ACCURACY, PROBABILISM AND BAYESIAN UPDATE IN INFINITE DOMAINS

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Abstract. Scoring rules measure the accuracy or epistemic utility of a credence assignment. A significant literature uses plausible conditions on scoring rules on finite sample spaces to argue for both probabilism—the doctrine that credences ought to satisfy the axioms of probabilism—and for the optimality of Bayesian update as a response to evidence. I prove a number of formal results regarding scoring rules on infinite sample spaces that impact the extension of these arguments to infinite sample spaces. A common condition in the arguments for probabilism and Bayesian update is strict propriety: that according to each probabilistic credence, the expected accuracy of any other credence is worse. Much of the discussion needs to divide depending on whether we require finite or countable additivity of our probabilities. I show that in a number of natural infinite finitely additive cases, there simply do not exist strictly proper scoring rules, and the prospects for arguments for probabilism and Bayesian update are limited. In many natural infinite countably additive cases, on the other hand, there do exist strictly proper scoring rules that are continuous on the probabilities, and which support arguments for Bayesian update, but which do not support arguments for probabilism. There may be more hope for accuracy-based arguments if we drop the assumption that scores are extended-real-valued. I sketch a framework for scoring rules whose values are nets of extended reals, and show the existence of a strictly proper net-valued scoring rules in all infinite cases, both for f.a. and c.a. probabilities. These can be used in an argument for Bayesian update, but it is not at present known what is to be said about probabilism in this case.

1. Introduction

Scoring rules measure the accuracy of a credence assignment, the match between that assignment and reality. Strictly proper scoring rules have the property that for any probabilistically coherent credence assignment p, the expected value of the score of p by the lights of p is better than the expected value of the score of any other credence q by the lights of p.

Under continuity conditions, the score of a credence that does not satisfy the axioms of probability is strictly dominated by the score of a credence that does satisfy these axioms ([12], [10], [15] and [17]; cf. [14]). This result has been interpreted philosophically as an argument for probabilism, the doctrine that our credences rationally ought to satisfy the axioms of probability (e.g., [8], [13]).

The domination results invoked in the arguments for probabilism are typically based on finite spaces, though recently Kelley [9] and Nielsen [11] have found domination theorems in infinite cases. I shall argue that, notwithstanding these results, the prospects for extending the accuracy-based arguments for probabilism to infinite cases are slim.

Greaves and Wallace [7] have given an expected accuracy based argument for the optimality of Bayesian update: they showed that the evidence-response strategy that optimizes expected accuracy proceeds by conditionalizing on evidence. Here, the prospects for an extension to infinite spaces will be seen to be mixed.

More explicitly, first, I show that if probabilities are understood to be countably additive, then there does exist a strictly proper scoring rule on a *countably* infinite space, continuous in one plausible sense, and an accuracy argument for Bayesian update can be given. However, strictly proper scoring rules on an infinite space do not in general have a domination theorem in the countable additivity case as needed for the arguments for probabilism.

Second, I give a cardinality argument to show that there are no strictly proper scoring rules on an infinite space if probabilities are understood to be merely finitely additive, and discuss the consequences for arguments for probabilism.

Moreover, I show that in a number of cases, including countably and finitely additive ones, one cannot use accuracy-based arguments to argue for Bayesian update. Indeed in these cases one cannot even argue for the superiority of Bayesian update to anti-Bayesian update where one conditionalizes on the complement of the evidence. This fact may make one suspicious of extending accuracy-based reasoning to infinite cases.

I also discuss how one might attempt to make use of Nielsen's [11] domination theorem for quasi-strictly proper continuous scoring rules in the finitely additive case to generate an argument for probabilism, but argue that these scoring rules treat scores of non-probability credences "unfairly", by requiring that they satisfy a condition that the probability credences are not required to satisfy (and cannot be required to, given the earlier cardinality argument). And I argue that Kelley's [9] restriction to countable opinion sets is unlikely to yield a compelling argument for probabilism in infinite cases. I also briefly discuss our current insufficient state of the art regarding scoring rules with hyperreal values and sketch a promising approach using scores whose values are nets of extended-real values. That approach allows us to define strictly proper scoring rules in all the infinite cases, and these scoring rules support the expected inaccuracy argument for Bayesian update in all infinite cases.

Let us review the formal aspects of the finite case first. Let Ω be a finite sample space, encoding the possible situations that the credences concern. Let $\mathcal{C}$ be the set of all functions from the power set of Ω to $[0,1]$: these we call credence functions. Let $\mathcal{P}$ be the subset of $\mathcal{C}$ which consists of the functions satisfying the axioms of probability. An *inaccuracy scoring rule* is a function s from a set $\mathcal{F} \supseteq \mathcal{P}$ of credence function to $[M, \infty]^\Omega$ for some finite M, where A^B is the set of functions from B to A. Then $s(c)(\omega)$ for $c \in \mathcal{F}$ measures the inaccuracy of the forecast c when we are in fact at $\omega \in \Omega$, with lower values being more accurate.

Given a probability $p \in \mathcal{P}$ and an extended real function g on Ω, let $E_p g$ be the expected value with respect to p defined in the following way to avoid multiplying infinity by zero:

$$E_p g = \sum_{\omega \in \Omega, p(\{\omega\}) \neq 0} p(\{\omega\}) g(\omega).$$

We then say that an scoring rule s is *proper* on $\mathcal{F} \supseteq \mathcal{P}$ provided that for every $p \in \mathcal{P}$ and every $c \in \mathcal{F}$, we have $E_p s(p) \leq E_p s(c)$, that it is *strictly proper* provided the inequality is always strict when p is not identical with c, and that it is *quasi-strictly proper* provided that it is proper and the inequality is strict when $p \in \mathcal{P}$ and $c \in \mathcal{C} \backslash \mathcal{P}$.

Propriety captures the idea that if an agent adopts a probability function p as their credence, then by the agent's lights there can be no improvement in the expected score from switching to a different credence. Strict propriety adds that by the agent's lights it would be actually harmful to switch, while quasi-strict propriety holds that switching to a non-probability credence would be harmful by the agent's lights. Proper and strictly proper scoring rules have been widely studied: for a few examples, see [2], [6], [13], [14], [19].

A scoring rule is *probability-continuous* provided that the restriction of s to $\mathcal{P}$ is a continuous function to $[M, \infty]^\Omega$, where $\mathcal{P} \subseteq [0, 1]^{\mathscr{P}\Omega}$ and $[M, \infty]^\Omega$ are equipped with the Euclidean topologies, and where $\mathscr{P}A$ is the powerset of A. This is equivalent to requiring the mapping $p \mapsto s(p)(\omega)$ to be a continuous function from $\mathcal{P}$ to $[M, \infty]$ for every $\omega \in \Omega$.

Say that an score $s(c_1)$ is *non-strictly dominated* by a score $s(c_2)$ provided that $s(c_2)(\omega) \leq s(c_1)(\omega)$ for all $\omega \in \Omega$, *weakly dominated* provided that additionally strict inequality holds for at least one ω, and *strictly dominated* if the inequality is always strict.

