What We Talk About When We Talk About Mathematics

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What do we talk about when we talk about mathematics? Numbers and functions, certainly. Algebraic structures sometimes. The structures can get pretty complicated. But these are really things that we talk about when we *do* mathematics. What do we talk about when we talk *about* mathematics, which has been around for a long time?

Philosophers have two ways of talking about mathematics. Some of them talk about what mathematicians say and do. When these philosophers talk about mathematics, they talk about definitions, theorems, and proofs, and sometimes calculations, questions, and conjectures. They also talk about methods, intuitions, and ideas, and maybe it is not so clear what those are. But even methods, intuitions, and ideas are found in what mathematicians say and do. So when these philosophers talk about mathematics, they are talking about mathematical talk.

The other way of answering the second question is to repeat the answer to the first question with more emphasis. When some philosophers talk about mathematics, they talk about numbers, functions, and algebraic structures, and about how our ordinary mathematical talk latches on to those things. When these philosophers talk about mathematics, they are talking about what mathematical talk is about.

Logicians make a big deal about the difference between *syntax* and *semantics*. When logicians talk about syntax, they talk about the rules of a language. When they talk about semantics, they talk about what a language means. So it's tempting to say that the first bunch of philosophers are interested in the syntax of mathematics and the second bunch of philosophers are interested in its semantics.

The distinction between syntax and semantics is useful in logic, but it is not very useful in philosophy. In fact, it causes a lot of problems. We ought to think long and hard about how we got stuck with these problems, and then we ought to figure out how to get out of the mess we are in.

1 The problem

It helps to think about the ways we talk about mathematics. That's how we got axiomatic foundations like set theory and type theory. Foundations helped clarify the ways we talk about numbers and functions and algebraic structures. They also helped us think about important questions, like, is it o.k. to talk about functions we cannot compute? Is it o.k. to use the axiom of choice?

People came up with formal languages for mathematics before there were programming languages, or languages for databases and expert systems. Those are also formal languages, but formal languages for mathematics were there first, and we got them by thinking about the ways we talk about mathematics.

So theories of mathematical language are useful. What have philosophical theories of mathematical objects given us? Not much. Mathematicians did not decide that it was o.k. to use the axiom of choice because philosophers were able to tell them what it means. Philosophers can't even agree about what it means. That's not a problem, because you don't have to say what the axiom of choice means to do mathematics. You *do* have to decide whether to use the axiom of choice. If you are trying to do that, you ought to talk it over with people you know, especially if you want them to listen to you later on. It's probably better not to talk to philosophers.

I am not saying that semantics isn't useful. It's really useful in logic. There are a million proof systems for propositional logic, and what they all have in common is that they prove formulas that are always true, no matter how you interpret the variables. Without knowing what it means for a propositional formula to be true, you can't say what it means for a proof system to be correct, and then it's really hard to explain what all the good proof systems have in common.

It is also useful in computer science. The semantics of a programming language tells you what the programming language is supposed to do and what it means for a compiler to be correct. We use programming languages all the time. If we couldn't think about whether we are implementing them correctly, we would be in pretty bad shape.

Semantics is even useful in mathematics. On the face of it, a polynomial is an expression. It has terms, maybe a constant term, and a term of highest degree, and those are expressions too. But a polynomial can also be a function on the real numbers, which is a thing that the expression describes. A polynomial can also be an element in a polynomial ring. Polynomial rings give us ways of thinking about polynomials without worrying about whether $x+1$ and $1+x$ are the same. At some point, we have to say what the elements of a polynomial ring are, and one way is to do it is to say that they are expressions, maybe up to an equivalence relation. Knowing how to reason about expressions and how the expressions are related to what they express is generally helpful.

In all these situations, we have expressions that describe things and we have other ways of thinking about the things that the expressions describe. Semantics fits the pieces together.

When we talk about mathematical statements, how do we talk about the things that they describe? With mathematical statements, of course. We use mathematics to talk about things like numbers and groups and spaces. When we talk about them we are just doing mathematics. It's not like we have some other way of talking about them. Logicians and computer scientists have special-purpose languages, like the language of an algebraic structure or a programming language. But when they talk about those languages and what they mean, they use mathematics. There is no magical philosophical language that tells us what things *really* are, and we don't need one.

