From Philosophical Traditions to Scientific Developments: Reconsidering the Response to Brouwer's Intuitionism

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Abstract

Brouwer's intuitionistic program was an intriguing attempt to reform the foundations of mathematics that eventually did not prevail. The current paper offers a new perspective on the scientific community's lack of reception to Brouwer's intuitionism by considering it in light of Michael Friedman's model of parallel transitions in philosophy and science, specifically focusing on Friedman's story of Einstein's theory of relativity. Such a juxtaposition raises onto the surface the differences between Brouwer's and Einstein's stories and suggests that contrary to Einstein's story, the philosophical roots of Brouwer's intuitionism cannot be traced to any previously established philosophical traditions. The paper concludes by showing how the intuitionistic inclinations of Hermann Weyl and Abraham Fraenkel serve as telling cases of how individuals are involved in setting in motion, adopting, and resisting framework transitions during periods of disagreement within a discipline.

1. Introduction

In the Kant lectures series that took place at Stanford University in 1999, philosopher Michael Friedman introduced an elaborated account of the pivotal role of philosophy in rendering scientific revolutions rational (later matured into his book *Dynamics of Reason*). Friedman's central argument is that we should consider parallel developments in scientific philosophy in order to provide a philosophical foundation for replacing an existing framework with a radical, new one. According to Friedman, philosophy's role is to provide a source of new ideas in order to stimulate the creation of new frameworks or paradigms.

Throughout the book, Friedman presents several examples of scientific developments that were conceivable due to prior philosophical advances, at least to some extent. One of them, the one I will focus on in this paper, is Einstein's theory of relativity. However, Friedman nowhere engages failed attempts at revolutionizing a science. The current paper sets forth to put such an attempt, namely, Brouwer's intuitionistic program, under the lens of Friedman's model. The aim of this paper is to better understand, by means of Friedman's theory, why Brouwer's intuitionism eventually did not prevail.

Brouwer was familiar with Kant's, Poincaré's and the semi-intuitionists' views on mathematics and mathematical entities. Building on Friedman's model, this paper examines how such philosophical developments may have provided Brouwer with an incentive to develop his intuitionistic theory, and to what extent, if at all, they can be considered as philosophical traditions from which Brouwer's intuitionism may have evolved.

During the 1920s, Brouwer's intuitionism played a significant role in the debate about the foundations of mathematics (Hesseling 2003). A decade later, in the 1930s, the debate started to fade, alongside the discussion about intuitionism. Despite being abandoned by the mathematical community, intuitionism continued to be practiced and developed by several mathematicians who were deeply influenced by Brouwer's ideas. Some of these developments are still being discussed (van Atten, Boldini, Bourdeau & Heinzmann 2008; Posy 2020). Therefore, a related question that will be analyzed within the scope of this paper is how Brouwer's intuitionistic program was appealing enough to several well-known mathematicians, such as Abraham Fraenkel and Hermann Weyl, to consider it a viable option to replace classical mathematics.

2. Friedman's path to scientific philosophy

Friedman's theory of scientific philosophy builds upon the accounts of Kant, Schlick, Reichenbach, Carnap, and Kuhn and their positions regarding the role of philosophy in scientific developments. Friedman argues that, taken alone, none of their views fully captures the way prior philosophical work can set the stage for a scientific framework transition. To understand the underlying meaning of Friedman's notion of scientific philosophy, and in particular what he adopts and where he departs from each, let us briefly examine these accounts through Friedman's perspective.

2.1 From Kant to Kuhn and beyond

The status of philosophy, as Friedman describes Kant's viewpoint, is entirely different from all empirical sciences. Empirical sciences, such as psychology or mathematics, as well as the elements of pure a priori knowledge, such as geometry, are first-level sciences. Philosophy is a second-level discipline that enables us to know and have representations of these first-level objects. To use Friedman's words, Kant viewed philosophy as a "transcendental inquiry into the conditions of possibility of our first-level knowledge of objects in space and time (the only genuine objects of knowledge there now are) supplied by mathematical natural science." (Friedman 2001, 9)

Friedman claims that Newton's physics is a telling example of the conceptual problem Kant was trying to deal with. Newton's physics was considered a successful theory both from empirical and mathematical perspectives, but Kant was concerned with how such a theory made rational sense. As Friedman describes it, Kant's answer was that concepts such as space, time, motion, action, and force are not mere abstractions from our experience portraying a metaphysical realm that exists behind the phenomena, but rather they are constitutive givens of human comprehension (Friedman 2001, 11). According to Friedman, this is the only way Kant sees possible for one to rationally explain how such pragmatic empirical success is actually possible in the first place. Thus, for Kant, what rendered Newtonian physics rational was that it be shown to rest on necessary, a-priori mind-given truths (Friedman 2001, 26).

However, the development of non-Euclidean geometries posed a significant challenge for Kant's view of a priori knowledge. If there is a possibility that space can be non-Euclidean, and we are able to conceive this option in our minds, then Kant's view of Euclidean geometry as a-priori mind-given truth that is built into our mental capacities no longer holds. In light of the emerging difficulty with Kant's concept of a priori knowledge, Friedman rejects the idea of scientific frameworks as "fixed and unrevisable a priori principles" (Friedman 2010, 30) in favor of "a relativized and dynamical conception of a priori mathematical-physical principles, which change and develop along with the development of the mathematical and physical sciences themselves" (Friedman 2001, 31) which he finds in the works of Schlick, Reichenbach, Carnap, and of course Kuhn.

Schlick and Reichenbach are credited by Friedman for continuing the Kantian project of a priori knowledge without forcing it to be unrevisable or fixed for all time. Schlick is described by Friedman as the "very first professional scientific philosopher" (Friedman 2001, 12). According to Friedman, Schlick thought that philosophy underlies every scientific problem, and philosophical theories should be considered general theories of science, deeply interconnected with the scientific disciplines themselves (Friedman 2001, 13). Thus, philosophy for Schlick is concerned with the changing foundations of every scientific discipline.

Friedman describes Reichenbach's *The Theory of Relativity and A Priori Knowledge* as the "clearest articulation" of the logical empiricists' new view of a priori principles. Reichenbach differentiates between two meanings of the Kantian a priori: fixed, on the one hand, and "constitutive of the concept of the object of [scientific] knowledge" on the other (Friedman 2001, 30). The theory of relativity favors the latter meaning, so claims Reichenbach, as it involves a priori constitutive principles as necessary presuppositions, but those principles are not necessarily fixed: they changed during the transition from Newtonian physics to general relativity (Friedman 2001, 30–31).

It is in Carnap's *Logical Syntax of Language* (Carnap 1934) that Friedman finds a compelling expression of the logical empiricists' new approach. Logical rules, according to Carnap, are constitutive of the concepts of "validity" and "correctness", thereby relative to the linguistic framework in which they operate. Once the framework is chosen, its logical rules are a priori rather than empirical. (Friedman 2001, 31). In the late 1920s, Carnap was immersed in Hilbert's program to establish a logical discipline named "meta-mathematics", and claimed that Hilbert's approach should be extended from logic to the whole discipline of philosophy (Carnap 1934). "Meta-logical investigations" of logical structures and relations in the language of science represent the new role of scientific philosophy, and philosophy should be regarded as a branch of mathematical logic (Friedman 2001, 16). According to Friedman, Carnap conceived philosophy as a meta-science, just like Kant did; but unlike Kant, Carnap considered philosophy to be "a branch or part of science as well - this time a branch of formal or a priori (as opposed to empirical) science" (Friedman 2001, 17).