Predd, et al. [14] showed that if s is a probability-continuous additive[1] strictly proper scoring rule, then for any non-probability c, there is a probability p such that $s(c)$ is strictly dominated by $s(p)$. In other words, any forecaster whose forecast fails to be a probability can find a forecast that is a probability and that is strictly better according to s, without any further evidential input. This strongly suggests that the non-probabilistic forecast is irrational. Recently, Pettigrew [12] announced that this result holds without the assumption of additivity, merely assuming probability-continuity. This proof was shown to have flaws [10], but correct proofs were found by Nielsen [10] and Pruss [15]. Nielsen's proof also extended the result to the

[1] A scoring rule is additive if the score at $\omega \in \Omega$ is the sum of scores each of which depends only on the credence for a single event E and whether that event occurs at ω. See Section 2.3, below, for an explicit definition.

quasi-proper case. Pruss [17] has also shown that some kind of probability-continuity assumption is necessary, and analyzed how far the this assumption can be weakened.

We now introduce our infinite setup. Suppose that Ω is an infinite space. Let $\mathcal{C}$ be the set of credence functions on the power set of Ω, i.e., functions from $\mathscr{P}\Omega$ to $[0,1]$. Let $\mathcal{P}_{\mathrm{c}}$ and $\mathcal{P}_{\mathrm{f}}$ be respectively the sets of countably and finitely additive probabilities on the power set of Ω. Thus, $\mathcal{P}_{\mathrm{c}} \subseteq \mathcal{P}_{\mathrm{f}} \subset \mathcal{C}$.[2]

An inaccuracy scoring rule on a set $\mathcal{D}$ of credences is a function s from $\mathcal{D}$ to $[M,\infty]^{\Omega}$ for some finite M. Given a $p \in \mathcal{P}_{\mathrm{f}}$ and a function g on Ω, let $E_p g = \int_\Omega g\,dp$, with the integral being a standard Lebesgue integral if p is countably additive and defined the same way as a Lebesgue integral—by splitting g into positive and negative parts and approximating each by simple functions—if p is finitely additive (for details, see [11]).

If $\mathcal{P} \subseteq \mathcal{D} \cap \mathcal{P}_{\mathrm{f}}$, then we say that a scoring rule s on $\mathcal{D}$ is $\mathcal{P}$-proper on $\mathcal{D}$ provided that for every $p \in \mathcal{P}$ and $u \in \mathcal{D}$ we have $E_p s(p) \le E_p s(u)$. We say that the propriety is strict provided that the inequality is strict whenever $p \ne u$, and that it is quasi-strict provided that the inequality is strict whenever $u \notin \mathcal{P}$. The cases of interest to us will be when $\mathcal{P}$ is one of $\mathcal{P}_{\mathrm{f}}$ and $\mathcal{P}_{\mathrm{c}}$, in which cases we will respectively talk of s being finitely-additively (f.a.) and countably-additively (c.a.) proper.

We will say that f.a. or c.a. probabilism is the respective thesis that rational credence assignments are f.a. or c.a. probabilities.

Our definitions of types of domination extend directly from the finite case.

2. Countably additive probabilities

2.1. **A strictly proper scoring rule.** We now show that the analogue of the Pettigrew-Nielsen-Pruss strict domination theorem is in general false in the infinite c.a. case. Suppose Ω is countably infinite and hence identified with the natural numbers $\mathbb{N}$. For a c.a. probability p, let $\hat{p}$ be the sequence $(\hat{p}_n)_{n\in N}$ where $\hat{p}_n = p(\{n\})$. Let $\|v\|_q = (\sum_n |v_n|^q)^{1/q}$ be the ℓ^q-norm of a sequence v of real numbers, and let $\ell^q = \ell^q(\mathbb{N})$ be the set of sequences with finite ℓ^q norm.

Note that for any c.a. probability p, we have $\|\hat{p}\|_1 = 1$, and conversely for any non-negative sequence v with $\|v\|_1 = 1$, there is a unique c.a. probability p such that $v = \hat{p}$. We can thus define a topology on $\mathcal{P}_{\mathrm{c}}$ by identifying it with the non-negative functions in the unit sphere of ℓ^1.

Consider this scoring rule:

$$\mathrm{Sph}^*(c)(n) = \begin{cases} \frac{-\hat{c}(n)}{\|\hat{c}\|_2} & \text{if } c \in \mathcal{P}_{\mathrm{c}} \\ \frac{-1}{2(n+1)} & \text{otherwise.} \end{cases}$$

Restricted to the probabilities, this is the negative of the spherical accuracy score. The score is uniformly bounded: the score of any credence c is a function from $\mathbb{N}$ to $[-1,0]$. Moreover, because the ℓ^2-norm is continuous on

[2]Assuming the Axiom of Choice, the first inclusion is also strict.

ℓ^1, the score restricted to $\mathcal{P}_c$ (with the topology coming from identification with the non-negative members of ℓ^1) is a continuous function to $[-1,0]^{\mathbb{N}}$, i.e., for each fixed n, $p \mapsto \mathrm{Sph}^*(p)(n)$ is continuous. Slightly more strongly, for any fixed $q \in \mathcal{P}_c$, the function $p \mapsto E_q \mathrm{Sph}^*(p)$ is continuous on $\mathcal{P}_c$, being the product of the continuous function $p \mapsto -1/\|\hat{p}\|_2$ and the function $p \mapsto \langle \hat{q}, \hat{p} \rangle$ (where $\langle \cdot, \cdot \rangle$ is the inner product on ℓ^2), which function is continuous because ℓ^1 continuously embeds in ℓ^2 (this is like the continuity condition used by [11] in the f.a. case).

In the Appendix, I prove that Sph^* is strictly c.a.-proper (which is why I defined Sph^* on $\mathcal{P}_c$ as I did, rather than just setting it equal to some constant like -2 outside of $\mathcal{P}_c$). We thus have a continuous uniformly bounded strictly c.a.-proper scoring rule. But no member of $\mathcal{C} - \mathcal{P}_c$ has a score that is even non-strictly dominated by the score of a member of $\mathcal{P}_c$. For if $c \in \mathcal{C} - \mathcal{P}_c$, then

$$\sum_{n=0}^{\infty} \mathrm{Sph}^*(c)(n) = \sum_{n=0}^{\infty} \frac{-1}{2(n+1)} = -\infty,$$

while for any $p \in \mathcal{P}_c$ we have:

$$\sum_{n=0}^{\infty} \mathrm{Sph}^*(p)(n) = \frac{-\sum_{n=0}^{\infty} \hat{p}(n)}{\|\hat{p}\|_2} = \frac{-1}{\|\hat{p}\|_2} > -\infty,$$

which makes it impossible to have $\mathrm{Sph}^*(c)(n) \geq \mathrm{Sph}^*(p)(n)$ for all n. In particular, the Pettigrew-Nielsen-Pruss strict domination theorem does not extend to the infinite c.a. case.

We thus cannot give an argument for probabilism in the c.a. case based on strict domination or even one based on weak domination using strictly proper scores continuous on $\mathcal{P}_c$.

2.2. **Continuity on all credences.** One may object, however, that although Sph^* is continuous on the c.a. probabilities, it is not continuous on the whole set of credences $\mathcal{C}$, since as soon as one moves away from $\mathcal{P}_c$, the score jumps immediately to the function $n \mapsto -1/(2(n+1))$. The same intuition that leads us to think that probability-continuity of a scoring rule is a reasonable assumption in the context of arguments for probabilism—namely, that close-by credences should have close-by scores—suggests that we should more generally require continuity on $\mathcal{C}$.

It is not immediately clear, however, what continuity means here. The scoring rule s is a function from $\mathcal{C}$ to $[M,\infty]^{\Omega}$. It is natural to impose the product topology on $[M,\infty]^{\Omega}$, which is equivalent to requiring $c \mapsto s(c)(\omega)$ to be a continuous function from $\mathcal{C}$ to $[M,\infty]$ for each fixed $\omega \in \Omega$, or to require more strongly that $c \mapsto E_q s(c)$ be continuous for each fixed $q \in \mathcal{P}_c$. But the question of the appropriate topology on $\mathcal{C}$ is less clear.