It's not just that worrying about what mathematical objects are isn't helpful. It causes a lot of problems. One of them is Benacerraf's problem, which goes like this. Scientists learn about the world by poking it and seeing what happens. They do experiments and measure things. But then how do they learn about numbers? You can't poke them and measure them because they aren't anywhere. If we can't learn about them in a scientific way, it's hard to say how they can be useful for science. Maybe you want to say that mathematics isn't part of science, but even so, if you are a responsible scientist and you use numbers, you ought to say how you know what you think you know about them.

This way of thinking about mathematics has to be wrong. Of course we can learn mathematics. That's one of the things we do in school and also when we get older. And of course mathematics is useful for building airplanes and bridges and for making our tax forms come out right. That's why we learn it. It's just that numbers and triangles are not like rocks and trees and sofas. We don't learn about them by bumping into them. They aren't even like atoms and magnetic fields. We learn about them in different ways.

I don't blame Paul Benacerraf for saying what was on his mind. Sometimes you have to talk about the things that are bothering you and get them out in the open. Then you can take a serious look at them and realize that you don't have to worry about them. The problem is, a lot of philosophers can't get over it.

There are many important questions about mathematics. Should we use computers to do proofs? Is mathematics on the right path? Is it getting too abstract? Is it getting too applied? How can we tell whether something is good mathematics? Does statistics count? Is AI going to change everything? Should we be worried? But now there is nobody left to talk about things like that. Mathematicians are too busy trying to prove their theorems and philosophers are too busy trying to figure out what numbers really are. Nobody wants to be a bad mathematician or a bad philosopher so they stick to what they are doing.

2 What went wrong

The philosophy of mathematics took a bad turn sometime in the twentieth century. The first half of the twentieth century was pretty good for philosophy of mathematics. It was bad for humanity, especially in Europe, but it was good for philosophy of mathematics. It was good for philosophy of science too.

The logical positivists got things going by telling everyone how science works. It is not as simple as they made it out to be, but one of the things they got more or less right is that the way we talk about things is important. Rudolf Carnap said a lot about "linguistic frameworks," but that just means ways of talking about things.

The logical positivists said that mathematics comes down to a choice of linguistic framework. This makes sense when you think about it. Doing mathematics means coming up with ways of thinking about things, which is pretty much the same as coming up with ways of talking about things. Having good ways of thinking about things means having good ways of thinking about the world. But mathematics is about the ways of thinking and not about the world. When we do science we choose a mathematical description a lot like the way we buy a car to get around. If the car doesn't work, we unload it on someone and get a better one.

But even though the logical positivists thought that mathematics comes down to making choices—they called them *pragmatic* choices—they also said that we shouldn't talk about the reasons for making those choices. They are outside the framework. That makes them metaphysical questions, which means that they are not scientific, which is bad.

If we are going to do mathematics, why shouldn't we talk about how we are doing it? If we have reasons for our choices, why can't they be scientific? If I want to decide what car to buy, I am sure as hell going to talk about it. I am going to think about all the things I want to do with the car. I am going to go to the library to look at all the car magazines and I am going to ask my friends for advice. It's hard to make decisions. It helps to talk about them.

Eventually W. V. O. Quine came along and said that the logical positivists were wrong. It's not just mathematics that is determined by language. All of science is determined by language, and there is nothing special about mathematics. It is all one big web of beliefs. Everything has to do with how we talk about things, and we had better make good choices about how we talk about things if we want science to work out.

But even though Quine thought we had to make choices, he also thought we shouldn't talk about them much. When we talk about science and do it right, we are just *doing* science. He also said that there isn't a principled distinction between talking about things and coming up with ways of talking about them. This was a jab at the logical positivists, who thought that this was exactly the difference between science and mathematics. Science is about things, and mathematics is about how we talk about them.

Philosophers like to talk about principled distinctions, but mathematics is different from science and it doesn't make sense to pretend they are the same. The logical positivists said that mathematics is different because it is analytic, which means that mathematicians define everything. Or stipulate everything. Definitions determine what words mean because that's what definitions do. Axioms are true because we decide that they are true, not because of the way the world is. Axioms are like definitions. They define the things they talk about.

