mass gives a state whose existence in Nature would contradict Galilean invariance (Bargmann's Superselection Rule)." A solution, apparently intended by Wightman, was widely adopted as a valid proof of mass superselection.[21] Subsequently, however, at-

[21] I say apparently intended by Wightman because, on the one hand, he never objected to the standard solution in print; but, on the other hand, the standard solution does not

tacks on the rule were launched from various directions but, arguably, none of them found their mark. Nevertheless, Wightman's exercise and its standard solution can be faulted for producing a temptation to a misperception of the nature of mass superselection in NRQM. The standard argument for mass superselection has a form similar to the argument for superselection of whole-integer/half-integer spin. The latter superselection rule *is* an example par excellance of superselection in the sense intended by Wightman and Wigner—namely, a restriction on what is observable in the theory. So it is tempting to think, wrongly, that the argument for mass superselection also demonstrates the same kind of restriction on what is observable in the theory. Correcting this misimpression does not undermine mass superselection in the sense of prohibition on unstable particles in NRQM.

Having disentangled the different senses in which there is and there isn't mass superselection in NRQM, perhaps the controversy over this rule can be allowed to rest.

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conform to his high standards of clarity and rigor.

## Appendix: Infinite superselection

The idea of superselection was illustrated in terms of two superselection sectors. The direct sum construction can be generalized to handle infinities. The direct sum $\oplus_{a\in\mathcal{I}}\mathcal{H}_a$ of the Hilbert spaces $\mathcal{H}_a$ is defined for an index set $\mathcal{I}$ that may be finite, denumerable infinite, or even non-denumerable. It consists of vectors $\oplus_\vartheta := \oplus_{a\in\mathcal{I}}\vartheta_a$ defined by a family $\vartheta := \{\vartheta_a\}$, $a \in \mathcal{I}$ and $\vartheta_a \in \mathcal{H}_a$, provided that $\sum_{a\in\mathcal{I}} ||\vartheta_a||_{\mathcal{H}_a} < \infty$. When the index set in non-denumerable this sum is understood as $\lim_F \sum_{\alpha\in F} ||\vartheta_\alpha||_{\mathcal{H}_\alpha}$ where the $F$ are finite subsets of $\mathcal{I}$, and $\lim_F \sum_{\alpha\in F} ||\vartheta_\alpha||_{\mathcal{H}_\alpha} = L$ means that for any $\epsilon > 0$ there is a finite $F_0 \subset \mathcal{I}$ such that for any finite $F$ with $\mathcal{I} \supset F \supset F_0$, $|\sum_{\alpha\in F} ||\vartheta_\alpha||_{\mathcal{H}_\alpha} - L| < \epsilon$. The rules for scalar multiplication and vector addition are given respectively by $\alpha\oplus_\vartheta = \oplus_{a\in\mathcal{I}}\alpha\vartheta_a$, $\alpha \in \mathbb{C}$, and $\oplus_\vartheta + \oplus_\zeta = \oplus_{a\in\mathcal{I}}(\vartheta_a + \zeta_a)$. This direct sum vector space is complete in the norm derived from the inner product $\langle\oplus_\vartheta, \oplus_\zeta\rangle := \sum_{a\in\mathcal{I}}\langle\vartheta_a, \zeta_a\rangle_{\mathcal{H}_\alpha}$ and is, therefore, a Hilbert space.

If $\dim(\mathcal{H}_a) = D$ for all $a$ then $\dim(\oplus_{a\in\mathcal{I}}\mathcal{H}_a) = D \cdot |\mathcal{I}|$. In particular, if the index set $\mathcal{I}$ is denumerable and the $\mathcal{H}_a$ are all separable then so is their countable direct sum; but if $\mathcal{I}$ is non-denumerable then the direct sum over $\mathcal{I}$ of separable spaces produces a non-separable Hilbert space. In much of the physics community non-separable Hilbert spaces are thought to give off a bad odor, but there are some cases where they come in handy. Cases where there is superselection for a quantity with continuous spectrum is a case in point.[22]

