Open Texture, Rigor, and Proof

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Abstract

Open texture is a kind of semantic indeterminacy first systematically studied by Waismann. In this paper, extant definitions of open texture will be compared and contrasted, with a view towards the consequences of open-textured concepts in mathematics. It has been suggested that these would threaten the traditional virtues of proof, primarily the certainty bestowed by proof-possession, and this suggestion will be critically investigated using recent work on informal proof. It will be argued that informal proofs have virtues that mitigate the danger posed by open texture. Moreover, it will be argued that while rigor in the guise of formalisation and axiomatisation might banish open texture from mathematical theories through implicit definition, it can do so only at the cost of restricting the tamed concepts in certain ways.

Keywords: Mathematical Concepts, Rigor, Proof, Implicit Definitions

1 Introduction

Mathematical concepts are usually considered to be completely sharp, without vagueness or indeterminacy. Recently, this Fregean dogma has been under critical scrutiny. While making an argument for the applicability of the methods of conceptual engineering in the philosophy of mathematics, Tanswell argues that mathematical concepts are subject to open texture, a kind of semantic indeterminacy (Tanswell, 2018). Coming from a somewhat different angle, Shapiro and Roberts (2021) also investigate open texture in mathematics, arguing that it threatens the traditional role of proof, but that rigorously defined mathematical concepts are not subject to open texture. In this paper, several definitions

of open texture will be compared and contrasted, and their connection with proof and rigor will be examined.

The paper is structured as follows: After this introduction, section 2 surveys definitions of open texture to be found in the literature, comparing and contrasting them. Then, section 3 explores the consequences of open texture in mathematics. After that, section 4 is devoted to open texture and rigor, drawing on recent work on implicit definitions to see to what extent and in what sense rigor can banish open texture from mathematics (Giovannini & Schiemer, 2021).

Before beginning with the text proper, a few words abut concepts more generally should be said. As is established in the literature on open texture, I will not supply a precise definition of concept. Underlines will be used to mark mention of concepts - for instance, <u>circle</u> refers to the circle-concept. Due to to the current interest in conceptual engineering, there are quite a few recent papers about what the best concept of concept is (e.g., Isaac, 2020; Koch, 2021). I am partial to concepts being the meanings of words, a position recently defended by Thomasson (2021). In particular, this means that I think that concepts can be patchwork, in the sense of there not necessarily being necessary and sufficient conditions on the application of concepts. The patchwork nature of concepts has recently been vigorously explored and defended by Mark Wilson (2008)(see also Haueis, 2022). The concept of group, for instance, could have as patches the theory of groups in some logic (classical, relevant, second-order...), groups as internalised to some set theory, or group objects in certain categories. With that said, not much of what is to follow will depend on a specific account of what concepts are. Furthermore, concerns about the ontological nature of mathematical objects will be bracketed here. For an opinionated survey, the reader is directed to Cellucci (2020).

2 Defining open texture

The notion of open texture was introduced into philosophy by the logical positivist Friedrich Waismann in a series of papers attacking the analytic/synthetic distinction and thereby versions of crude verificationism about meaning (Russell, 2019). Open texture is a form of semantic indeterminacy. Waismann thought that the open texture of our empirical concepts hangs together with them not being precisely delimited in all directions. Before getting into the thick of it, it is worth citing one of the examples that Waismann uses to motivate the idea of open texture at length:

> "Suppose I have to verify a statement such as "There is a cat next door"; suppose I go over to the next room, open the door, look into it and actually see a cat. Is this enough to prove my statement? Or must I, in addition to it, touch the cat, pat him and induce him to purr? And supposing that I had done all these things, can I then be absolutely certain that my statement was true? . . . What, for instance, should I say when that creature later on grew to a gigantic size? Or if it showed some queer behaviour usually not to be found with cats, say, if, under certain conditions, it could be revived from death whereas normal cats could not? Shall

I, in such a case say that a new species has come into being? Or that it was a cat with extraordinary properties?" (Waismann, 1968)

The purpose of this little story is to demonstrate that the concept <u>cat</u> is open-textured. Using it as an inspiration, there are different ways to sharpen this idea.

The first would be to take the term "cat" to exhibit a kind of generalised vagueness (Shapiro, 2006, appendix), such that there are clear cases (the ordinary cat first encountered), a penumbra (a gigantic cat of perhaps variable size), and clear non-cases (gigantic undead cat). Ordinary language doesn't prescribe a use in the penumbra, leaving open whether the concept cat still applies to the suddenly enlarged cat. On this reading of Waismann's story, one can assimilate open texture into vagueness.[1] In his book on vagueness, Shapiro (2006), aware that this is not quite what Waismann most likely had in mind, offers a definition of open texture in this spirit:

Definition 1 (OTV) A concept C has open texture iff there are cases for which a competent, rational agent may acceptably assert either that the concept applies or that it disapplies.

However, this definition arguably misses the core of the dynamic nature of Waismann's example: The cat changes, and the observer is confronted with a new situation. Tanswell (2018) argues that while Shapiro captures one sense in which a concept can be open, it misses out on an important aspect of open texture. Shapiro is concerned with agents' decisions being open in certain extant hard cases, while Waismann was more concerned with the possibility of genuinely new cases arising. Intending to capture this thought, Tanswell defines a concept as open-textured iff there are always possible objects outside the standard domain of application for which there is no fact of the matter as to whether the concept applies to them.[2] Seemingly independently of Tanswell, Roberts and Shapiro adopt a similar perspective and define open texture as follows (Shapiro & Roberts, 2021):

Definition 2 (OTN) A natural language concept C displays open texture iff it is always possible for there to be an object a such that nothing concerning the established use of C, and nothing concerning the nonlinguistic facts, determines that a falls under C, nor does anything determine that a does not fall under C.