As promising as Carnap's picture of dynamic yet constitutive principles of knowledge appeared, Friedman points to a lacuna in Carnap's account. Carnap argued that a framework, as a whole, cannot be judged as true or false but only as being fruitful or conducive to its purpose since a higher-level framework does not exist. In Friedman's view, Carnap completely overlooked "the relativistic predicament arising in the wake of Kuhn's work on scientific revolutions" (Friedman 2001, 57). Friedman criticizes Carnap for his lack of engagement with Kuhn's theory despite being the one who commissioned it, and probably one of the first to read it, which distances Carnap's theory from accounting for Kant's original questions regarding the possibility of mathematics and the natural sciences.

Next, Friedman turns to Kuhn and calls him to task on his theory of paradigm shifts in science, as it fails to show how a framework transition can be rational (Friedman 2001, 99). The difference between instrumental rationality and communicative rationality, a distinction that Friedman borrows from Habermas, plays a significant role in his criticism of Kuhn's universal rationality (Friedman 2001, 53–56). Instrumental rationality refers to our capacity to reason when the ultimate goal is to pursue one's own subjective point of view, whereas communicative rationality refers to our ability to engage in argumentative deliberation in order to reach "an agreement or consensus of opinion" (Friedman 2001, 54). According to Friedman, "there can be no ground for a truly universal rationality within purely instrumental reason" (Friedman 2001, 55), as the latter is personal and involves diverse goals that are subjective, thus differ from one individual to another. Since Kuhn fails to clearly distinguish between these two aspects of human rationality, his attempt to "find permanent criteria or values held constant throughout the development of science necessarily fails" (Friedman 2001, 55). It is communicative rationality that holds the key to explaining the rationale behind scientific framework replacement[1].

The theory Friedman proposes rests upon the notion of communal rational consent, transmitted from an old framework to a new one through a "higher-level meta-framework", whose function is to mediate between the two (Friedman 2001, 105). From Friedman's perspective, the constitutive principles of the new framework evolve from the old framework by the process of interframework mediation. Thus, we can view "the evolution of succeeding paradigms or frameworks as a convergent series, as it were, in which we successively refine our constitutive principles in the direction of ever greater generality and adequacy" (Friedman 2001, 63). Moreover, the new framework can always yield the old framework as a limiting case, thereby placing earlier frameworks alongside their successors on a continuum. Friedman explicitly claims that his philosophical model applies to science, and devotes the last chapter to briefly discuss three examples from biology, quantum mechanics, and chemistry, in his attempt to explain "how the present philosophical account bears on other cases of scientific revolutions standardly so-called" (Friedman 2001, 119).

Friedman maintains that in philosophy, there will never be a state of stable consensus on a common paradigm, but only a constantly shifting dialectic of thought between philosophical positions and schools (Friedman 2001, 21). However, it does not imply that developments in science should be viewed as disassociated from developments in philosophy, but quite the opposite. As Friedman writes:

> For, at moments of scientific revolution, the scientific transitions themselves (the transitions to a new paradigm) are actually quite inconceivable without the parallel developments in philosophy taking place at the same time, and, as it were, on a different level. (Friedman 2001, 22)

During periods of consensus within a discipline, practitioners operate according to the agreed upon framework that defines the norms, standards, and the rules of the game. In normal science, these norms and standards are not called into question, but in times of

[1] It should be pointed out that the critical role Friedman ascribes to communicative rationality in *Dynamics of Reason* is minimized in Friedman's later writings, as can be seen in his 300-pages-long response paper in Dickson & Domski's edited volume *Discourse on a New Method: Reinvigorating the Marriage of History and Philosophy of Science* (Dickson & Domski 2010).

revolution attempts, it is precisely such a formerly agreed upon framework that is being reconsidered (Friedman 2001, 22). During framework transitions, practitioners no longer operate within normal science since the questions they are dealing with address the framework itself, rather than the content it governs. Philosophy's role, according to Friedman, is to provide a source of new ideas in order to stimulate the creation of new frameworks or paradigms that will be able to settle the practitioners' concerns and replace the currently working frameworks, thus enable science to continue to progress through revolutions (Friedman 2001, 23).

Friedman's notion of scientific philosophy depicts philosophical arguments as entangled with and indispensable to revolutionary innovations in science. There is a special role Friedman assigns to philosophy as a meta-framework where questions about the normative boundaries of the scientific discipline can arise: questions that cannot arise from within the discipline itself, and that create the ground for new ideas to emerge and novel theories to develop. The exact dynamic relationship between scientific progress and philosophical ideas will be analyzed further in the next section, examining Friedman's example of the development of Einstein's relativity theory and the philosophical discussion that facilitated it.

2.3 Einstein, Helmholtz, and Poincaré: How a new conceptual framework becomes a viable alternative

The story of Einstein's general theory of relativity is brought forth by Friedman to demonstrate how parallel developments in scientific philosophy can provide a philosophical foundation for replacing an existing framework with a radically new one. According to Friedman (Friedman 2002, p. 193–95, 202–7), the philosophical debate on the foundations of geometry between Helmholtz and Poincaré provided Einstein with an incentive for considering the possibility that gravity may be represented by non-Euclidean geometry as a live option for his physics[2].

From a Newtonian perspective, Einstein's general theory of relativity is neither mathematically nor physically possible since the necessary developments in either discipline did not exist before the late nineteenth century. Even after the required mathematics was developed, it was only Einstein's own work on the "principle of equivalence" that made his new theory physically possible as well. How did Einstein come to think that an abstract mathematical structure, such as a four-dimensional semi-Riemannian manifold, can achieve objective physical meaning by gravity? Friedman (Friedman 2010, p. 499) claims that Einstein's theory "required a genuine expansion of our space of intellectual possibilities" and wonders how such an expansion can be rationally justified.

Friedman suggests that in order to understand the transition from Newton's theory to Einstein's theory, we must take into account the developments in contemporary scientific philosophy that were happening around the same period. Kant was the first to provide Newtonian physics with a philosophical foundation that was reconfigured over the years by

[2] There are contradicting views regarding the influence Poincaré and Helmholtz had on Einstein's theory of relativity. The Einstein expert and philosopher John D. Norton (Norton 2010; 2020) argues that there is little supporting historical evidence to Friedman's story and that Einstein himself points to Hume and Mach as his influences for special relativity.

“scientific philosophers” (to use Friedman’s phrase) such as Helmholtz and Poincaré. Einstein -- who was immersed in the debate on the relativity of space, time, and motion and was also familiar with the philosophical debate on the foundations of geometry between Helmholtz and Poincaré -- was able to connect the two separate lines of thought and create a new kind of geometry[3].

Even though Einstein’s theory was incommensurable with the Newtonian theory it replaced, it had firm roots in the same philosophical tradition from which the Newtonian framework evolved. The existence of those roots can explain how such a transition was rationally possible. Dating back to the seventeenth century, the philosophical discussions and disagreements between Descartes, Huygens, and Leibniz about concepts such as absolute and relative motion paved the path for the development of Newton’s ideas. Furthermore, these same problems and disagreements have contributed to the debate between Samuel Clarke and Leonhard Euler in the eighteenth century. In the nineteenth century, the debate continued to evolve both from a philosophical perspective (by Mach) and a scientific one, and this historical and philosophical background set the stage for the introduction of Einstein’s new theory.