The relevant non Neumann superselection algebra to go along with $\oplus_{a\in\mathcal{I}}\mathcal{H}_a$ is not $\mathfrak{B}(\oplus_{a\in\mathcal{J}}\mathcal{H}_a)$ but the smallest subalgebra $\oplus_{a\in\mathcal{I}}\mathfrak{B}(\mathcal{H}_a)$ generated by operators of the form $\oplus_{a\in\mathcal{I}}A_a$, where $A_a \in \mathfrak{B}(\mathcal{H}_a)$ and $\{||A_a||\}$ is bounded, acting on the direct sum space $\oplus_{a\in\mathcal{I}}\mathcal{H}_a$. For NRQM the superselection for mass corresponds to the Hilbert space $\oplus_{a\in\mathbb{R}^+}\mathcal{H}_m$, where the $\mathcal{H}_m$ are the $L^2_{\mathbb{C}}$ spaces constructed from solutions to the Schrödinger equation for mass $m$. The mass operator $M := \oplus_{a\in\mathbb{R}^+}mI_m$ acting on $\oplus_{a\in\mathbb{R}^+}\mathcal{H}_m$ is $M := \oplus_{a\in\mathbb{R}^+}mI_m$, where the $I_m$ is the identity operator on $\mathcal{H}_m$. Vectors in $\oplus_{a\in\mathbb{R}^+}\mathcal{H}_m$ lying entirely in $\mathcal{H}_m$ are eigenvectors of $M$. Yes, self-adjoint operators with a continuous spectrum can have eigenvalues when operating on a non-separable Hilbert space. Since $M$ is not a bounded operator it is not observable in the sense of being in $\oplus_{a\in\mathcal{I}}\mathfrak{B}(\mathcal{H}_a)$, but it is an observable in the extended sense

[22] Other cases include idealizations such as infinite spin chains; see Earman (2020).

that its spectral projections, i.e. the projections $E_m$ onto the $\mathcal{H}_m$ are all in $\oplus_{a\in\mathcal{I}}\mathfrak{B}(\mathcal{H}_a)$.

In the literature most of the discussion of continuous superselection rules is in terms of the direct integral construction, which may be viewed as a generalization of the direct sum construction wherein the index set $\mathcal{I}$ of the direct sum is replaced a measure space $(X, \mu)$. The component Hilbert spaces $\mathcal{H}_x$ of the direct integral Hilbert space $\mathcal{H}^\oplus = \int_X^\oplus \mathcal{H}_x d\mu(x)$ are indexed by points $x \in X$. An element of $\mathcal{H}^\oplus$ is a function $f : X \to \cup_{x\in X}\mathcal{H}_x$ such that $f(x) \in \mathcal{H}_x$ for all $x \in X$ and $x \longmapsto \langle f(x), g(x)\rangle_{\mathcal{H}_x}$ is $\mu$-integrable. The inner product on $\mathcal{H}^\otimes$ is given by $\langle f, g\rangle_{\mathcal{H}^\oplus} := \int_X \langle f(x), g(x)\rangle_{\mathcal{H}_x} d\mu(x)$. Two measures that are absolutely continuous with respect to one another give rise to isomorphically isometric direct integral spaces.[23] The main use for the direct integral construction in mathematics is in proving results about von Neumann algebras, e.g. every von Neumann algebra acting on a separable Hilbert space is a direct integral of factor algebras.

Note that if the $\mathcal{H}_x$ are separable and $(X, \mu)$ is a standard Borel space then $\mathcal{H}^\oplus$ is separable (Dixmier 1984 II.1.6 Corollary).[24] This is awkward for treating mass superselection in NRQM. The $\mathcal{H}_x$s feeding into the direct integral construction are $\mathcal{H}_m$s which are supposed to be the eigenspaces of the mass operator. But a self-adjoint operator acting on a separable Hilbert space has no eigenvalues if it has a purely continuous spectrum. As another manifestation of this problem, the projections onto the $\mathcal{H}_m$s should count as observables, but these projections are not all in $\mathfrak{B}(\int_{\mathbb{R}+}^\oplus \mathcal{H}_m d\mu(m))$ if the direct integral $\int_{\mathbb{R}+}^\oplus \mathcal{H}_m d\mu(m)$ is taken in its usual meaning; for a von Neumann algebra acting on a separable Hilbert space is sigma-finite, i.e. there are at most a countable infinity of mutually orthogonal projections. It is something of a mystery of why physicists use the direct sum construction to treat superselection rules when the index set $\mathcal{I}$ is finite or countably infinite but abandon the direct sum construction when $\mathcal{I}$ is non-countable.

[23]See Takesaki (2001) and Dixmier (1984) for more details.

[24]Standard Borel means that there is a metric on $X$ that makes it a complete separable metric space in such a way that the the $\mu$-measurable sets are the Borel $\sigma$-algebra.

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