Roberts and Shapiro note that most philosophers of science nowadays would agree that most, if not all, empirical terms display open texture in this sense. They point to one possible exception, namely natural kind terms, where

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[1]This is also how, following H.L. Hart, open texture has often been understood in the philosophy of law (see e.g., Bix, 2013).
[2]Strictly speaking, the 'always' is an addition due to Vecht (2020), who adds it to account for the persistent nature of open texture.

adherents of something like reference magnetism in the Lewisian sense[3] seem to think that concepts corresponding to natural kind terms do not display open texture in either of the two senses above.

Coming back to empirical terms, whatever reasons philosophers might have nowadays for taking them to be open-textured, the primary reason for Waismann is what he calls "the essential incompleteness of description" (Waismann, 1968, p.43): The fact (or postulate) that no description of an empirical object is ever complete in the sense of supplying all possible details. John Horty ascribes to Waismann the view that concept ascriptions are made on the basis of descriptions of objects. He then goes on to provide another definition of open texture, in my opinion coming closest to capturing the idea of open texture that seems to lie beneath Waismann's examples (Horty, 2020):

Definition 3 (OTD) A concept C is open-textured iff for any object a and any description of that object D on the basis of which one can apply C to a, it is always possible to add further details to D such that one can no longer apply C to a on the basis of D.

To my mind, OTD differs from the previous definitions mainly in two respects: Firstly, it explicitly incorporates an account of concept ascription. Concepts can be ascribed on the basis of descriptions. One way to think about this is that a description D of some object a might offer defeasible reasons for judging that a falls under the concept C. The plausibility of this account will depend on the precise circumstances and objects under consideration.

For mathematical objects, a platonist might insist that some of our knowledge of mathematical objects comes about by faculties like Gödelian intuition. Be that as it may, I think that this descriptivist account of concept ascription has some intuitive plausibility, especially in the light of a moderate naturalism about mathematical objects. Some of its consequences will be explored in the next section. Note that a descriptivist account of concept ascription need not come together with a descriptivist account of reference, which would be problematic. Such accounts are known to suffer from various problems, most importantly Putnam's model-theoretic arguments (Lewis, 1984; Putnam, 1985).

Secondly, this definition again changes focus: Understanding open texture as a species of vagueness directs the attention to hard cases, understanding it as the possibility to go on in either way in the presence of new objects directs the attention to new cases, and Horty's definition of vagueness as connected to the indefinite extensibility of descriptions widens focus to include the dynamics within a single case. In this sense, it fits Waismann's story above most snugly, given that that it is somehow the same initially cat-like entity that undergoes changes, making us doubt our initial conceptual ascriptions. But of course it seems to presuppose that extending descriptions doesn't affect the ascription of objecthood and identity of objects. It is easy to continue Waismann's story

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[3]The locus classicus for reference magnetism is Lewis (1984).

in such a way as to make us doubt that the cat is one entity, and thus maybe a slightly weakened version of the above definition would be more apt. I don't think this matter is especially important for our purposes here.

Before concluding this survey of extant definitions of open texture, it is worthwhile to consider a worry about ascriptions of concepts on the basis of definitions: One might think that it is always possible to give descriptions of objects that immediately and indefeasibly license the ascription of some concept. For instance, I might describe the cat-like entity a next door by "The cat next door", and it would follow from that that a is a cat. But this description does not provide good reasons to think that a is a cat beyond some form of testimony. If challenged on the entity being a cat, I might offer other descriptions, such as "It looks like a cat, moves like a cat, and purrs like a cat if induced to do so.". But the link from this description to ascriptions of <u>cat</u> can be severed: It might turn out that the cat is some kind of highly advanced robot, for instance. Part of Waismann's point in his investigations into open texture was that all such links between empirical concepts, if they are not purely stipulative, can be severed or defeated. He was working on an attack on verificationism, and it is easy to see how the considerations above regarding open texture could be used to attack a proponent of the view that all descriptions of objects on the basis of which one ascribes concepts have to be given in terms of what is somehow immediately accessible to experience. To rescue OTD from the threat of triviality posed by direct descriptions of the form "a is C", one should maybe amend the definition of OTD to say that the description on the basis of which one ascribes some concept has to provide non-stipulative reasons for the ascription. Waismann's insight is then that these reasons for the ascription of empirical concepts are defeasible.

3 Open texture and mathematical concepts

For Waismann, mathematical concepts were paradigmatic examples of concepts with closed texture, in virtue of mathematical objects being completely describable. In his writings on the philosophy of mathematics and the evolution of mathematical concepts, Waismann (1936) presented himself as an advocate of the Hilbertian axiomatic method, but was also influenced greatly by the middle Wittgenstein. Plausibly, he thus considered it possible to describe any given mathematical object completely in some formal system or calculus. However, my goal here is not to engage in Waismann scholarship, but rather to take his ideas on open texture as a starting point for investigations into matters closer to contemporary philosophy of mathematics. The aim of this section will be to first describe possible consequences of the presence of open-textured concepts in mathematics. After that, some space will be devoted to an initial assessment of the texture of mathematical concepts according to the definitions of open texture given in the last section.