According to Friedman, it was the way that Einstein had positioned his new theory along the historical trajectory within the tradition of scientific philosophy that rationally justified Einstein’s new framework. For a theory to be rationally justified in Friedman’s model, it does not mean that it has to be accepted, only that it has to be rationally possible, in the sense that a practitioner who is deeply engaged with a specific working framework might be able to consider the new suggested theory as a viable alternative to the one to which he is currently committed. As Friedman puts it:

> What Einstein did, in creating the new spatiotemporal coordination effected by the special theory of relativity, was to put this contribution into interaction with recently established empirical facts concerning the velocity of light in a striking and hitherto unexpected manner. Then, in creating the new spatiotemporal coordination effected by the general theory, Einstein, even more unexpectedly, put these two scientific accomplishments, together with the entire preceding philosophical debate on absolute versus relative motion, into interaction with a second already established empirical fact concerning the equality of gravitational and inertial mass. Since Einstein's introduction of a radically new conceptual framework was thus seriously engaged with both the established philosophical or meta-scientific tradition of reflection on absolute versus motion which had surrounded classical physics since its inception, and also with already established empirical and conceptual results at the scientific level, a classical physicist, on his own terms, had ample reasons seriously to consider Einstein's work. He did not, of course, need to adopt Einstein's new paradigm as correct, but he would have been irrational, unreasonable, and irresponsible (again on his own terms) to fail to consider it as a live alternative. (Friedman 2001, p. 108)

[3] Friedman speaks here about special relativity, as he emphasizes Einstein's "great innovation […] in the special theory of relativity", which Einstein then applies to "the concepts of time and simultaneity" (Friedman 2001, 23).

This paragraph raises a few questions regarding the process of how a new theory becomes a "live option", and how it is transformed from merely an alternative theory to a widely accepted theory. First, for a new framework to be scientifically viable, it must satisfy scientific standards. However, in cases of scientific revolutions, the newly suggested framework often pertains to standards radically different from those obtaining in the science, begging the question of how it can possibly be considered a viable alternative (Fisch 2017; Fisch and Benbaji 2011). Secondly, In Friedman's example, the "classical physicist" is willing to consider Einstein's work because of the connections Einstein made between his new theory and previous discussions in philosophy (and science). It implies that Friedman's "classical physicist" is familiar with philosophical discussions about his research interests and finds them relevant and important to his scientific line of work. However, one should take into account that not all practitioners are immersed in philosophical debates about their own disciplines, nor do all of them believe that philosophical roots provide solid reasons for considering new scientific ideas.

The story of Brouwer's intuitionism serves as a good example of the relation Friedman describes between science and philosophy, as Brouwer's intuitionistic program was primarily developed out of philosophical considerations. It is also an intriguing example of how a new mathematical theory was perceived as alien to practicing mathematicians. Even those who were willing to consider Brouwer's intuitionism as a possible alternative to classical mathematics, such as Hermann Weyl and Abraham Fraenkel, eventually found his new theory too restricting or too philosophical for practical everyday work (Iemhoff 2019; van Dalen 1995).

Brouwer's intuitionistic theory provided a way to avoid the paradoxes of set-theory by utterly changing the way mathematical entities were perceived. According to Brouwer, mathematical objects are not merely symbols and formulas written on a piece of paper, but creations of the human mind constructed through mental activity (Brouwer 1907; 1912; Dummett 1977; Troelstra and van Dalen 1988). Following this new conceptual framework, Brouwer's solution to the foundational problem required a massive reformation to the discipline of mathematics which obliged practitioners to renounce widely acceptable concepts and theories. I return to discuss the motivation and implications of Brouwer's approach in section 3.1.

Read through the prism of Friedman's model, and alongside Einstein's story, arises the question of the possibility of tracing the philosophical roots of Brouwer's intuitionism to any previously established philosophical traditions. If such a connection exists, how profound should it be? To properly address this question, in the following sections I examine the possible philosophical roots of Brouwer's intuitionism in light of Friedman's theory.

3. Which philosophical traditions may be ascribed to Brouwer's intuitionism?

3.1 Kant and the French semi-intuitionists

According to several Brouwer scholars and biographers, the philosophical origins of Brouwer's intuitionistic thinking are to be located in Kant's philosophy. Dirk van Dalen (1978, p. 298–99) refers to a correspondence between Brouwer and Korteweg, in 1906, as first-hand evidence that Brouwer had read Kant's *Critique of Pure Reason* and that it had shaped his approach to mathematics:

> Korteweg: Can one mention the name of Kant in a mathematical article?
> Brouwer: Yes, Russell and Couturat did so, and the subject forces me to do so.
> Korteweg: Did you study Kant sufficiently thoroughly to form an opinion?
> Brouwer: I cannot prove that I did, but I have read *Die Kritik der Reinen Vernunft* repeatedly and seriously studied the passages I need. (van Dalen 1978, p. 298–99).

Walter van Stigt (1990, p. 127) argued that Brouwer himself felt that at least part of his own philosophical viewpoint of the a-priori could be traced back to Kant's philosophy, pending some adjustments to Kant's concept of the a-priori. The a-priority of space and time is discussed in the second chapter of Brouwer's dissertation, where Brouwer explicitly admits that he accepts Kant's a-priority of time:

> Next the apriority; this can mean one of two things:
> 1. Existence independent of experience.
> 2. Necessary condition for the possibility of science.
>
> If the first alternative is meant, then it follows from its intuitive construction that all mathematics is a priori, e.g. the non-Euclidean geometry as much as the Euclidean, the metrical as much as the projective. Now let the second alternative be meant. Where scientific experience finds its origin in the application of intuitive mathematics to reality and, apart from experimental science, no other science exists barring only the properties of intuitive mathematics, we can call a priori only that one thing which is common to all mathematics and is on the other hand sufficient to build up all mathematics, namely the intuition of the many-oneness, the basic intuition of mathematics. And since in this intuition we become conscious of time as change per se, we can state: The only a priori element in science is time. (Brouwer 1907, p. 99)

Later in the same chapter, Brouwer (1907, p. 114) specifically addresses Kant's notion of space and states that his dissertation's principal goal is to "rectify Kant's point of view on apriority in the experience and bring it up to date". The footnotes throughout his dissertation indicate that Brouwer read Karl Kehrbach's edited version of Kant's *Critique of Pure Reason*, and his understanding of Kant's apriority of space is as follows:

> Kant defends the following thesis on space: The perception of an external world by means of a three-dimensional Euclidean space is an invariable attribute of the human intellect; another perception of an external world for the same human beings is a contradictory assumption. (Brouwer 1907, p. 114)

According to Brouwer's comprehension of Kant, Kant proves his argument by decomposing it into two separate assumptions:

1. We obtain no external experiences barring those placed in empirical space, and cannot imagine those experiences apart from empirical space.
2. For empirical space the three-dimensional Euclidean geometry is valid. (Brouwer 1907, p. 114)

Therefore, it follows that “the three-dimensional Euclidean geometry is a necessary condition for all external experiences and the only possible receptacle for the conception of an external world so that the properties of Euclidean geometry must be called synthetic judgments a priori for all external experience” (Brouwer 1907, p. 114). However, claims Brouwer, it can immediately be objected that “we obtain our experiences apart from all mathematics, hence apart from any space conception; mathematical classifications of groups of experiences, hence also the creation of a space conception, are free actions of the intellect, and we can arbitrarily refer our experiences to this catalogization, or undergo them unmathematically” (Brouwer 1907, p. 115). Since Brouwer’s objection maintains that it is possible to imagine external experiences apart from our conception of space and that the creation of the image of space is a free act of the intellect, space cannot be part of the a priori as Brouwer sees fit.