One option is to think of $\mathcal{C}$ as the infinite product space $[0,1]^{\mathscr{P}\Omega}$, equipped with the product topology. A basis for this topology can be given by the family of sets of the form

$$N_{F,x,r} = \{c \in [0,1]^{\mathscr{P}\Omega} : \forall A \in F[c(A) \in (x(A) - r(A), x(A) + r(A))]\},$$

where F is a finite subset of $\mathscr{P}\Omega$, x is a function from F to $[0,1]$, and r is a function from F to $(0,1]$.

Say that a scoring rule s is *$\mathcal{P}$-distinguishing* provided that there isn't a $p \in \mathcal{P}$ and a $c \notin \mathcal{P}$ such that $s(p) = s(c)$ everywhere on Ω. Any $\mathcal{P}$-quasi strictly proper scoring rule is $\mathcal{P}$-distinguishing. A score's being probability distinguishing (in the relevant sense of "probability") is a necessary condition for the score to have any hope of yielding an accuracy argument for probabilism (in the respective sense), since accuracy arguments for probabilism depend on the idea that the scores of non-probabilities are worse than those of probabilities.

Nielsen's work [11] in the infinite case uses the product topology. In the Appendix, we will prove the following using cardinality considerations which shows a massive failure of continuity in the product topology on $[0,1]^{\mathscr{P}\Omega}$.

Proposition 1. *Assume the Axiom of Choice. Let $\mathcal{P}$ be $\mathcal{P}_{\mathrm{f}}$ or $\mathcal{P}_{\mathrm{c}}$. If Ω is infinite, then for any $\mathcal{P}$-distinguishing scoring rule s and every $p \in \mathcal{P}$, the scoring rule s fails to be continuous at p with respect to the product topology on $\mathcal{C} = [0,1]^{\mathscr{P}\Omega}$ and the product topology on $[M,\infty]^{\Omega}$.*

It follows that there is no product-topology continuous quasi-strictly proper scoring rule on $\mathcal{C}$, and no accuracy argument for probabilism on the basis of a product-topology continuous scoring rule on $\mathcal{C}$. The Proposition applies both in the c.a. case that we are discussing in this section as well as in the f.a. case.

Here there is some room for further research, however. An opponent of probabilism is unlikely to hold that *every* credence in $\mathcal{C}$ is rationally acceptable. Instead they are likely to say that there is some set $\mathcal{R}$ of rationally acceptable credences that includes some but not all non-probabilistic credences. For instance, Pruss [16] finds a way of avoiding some pragmatic arguments against inconsistent credences in the special case of credences that satisfy the zero ($p(\varnothing) = 0$), normalization ($p(\Omega) = 1$) and monotonicity (if $A \subseteq B$, then $p(A) \le p(B)$) axioms. Further research is needed to determine if Proposition 1 can be extended to scoring rules that merely distinguish $\mathcal{P}$ from $\mathcal{R} - \mathcal{P}$ instead of distinguishing $\mathcal{P}$ from $\mathcal{C} - \mathcal{P}$.

Next note that there is another natural topology on $[0,1]^{\mathscr{P}\Omega}$. We can think of $[0,1]^{\mathscr{P}\Omega}$ as embedded in the vector space space $\ell^\infty(\mathscr{P}\Omega)$ of uniformly bounded functions from $\mathscr{P}\Omega$ to $\mathbb{R}$ with the norm $\|f\|_\infty = \sup_{A\in\mathscr{P}\Omega} |f(A)|$. This norm appears natural in multiple ways.

First, for countable Ω, the $\ell^\infty(\mathscr{P}\Omega)$ topology restricted to $\mathcal{P}_{\mathrm{c}}$ is equivalent to the very natural $\ell^1(\Omega)$ topology on $\{\hat{p} : p \in \mathcal{P}_{\mathrm{c}}\}$ under the correspondence $p \mapsto \hat{p}$. This follows from Lemma 1 in the Appendix. (Note that Lemma 2 in the Appendix tells us that Sph^* is continuous on $\mathcal{P}_{\mathrm{c}}$ in this topology.)

Second, for finitely additive probabilities, the distance in this norm is the total-variation distance between finitely additive probability measures, an accepted way to measure closeness of measures at least in the countably additive case.

Third, the space of finitely additive signed measures on Ω with the total-variation norm is the dual of $\ell^\infty(\Omega)$ and forms a subspace of $\ell^\infty(\mathscr{P}\Omega)$, and the total-variation norm (now in the signed case) on such measures is equal to the $\ell^\infty(\mathscr{P}\Omega)$ norm [5, p. 296].

Now it turns out that there *is* a c.a.strictly proper score that is continuous in the $\ell^\infty(\mathscr{P}\Omega)$ but does not satisfy a strict domination result.

To see this, say that a scoring rule s on a set A of credences is uniformly bounded if there is a finite K such that $s(c)(\omega) < K$ for all $c \in A$ and $\omega \in \Omega$. Say that the $\mathcal{P}$-weak topology on $[M,\infty]^\Omega$ is the weakest topology that makes $E_q(\cdot)$ continuous on $[M,\infty]^\Omega$ for each fixed $q \in \mathcal{P}$. Recall that we have shown that Sph^* is continuous on $\mathcal{P}_\mathrm{c}$ with respect to the $\mathcal{P}_\mathrm{c}$-weak topology on $[M,\infty]^\Omega$.

In the Appendix, we will prove the following.

Proposition 2. *Let $\mathcal{P}$ be one of $\mathcal{P}_\mathrm{c}$ and $\mathcal{P}_\mathrm{f}$. Let s be any uniformly bounded proper score on $\mathcal{P}$ that is continuous in the $\ell^\infty(\mathscr{P}\Omega)$ topology on $\mathcal{P}$ and the $\mathcal{P}$-weak topology on $[M,\infty]^\Omega$. Let $f_0 : \Omega \to [M,\infty]$ be any function such that $E_p f_0 > E_p s(p)$ for all $p \in \mathcal{P}$. Then for any $c_0 \in \mathcal{C} - \mathcal{P}$, there is an extension of s to a $\mathcal{P}$-quasi-strictly proper score $\bar{s}$ on $\mathcal{C}$ such that $\bar{s}(c_0) = f_0$ and $\bar{s}(p) = s(p)$ for all $p \in \mathcal{P}$.*

Note that if s is strictly proper on $\mathcal{P}$, then $\bar{s}$ will be $\mathcal{P}$-strictly proper on $\mathcal{C}$. Given the above, let s be the spherical inaccuracy score on $\mathcal{P}_\mathrm{c}$ (i.e., the restriction of Sph^* to $\mathcal{P}_\mathrm{c}$), let c_0 be any credence in $\mathcal{C} - \mathcal{P}_\mathrm{c}$, let $f_0 = \mathrm{Sph}^*(c_0)$, and we will have an $\ell^\infty(\mathscr{P}\Omega)$-continuous c.a. strictly proper scoring rule $\bar{s}$ where $\bar{s}(c_0) = f_0$ is not (even non-strictly) dominated by any score of a c.a. probability, for the same reason as was given for the failure of domination in the case of Sph^*.

So, in the countably additive case, if we understand continuity on $\mathcal{C}$ with respect to the product topology, we do have a strict domination theorem for any quasi-strictly proper continuous score, but only trivially because there are no such scores. And if we understand continuity using the $\ell^\infty(\mathscr{P}\Omega)$ topology, we do not have a corresponding domination theorem (even of a non-strict variety). In neither case do we have a viable accuracy-based argument that probabilities are rationally preferable to non-probabilities.