3.1 Open texture, proof, and defeasibility

According to Shapiro and Roberts, open texture threatens to undermine one of the traditional virtues of mathematics, namely the certainty established by proof. Proof, as traditionally conceived, is the ultimate guarantee of the truth of a proposition, removing all possible doubt. Once a proposition is proven, it can be used freely and without second-guessing in all other parts of mathematics. Open texture poses a threat to this picture:

> "And, for this purpose, open texture is a flaw. We would have to constantly check every lemma used in every proof, to make sure it is still good, that some unforeseen case has not arisen to undermine the proof, at the level of generality needed." (Shapiro & Roberts, 2021, p.179)

Their main example for something like this happening is Lakatos' well known study of the proof of Euler's conjecture. Euler's proof is threatened by the open texture of polyhedron (Lakatos, 2015), since monsters such as the urchin or the hollow cube were initially in the penumbra of the polyhedron-concept, and admitting them into the class of polyhedra would have vitiated the proof. These monsters then led to changes in the definition of polyhedron. By the examples they choose, Roberts and Shapiro make the threat posed by open texture seem like it is primarily due to unconceived counterexamples.

To assess this threat, it is best to be precise about the definition of OT used. On OTV, open texture is something like the possibility of rational disagreement about the extension of concepts. While this seems to make proof impossible in some cases, it doesn't necessarily seem to provide unconceived counterexamples. On OTN, the definition Roberts and Shapiro are using in the above quote, there is the possibility of new objects for which it is open whether the concept under consideration applies to them. Imagine, for instance, a proof trying to establish something about polyhedra, with a step in which it is used that polyhedra are always triangulable. But then an interlocutor presents a new non-triangulable solid S, for which it is open whether it is a polyhedron. It now seems that the mathematicians considering the proof have a choice: If they accept S as a polyhedron, the proof does not work as intended. If they do not, the proof works.

There might be independent reasons to accept or not accept S as a polyhedron: S might satisfy important characteristica for polyhedra, be a polyhedron according to some working definition, or it might void other proofs about polyhedra. Mathematicians are thus not quite free in their choice, as they might be constrained by, for instance, consistency in their reasons for calling something a polyhedron or by means-ends reasoning. Nonetheless, unconceived counterexamples are in general importantly different, in that there might be a fact of the matter as to whether they fall under the concept under consideration. So open texture does pose a threat to the certainty established by proof, but is a threat of a slightly different nature than that posed by ordinary unconceived counterexamples. This is because when confronted with a penumbral potential counterexample, mathematicians must engage in a deliberative process to

decide whether to take it to be a counterexample. Roberts and Shapiro suggest that open texture in this sense is not a problem for rigorous mathematics, a claim that will be examined later.

Interpreting open texture as the defeasibility of concept ascription (OTD), the threat is even more severe: If concept ascriptions can be overturned by additional descriptive detail, even statements about familiar objects might turn out false after adding in additional detail. Moreover, if a proof about some object proceeds on the basis of some set of properties P, the same proof might not be applicable to an object with properties P, because it might have additional properties overturning concept ascriptions made on the basis of P. Put succinctly, reasoning on the basis of concepts subject to OTN is defeasible, i.e. non-monotonic. Now, in informal mathematical reasoning it might of course happen that some proof attempt is defeated by pointing out additional properties some object has, for instance because these properties deductively entail other statements about the object incompatible with the desired result. In that case, I think there is a strong intuition that the prover just made a mistake, in contrast with the relevant cases for OTV and OTN. Moreover, it should be noted that classical logic is monotonic by assumption, so formal derivations cannot be overturned by adding additional premises. Whether this is relevant for informal proof depends on the view one adopts towards the relationship between informal proof and formal derivations, which will be discussed now.

The preceding paragraphs let the contention that mathematical proofs are what grants mathematical knowledge (or even certainty) go unchallenged. Before challenging it, something should be said about the relationship between formal derivations and informal proof. In the philosophical literature, it is nowadays acknowledged that mathematicians usually do not prove theorems in formal logic, working instead in natural language with various symbolic additions (Rav, 1999). Moreover, mathematicians often (intentionally) leave various gaps in proofs, work with diagrams, and heavily rely on background knowledge of their audience (De Toffoli & Giardino, 2015; Fallis, 2003). So proof is not just formal derivation, but the precise relationship between proof and formal derivation is still contested. Some think that informal proofs serve as abbreviations or indications for formal derivations (Azzouni, 2004), while others think that proofs are basically recipes or algorithms for proving activities (Azzouni, 2020; Larvor, 2012). Now, formal derivations are in some sense not susceptible to open texture: Monotonicity is a structural feature of classical logical consequence. Moreover, the extensions of predicates in classical logic are always determinate, there is no penumbra. But this does not tell us anything about whether our mathematical concepts can be captured formally, and also little about informal proof and open texture. The former will be discussed later. As to the latter, one could now go through all extant accounts of informal proof to assess their relationships with open texture, but this would necessitate work beyond the scope of this paper. But in any case, the main aspect of

proof at stake here is the justification imparted by proofs, and whether that is threatened by open texture.

In recent work, De Toffoli (2021) sketches a fallibilist account of mathematical justification through proof. She argues that proof-possession is an overly demanding standard of mathematical justification, and that instead mathematical justification can be attained in virtue of having something that looks like a proof to relevant experts, a simil-proof. She distinguishes between traditional proof and simil-proof, where it is the former that has traditionally been thought to justify mathematical knowledge. Building on previous work, she defines a proof to be a correct deductive argument for a mathematical conclusion from acceptable premises that is shareable. If one wants to be more demanding, a condition on transferability or a priori verifiability can be added. The details of what shareability means are not important here, but transferability matters for the purposes of dealing with open texture.[4] Roughly, an argument is transferable if consideration of the steps of the argument alone is enough to convince a relevant expert of the validity of its steps: No testimonial knowledge or knowledge about the generation of the argument is necessary. OTV seems to threaten transferability, while OTN and OTD threaten correctness of deductive inference by making inference defeasible. However, as mentioned above, De Toffoli holds that proof-possession is too demanding a standard for mathematical justification for fallible agents - possession of a simil-proof is enough, so the question now is whether open texture threatens simil-proofs similarly.