Such a reading raises the question of whether Brouwer’s notion of “intuition of time” is exactly the same as Kant’s. Mark van Atten (2004, p. 2–3) argues that Brouwer and Kant view the function of intuition of time in mathematics in the same manner, but the way this intuition comes about is different. He writes:

> The constructions that Brouwer has in mind are based on the intuition of time, as in Kant. Mathematics deals with purely formal objects; like Kant, Brouwer held that time is required for thinking of any object whatsoever. It then becomes plausible to base mathematics on time. Unlike Kant, Brouwer made a sharp distinction between subjective time (inner time consciousness) and objective or scientific time (time as it figures in physics and that you can see on a clock). Mathematics according to Brouwer is based on subjective time. […] what should be said now is that, in spite of certain differences between their conceptions of the intuition of time, the function of this intuition with respect to mathematics is the same for Kant and Brouwer. It is the fundamental given out of which all the rest is developed. (van Atten 2004, p. 2–3)

It should be pointed out that the purpose of this paper is *not* to account for the accuracy of Brouwer’s understanding of Kant. Brouwer read Kant the way he read it, and he may have totally misunderstood Kant, yet he still considered Kant’s apriority of time as his inspiration for developing his intuitionistic program. To use Brouwer’s own words:

> However weak the position of intuitionism seemed to be after this period of mathematical development, it has recovered by abandoning Kant’s a-priority of space but adhering the more resolutely to the a-priority of time. This neo-intuitionism considers the falling apart of moments of life into qualitatively different parts, to be reunited only while remaining separated by time as the fundamental phenomenon of the human intellect, passing by abstracting from its emotional content into the fundamental phenomenon of mathematical thinking, the intuition of the bare two-oneness. This intuition of two-oneness, the basal intuition of mathematics, creates not only the numbers one and two, but also all finite ordinal numbers, inasmuch as one of the elements of the two-oneness may be thought of as a new two-oneness, which process may be repeated indefinitely; this gives rise still further to the smallest infinite ordinal number ω. Finally this basal intuition of

> mathematics, in which the connected and the separate, the continuous and the discrete are united, gives rise immediately to the intuition of the linear continuum, i.e., of the "between," which is not exhaustible by the interposition of new units and which therefore can never be thought of as a mere collection of units. (Brouwer 1912, p. 11–12)

This quote describes what Brouwer calls the “first act of intuitionism,” which separates mathematics from mathematical language and places the origins of intuitionistic mathematics in the perception of a move of time (van Atten, van Dalen & Tieszen 2002). The first act of intuitionism is founded on the idea of ‘two-ity,’ which generates the natural numbers. After becoming aware of a sensation passing into another sensation and of retention in memory of what was sensed earlier, we can add unity to unity and hold them together in our consciousness. Put differently, the intuitionistic concept of natural numbers obtains an immediate certainty as to what is meant by the number 1 and that the mental process which goes into the formation of the number 1 can be repeated. By repeating the process, we get the concept of the number 2, and by repeating it again, we get the concept of the number 3, and so on.

Following Brouwer’s perception of time, finite numbers and constructively given denumerable sets are objects that we can intuitively grasp, while the Cantorian collection of all real numbers, which is an infinite entity, is an object that exceeds our grasp (Brouwer 1952). Brouwer’s intuitionism acknowledges potential infinity as a legitimate object since it can be constructed, at least to some extent, but it excludes actual infinity. According to Brouwer, we have an intuitive grasp of the continuum as a whole, which means that no set of points can exhaust the continuum. As a result, a continuum that is constructed out of a set of independently given points (like the Cantorian continuum) cannot be considered a legitimate mathematical entity (Brouwer 1907, 8-9, 62). The implications of Brouwer’s intuitionism on the practice of mathematics were substantial: core notions like the principle of excluded middle and the concept of negation were deemed unacceptable, and central mathematical theories, such as Zermelo’s axiom of choice, were excluded or extensively altered.

Walter van Stigt (1990, p. 127–32) suggests that more than Kant had influenced Brouwer, the development of Brouwer’s intuitionistic thought is deeply rooted in another philosophical movement: the French intuitionist school. The pre-intuitionist school, later known as the French semi-intuitionists, mainly consisted of the mathematicians Rene Baire, Emile Borel, Henri Poincaré, and Henri Lebesgue. Baire, Borel, and Lebesgue[4] held constructive views about the foundations of mathematics, but none of them formulated a coherent philosophy of mathematics – they left this job for "philosophers and psychologists" (Hesseling 2003). Borel, who was the most prominent of the three in expressing his views about foundational issues, believed that mathematical ideas could be presented with utmost clarity not through logical axioms or techniques but through subjective reconstruction in the human mind (Bru, Bru, and Chung 2009). Mathematics,

[4] Throughout the paper, I distinguish between Poincaré and the other semi-intuitionists as there is a considerable difference in how each of them had shaped Brouwer's philosophical views on mathematics (Brouwer 1952; van Dalen 2013; Michel 2008). This section will discuss Baire, Borel, and Lebesgue's influence on Brouwer, and the next section is devoted solely to the influence of Poincaré on the development of Brouwer's intuitionistic thought.

according to Borel (1907), is a human mental activity expressed by language, and mathematical proofs cannot rely solely on logical rules since logic can provide the working framework but not the elements themselves. The role of logic is "limited to supplying the material for it, and one does not confuse the mason with the architect" (Borel 1907, p. 279).

Borel (1914) argued that consistency is a necessary demand for mathematical existence but not sufficient. According to Borel, it is not enough for an object to be part of a consistent system in order for it to exist mathematically; the object also has to be effectively definable, namely, that it is possible to define it with a finite number of words. If a number cannot be defined in a finite number of words within a consistent system, it is considered unrealized or non-existent. As an example of a non-existent number, Borel (1928, p. 154) describes the scenario of a denumerable number of people choosing a digit, one after the other. Since there is a denumerable number of choices, the number created cannot be defined in a finite number of words; therefore, it cannot be considered realized and hence, does not exist.

It should be noted that Borel does not exclude the use of words like ‘infinite’ when describing mathematical objects, but only that such a definition should be clearly described by a law, in a finite number of words, and should have clear instructions in which order they should be executed. Borel’s definition of mathematical existence resembles, at least to some extent, Brouwer’s construction of choice sequences, one of the essential elements of his intuitionistic theory. Similar to Brouwer, Borel accepted as legitimate sets only the denumerable ones since they are the only ones that can be defined. Consequentially, Borel differentiates between the geometrical continuum that is given to us in nature and is accessible through our intuition and the numerical continuum, which is an artificial creation that is uncountable, hence undefinable.

The concept of intuition is omnipresent in Borel's work, even though he does not account for the term's exact nature. According to Borel (1898, p. 176–222), mathematical reality cannot be built solely upon logical arguments; it must be linked to intuition. As for the concept of the natural number and the set of natural numbers, Borel (1914, p. 179) accepted both as “clear notions” for mathematicians since there exists a “practical agreement among mathematicians in the use of these notions”. Here Brouwer and Borel differ, as Borel felt that it is the work of philosophers and psychologists to understand why some mathematical notions are clearer than others to mathematicians, while Brouwer placed such philosophical distinctions at the core of his theory.

There is overtly a firm connection between the French semi-intuitionists’ ideas and the development of Brouwer’s intuitionistic thought, as the two address concepts such as logic, mathematical proofs, and mathematical existence in a similar manner. Nevertheless, can one regard their views as a philosophical tradition from which Brouwer’s intuitionism has evolved, in a way that resembles the profound engagement of Einstein’s theory with the philosophical debate on the foundations of geometry, as described by Friedman?