Quine said that has to be wrong because when someone writes a dictionary and says that some word means something or other, they aren't supposed to make it up. The definition is supposed to describe something that is already there, and the logical positivists didn't explain how the way mathematics got to be there is any different from the way that science got to be there. Then he backed up a little and allowed that sometimes definitions do other things. Some definitions clarify the meaning of words and other definitions are abbreviations. But even the definitions that are supposed to clarify are supposed to clarify things that are already there, and Quine didn't think that the abbreviations were all that interesting.

But that's the whole point. When you clarify something enough so that you know what the rules are, that's when you have mathematics. Any mathematician will tell you that coming up with good definitions is very hard to do. So Quine took the most interesting part of mathematics and made it sound too boring to talk about.

Mathematicians still think about important questions, but they are afraid that if they talk about them, they are doing philosophy, which is a waste of time. Philosophers think that if they try to be scientific about mathematics, then they are *not* doing philosophy, which, for them, is also a waste of time. So mostly we talk about things that don't matter, and when we talk about things that matter, we don't do it well. At least the mathematicians can go back to doing mathematics.

3 How to fix things

How can we avoid talking about things that don't matter? Sometimes it helps to look around and notice that nobody cares what we are saying. But that doesn't always work. Sometimes we tell ourselves that the reason nobody cares is that we are talking about things that are so deep and important that nobody else can appreciate them.

Another thing we can do is look at the history of mathematics. History is really interesting. When you read about how mathematics was done you see that people had very good ideas. They had to work hard to come up with the ideas. You can think about why they decided to talk about things the way they did and you can think about what makes the ideas they had so good.

It doesn't help to think about whether the numbers people used to talk about are the same as the numbers we talk about today and whether their words latched onto them in the same ways that ours do. It *is* interesting to think about how people used to talk about numbers and how we talk about numbers today and how our talk has changed. But that's not the same as thinking about how numbers have changed.

There is a guy I know who writes about mathematics. He has written about the history of analysis and the history of algebra and the history of geometry in the nineteenth century. He has written about where mathematics comes from and how it gets used in physics. He has also written about famous mathematicians and big ideas like modernism. Mostly he writes about what mathematicians thought and what they did. Sometimes he uses words like "ontology" and "epistemology." By that he just means the way people talk about things and think about them.

It would be nice if more mathematicians read about the history of mathematics. It would be nice if some philosophers read about it too, and even some people who aren't mathematicians or philosophers. Then we could all get together and talk about it. We could talk about mathematics and how it got to be the way it is. We could talk about why we like mathematics so much and what we like about it. We could even talk about how it might be different by the time our children and grandchildren are grown up and are doing mathematics on their own. It would be nice to talk. I am pretty sure we would like it.

Appendix

I am grateful to the editors for accepting this unconventional contribution to an otherwise scholarly collection. The title is an homage to Raymond Carver's short story, "What We Talk About When We Talk About Love." I originally set out to write an ordinary philosophical essay about the role of syntax and semantics in the philosophy of mathematics, but having chosen the title, it was hard to resist emulating Carver's narrative style. Doing so was liberating because it encouraged me to avoid overworn philosophical tropes and to formulate ideas as clearly and simply as I could.

The history of mathematics is a powerful philosophical tool, and thinking about what has changed and what has remained constant provides critical insights as to why we do mathematics the way we do. The guy in Section 3 who writes about mathematics is, of course, Jeremy Gray, who has always treated the history of mathematics as a history of ideas. His respect for the power of those ideas animates his work. I have learned a lot from him, and if this essay brings him a bit of enjoyment in return, it will have served its main purpose.

A secondary purpose was to explore the way that certain developments in twentieth century philosophy of mathematics have shaped the way we think about the subject. Carnap's influential "Empiricism, Semantics, and Ontology" [2] can be taken as an exemplar of the views attributed to the logical positivists in Section 2, and, of course, the counterpoint provided there is a summary of Quine's "Two Dogmas of Empiricism" [5]. Volumes have been written about the issues raised in these two publications, and readers can turn to the *Stanford Encyclopedia of Philosophy* [3] for details and references. I also recommend Edmonds' recent book, *The Murder of Professor Schlick: The Rise and Fall of the Vienna Circle* [4], for an engaging exposition of the historical context.