2.3. **Other conditions.** There is no domination result for strictly proper scores with $\ell^\infty(\mathscr{P}\Omega)$ continuity where Ω is countably infinite. But maybe adding some other condition will yield a domination result. The original Predd *et al.* strict domination theorem was for additive scores. Perhaps we have some hope of a domination theorem in our infinite case for additive scores. Specifically, an additive score will be of the form:

$$s(c)(\omega) = \sum_{A \in \mathscr{P}\Omega} s_A(c(A), 1_A(\omega)),$$

for some family of functions $\{s_A\}_{A\in\mathscr{P}\Omega}$, where $1_A(\omega)$ is 1 if $\omega \in A$ and 0 otherwise. Now, $\mathscr{P}\Omega$ is an uncountable set. Note that for the infinite sum to make sense and have a value in $[M, \infty]$, for each ω, at most countably many summands can be negative.

However, we have the following no-go result that shows that additive scores fail to be probability distinguishing:

Proposition 3. *Assume the Axiom of Choice. If Ω is countably infinite, then any additive score s has the property that there is a credence $c \notin \mathcal{P}_{\mathrm{f}}$ and a probability $p \in \mathcal{P}_{\mathrm{c}}$ such that $s(c) = s(p)$ everywhere on Ω. In particular, no additive score is c.a. or f.a. quasi-strictly proper.*

The proof is given in the Appendix. More generally, it follows from Proposition 3 that no argument based on additive scores will allow one to say that probabilities (either of the c.a. or the f.a. sort) are always rationally superior to non-probabilities for a countably infinite Ω.

Another somewhat plausible condition on a scoring rule from the literature that might be used in an argument for probabilism would be strict truth-directedness. A scoring rule s is strictly truth directed on a set of credences provided that if c' is truer than c at ω, then $s(c')(\omega)$ is better than $s(c)(\omega)$, where c' is truer than c at ω provided that $c'(A) \geq c(A)$ whenever A contains ω and $c'(A) \leq c(A)$ whenever A does not contain ω, with strict inequality in at least one case. Campbell-Moore and Levinstein [1] have argued that truth-directedness is more initially plausible than strict propriety, though it can be used to prove strict propriety given propriety and additivity. We know that additivity is not a tenable condition in infinite cases (Proposition 3, above). We also know that continuity, propriety and strict truth-directedness are insufficient for a (strict) domination theorem for finite Ω.[3]

This might lead us to conjecture that in the infinite c.a. case we could run an argument for probabilism on the basis of scoring rules that are continuous (either just on the probabilities or fully $\ell^\infty(\mathscr{P}\Omega)$ continuous), strictly proper and strictly truth-directed. But this is hopeless, as there are no such scoring rules. More generally:

Proposition 4. *Assume the Axiom of Countable Choice.*[4] *Then there is no strictly truth-directed scoring rule on the set of all extreme credences.*

Here, a credence is *extreme* provided that the credence of every event is either zero or one. Note that one extreme credence is truer than another just

[3]See [Anonymized].

[4]The Axiom of Countable Choice is the special case of the Axiom of Choice where the collection of non-empty disjoint sets for which a choice function is sought is countable.

in case the former gets right all the events the latter does but not conversely. The proof is given in the Appendix.

2.4. **Larger spaces.** Our example of a strictly proper scoring rule that does not support any domination result was on a countably infinite space. What can we say if our space has larger cardinality, such as the interval $[0,1]$?

First note that while the powerset algebra is very natural for a countably infinite space, it is not a natural σ-algebra for c.a. probabilities on typical uncountable spaces such as $\Omega = [0,1]$. For instance, if Ω is a topological space, then a very natural σ-algebra is the Borel σ-algebra $\mathcal{B}$ generated by the open sets. Note that when our σ-algebra is not $\mathscr{P}\Omega$, we additionally require of a scoring rule s that the score of any credence be a measurable function on Ω with respect to the σ-algebra. While our definitions of propriety and domination were given in the special case of probabilities on the σ-algebra $\mathscr{P}\Omega$, they naturally generalize to any σ-algebra $\mathcal{F}$. Let $\mathcal{P}_{\mathrm{c}}(\mathcal{F})$ be the set of all c.a. probabilities on $\mathcal{F}$ and let $\mathcal{C}(\mathcal{F}) = [0,1]^{\mathcal{F}}$.

We then have the following result, with proof sketched in the Appendix.

Proposition 5. *Assume the Axiom of Countable Choice. If $\mathcal{F}$ is countably generated and contains all singletons, then there is a strictly proper uniformly bounded scoring rule s on $\mathcal{P}_{\mathrm{c}}(\mathcal{F})$ such that $p \mapsto s(p)(\omega)$ is continuous with respect to the $\ell^\infty(\mathcal{F})$ topology on $\mathcal{C}(\mathcal{F})$ for each fixed ω and there is a credence $c \notin \mathcal{P}_{\mathrm{c}}(\mathcal{F})$ with c not even non-strictly s-dominated by any c.a. probability p.*

This yields a very broad family of examples. The Borel σ-algebra on any Polish space (a metric space with a countable dense subset) is countably generated. In particular, there will be a counterexample to domination of non-probabilities by probabilities for the Borel σ-algebra on $\mathbb{R}^n$ for any n.

On the other hand, there are cases where there are no strictly proper scoring rules. Let κ be a cardinal, and consider the space $\Omega = \{0,1\}^\kappa$, the set of all κ-long sequences of zeroes and ones. One can think of this space as recording outcomes of κ coin flips, though depending on the probability function, these coin flips may not be fair or independent. Say that a subset A of Ω depends only on coordinates in $C \subseteq \kappa$ provided that for any ω and ω' in Ω, if $\omega(x) = \omega'(x)$ for all $x \in C$, then $\omega \in A$ if and only if $\omega' \in A$. Then the product σ-algebra $\mathcal{F}$ is generated by the subsets of Ω that depend only on finitely many coordinates. Note that each member of the product σ-algebra depends only on countably many coordinates.

For a fixed ω define the extreme (c.a.) probability u_ω concentrated on ω by $u_\omega(A) = 1$ if $\omega \in A$ and $u_\omega(A) = 0$ otherwise. Note that if κ is uncountable, then no singleton $\{\omega\}$ is a member of $\mathcal{F}$ (since all the coordinates are needed to specify a singleton, but the members of $\mathcal{F}$ depend only on countably many coordinates), but nonetheless u_ω and $u_{\omega'}$ are not identical if $\omega \neq \omega'$, since in the latter case there is some subset A that depends on only one coordinate (namely, a coordinate x such that $\omega(x) \neq \omega'(x)$), and hence is in $\mathcal{F}$, that contains ω but not ω' so that $u_\omega(A) \neq u_{\omega'}(A)$. In the Appendix, the following is proved by showing that there are at most κ^ω measurable functions on Ω.

Proposition 6. *Assume the Axiom of Choice. If κ is such that $2^\kappa > \kappa^\omega$, and $\Omega = \{0,1\}^\kappa$ and $\mathcal{F}$ is the product σ-algebra, then for any scoring rule, there are extreme probabilities concentrated on different points that have the same score.*

And hence there is no c.a. or f.a. strictly proper scoring rule on this Ω (extreme probabilities are automatically c.a.). Note that the inequality $2^\kappa > \kappa^\omega$ holds if $\kappa = \mathfrak{c}$ or more generally if $\kappa = \lambda^\omega$ for any $\lambda \geq 2$,[5] and that a necessary condition for $2^\kappa > \kappa^\omega$ is that κ is uncountable.[6]

2.5. **Bayesian update.** Any strictly proper scoring rule supports an argument for Bayesian update in the infinite case exactly as in the finite case, and we have seen that there is a strictly proper scoring rule on a countable Ω, and more generally on the Borel σ-algebra of any Polish space.