Several scholars and historians, such as Dennis Hesseling and Alain Michel, would probably disagree with such a characterization. Hesseling (2003) and Michel (2008) argue that the French semi-intuitionists never formed their ideas into a systematic philosophical stream and that they did not even refer to themselves as “intuitionists” but as “empiricists” or as “realists”; it was Brouwer (1952) who was the first to refer to their theories as “preintuitionistic”. Michel (2008) claims that the French semi-intuitionists’ philosophical doctrine

> is initially defined in a negative manner, by the absence of doctrinal fixation. […] Lack of interest in foundational problems is, in the case of our French authors, of course not a mere oversight. It is actually a deep feature, which reflects the way in which they apprehend mathematical activity. What Brouwer sometimes calls ‘the Paris school’ never constituted a school. Between our authors, there were exchanges and debates, as in the famous ‘five letters on set theory’, but no attempt to line up under a common banner. Nothing in any case to compare with what took place in Amsterdam or even in Göttingen. None of our authors taken in isolation ever claimed to offer a doctrine, and it would be difficult to connect them to a distinctive philosophy, the way Brouwer or Hilbert were connected to Kant. Therefore, when they were summoned to do so, they rather claim their commitment to ‘empiricism’ or to ‘pragmatism’, a simple way of appealing to a standard of judgement lying outside of theoretical or doctrinal speculations. (Michel 2008, p. 160)

Let us now focus on how Henri Poincaré's work and philosophical position shaped Brouwer's intuitionistic view. Even though Poincaré was considered (according to Brouwer's own classification) a French semi-intuitionist, his entanglement in the philosophical debate on the foundations of mathematics was much more profound and substantial than his French colleagues. The next section will examine the similarities and differences between Brouwer's intuitionism and Poincaré's philosophy of mathematics and whether Brouwer's intuitionistic ideas can be traced to the type of philosophical stance presented by Poincaré.

3.2 Poincaré

Throughout the years, historians, philosophers, and mathematicians have ascribed to Poincaré several distinct, sometimes even opposing, philosophical views. Hesseling (2003) described him as a neo-Kantian, and van Dalen (2013) and Michel (2008) portrayed him as an eminent member of the French semi-intuitionist school. On the other hand, Arend Heyting (1934) implied that Poincaré’s and the semi-intuitionists’ views differ remarkably, to the point that he did not regard Poincaré as part of that school at all. Heinzmann and Stump (2017) claim that Poincaré was neither formalistic nor intuitionist nor empiricist, but somewhere between, and Solomon Feferman (1998; Feferman and Hellman 1995) argued that Poincaré was one of the first mathematicians to embrace the philosophical position of Predicativism.

Poincaré is also considered among the first to articulate a conventionalist view, namely, that certain concepts and principles of science, like geometry, are not subject to empirical basis and should not be considered a-priori truths (Ben-Menahem 2001; Folina 1992; Zahar 2001). According to Poincaré (1902, p. 50), “experience does not relate to space, but to empirical bodies. Geometry deals with ideal bodies, and it can therefore be neither proved nor disproved by experience. Since the propositions of geometry cannot be analytical either, these propositions must then be conventions, neither true nor false”. Put differently, for Poincaré, events do happen objectively in the world, but the ways in which these events are ordered in space and time (that is, the specific geometry we take them to instantiate) is a matter of convention. Poincaré’s conventionalism inspired several schools and practitioners (such as the logical positivists, Einstein, Quine, and Putnam, among others) and made a significant impact on the philosophy of science and mathematics (Ben-Menahem 2006).

Regardless of which characterization best represented his actual philosophical position, it is clear that unlike the French semi-intuitionists, Poincaré considered himself and was considered by others to be a philosopher and not only a mathematician. To what extent, if at all, can Brouwer's intuitionism be traced to Poincaré's philosophy of mathematics?

Brouwer and Poincaré shared similar views regarding several foundational issues. Concerning the debate about the nature of mathematical reasoning, Poincaré viewed it as involving epistemic content over and above its purely logical dimensions, thereby stressing the synthetic rather than the analytic character of mathematics. Mathematics, so argued Poincaré (1894), cannot be built solely on logical principles since he believed that logic could never derive a general statement from a particular one; mathematics was in principle irreducible to pure logic. As he put it:

> The syllogism cannot teach us anything essentially new. It must be conceded that mathematical reasoning has of itself a sort of creative virtue, and consequently differs from the syllogism. (Poincaré 1894, p. 371–72)

Nevertheless, Poincaré does not exclude logic from mathematics, but quite the opposite. As Brouwer (1952) rightly noticed, Poincaré

> […] reestablished on the one hand the essential difference in character between logic and mathematics, and on the other hand the autonomy of logic as a part of mathematics. (Brouwer 1952, p. 140)

Poincaré's views on mathematical logic were less restrictive than Brouwer's intuitionistic approach, as he did not reject classical logic altogether; nonetheless, according to Detlefsen (1993, p. 270) Poincaré maintained that mathematical reasoning should not be seen as a "primarily logical relationship between propositions, but rather as an epistemic relationship between judgments".

As for the antinomies in set theory, Poincaré (1905) thought that they arise primarily if one applies logic, which can only be applied on finite sets, onto infinite sets. Brouwer's approach (1919) is almost identical:

> In my opinion the Solvability Axiom [also known as 'Hilbert's Dogma'] and the principle of the excluded third are both false, and the belief in these dogmas historically is the result of the fact that one at first abstracted classical logic from the mathematics of subsets of a particular finite set, and next ascribed an a priori existence, independent from mathematics, and finally, on the basis of this alleged apriority, applied it to the mathematics of infinite sets. (Brouwer 1919, p. 204)

However, there is a profound difference between Brouwer's and Poincaré's arguments about logic and their implications on mathematics. Whereas Brouwer saw in the set-theoretical paradoxes an indication that something was sufficiently wrong with the foundations of mathematics to require significant reform of the discipline, Poincaré viewed them as mere technical flaws whose correction did not entail comprehensive revision (van Dalen 2012).

Intuition, according to Poincaré (1905), is a fundamental part of mathematics, just as logic is. Logical principles give us certainty, which is the “instrument of proving”, while intuition is the “instrument of inventing” (Poincaré 1905, p. 37). We need “a faculty that will show us from afar the final goal, and that faculty is intuition” (Poincaré 1905, p. 26).

Even though Brouwer and Poincaré agreed that intuition is a guarantee of the certainty inherent in mathematics, Brouwer regarded intuition as the basis of mathematical construction, a viewpoint that is not shared by Poincaré (Heinzmann and Stump 2017).

As several scholars, such as Gerhard Heinzmann (2008), Philippe Nabonnand (2008), and Dirk van Dalen (2012), have argued, Brouwer’s and Poincaré’s approaches to the foundations of mathematics are commonly viewed as closely related. However, the differences between Brouwer’s and Poincaré’s philosophical perspectives undermine any attempt to prove that Brouwer’s intuitionistic theory was rooted in Poincaré’s philosophy of mathematics.

According to Heinzmann and Nabonnand (2008), the most notable difference is the significant role experience plays in Poincaré’s theory as opposed to Brouwer’s and Heyting’s characterization of ‘contemporary intuitionists’, who deem mathematical knowledge to be independent of experience. Heinzmann and Nabonnand (2008, p. 177) present compelling evidence that for Poincaré, mathematical theories are dependent on external experience, and it is through experience that our minds can apply their intrinsic capacities.