My goal here has not been to add to the debates, but, rather, to reflect on the way they have influenced the philosophy of mathematics. I find the things that Carnap and Quine have in common to be more striking than their differences, and I hope this essay makes it clear that I take their shared focus on the communicative and inferential norms of mathematics and the sciences to be an important philosophical advance.

But, curiously, this focus is not what drives the philosophy of mathematics today. This essay offers one possible explanation as to why not. It is easy to interpret Carnap's and Quine's portrayal of the relationship between philosophy and science as an implicit affirmation that the best thing that philosophers can do is to respectfully step aside while mathematicians and scientists do their work. It is not surprising that philosophers since then have resisted that conclusion and have instead turned their attention to puzzles in metaphysics and epistemology, topics that are comfortably within their wheelhouse.

This perceived dichotomy between thinking about mathematics and thinking about meaning, reference, and the nature of mathematical objects is unfortunate. There is a lot to be learned by paying attention to mathematics itself, and philosophers are well positioned to help us make sense of the norms, values, and goals of the practice. Mathematicians may be very good at doing mathematics, but that doesn't necessarily imply that they are good at thinking about what they do. Philosophers' training puts them in a position to assess the mathematical literature critically, analyze the conceptual and inferential structure, make sense of the implicit norms and expectations, study the means that mathematicians employ, and understand how they are suited to their goals. That requires familiarity with the relevant mathematics but not the same type of expertise. We still have a lot to learn about the nature of mathematics and its applications to science, industry, and policy.

Instead, an interminable focus on disconnected technical problems has had a devastating effect on philosophy of mathematics. A recent analysis of tenure-track positions advertised in *Jobs for Philosophers* in the 2021–2022 academic year doesn't even mention philosophy of mathematics in its categorization.[1] Digging into the data shows that the

[1] https://philosopherscocoon.typepad.com/blog/2022/04/where-the-tt-jobs-werent-in-2021-22.html

phrase “philosophy of mathematics” occurs in only three of the 201 advertisements, in each case listed among multiple areas of potential interest. Surely this is an indication that the field is no longer viewed as important. It is sad that a discipline that was so central to the philosophical tradition from ancient times to the middle of the twentieth century now barely registers a pulse.

But let me temper this doom and gloom with some more positive notes. First, colleagues assure me that the outlook for philosophy of mathematics is more optimistic in Europe, and I would not be surprised to learn that other communities have also managed to escape the gravity of the Anglo-American analytic tradition.

Second, whatever their long-term career prospects may be, a number of talented young people are throwing caution to the winds and finding ways of doing important work in the field. Meetings of the Association for the Philosophy of Mathematical Practice, of which Jeremy was a founding member, are lively and well attended. I only wish the organization would drop the phrase “philosophy of mathematical practice” in favor of “philosophy of mathematics.” We should worry about philosophy of mathematics that *doesn’t* have anything to do with mathematical practice, and we should avoid depicting philosophy that does as anything less than the proper heir to a long philosophical tradition.

Third, there is still a lot to be done. Almost all of the philosophical papers I have written end with cheery exhortations to roll up our sleeves and get to work. See, for example, the last section of my “Reliability of Mathematical Inference” [1], which, incidentally, cites a number of the young philosophers alluded to in the previous paragraph.

Finally, there is considerable interest. At a time when it seems that every undergraduate is majoring in computer science, data science, or business, I still come across students from across the United States that are double-majoring in mathematics and philosophy. Rebelling against the segregation of science from the humanities, they are an encouraging reminder that there are still young people who find value in the scholarly traditions that have served us well for centuries. We would do well to support them.

At the end of the day, mathematicians are among the most philosophically inclined people on the planet. Dealing with creative flights of abstraction on a daily basis encourages constant reflection on the nature and meaning of their craft, so there is still room for a philosophy of mathematics that does the subject justice. All we need to do is take stock of where we are and where we want to be, and then figure out how to get from here to there.

References

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