To be explicit, consider a choice between update strategies that specify how one changes one's credence upon learning which event in a finite partition $\{F_1, F_2, \ldots, F_n\}$ obtains, where for each i we have $p(F_i) > 0$ for one's prior credence p, which we assume to be a probability, countably additive in this case. On Bayesian update, one's credence goes from p to $p_i = p(\cdot \mid F_i)$ if F_i is learned. But we can have some alternative update strategy where one's credence goes to some other set of credences depending on which of the F_i one learns. Let's consider a strategy where one's credence goes to c_i

[5]I am grateful to [anonymized] for pointing me to the case of λ^ω.

[6]In case one thinks that the lack of a strictly proper scoring rule in Proposition 6 is solely due to the fact that $\mathcal{F}$ lacks singletons, let $\mathcal{F}^*$ be the σ-algebra whose members are sets that differ from members of $\mathcal{F}$ at countably many points (i.e., $A \in \mathcal{F}^*$ if and only if there is a $B \in \mathcal{F}$ such that $A - B$ and $B - A$ are countable). Partition κ into two disjoint subsets κ_1 and κ_2 with κ_1 having the same cardinality as κ and κ_2 infinite (this uses the Axiom of Choice). For any $\alpha \in \{0,1\}^{\kappa_1}$, we define a c.a. probability q_α as follows. For any function f from κ to $[0,1]$, there is a unique c.a. probability r_f on the product σ-algebra that makes the κ coin tosses be independent and assigns probability $f(x)$ to the xth toss being 1, i.e., $r_f(\{\omega \in \{0,1\}^\kappa : \omega(x) = 1\}) = f(x)$. Let $f(x) = \alpha(x)$ for $x \in \kappa_1$ and $f(x) = 1/2$ otherwise. Given $A \in \mathcal{F}^*$, let A' be the unique member of $\mathcal{F}$ that differs from A in countably many points, and let $q_\alpha(A) = r_f(A')$. Note that q_α assigns zero probability to every singleton and hence every countable set. Strict propriety of a scoring rule s then requires that $s(q_\alpha)$ and $s(q_\beta)$, for $\alpha \neq \beta$, differ on some set that has non-zero q_α probability, and hence that they differ on some uncountable set. Define the equivalence relation $\sim$ on $\mathcal{F}^*$-measurable functions by $u \sim v$ if and only if u and v are equal outside a countable set. Any $\sim$-equivalence class contains exactly one $\mathcal{F}$-measurable function, and we have shown that by strict propriety $s(q_\alpha)$ and $s(q_\beta)$ must be in different $\sim$-equivalence classes if $\alpha \neq \beta$. Thus strict propriety requires that there be at least as many $\mathcal{F}$-measurable functions as $\alpha \in \{0,1\}^{\kappa_1}$. There are 2^κ such α, while the proof of Proposition 6 show that there are at most κ^ω real-valued functions that are $\mathcal{F}$-measurable.

on F_i. Then one's expected score will be:

$$E_p\left(\sum_{i=1}^n 1_{F_i} s(c_i)\right) = \sum_{i=1}^n p(F_i) E_{p_i}(s(c_i))$$
$$\geq \sum_{i=1}^n p(F_i) E_{p_i}(s(p_i)) = E_p\left(\sum_{i=1}^n 1_{F_i} s(p_i)\right).$$

The right hand side is the expected score of the Bayesian update strategy. By strict propriety, the inequality is strict unless $c_i = p_i$ for every i, and hence a Bayesian update strategy beats any other update strategy. Nothing in this argument depends on the finiteness of Ω (and if we like, we can even extend it to a countably infinite partition $\{F_i\}$).

3. Finitely additive probabilities

3.1. **Strict propriety.** An f.a. strictly proper scoring rule would have to assign a different score, i.e., a different extended-real function on Ω, to every different f.a. probability. But, given the Axiom of Choice, it can be shown that there are more f.a. probabilities than extended-real functions on Ω if Ω is infinite. Indeed, recalling that a credence function is *extreme* provided that it assigns 0 or 1 to every event, we have:

Proposition 7. *Assume the Axiom of Choice. Suppose Ω is infinite. Then for any scoring rule s on $\mathcal{P}_{\mathrm{f}}$, there are different extreme f.a. probabilities that have the same score everywhere on Ω.*

Proof. Let κ be the cardinality of Ω. Any ultrafilter $\mathfrak{U}$ on Ω defines a unique extreme f.a. probability measure $p_{\mathfrak{U}}$ such that $p_{\mathfrak{U}}(A) = 1$ if $A \in \mathfrak{U}$ and $p_{\mathfrak{U}}(A) = 0$ otherwise, and there are 2^{2^κ} ultrafilters on Ω by the Axiom of Choice [3, Theorem II.7.1]. On the other hand the cardinality of $[0,1]^\Omega$ is $(2^{\aleph_0})^\kappa = 2^{\aleph_0 \times |\Omega|} = 2^\kappa$ (where we also used the Axiom of Choice). □

It immediately follows that:

Corollary 1. *Assume the Axiom of Choice. Suppose Ω is infinite. Then no scoring rule is f.a. strictly proper on $\mathcal{P}_{\mathrm{f}}$.*

This result was first obtained by Michael Nielsen (correspondence) under additional continuity assumptions.

Taking the algebra on which our probabilities are defined to be $\mathscr{P}\Omega$ is very natural if Ω is countable, but less natural when Ω is not countable. But there is another natural case, that of the Borel sets in a Polish space. From Proposition 5 we learned that there is always a c.a. strictly proper scoring rule in such a case. This is not true in the infinite f.a. case (with the natural extension of definitions to f.a. probabilities defined over an algebra smaller than $\mathscr{P}\Omega$).

Proposition 8. *Assume the Axiom of Choice. If Ω is an infinite Polish space, and $\mathcal{B}$ are the Borel sets on Ω, then for any scoring rule s on $\mathcal{P}_{\mathrm{f}}(\mathcal{B})$, there are different extreme f.a. probabilities on $\mathcal{B}$ that have the same score on Ω.*

Proof. There are continuum many Borel measurable functions between two infinite Polish spaces, and $[M, \infty]$ is a Polish space. On the other hand, any countable set on a Polish space will be in $\mathcal{B}$. If A is a countably infinite subset of Ω, then there will be $2^{2^{\omega}}$ extreme f.a. probabilities (see the proof of Proposition 7) on A, each of which extends to an f.a. probability on $\mathcal{B}$. □

3.2. **Coincident scores.** Note that Propositions 7 and 8 are closely analogous to Proposition 6. All three tell us that in certain cases there are more (f.a. or c.a., respectively) extreme probabilities than possible scores, and hence for any scoring rule, there will be distinct probabilities, even extreme probabilities, that get the same score.

There is reason to think that this is bad news for the prospects of accuracy-theoretic analysis of credences in these cases. The f.a. cases appear somewhat more problematic than the c.a. cases, since the f.a. cases include the very natural case of the powerset algebra on a countably infinite Ω and that of Borel measures on Polish spaces, while the c.a. cases concern rather outré situations like those of uncountably many coin tosses, whereas the more common case of Borel measures on Polish spaces have proper scoring rules.