For Brouwer, mathematics is an activity of exact thinking. It is only through the activity of thinking that mathematical truths can be determined, and a proposition can be determined as true only if the subject has experienced its truth by having carried out an appropriate mental construction (Brouwer 1907; 1912; van Atten 2020). In Brouwer’s view, to experience means to carry out an appropriate mental construction in one’s mind and by no means to go beyond the realm of the human mind. In this respect, Brouwer and Poincaré clearly use the term “experience” differently. For Poincaré (1905, p. 24), it is the outer world through which one can exercise the faculty of intuition.

The role of language is another bone of contention between the two. According to Brouwer (1907, p. 98–100), mathematics is founded on the pure intuition of time, which he classed as a pre-linguistic form of consciousness. Language, therefore, applies to mathematical activity only after the fact; hence, it plays no significant role in mathematics (Brouwer 1907, p. 128–30). Poincaré held a completely different view. Not only did he not separate language and mathematics, but he also ascribed to language an essential role in mathematical reasoning (Heinzmann and Stump 2017).

As noted throughout this section, there are several aspects of Poincaré's, Kant's, and the semi-intuitionists' mathematical philosophies that reappear in Brouwer's work. However, there are also several major and unbridgeable differences between Brouwer's intuitionistic theory and each of these systems: Brouwer and Kant differ on one of the major cornerstones of Brouwer's theory, namely, the notion of intuition itself; the same is true for

Brouwer and Poincaré in regards to the concept of experience and the role of language in mathematics; and despite the similarities between Brouwer's ideas and the semi-intuitionists' approach, their work can hardly be considered as a solid philosophical tradition. Compared to Friedman's picture of Einstein's theory of relativity and its significant engagement with Helmholtz' and Poincaré's philosophical debate, Brouwer's intuitionism lacks such a deep connection to Kant's, Poincaré's or the semi-intuitionists' philosophical stances. Moreover, Brouwer himself insisted, in a letter to Abraham Fraenkel, that his ideas did not continue the work of any other mathematician or philosopher, complaining about the "expropriation, which the German-speaking review journals practised on him, by making me share with Poincaré, Kronecker and Weyl what is my exclusive personal and spiritual property"[5].

Therefore, when viewed through the prism of Friedman's theory, the reluctance of the mathematical community to accept Brouwer's intuitionistic approach may seem rational indeed. Unlike the "classical physicist" in Einstein's case, the analogous classical mathematician apparently did not have "ample reasons" to consider Brouwer's work seriously; on the contrary – Brouwer's intuitionism was perceived as an isolated theory, entirely disconnected and even contradictory to common mathematical practices, that shared some commonalities with several philosophical traditions (such as Kant's, the semi-intuitionists' and Poincaré's) but did not adhere to a single philosophical tradition[6].

However, even though Brouwer's intuitionism did not prevail, it created quite an extensive stir[7]. Within the boundaries of the foundational debate, intuitionism played a significant role, gaining over 250 reactions between 1921 and 1933 (according to the data presented by Hesseling (2003, p. 96)). Several mathematicians, including Hermann Weyl and Abraham Fraenkel, considered intuitionism a viable option to replace classical mathematics, even if only for a short period. How did individual mathematicians, such as Weyl and Fraenkel, come to consider Brouwer's intuitionistic notions as legitimate alternatives in the first place? The following section will focus on the intuitionistic episodes of Hermann Weyl and Abraham Fraenkel, in an attempt to account for their reasons for considering Brouwer's intuitionism as a viable option. It will also discuss the influence their sympathetic responses to it had on both the discipline and the members of the mathematical community.

4. The appeal of Brouwer's intuitionism

4.1 Abraham Fraenkel

Abraham Fraenkel came to engage Brouwer's intuitionistic ideas through his interest in set theory. Fraenkel earned his Ph.D. in mathematics from the University of Marburg in 1914, where he also submitted his habilitation thesis a year later. In Marburg, Fraenkel worked on

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[5] Letter from Brouwer to Fraenkel, 28 January 1927 (van Dalen 2013, p. 390).

[6] Brouwer's inability to convince the mathematical community to follow his intuitionistic footsteps derived mainly from the implications of embracing Brouwer's approach. It was not Brouwer's motivation to change mathematics that mathematicians rejected, but his intuitionistic program's outcome – namely, the necessity to completely abandon a significant part of mainstream mathematics.

[7] In certain circles, Brouwer's intuitionism never ceased to be discussed; there were always mathematicians who abode by it and never actually abandoned the idea. Brouwer's student Arend Heyting and his students Dirk van Dalen and Anne Sjerp Troelstra and their successors have developed streams of contemporary intuitionism that are still being discussed nowadays (McCarty 2005; Moerdijk 1998; Posy 2005).

g-adic systems and the theory of abstract rings, but after only a few years, he abandoned the study of rings and turned to set theory (Corry 2004). In 1919 he published the first edition of his booklet on set theory, *Introduction to set theory* (Fraenkel 1919; 1967). Fraenkel barely addressed intuitionism in the first edition of his booklet, but the second edition, published in 1923, contained an extensive treatment of intuitionistic concepts such as mathematical existence and the principle of excluded middle.

During the 1920s, Fraenkel’s work on set theory established his reputation for good, and his name came to be associated with the “Zermelo-Fraenkel set theory”, Fraenkel’s improvement of Zermelo’s axiomatic system (van Dalen 2000). What makes a notable mathematician, immersed in classical set theory, consider Brouwer’s approach as a viable alternative and devote an entire section of the 1923 edition of his book to examining it closely?

Fraenkel’s working knowledge of Dutch (due to his marriage Wilhelmina Malka A. Prins of Baarn), as well as his personal contacts with Brouwer, rendered the latter’s ideas available to him. During a visit with his in-laws in Amsterdam, Fraenkel scheduled a meeting with Brouwer that led to several others (van Dalen 2013). Brouwer and Fraenkel formed a relationship and commented on each other’s work: Brouwer assisted in proof-reading the second edition of Fraenkel’s book on set theory in 1923, and Fraenkel attended several of Brouwer’s lectures at the University of Amsterdam and continued to discuss them with Brouwer[8].

The interest Fraenkel showed in Brouwer’s intuitionism was genuine and contradicted the response of the mathematical community in Germany at the time. As Fraenkel wrote to Brouwer in 1923:

> Among other things, it was very interesting for me to observe the fresh life of intuitionism, which has been pronounced dead from many corners. In my own mind, these questions are still fermenting.[9]

Dirk van Dalen (2000, p. 286) describes Fraenkel’s exposition of Brouwer’s intuitionism in his book on set theory as courageous, given the mathematical community’s unfavorable attitude, especially in the light of Hilbert’s criticism of Brouwer and Weyl. Fraenkel made a point of including expositions of intuitionism in every book he wrote, thereby introducing intuitionism to a broader mathematical audience. Fraenkel’s sympathetic expositions of intuitionism were a boost to Brouwer’s efforts, who drew great encouragement from the fact that a mathematician of voice and standing was taking his views seriously.

Fifty years later, and eight years after his death, a new edition of Foundations of set theory was published (together with Yehoshua Bar-Hillel and Azriel Levy, 1973), in which an entire

[8] In the course of their correspondence, Brouwer and Fraenkel experienced some friction when Brouwer became agitated by reading Fraenkel’s (1927) sections about intuitionism in his *Ten Lectures on the Foundations of Set Theory*. It is unclear what exactly provoked Brouwer’s outburst, as documented in Brouwer’s letters (van Dalen 2013), which came as a surprise to Fraenkel, who admired Brouwer and his work. Eventually, they made up, and the correspondence between them continued. That same year Brouwer wrote Fraenkel that despite his stern remarks, “this letter is accompanied only by benevolent and friendly feelings towards you.“ (Brouwer to Fraenkel, 18 January 1927 (van Dalen 2013)).