To judge this matter, a more precise definition of simil-proof is needed. Here is De Toffoli on what a simil-proof is:

> "An argument is a Simil-Proof (SP) when it is shareable, and some agents who have judged all its parts to be correct as a result of checking accept it as a proof. Moreover, the argument broadly satisfies the standards of acceptability of the mathematical community to which it is addressed." (De Toffoli, 2021, p.13)

Note that for agents to be able to judge parts of an argument correct independently, it has to be transferable to some extent. Simil-proofs are clearly not as threatened by open texture, because they allow for the possibility of error, and thus also for agents to have missed some potential objects for which the argument does not work. The same holds for the defeasibility induced by open texture in the sense of OTN.

To take stock, while the traditional picture of proof as the ultimate guarantee of mathematical justification and knowledge is threatened by open texture, accounts of mathematical justification which allow agents to be fallible are not vulnerable in the same way, which is of course not surprising. While Shapiro and Roberts are right that the presence of open texture undermines the traditional role of proof, that traditional role was under heavy attack anyway.

Still, it is true that the presence of open texture makes using proved propositions as autonomous building blocks for mathematics harder, since

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[4] "Transferability" was coined by Easwaran (2009) in a study of what distinguishes probabilistic proofs from genuine proofs.

sharpening open-textured concepts involved in proofs of said propositions necessitates a re-appraisal of these proofs. Fortuitously, proofs (and simil-proofs) have features to make such re-appraisals easier, namely transferability and convertibility. An argument A for a proposition P is convertible if its structure is such that it allows the conversion of rebutting defeaters into undercutting defeaters.[5] Easwaran (2015) argues that reasons for mathematical propositions by published proofs are in general defeasible, but identifies convertibility as a desideratum on proofs that makes dealing with this defeasibility tractable. In the fallibilist context, the transferability of a simil-proof rules out that the SP uses testimonial or statistical arguments. Transferability thereby also serves as a minimal requirement for a simil-proof to be convertible.

Allow me to explain convertibility in more detail: A rebutting defeater for a proposition P is a reason for the negation $\neg P$ of P. In mathematics, a counterexample to some conjecture would for instance prima facie be a rebutting defeater. Given a reason A for P, an undercutting defeater for A does not necessarily provide reason for $\neg P$, but it provides reason to think that A is not a reason for P. In mathematics, an undercutting defeater for P might for instance be a counterexample for a lemma used in a (simil)-proof A of P. The main reason why convertibility is epistemically advantageous is that it allows one to turn counterexamples to propositions into counterexamples for lemmata used in proofs of these propositions. A simil-proof without this feature would not be as detailed as it should be, and would thus not fully lay down the relevant reasons for the proposition it aims to prove. The requirements that convertibility imposes on simil-proofs are a high level of relevant detail and the barring of certain other sources of evidence that cannot provide genuinely mathematical justification, such as testimony and statistical correlations, and perhaps even forms of inductive evidence and abductive reasoning. If Easwaran's observations are correct, convertibility is not an additional normative requirement on simil-proofs, but rather a common and desired feature of simil-proofs in mathematical practice. To put it into Lakatosian terms, convertibility is the feature of simil-proofs that enables lemma incorporation, that is, that feature which allows one to turn global counterexamples into local ones, which in turn can be disarmed by modifying lemmata used in the proof.[6]

Before concluding, some clarifying remarks are in order.[7] The first is that the preceding paragraphs are not meant to tie too close a connection between open texture and fallibility. The fallibility of human agents is, as far as I can tell, not directly connected to the open texture of terms in their language. All I mean to say is that some of the virtues of (simil-)proofs that mitigate human fallibility also serve to mitigate open texture. They mitigate it in the

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[5]This terminology stems from Pollock's work 1970.

[6]Thanks to a reviewer of this journal for encouraging me to be explicit about the connection of Easwaran's terminology to the that of Lakatos. For more on this connection, the interested reader is directed to Easwaran's paper, which is very explicitly motivated by Lakatosian concerns.

[7]Thanks to a reviewer for alerting me to the fact that pointing out that fallibility threatens the reliability of proof is not enough, since that appeal alone would leave me unable to explain what the problem with the other examples is.

sense that they make adapting proofs easier once some open-textured concept is sharpened, extended, or perhaps revised. The second remark has to do with the rarity of actual conceptual revisions caused by new cases for open-textured concepts in contemporary mathematics. In practice, mathematicians seem to worry less about open texture than about other threats to the reliability of proofs. Proofs of extreme length, such as the proof of the Feit-Thompson theorem, proofs that are the product of large-scale group collaborations, such as the classification of finite simple groups, or (purported) proofs that make use of esoteric techniques available only to small circles of insiders all pose threats to the reliability of proof (see Habgood-Coote & Tanswell, 2021, for a discussion of the first two examples). This is because their validity is excessively hard to check for a single agent, or even a community of agents. For these kinds of threats, it is much less clear whether and to what extent they are mitigated by the virtues Easwaran and De Toffoli emphasise. In this, open texture and these other threats differ,

But there is another reason why open texture does not seem like an actual threat to contemporary mathematical practice, and that is the adoption of set theory as a foundation for mathematics. Maddy (2019) argues that one of the functions of set theory as a foundation for mathematics is to provide a *generous arena* for mathematics: Set theory is expressive enough to provide representatives for all kinds of mathematical objects, and adopting set theory as a foundation seems to at least partially tame open texture. Set theory provides a shared language, thus lessening rational disagreement over the meanings of terms and leaving open texture as vagueness (OTV) less room. It restricts additional descriptive detail to that expressible in the language of set theory, thus restraining open texture as the defeasibility of concept ascription (OTD). And finally, it restricts the space of new candidate objects for a given concept to those to be found in the set-theoretic universe(s), thus giving open texture as the possibility of new cases (OTN) less room to work with.