In any case, why are these theorems bad news for accuracy-theoretic analyses?

First, there is some reason to think that scoring rules should be strictly proper and not just proper. It is plausible that any probability function could be the credence assignment of some rational agent given some collection of evidence[7] and hence is rationally acceptable under some circumstances. But it is also plausible that once one has a rational probability assignment, it is irrational by one's lights to evidencelessly switch to a different assignment. But if one's expected score for switching would be no worse than one's expected score for one's current credences, it would not be irrational to switch. Hence, we have reason to accept strict propriety if an accuracy-theoretic framework is to be plausible, and yet in our infinite finitely-additive case we cannot have strict propriety.

Second, Pruss has shown (after translating from his accuracy setting) in the finite case that if a continuous proper scoring rule on probabilities has the property that $E_r s(r) < \infty$ for all r, but fails to satisfy $E_p s(p) < E_p s(q)$ for some probabilities p and q, then $s(p)$ and $s(q)$ are identical (the condition

[7] This privileges probability functions over other credence assignments, but in a defensible way: it is highly controversial whether a non-probabilistic credence could be rational, but much more reasonable to think that any probabilistic one could be. Though, admittedly, it is *somewhat* controversial to think that a merely f.a. probability could be a rational assignment.

that $E_r s(r) < \infty$ is a kind of anti-pessimism condition: no probabilistic credence is such that by its own lights it is maximally inaccurate), and suggested [18] that this yields an argument for strict propriety: a scoring rule should be "sufficiently 'sensitive'" to assign different scores to different probabilistic forecasts. Pruss's result adapts readily to the infinite setting for an appropriate sense of continuity (see Proposition 9 in the Appendix for details), both in f.a. and c.a. cases. Assuming propriety and continuity are reasonable conditions (and some kind of continuity assumption is needed for domination theorems of the sort used for accuracy-based arguments for probabilism [17]), it follows that we have an argument for strict propriety on the probabilities.

Moreover, Pruss's argument is particularly plausible in the case where p and q are mutually singular in the sense that there is an event A such that $p(A) = 1$ and $q(A) = 0$. For then by the lights of p, the probability distribution q should intuitively be really terrible as compared to p, because it assigns probability zero to what by the lights of p is certain. Thus we would expect to have $E_p s(p) < E_p s(q)$ at least for mutually singular p and q. But assuming continuity and propriety, Propositions 6 and 7 shows that we cannot have this strict inequality for all mutually singular p and q, since any two distinct extreme probabilities are mutually singular.

Third, a very plausible condition on a score s, a condition much weaker than strict propriety, is what we might call the mutual singularity weak improvement condition (mswic): if we have probabilities p and q and an event A such that $p(A) = 1$ and $q(A) = 0$, then $s(p)(\omega)$ is better (i.e., smaller in our inaccuracy scoring case) than $s(q)(\omega)$ for some $\omega \in A$. It immediately follows from mswic that different extreme probabilities must get different scores, which is impossible in the infinite cases covered by the Propositions.

The Propositions also show that in the relevant cases we cannot give an accuracy-based argument for Bayesian conditionalization, unlike in the c.a. Borel-measure on Polish space case. For suppose s is any score. Let p and q be different extreme probabilities that get the same score everywhere. Let E be an event such that $p(E) = 1$ and $q(E) = 0$ (for any two different extreme probabilities there is such an event). Let r be the probability defined by $r(A) = (p(A) + q(A))/2$. Then $r(E) = 1/2 > 0$. Moreover, $r(A \mid E) = p(A)$ and $r(A \mid E^c) = q(A)$. The Bayesian update strategy then is to move from r to p or q respectively depending on whether E or E^c occurs. Let's say that the "anti-Bayesian" update strategy is to move from r to q or p respectively depending on whether E or E^c occurs—i.e., it is to conditionalize on E^c if E is observed and on E if E^c is observed! But since p and q have the same score, no matter whether we use Bayesian or anti-Bayesian update, and no matter whether we observe E or E^c, we get the same score. There is thus no accuracy based reason to prefer the Bayesian strategy to the anti-Bayesian one.

This argument should should make us suspicious of scoring rules as a tool for studying inaccuracy in general in infinite cases. For regardless whether one thinks Bayesian update is rationally required, few things in epistemology are as clear as the superiority of the Bayesian to the *anti*-Bayesian strategy. But let us put these suspicions aside for a moment, and now consider the prospects for an argument for f.a. probabilism based on quasi-strictly proper scoring rules.

3.3. **Quasi-strictly proper scoring rules.** We saw that no scoring rule is f.a. strictly proper on all the credences. There are, however, f.a. quasi-strictly proper scoring rules on all the credences. For instance, we have the following trivial scoring rule:

$$\mathrm{Triv}(c)(\omega) = \begin{cases} 0 & c \in \mathcal{P}_{\mathrm{f}} \\ 1 & \text{otherwise.} \end{cases}$$

(If we prefer, we can have a fairly trivial $\ell^\infty(\mathscr{P}\Omega)$-continuous quasi-strictly proper scoring rule by applying Proposition 2 with $\mathcal{P} = \mathcal{P}_{\mathrm{f}}$, $s(p)$ being the constant function equal to zero everywhere for $p \in \mathcal{P}_{\mathrm{f}}$, c_0 any member of $\mathcal{C} - \mathcal{P}_{\mathrm{f}}$, and f_0 being the constant function equal to one everywhere.)

Nielsen [11] has shown that any f.a. quasi-strictly proper scoring rule s that is continuous on $\mathcal{P}_{\mathrm{f}}$ with respect to the product topology satisfies the strict domination property that for any credence c outside $\mathcal{P}_{\mathrm{f}}$, there is a credence $p \in \mathcal{P}_{\mathrm{f}}$ that strictly s-dominates c.

Now an argument for probabilism on the basis of strict domination with respect to a scoring rule like Triv is singularly uncompelling. Any non-member of $\mathcal{P}_{\mathrm{f}}$ is Triv-dominated by any member of $\mathcal{P}_{\mathrm{f}}$ simply because we have stipulated that the score of a non-member of $\mathcal{P}_{\mathrm{f}}$ is everywhere 1 and that of a member of $\mathcal{P}_{\mathrm{f}}$ is everywhere 0. This is simply *ad hoc*. What we want for a compelling argument for probabilism are scoring rules that aren't *ad hoc*. It is, of course, difficult to say rigorously what makes a scoring rule *ad hoc*.

But in any case, an argument for probabilism on the basis of *merely* quasi-strictly proper scoring rules is uncompelling. Quasi-strict propriety treats non-probability credences from the outset as second-class citizens, by requiring that $E_p s(p) < E_p s(c)$ whenever p is a probability and c is not, but only requiring a non-strict inequality when both are probabilities (with c still distinct from p). It is unimpressive that a scoring rule chosen to satisfy a condition that disadvantages non-probability credences, and which does not satisfy a similar strictness condition in the case of probability credences, can be used to yield the conclusion that non-probability credences are rationally inferior.

Moreover, consider the intuitive idea that a scoring rule should yield the judgment that any rationally acceptable credence would make it irrational to evidencelessly switch credences. This idea yields strict propriety. To make it yield only quasi-strict propriety, one would need to make an *ad hoc* restriction that it only applies in cases where one is considering switching to a non-probability credence, but there is no reason—unless perhaps one already accepts probabilism—to make that restriction. And Pruss's [18] earlier discussed argument for strict propriety as it stands only yields strict propriety as restricted to the probabilities. Quasi-strict propriety, thus, seems to be unmotivated once one drops strict propriety.