[9] Letter from Fraenkel to Brouwer, 18/4/1923 (Hesseling 2003, p. 150).

chapter was devoted to intuitionism. This chapter was updated following notes left by Fraenkel and later revised by Dirk van Dalen with comments from Arend Heyting (Fraenkel, Bar-Hillel & Levy 1973, ix). As van Dalen (2013, p. 391) points out, Fraenkel "conscientiously strove to incorporate the basic ideas of mathematics in his books, including Brouwer's ideas on intuitionism." Fraenkel treated them as more than an episodic attack on traditional mathematics; although he did not endorse it, he saw intuitionism as playing a significant role in the debate on the foundations of mathematics that should not be overlooked.

Nonetheless, the restrictions Brouwer's intuitionism imposed on mathematics did not escape Fraenkel's attention. In a series of lectures on set theory that Fraenkel (1927, p. 58) gave in 1925 in Kiel, he accused intuitionism of amputating a critical part of analysis from the body of mathematics, a process whose outcome created a substantial difficulty for the discipline.

The correspondence between Fraenkel and Brouwer during the 1920s sheds some light on Fraenkel's deliberations: he clearly understood the extensive limitations Brouwer's intuitionism imposed on mathematics but still maintained that intuitionism deserved a place within the discipline (van Dalen 2000; 2013; Hesseling 2003). But there is more to Fraenkel's deliberations than his letter expresses. In a section on intuitionism from the 1923 edition of his book (Fraenkel 1923, p. 164), Fraenkel regarded Brouwer's intuitionism as a brave and revolutionary move on mathematics but remained skeptical as to its viability as an alternative system to the one to which he was committed. The reappearance of intuitionism in Fraenkel's posthumous works (Fraenkel 1967; Fraenkel et al. 1973) shows that he addressed intuitionism as part of mathematics long after the foundational debate had abated and the mathematical community's interest in Brouwer's theory decreased. Fraenkel was instrumental in disseminating Brouwer's ideas positively to the wider reaches of the mathematics community, though without recommending its endorsement.

A similar approach to intuitionism can be found in Fraenkel's views regarding the philosophy of mathematics. According to Yehoshua Bar-Hillel, Fraenkel was a man of vision who did not confine himself to the foundations of mathematics and was never "just a mathematician" (Fraenkel 2016, xix). He held non-mainstream philosophical views and was able to explain philosophical positions even though he did not endorse them, much like he did with the foundations of mathematics:

> He tended towards Platonism as a philosophy of mathematics, namely that mathematical entities fully exist as abstract objects, even at times when this view was not very popular. However, he also gave the best and clearest interpretation of intuitionist views, which he personally did not support. […] He was fully aware that it was impossible to prove Platonism to be the only tenable mathematical philosophy. This view appealed to him personally, and he managed to weather the various foundational crises rather well. (Fraenkel 2016, xix)

Nevertheless, Fraenkel's engagement with philosophy went beyond the philosophy of mathematics. In his 1930 essay "Beliefs and Opinions in Light of the Natural Sciences" (Fraenkel 1930), he noted that science evolves, an idea that Thomas Kuhn would develop thirty years later. However, unlike Kuhn, Fraenkel exploits the evolutionary features of the history of science to emphasize the important role philosophy of religion plays in light of the uncertainty of the natural sciences:

> […] in light of recent developments in the natural sciences, especially in light of modern physics, we see on the one hand the impermanence of the concepts and assumptions that once seemed timeless, strong, and irrefutable; on the other hand we also see that the words of Torah are not harmed or refuted by them, and it is our responsibility to hold on with an artist's faith, to the words of the sages: "Turn it over and over, for all is in it." (Fraenkel 1930)

As can be seen from this quote, Fraenkel's philosophical views were significantly different from Brouwer's. Nonetheless, despite the philosophical and foundational differences between the two, Fraenkel still found intuitionism to be a necessary part of his books. Fraenkel was not the only renowned mathematician who found himself attracted to Brouwer's intuitionism. The following section will briefly sketch Hermann Weyl's far more significant engagement with intuitionism in the course of which he was seriously motivated to adopt intuitionism as a preferable alternative to classical mathematics.

4.2 Hermann Weyl

Hermann Weyl is often viewed by scholars of Brouwer's intuitionism (and even by Brouwer himself) as Brouwer's first convert and his most radical follower (Mark van Atten 2017; Beisswanger 1965; Bell 2000; van Dalen 1995; 2000; 2013; Hesseling 2003; Rosello 2012; Scholz 2004; van Stigt 1990). Weyl was Hilbert's prominent student and a well-established mathematician who, like Hilbert, was also immersed in the philosophical discussion of the foundations of mathematics. His deep interest in philosophy, especially in Husserl's phenomenology and Fichte's metaphysical idealism, provided the grounds for his eventual inclination towards Brouwer's approach (Scholz 2004; Sieroka 2009)[10].

It was set theory that led Weyl, like Fraenkel, to consider Brouwer's intuitionistic ideas in the first place, but Weyl's engagement with set theory derived from different reasons. Weyl saw the antinomies of set theory not as marginal esoteric philosophical conundrums but as evidence of severe problems that plagued mathematics' very core (Weyl 1921, p. 86). Brouwer and Weyl's meeting in 1919 during a summer vacation in the Engadin prompted Weyl to turn to Brouwer's intuitionism in his search for a solution to the foundational problem. Dirk van Dalen (2013, p. 299) claims that this meeting "must have opened Weyl's eyes to the deeper issues of constructive mathematics, for he immediately mastered the Brouwerian insights and started to present them in his own way".

In 1920, shortly after they met, Weyl sent Brouwer a copy of his new paper "On the New Foundational Crisis in Mathematics". In it, Weyl declares that Brouwer's intuitionistic theory

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[10] It should be noted that Weyl's philosophical approach and Brouwer's philosophical approach were quite different: Brouwer's philosophy amounted virtually to solipsism, while Weyl still seemed to have cleaved to phenomenology (Mancosu and Ryckman 2002). In a lecture delivered in 1954 titled "Insight and Reflection," Weyl portrayed his long philosophical voyage that had begun with Kant, moved on to idealist phenomenology, and ended with Weyl's recognition that the latter entails several problematic consequences (Weyl 1955). Hence, Brouwer and Weyl may have both seen intuitionism as profoundly connected to a philosophical tradition, but it was not *the same philosophical tradition*.

is the most viable alternative for rebuilding the foundations of mathematics on solid ground. But as supportive as he was towards Brouwer's ideas, he was also aware of its problematic aspects. Within the same paper, Weyl acknowledged the far-reaching restrictions Brouwer's intuitionism imposed on the everyday practice of mathematics (Weyl 1921, p. 109). A few years later, Weyl changed his mind and turned away from intuitionism (only to return to it a decade later)[11]. Nevertheless, Weyl's 1921 paper did for intuitionism what Brouwer himself struggled to do: it presented Brouwer's theory clearly and coherently[12], thereby enabling it to reach a wider audience.

The scientific community responded to Weyl's paper; if until 1921 intuitionism was discussed in small private circles, mainly between Brouwer, Weyl, Fraenkel, and Hilbert, from 1921 onwards, the number of reactions to intuitionism grew extensively, and intuitionism was given a leading role in the foundational debate, encouraging mathematicians, logicians, and philosophers to react (Hesseling 2003).