As good as this may sound, there is of course room to object: Maddy notes that it is an empirical fact that all mathematical objects of interest can be represented using set theory, and there are some who take the developments in homotopy theory to demonstrate that this will soon no longer be the case (Marquis, 2013). And then there is the question of whether mathematicians have actually adopted set theory as a foundation, or whether that is merely the mathematical community's official stance on these matters. And last but not least, there is the texture of the concept set itself to worry about. The latter two of these issues will be taken back up in section 4.

To sum up, the presence of open texture makes some mathematical justification defeasible and thus threatens the traditional role of proof as pristine guarantee of certainty. But this impossible standard is rendered untenable by the fallibility of human agents independently of open texture. Moreover, some of the features of (simil-)proofs that mitigate fallibility also serve to lessen the impact of open-textured concepts. This is not necessarily the case for all threats to the reliability of proofs. Finally, and this will be examined further in section

4, the adoption of set-theoretic rigor can explain the fact that open-textured concepts seem to pose little threat to the reliability of proof for contemporary mathematics. There is, however, no guarantee that set theory will be able to play this role forever.

3.2 The extent of open texture in mathematics

In this subsection, the extent of open texture in mathematics is investigated. That is, it is investigated whether and which mathematical concepts are open-textured, and if so, to what extent. For the sake of clarity, the different definitions of open texture will be discussed successively.

3.2.1 Extent of open texture as vagueness (OTV)

In a sense, OTV is the least demanding definition of open texture. If one adopts OTV, many mathematical concepts used in informal mathematical practice will come out as having open texture, such as concepts like number, space, or function. For all of these concepts, it was at some point in time the case that rational, competent mathematicians debated over whether these concepts should be ascribed to certain entities. Debates about whether negative or complex numbers should be called numbers were once fierce (Wagner, 2017, ch.2), but nowadays, not much mathematical significance is attached to something being called a number. In the case of sets, these debates are still ongoing, in their latest incarnation as a debate between those who think that the set-concept is univocal, i.e. that there is one determinate universe of sets, such as Steel (Bagaria & Ternullo, 2020), and those who think that there are many universes of sets, such as Hamkins (2012) and Priest (2021). Wrapping this discussion up, Shapiro's first definition of open texture as something like the possibility for reasonable disagreement about concept ascription seems to be quite easily satisfiable even for mathematical concepts.

3.2.2 Extent of open texture as defeasibility (OTD)

On OTD it is less clear in what sense mathematical concepts have open texture. Recall that according to OTD, a concept has open texture if ascriptions of that concept on the basis of a description can be defeated by adding further detail to the description. As mentioned already, Waismann thought that mathematical objects, in contrast to physical ones, could be described completely. For instance, describing a triangle in Euclidean space can be done completely by giving its three vertices. But of course one would have to then go on to describe Euclidean space, and so on, until one either hits descriptive bedrock, whatever that may be, or finds some other way out of this regress. This idea will be investigated in section 4.

Perhaps the idea that mathematical objects can be described completely can be sustained with a deflationary attitude towards mathematical objects: A mathematical object is just a singular term in some formal framework, i.e. a constant symbol that can be defined in that framework. Put in Carnapian

terms, the description of a mathematical object is an internal affair, possible only within some formal language. One problem with this is that in mathematical practice, one usually doesn't work strictly within a formal language, but rather in natural language augmented symbolically. Moreover, most mathematical objects of interest have a long and complex history, and don't seem to be exhausted by mere description in some formal framework.[8] There are thus some large issues here, but working out what a complete description might be in a formal framework has some interesting consequences nonetheless.[9]

Restrict attention to some signature Σ in classical logic (first- or second-order). Let $a_1, \ldots a_n$ be constant symbols in Σ, and let $\phi(x_1, \ldots, x_n)$ be an open n-ary formula in the language $\mathcal{L}_\Sigma$ based on Σ. Intuitively, ϕ completely describes $a_1, \ldots, a_n$ if asserting ϕ of $a_1, \ldots, a_n$ determines the truth value of all other n-ary formulas $\psi(x_1, \ldots, x_n)$ in $\mathcal{L}_\Sigma$. Here, ψ is of course a formula built up from the signature Σ, and the following definitions are thus sensitive to the signature chosen. Depending on the this, one might want to restrict the class of formulas which are determined by ϕ. One might, for instance, not want to insist on all set-theoretic formulas being settled by the axioms of Peano arithmetic.

With this setup, one way to explicate this idea syntactically is as follows, assuming that $\vdash$ is the classical syntactically determined consequence relation on $\mathcal{L}_\Sigma$:

Definition 4 (Complete Description) A formula $\phi(x_1, \ldots, x_n)$ is a complete description iff for all formulas $\psi(x_1, \ldots, x_n)$ in $\mathcal{L}_\Sigma$ either $\phi(x_1, \ldots, x_n) \vdash \psi(x_1, \ldots, x_n)$ or $\phi(x_1, \ldots, x_n) \vdash \neg\psi(x_1, \ldots, x_n)$ holds.

If one thinks of ϕ as codifying a theory about some concept, one way to parse this is as saying that said theory is complete. Supposing for a moment that concepts are ascribed on the basis of single descriptions, this tells us that a concept is closed-textured if the corresponding description is complete. Many important theories, such as Peano arithmetic or ZFC are not complete in this sense, no matter whether one formulates them in first- or second-order logic. Formulating them in first-order logic does not yield a complete theory because of non-standard models, and the second-order treatment likewise fails to deliver completeness because of Gödel's incompleteness theorems.[10] Of course, there are interesting complete theories, so this is notion is not vacuous.