For purposes of an argument for probabilism, we might try to replace the unachievable strict propriety condition with propriety conjoined with some intuitively compelling conditions that suffice for a domination argument for probabilism but do not bake-in an *ad hoc* preference for probabilities over non-probabilities in the way quasi-strict propriety does. Two candidates for such conditions would be additivity and strict truth-directedness. However, by Proposition 3 we know that no additive score yields an accuracy argument for f.a. or c.a. probabilism in infinite cases, since additive scores fail in general to distinguish probabilities (in either sense) from non-probabilities, while Proposition 4 showed that there are no strictly truth-directed scoring rules on an infinite space.

It is also worth noting that Nielsen's domination theorem involves a second kind of unjustified distinguishing of non-probabilistic credences. Nielsen assumes the scoring rule is continuous on $\mathcal{P}_{\mathrm{f}}$ with respect to the product topology on $[0,1]^{\mathscr{P}\Omega}$. But while continuity is a plausible condition on an epistemic utility or scoring rule, requiring a scoring rule's continuity on $\mathcal{P}_{\mathrm{f}}$ without requiring continuity on all of $\mathcal{C}$ is difficult to justify. But it follows from Proposition 1 that no scoring rule that is continuous with respect to the product topology on $\mathcal{C}$ is either f.a. or c.a. quasi-strictly proper.

4. Two potential ways out of cardinality problems

4.1. **Countable opinion sets.** Some of our results are based on cardinality considerations. For instance, it is because there are more finitely additive probabilities than scores that there is no f.a. strictly proper scoring rule for infinite Ω (Corollary 1), because $\mathscr{P}\Omega$ is uncountable that there are no quasi-strictly additive scoring rules for countably infinite Ω (Proposition 3), etc. But if we restrict ourselves to f.a. probability functions defined over a countable algebra of events there is much more hope. For instance, if $\mathcal{F}$ is a countably infinite algebra of events, then we can define a weighted Brier inaccuracy score by enumerating $\mathcal{F} = \{F_1, F_2, \dots\}$ and letting

$$B(p)(\omega) = -\sum_{n=1}^{\infty} 2^{-n}(1_{F_n}(\omega) - p(F_n))^2.$$

This will be additive, obviously strictly truth-directed and proper since it's a weighted sum of Brier scores. Moreover, it is elementary to verify that if we let $B_n(p)(\omega) = (1_{F_n}(\omega) - p(F_n))^2$, then $E_p B_n(p) \le E_p B_n(c)$ with equality if and only if $p(F_n) = c(F_n)$ (cf. the proof of Proposition 5 in the Appendix).

Using Kelley's [9] domination results, one can then run an argument for f.a. probabilism on $\mathcal{F}$.

However, the denial of probabilism is simply the claim that there is *at least one* rationally acceptable non-probabilistic credence. Nor need the denier of probabilism hold that there is such a credence on every algebra of events, just that there is one on *some* algebra of events. If the arguments for probabilism succeed in the finite case, we already know that it is not rational to have a non-probabilistic credence on a finite algebra $\mathcal{F}$ of subsets of Ω. (Normally that is argued in the case where Ω itself is finite, but the arguments trivially extend to the case where Ω is infinite but $\mathcal{F}$ is finite.) And presumably everyone agrees that probabilism is the correct view for probabilities defined on the algebra $\{\varnothing, \Omega\}$. If we are going to extend the arguments for probabilism to infinite cases, we should extend them to all of them—including ones where $\mathcal{F} = \mathscr{P}\Omega$—or at least all the ones on spaces whose cardinality is small enough for them to come up in practice (in science, one rarely if ever deals with spaces of cardinality greater than that of the continuum).

It is also worth noting that in classical probability theory, any c.a. probability function is required to be defined on a σ-algebra, and any infinite σ-algebra is well known to be automatically uncountable.

One may respond that it is not possible for human-like epistemic agents to have a credence assigned to an uncountable set of events, and hence that we only need to establish probabilism for countable sets of events. Now if assigning a credence to a set of events involves having some mental credence value stored separately for each event, then the finitude of ours minds does rule out having a credence assigned to an uncountable set of events—but it also rules out credences being assigned to a countably infinite set of events, and hence there is little point to extending probabilism beyond finite cases if our concern is with human-like agents.

But even in finite cases it is unrealistic to think of credence assignments as involving separate mental credence values for each event. For instance, as soon as I hear of a fair lottery with tickets numbered $1, 2, 3, ..., 20$, I assign credence $1/20$ to each ticket winning. But I do not think through each of the twenty atomic outcomes and gain a separate mental property corresponding to that outcome's having credence $1/20$. Rather, I endorse some sort of rule like "Each one gets $1/20$." And I certainly do not have a separate mental state for each of the 2^{20} events on the space, but I do endorse a rule like "Each event A gets probability $|A|/20$." And if it suffices for the assigning of a credence that one endorse some rule, then it is quite possible to assign a probability to an uncountable set of events. For instance, in an infinite lottery with tickets numbered $1, 2, 3, ...$, I can by a single rule assign the probabilistic credence $\sum_{n \in A} 2^{-n}$ to each of the uncountably many events A in $\mathscr{P}\{1, 2, 3, ...\}$. And I can assign a non-probabilistic credence by having a rule that where the above rule holds except when $A = \{2, 3\}$, in which case the credence is $1/3$ instead.

Thus, likely, we not only want probabilism in finite cases, but also in infinite cases, including cases where the set of events is the powerset of a countably infinite set as in some of the cases in this paper. But the accuracy arguments fail for such cases.

4.2. **Beyond extended real values.** What underlies the proof of Corollary 1 is the observation that there are more f.a. probability functions on $\mathscr{P}\Omega$ than possible extended real-valued scores on Ω when Ω is infinite. The approach just considered circumvented this by constricting the set of probability functions by considering only functions defined on a countable algebra of events. A complementary approach is to enrich the set of possible scores by replacing real numbers with a real closed field of larger cardinality, say a large field of hyperreals.

At this point, I cannot definitely rule out the possibility of an accuracy-based argument for probabilism based on hyperreal scores. However, two technical difficulties would need to be overcome.

First, some kind of continuity assumption is known [17] to be needed to prove strict domination theorems for a finite Ω. But it is not clear what kind of analogue to continuity we would have for hyperreal-valued scores.

Second, the concept of propriety depends on expected values of the score with respect to a probability. On a finite space, there is no difficulty defining an expected value of a hyperreal score—one just does a finite weighted average. But we do not know how to compute the expected value of a hyperreal score with respect to a standard probability measure. Even the concept of measurability of scores, essential when the probabilities are defined on an algebra smaller than $\mathscr{P}\Omega$, is problematic.

It is worth noting that if one can make sense of expectations of hyperreal valued scores and define a strictly proper score, then the standard expected inaccuracy argument for Bayesian update will work, as it does not require any continuity.

But while hyperreals have often been invoked by formal epistemologists, there is another approach that may be a better fit here and that does yield an argument for Bayesian update: scores whose values are nets.