Despite the overall popularity of Weyl's paper, the intuitionistic version Weyl presented in his paper is, to a large extent, different from the intuitionistic version that Brouwer had in mind. In the third part of his 1921 paper, Weyl articulates precisely the ideas that he borrowed from Brouwer: "The idea of the developing sequence, […] the doubt in the principium tertii exclusi, and the concept of the functio continua […]", and the intuitionistic ideas that are exclusively his own: "(1) the concept of a sequence alternates […] between "law" and "choice," […] (2) universal and existential theorems are not judgements in the proper sense […] (3) arithmetic and analysis merely contain general statements about numbers and freely developing sequences." (Weyl 1921, p. 109–10). Hence, while the influence of Brouwer is evident throughout Weyl's paper, Weyl eventually develops a somewhat different version of intuitionism.

One of the mathematicians influenced by Weyl's turn to intuitionism was Fraenkel. Fraenkel's choice of words in *Introduction to Set Theory* and *Ten lectures on the Foundation of Set Theory* indicates that he had borrowed at least some of Weyl's provocative metaphors. Fraenkel's image of mathematics (1923, p. 164) as an empire whose foundations are unstable and intuitionism as an attack with "whetted arms" bear clear signs of Weylian influence (van Dalen 2000; Weyl 1921). Moreover, as van Dalen (2000) points out, Fraenkel "heavily relied" on Weyl in his presentation of choice sequences.

The brief presentation of Abraham Fraenkel's and Hermann Weyl's stories suggests that when the focus shifts from ideas and conceptual structures to deliberating individuals, framework transitions are bound to look very different. An individual's reasons to act in certain ways, to devote his efforts to considering certain theories in his published work while ignoring others, are a matter of personal exposure and circumstance. Weyl and Fraenkel, each in his own way and for his own personal reasons, seriously engaged

[11] A further discussion on the topic can be found in van Dalen (van Dalen 1995).
[12] In contrast to Brouwer's own too technical and unclear writing style that several competent readers, such as the mathematician Bartel van der Waerden, complained about (Hesseling 2003, p. 61–62).

Brouwer's intuitionism and produced colorful, clear, positive, and elaborate expositions of it. Those expositions contributed to the dissemination of Brouwer's ideas, thereby affecting the trajectory of the debate and the development of intuitionism during one of the most intriguing episodes of disagreement in the discipline of mathematics. According to Friedman, framework transitions are rooted in new ideas. But how new ideas spread, at least to some extent, owes to the place that they come to occupy in the deliberations individual practitioners conduct among themselves and others.

The environmental cost of transitioning to a new framework is another aspect of Friedman's theory that might shed new light on the lack of reception of Brouwer's intuitionism. Friedman argues that successive scientific revolutions resemble a series of nested developments and that "earlier constitutive frameworks are exhibited as limiting cases" of the new ones (Friedman 2001, p. 63). Hence, during framework transitions, the price practitioners committed to the old framework have to pay is relatively low. Following this line of thought, the disapproval of Brouwer's intuitionism did not occur merely because it lacked a solid philosophical tradition but also due to the enormous price intuitionistic mathematics coerced mathematicians to pay. Even the mathematicians who were willing to seriously reconsider their commitments to the old framework found themselves caught between a rock and a hard place; they were aware of the foundational problems that intuitionism seemed to solve, but they could not continue their scientific work without the mathematical theories that Brouwer's intuitionism forced them to renounce. Eventually, despite not having any concrete solution to the foundational problem, most practicing mathematicians chose to continue their everyday mathematical work and lived with the contradiction. To the community's eyes, the overall damage Brouwer's intuitionism caused to mathematics was greater than its achievements. This aspect of Friedman's theory provides yet another possible explanation of the reluctant response of the mathematical community to Brouwer's intuitionism.

5. Concluding Remarks

This paper has taken a closer view of Brouwer's intuitionism through the lens of Michael Friedman's theory of framework transitions. The juxtaposition of Einstein's story (as told by Friedman) next to Brouwer's story suggests at least one considerable difference between the two, namely, the necessity, from Friedman's perspective, of a solid philosophical tradition, to which new ideas can be traced, and from which new theories can evolve. Even though Brouwer's ideas share some commonalities with Kant, the French semi-intuitionists, and Poincaré's philosophical views, none of them can be said to have provided Brouwer's readers with a reliable and exhaustive source of key elements necessary for deeming his proposal a live and viable option.

From Friedman's point of view, the fact that Brouwer's intuitionism was not rooted in an established philosophical tradition contributed to the reluctance of the mathematical community to accept it as a legitimate alternative to the then-current working framework. If philosophical developments can help explain the rationality of scientific progress, as in Friedman's story of Einstein, then in Brouwer's case, the lack of solid philosophical origins explains, at least partially, the rationality behind its abandonment[13]. In this sense, Friedman's model provides a new perspective on the possible reasons behind the community's reluctant response to Brouwer's ideas.

Despite the community's overall negative response to intuitionism, during the 1920s Brouwer's ideas aroused controversy and substantial interest. Intuitionism gained more and more attention within the foundational debate thanks to several individual mathematicians whose work had an impact. The short conversion of Hermann Weyl to intuitionism and the continuous interest shown by Abraham Fraenkel throughout his career (and their mutual influence and influence of others) indicate that individuals played a significant role in transitioning intuitionistic ideas.

Retold from Friedman's perspective, the story of Brouwer's intuitionism reveals that even in unsuccessful revolution attempts that cannot be fully traced to solid philosophical ground, ideas can travel, and people can still change their minds in favor of less acceptable theories. The stories of Hermann Weyl and Abraham Fraenkel are telling cases of how controversial and non-mainstream ideas made an impact on individual mathematicians. Moreover, it should be noted that in some circles, intuitionism continues to be discussed until today, and different extensions of intuitionism have continued to evolve throughout the years ranging over various disciplines[14]. One of those extensions derived from Weyl's interest in intuitionism and the foundations of mathematics, specifically in his The Continuum (Weyl 1918). Weyl's work gave rise to a predicativist point of view, which Solomon Feferman extended into a mathematical school of its own, namely, Predicativism (Feferman 1998, 2000, 2005; Feferman and Hellman 1995). Hence, the current paper allows, on the one hand, to better understand why Brouwer's intuitionism did not prevail and, on the other hand, to point out the significance of individual's deliberations when considering models of scientific framework transitions.

[13] This is not the case when considering Hermann Weyl's conversion to intuitionism. Weyl found philosophical enlightenment in Husserl's phenomenology and Fichte's metaphysical idealism. Brouwer's philosophical agenda was deeply influenced by Gerrit Mannoury's ideas of mathematics as a human creation, an idea that set the stage for the development of Brouwer's idealistic philosophy (van Dalen 1999; Mannoury 1909). This common philosophical ground, alongside their shared view about the primary intuition of time, has contributed to Weyl's inclination towards intuitionism. Having said that, it should be clear that Weyl's story is the exception: he is one of the few mathematicians who attempted to understand Brouwer's philosophical ideas and to follow his scattered and sometimes radical line of thought. Most mathematicians did not share Brouwer's and Weyl's deep interest in philosophy and gave up the challenging task of understanding Brouwer's philosophical motivations.

[14] In mathematics, Heyting's student, Anne Sjerp Troelstra, continued his supervisor's work in formalizing intuitionistic logic and choice sequences (Troelstra 1969, 1977), and philosopher Michael Dummett developed a philosophical basis for intuitionism by extending Heyting's approach (Dummett 1973; Posy 2020). More recent developments include semantic interpretations of intuitionism (Bezhanishvili and Holliday 2019), connections to type theory and computer science (Martin-Löf 1984), and employment of choice sequences to model indeterminacy in physics (Gisin 2019).

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