One might think that the definition above is somewhat too demanding, in that it requires a complete description to provably determine every other formula. In response, one could weaken this requirement in two ways, all the while sticking to second-order logic. One could demand semantic completeness: Let $\models$ denote the semantic consequence relation of classical second-order

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[8]See for instance chapter 1.2 of Grosholz (2016) for a historical discussion of <u>circle</u>.

[9]The following makes no claims whatsoever that this is what Waismann had in mind.

[10]Thanks to a reviewer for reminding me that the categoricity of second-order Peano arithmetic does of course not entail its completeness in the sense needed here.

logic, defined in the usual way as model-theoretic truth preservation. Then the corresponding definition of complete description would be:

Definition 5 (Complete Description, semantic) A formula $\phi(x_1,\ldots,x_n)$ is a complete description iff for all formulas $\psi(x_1,\ldots,x_n)$ in $\mathcal{L}_\Sigma$ either $\phi(x_1,\ldots,x_n) \models \psi(x_1,\ldots,x_n)$ or $\phi(x_1,\ldots,x_n) \models \neg\psi(x_1,\ldots,x_n)$ holds.

Now, in this sense, second-order Peano arithmetic for instance is a complete description, because it is categorical, that is, it has only one model, up to isomorphism. Second-order ZFC is only quasi-categorical, fixing a unique model only up to height and width of the cumulative hierarchy (see Incurvati, 2016, for a critical discussion). But once one has fixed these, one obtains a categorical description of the universe of sets, and ZFC thus becomes a complete description in the semantic sense. Now, to think that one can somehow grasp these descriptions and use them to ascribe concepts and other properties, one needs an antecedent grasp on the semantic consequence relation of second-order logic, and there are of course notorious philosophical difficulties with this (see e.g., Shapiro, 1991).

Partly because of these difficulties, and building on the work of others, Button and Walsh develop the idea of internal categoricity, which doesn't make use of semantical notions. Roughly, the idea is to mimic the definition of categoricity in the object language of second-order logic without semantic ascent to the set-theoretic metalanguage. Internally categorical theories are intolerant in a specific sense. Intolerance could be taken to provide another sharpening of complete describability:

Definition 6 (Complete Description, intolerant) A formula $\phi(x_1,\ldots,x_n)$ is a complete description iff the following holds for all formulas $\psi(x_1,\ldots,x_n)$ in $\mathcal{L}_\Sigma$:
$$\vdash \forall x_1,\ldots,x_n(\phi(x_1,\ldots,x_n) \to \psi(x_1,\ldots,x_n))$$
$$\vee \forall x_1,\ldots,x_n(\phi(x_1,\ldots,x_n) \to \neg\psi(x_1,\ldots,x_n))$$

This definition again gives no reason to think that all mathematical objects can be described completely, but it is noteworthy that a version of second-order Peano arithmetic and a version of second-order set theory are complete descriptions in something like this sense (ch. 10-11, Button & Walsh, 2018, see especially Theorem 10.3).[11] This fact seems to provide some reason to think that the natural numbers and the universe of sets (as described by ZFC) are not subject to open texture in the sense of OTD. There are two major caveats here: The first is that the definition above is relativised to some signature. To fully capture the idea that the provided description is complete, one would have to strengthen the above definition so as to let ϕ retain determinacy over perhaps all extensions of $\mathcal{L}_\Sigma$. The second caveat is that this cannot truly be

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[11]The determinacy of arithmetic more generally is a hotly debated topic (e.g., Goodsell, 2021; Warren & Waxman, 2020).

used to capture what it means for some concept to be open-textured in the sense of OTD, because the indefeasibility of consequence is in any case a structural feature of classical logic. Moreover, to show that important mathematical concepts are completely describable in any of the senses above, it seems like one would have to rely on categoricity theorems, and almost all philosophical applications of such theorems are intensely disputed (Maddy & Väänänen, 2022). But more importantly, these approaches to dealing with open texture fall under the broader category of approaches using rigor to deal with open texture, as they rely on capturing the concept in first- or second-order logic, be it syntactically or semantically. The goal of the last section will be to investigate whether and to what extent such approaches can be successful, and at what cost.

These negative considerations notwithstanding, it is certainly the case that in the context of proof, mathematicians have a certain amount of control over properties of the objects they are considering, in the sense that they are free to chose axioms from which to start, and insofar as these axioms describe objects, they have control over these descriptions. If a mathematician begins a proof with something like "Let G be a group", then nothing in the proof defeats this description of G in and of itself. Of course, the mathematician is likewise free to change their mind about what object they want to consider, but that is not a consequence of adding descriptive detail.

3.2.3 Extent of open texture as the possibility of new cases (OTN)

Turning to the second definition of open texture as the possibility of new, unsettled cases arising, we see that this definition is somewhat more demanding: For a concept to display open texture in this sense, it always has to be possible for there to be genuinely new cases. Since unsettledness presumably implies the possibility for rational disagreement, OTN is stricter than OTV. However, very similar examples demonstrate that there are mathematical concepts that have open texture according to OTN, such as the concept of number, space, or perhaps algebraic object. But again, these kinds of examples don't seem particularly worrisome. According to Shapiro and Roberts, there are also clear cases of concepts with closed texture, such as prime natural number or Euclidean triangle.

Vecht (2020) offers a precise delimitation of open texture under OTN. He argues that concepts with closed texture correspond to algebraic theories. For example, the theory of groups corresponds to a closed concept: Something is a group if and only if it satisfies the axioms of group theory, leaving room neither for reasonable disagreement nor unsettled cases arising. Implicit here is the thought that it can be settled once and for all what could count as in principle satisfying the axioms of group theory. No matter his exact argument[12], I think

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[12]Roughly, the idea is as follows: Non-algebraic (assertory) theories make existence claims, and once the existence of some objects is asserted, Waismann's essential incompleteness kicks in, yielding open texture. I don't follow Vecht's reasoning that algebraic theories make no existence claims, especially because he takes Peano arithmetic to be an algebraic theory, and in any case

that this assumption is unwarranted, see section 4. But in spirit his approach is similar to a suggestion by Roberts and Shapiro, who argue that rigorously defined concepts are not subject to open texture (Shapiro & Roberts, 2021, p.121).

To sum the discussion of this section up, some mathematical concepts clearly have open texture on OTV and OTN. On OTD, matters are somewhat more complicated, but here we have at least disambiguated matters somewhat and discussed in what sense mathematical objects in a formal framework can be completely described. There is one large issue left open by this section, namely whether rigor, as Shapiro and Roberts suggest, can actually banish open texture, and it is to this that we will turn to next.

4 Open texture and rigor

In a recent study, Burgess (2015) characterises mathematical rigor as requiring that every proof of a new proposition uses only that which has already been proven, and that every newly defined term be defined using previously explained or defined terms. Now, on pain of infinite regress, rigor also requires axioms, serving as the inferential bedrock for deductions of new propositions, and primitives, supplying the basic vocabulary in terms of which new terms can be introduced. At this level of grain, this description is incomplete, leaving open for instance the notion of logical consequence employed in deduction.[13]

This section's object will be to see to what extent the use of axiomatisation, formal languages and formal theories can close off the texture of mathematical concepts. Roughly, the idea that Shapiro and Roberts sketch is that the primitives of formal theories can be thought of as being implicitly defined in a Hilbertian sense, whence they are closed. Therefore, derivatively defined terms in these theories are closed too.[14] Note that this strategy seems prima facie promising for terms like "prime natural number" or "Euclidean triangle", but less so for terms designating structures, like "group" or "ring". The latter are usually defined not as corresponding to the extensions of terms of formal theories, but rather to correspond to whole models of theories, like group or ring theory, and we will see how that actually pans out in the following.

But before that, an argument due to Tanswell (2018) to the effect that for instance set-theoretic foundations do not protect mathematical concepts from open texture deserves to be considered. Briefly, Tanswell appeals to Haslanger's

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think that the distinction between algebraic and assertory/univocal theories should be understood in terms of attitudes towards theories: Univocal theories have an intended model.

[13]Burgess takes the underlying axioms to be those of ZFC and the underlying logic to be classical first-order logic. Burgess' picture is clearly wedded to the traditional picture of proof discussed in the last section, but for a critique of Burgess account in particular see Brown (2021). A more general critique of the epistemological benefits of inferential rigor can be found in Paseau (2016). It should be noted, however, that Paseau focuses on rigor in arguments and proofs, while the main concern in this section is rigor in conceptualisation.

[14]I think that if one were to systematically investigate how open texture propagates using normal forms for definitions, one would get something like a strong Kleene logic of indeterminacy(Belnap, 1993; Kleene, 1952).

manifest/operative distinction to argue that while mathematicians often profess to work in set theory, it does not seem to be the case that the actual operative concepts used by mathematicians in all fields of mathematics are those of set theory (Haslanger, 2012). On the one hand, it is not plausible that all mathematical concepts are actually set-theoretical concepts for familiar Benacerrafian reasons. On the other, there are areas of mathematics, such as low-dimensional topology, where set theory seems to play almost no role, even in published proofs (De Toffoli & Giardino, 2015).[15] Tanswell makes a convincing case, but does not directly touch on the central question here, as the question at hand is whether rigor can in principle be used to close off texture, not whether that currently is the case in mathematical practice.

As is customary[16], the Frege-Hilbert debate on the axiomatic method provides the background for this discussion of implicit definition (Frege, 1976). According to Frege, Hilbert's notion of implicit definition could not serve to define the meanings of the primitives of a theory. He instead thought that the basic terms can only be elucidated or explained. If one follows Frege in that, the strategy of getting rid of open texture in the primitives used via open texture is of course a non-starter. One hope for evading open texture now would be direct contact with perfectly sharp mathematical primitives or universals, maybe through something like Gödelian intuition or Russellian acquaintance. This situation can be fruitfully compared to reference to natural kind terms via reference magnetism. Roughly, Lewis' reference magnetism holds that some things (water, gold, etc.) are more eligible for being referents than others, aiding reference to them (Lewis, 1984). So maybe one way out of this predicament would be to hold that some mathematical terms, like "natural number" or "point" refer magnetically.[17] I don't regard this avenue as particularly promising, and even if it were to succeed, it would not help with answering the question at hand, since open texture would then not be closed off by rigor.

Given that the Fregean response seems to require some heavy-duty philosophical commitments, it is natural to turn towards a Hilbertian view: The primitive terms of some theory T in classical logic are implicitly defined by the theory. At this point, the recent paper by (Giovannini & Schiemer, 2021) is helpful. They develop two explications of open texture, and thus two possible ways in which implicit definition might close off texture. The first way goes as follows: Adopting a referential theory of meaning, the theory T fixes the meaning of its primitives by defining the class of models making T true. In contemporary mathematics, these models are usually understood to be set-theoretic structures, that is, sets with relations on them. But if the concept of set has open texture, the concept of set-theoretic structure has open texture too. Therefore, to banish open texture, one would have to specify a specific set

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[15]In recent work, McLarty (2018) observes that even the set theory used in well-known textbooks on point-set topology is underdetermined in the sense that it is compatible with various formal explications, among them both ZFC and categorical set theories.

[16]Both Shapiro and Roberts and Giovannini and Schiemer discuss the Frege-Hilbert debate extensively.

[17]See Marquis (2013) for a classification of mathematical concepts into natural-kind-like concepts and artefactual tools.

theory, say ZFC, which is then taken to provide the set-theoretic structures needed for interpreting T. But the same strategy, on pain of infinite regress, will not work for ZFC itself, unless one thinks that ZFC can somehow provide its own models. Ignoring this problem for a moment, this approach is well-suited to capturing structural concepts like group or ring, and a bit less well-suited to capturing primitives, like unit of a ring: The primitives of the theory T can be defined only intensionally, that is, by their extension in any given model of T.

With this approach, the open texture of the concept T is intended to capture gets pushed down a level, to the ambient set theory used. Even if some other way to close off the texture of the concept of set were to be found, the obvious worry is that T now just captures a different concept than the one it was originally intended to capture. This worry might already arise when T is formulated formally in some logic with a given vocabulary, outside of natural language. While rigor seems to provide a way to produce concepts (in the sense of terms whose proper use is described by the ambient logic and/or set theory) that are closed, one might object that these closed concepts are not the same as the ones captured by natural language. This is a serious concern and essentially a version of the continuity objection to conceptual engineering (Prinzing, 2018). In the mathematical context, one worry would be that there is not one version of rigor, and as standards of rigor change, the logic or set theory used might change. Moreover, one might want to interpret the theory T using bearers of meaning other than set, as is done in categorical logic. In its most acute form, the worry here would be that rigor can close off concepts only at the cost of hobbling them.

It is best to discuss the other proposal for explicating implicit definition before responding to this worry. The second precisification of implicit definitions proceeds by adopting a use-based, inferentialist theory of meaning, according to which the meaning of some term is given by the inferential potential of sentences it occurs in, that is, by what follows from them and from what they can be deduced. A theory T in some signature then implicitly defines its primitives by settling the deductive consequences of sentences in which they occur. This approach seems to be well-suited to defining primitives. The meaning of "unit" in group theory for instance is then exhausted by the inferential behaviour of the sentences containing "unit", such as "Every group has a unit" or "Units are neutral with respect to multiplication". Moreover, the approach does not suffer from the same dependence on a set theory, but is instead heavily dependent on the chosen logic and vocabulary. In part because of this, it seems less well-suited towards capturing structural concepts.

If one for instance defines the natural numbers using Peano arithmetic, it makes a difference whether the chosen logic is first- or second-order. The latter makes Peano arithmetic determinate, the former doesn't. Moreover, the deductive consequences of the theory will depend on the chosen vocabulary. Now, this doesn't really seem to pose a problem for the definition of the theory's primitives, as one might say in good structural fashion that for instance

the individual natural numbers are exhausted by their position in the structure of the natural numbers. However, such restriction to a given vocabulary again seem to hobble the concept of the natural number sequence as a whole: In mathematical practice, the natural numbers can be used to index and count all kinds of things, and induction can be applied to mathematical statements irrespective of non-arithmetic vocabulary they contain.[18] Moreover, it might again be that the logic used by the mathematical community changes. A possibly larger worry for the definition of open texture OTN is that on this purely inferential framework, it is not quite clear in what sense objects fall under the involved predicates, although one might again adopt a deflationary perspective towards this. But even then, the worry that the pre-theoretic concept is just replaced by this process looms large.

Two possible responses are available. One appeals to (enhanced) If-Thenism or versions of Priest's mathematical pluralism, arguing that this replacement is just how mathematics proceeds, teasing out logical consequences in some freely chosen but mathematically motivated logic from freely chosen but mathematically motivated axioms (Maddy, 2022; Priest, 2021). The other appeals to the patchwork nature of concepts. In general, concepts cannot be captured by simple definitions, and a concept is therefore not necessarily replaced by producing a formal theory and relativising it to some logic or set theory. In investigating such a formalised theory, one just moves to a different patch of the concept and draws consequences for that patch. Some of these might be transferable to other patches, and some not. For instance, adequacy theorems for first-order classical logic show that it doesn't matter whether one investigates ZFC using the syntactic or the semantic consequence relation. In contrast, the move to paraconsistent or intutionistic logic might make some investigations harder and some easier, and transfer between patches is not guaranteed, but perhaps still fruitful. Obviously these are mere gestures, but I think taking the recent work by Wilson on patchwork concepts seriously might pay serious dividends for discussions about mathematical concepts (Wilson, 2008). But that will have to be left for further work. Overall, the discussion here is inconclusive: Both approaches have their problems, but especially approach number two produces concepts with closed texture, whose relationship with the original concepts is unclear.

5 Conclusion

This paper compared and contrasted several definition of open texture. On the definition favoured by Roberts, Shapiro, and Tanswell, which focuses on the possibility of unsettled new cases, many mathematical concepts turn out to have open texture. Adopting Horty's definition, perhaps closest to Waismann's thoughts, makes matters less clear, because it comes with an account of concept ascription based on description that is difficult to apply to mathematical

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[18]Such considerations, i.e. the open-endedness of induction, play an important role in some strategies for proving the categoricity of the natural numbers (McGee, 1997; Warren, 2020, ch.10).

objects. No matter which definition one adopts, the traditional role of proof is threatened, but it is also threatened by the fallibility of human agents. Some of the virtues mitigating this fallibility also serve to mitigate the impact of open texture. Moreover, the great de facto reliability of mathematical proof in contemporary mathematics in spite of open texture can in part be explained by the adoption of set-theoretic rigor. The last section of the paper investigated how and in what sense rigor can close off open texture. The results here were ambiguous: Rigor seems to produce concepts with closed texture, but it is not clear that these are as flexible as the natural language concepts that mathematicians actually use. But perhaps they don't have to be, because the role of rigor in mathematics is arguably not primarily to aid in discovery, but rather to ensure consensus when justificatory problems arise.

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