Let $\mathfrak{A}$ be a directed set: a set with a partial preorder $\trianglelefteq$ and such that for any $a, b \in A$ there is a c with $a \trianglelefteq c$ and $b \trianglelefteq c$. A net in a set X is a function from a directed set to X. Let $X^{\mathfrak{A}}$ be the set of $\mathfrak{A}$-nets in X, i.e., functions from $\mathfrak{A}$ to X. We define the transitive relations $\prec$ and $\preceq$ on $[-\infty, \infty]^{\mathfrak{A}}$ as follows: $x \prec y$ if and only if there is an $a \in \mathfrak{A}$ such that for all $b \trianglerighteq a$ we have $x(a) < y(a)$ and $x \preceq y$ if and only if there is an $a \in \mathfrak{A}$ such that for all $b \trianglerighteq a$ we have $x(a) \leq y(a)$. Note that $x \prec y$ is in general a stronger claim than the conjunction $x \preceq y$ and $y \not\preceq x$.

Suppose Ω has an algebra $\mathcal{F}$. Given a function $f : \Omega \to [M, \infty]^{\mathfrak{A}}$, say that f is $\mathcal{F}$-measurable if and only if $\omega \mapsto f(\omega)(a)$ is $\mathcal{F}$-measurable for any fixed $a \in \mathfrak{A}$. For a f.a. probability p, define $E_p f$ to be the net such that $(E_p f)(a)$ is the expectation of $\omega \mapsto f(\omega)(a)$ (which function will sometimes also be written $f(\cdot)(a)$) for any fixed $a \in \mathfrak{A}$. An $\mathfrak{A}$-net-valued scoring rule $\mathfrak{s}$ is then a function from a set $\mathcal{D}$ of credences on $\mathcal{F}$ to $[M,\infty]^{\mathfrak{A}}$. If $\mathcal{P}$ is a subset of the f.a. probabilities on $\mathcal{F}$, the scoring rule is $\mathcal{P}$-proper provided that $E_p\mathfrak{s}(p) \preceq E_p\mathfrak{s}(c)$ for any $p \in \mathcal{P}$ and any $c \in \mathcal{D}$. The propriety is strict provided that we can replace $\preceq$ with $\prec$ whenever $p \neq c$.

We do not know what we can say about probabilism and domination in this setting given finite additivity. Given countable additivity, we had counterexamples to domination results for real-valued scores, and these of course apply just as much in this setting, since real numbers can be identified with $\mathfrak{A}$-nets in the reals for any singleton $\mathfrak{A}$.

However, we can say that a strictly proper net-valued scoring rule yields an expected inaccuracy argument for the optimality of Bayesian update. In the setting of Section 2.5, given the Bayesian update strategy where the credence goes to p_i on F_i and the non-Bayesian update where the credence goes to c_i on F_i, let $I = \{i : p_i \neq c_i\}$ be the nonempty set of indices for which the non-Bayesian update deviates from Bayesian update. For each $i \in I$, there is an $a_i \in \mathfrak{A}$ such that

$$E_{p_i}(\mathfrak{s}(p_i)(\cdot)(b)) < E_{p_i}(\mathfrak{s}(c_i)(\cdot)(b))$$

whenever $b \trianglerighteq a_i$, and of course for $i \notin I$, we have equality for all b. Because $\mathfrak{A}$ is a directed set and I is finite, there is an $a \in \mathfrak{A}$ such that $a \trianglerighteq a_i$ for all $i \notin I$. Then for any $b \trianglerighteq a$, we have:

$$\begin{aligned} E_p\left(\sum_{i=1}^n 1_{F_i}\mathfrak{s}(c_i)(\cdot)(b)\right) &= \sum_{i=1}^n p(F_i)E_{p_i}(\mathfrak{s}(c_i)(\cdot)(b)) \\ &> \sum_{i=1}^n p(F_i)E_{p_i}(\mathfrak{s}(p_i)(\cdot)(b)) = E_p\left(\sum_{i=1}^n 1_{F_i}\mathfrak{s}(p_i)(\cdot)(b)\right), \end{aligned}$$

because we have strict inequality for any an $i \in I$ and equality for the other indices. It follows that

$$E_p\left(\sum_{i=1}^n 1_{F_i}\mathfrak{s}(p_i)\right) \prec E_p\left(\sum_{i=1}^n 1_{F_i}\mathfrak{s}(c_i)\right),$$

and hence the expected value of the net-valued inaccuracy for Bayesian update beats that for the non-Bayesian update for any strictly proper net-valued score $\mathfrak{s}$.

And there will always be a strictly proper net-valued score. Let $\mathfrak{A}$ be the set of all finite subalgebras of $\mathcal{F}$ ordered by inclusion. For any $\mathcal{G} \in \mathfrak{A}$, define the normalized Brier score:

$$\mathfrak{b}(c)(\omega)(\mathcal{G}) = -\frac{1}{|\mathcal{G}|}\sum_{A\in\mathcal{G}}(1_A(\omega) - c(A))^2.$$

Then $\mathfrak{b}(c)(\omega)$ is a member of $[-1,0]^{\mathfrak{A}}$, and $\mathfrak{b}$ is a strictly proper net-valued scoring rule. Propriety follows from the propriety of Brier scores. To demonstrate strict propriety, note that if p is a probability and c is a credence, and

$p \neq c$, then the E_p expectation of $\omega \mapsto \mathfrak{b}(p)(\omega)(\mathcal{G})$ is strictly less than that of $\omega \mapsto \mathfrak{b}(c)(\omega)(\mathcal{G})$ as long as $\mathcal{G}$ contains at least one set A on which p and c disagree, by the strict propriety of Brier scores. Hence we will have a strict inequality as long as $\mathcal{G} \supseteq \mathcal{H}$ where $\mathcal{H} = \{\varnothing, \Omega, A, A^c\}$, and so we have $E_p\mathfrak{b}(p) \prec\!\!\cdot\, E_p\mathfrak{b}(c)$.[8]

There is also a weaker version of strict propriety in the net-valued case where instead of the relation $\prec\!\!\cdot$, we use the weaker relation $\prec$ where $x \prec y$ if and only if $x \preceq y$ but not $y \preceq x$. Then $x \prec y$ just in case $x \preceq y$ and for any $a \in \mathfrak{A}$ there is a $b \trianglerighteq a$ such that $x(b) < y(b)$. The above argument for strict propriety generalizes to yield a $\prec$-inequality for expected inaccuracies if we assume $\prec$-inequality in our definition of scoring rules.

Further research is needed to determine whether either of the definitions of strict propriety yields a plausible condition on net-valued scoring rules, and whether the corresponding inequality for expected inaccuracies yields a serious problem for the rationality of non-Bayesian update.

5. Conclusions

In the countably additive case on a countably infinite space, we have a strictly proper scoring rule that is continuous in a reasonable topology on the space of all credences, but that does not support any domination result, and hence does not yield an argument for probabilism. The same is true in the case of the Borel σ-algebra on any Polish space (metric space with countable dense subset).

One might hope to gain an argument for probabilism by adding further intuitively plausible conditions on the scoring rule, but we do not at present know what condition might do the job. Two conditions from the literature fail to help. Additivity does not do the job, at least for countably infinite Ω, as for any additive scoring rule there will be a probability and a non-probability (in either sense of probability) that get the same score, and strict truth-directedness does not help as on an infinite space there is no strictly truth-directed scoring rule defined on the powerset of the sample space.

A standard accuracy argument for Bayesian update runs just as well in the countably additive infinite case as in finite cases when there is a strictly proper scoring rule, and as we saw, in some natural cases there is such a scoring rule. That said, there isn't a strictly proper scoring rule in every case: for instance, there is a σ-algebra representing continuum-many coin flips where there is no strictly proper scoring rule.

In the finitely additive case, there are no strictly proper scoring rules when the probabilities are defined on the powerset, or when they are defined on the Borel sets of a Polish space. Nielsen [10] has proved an interesting domination result for quasi-strictly proper scoring rules, but requiring quasi-strict propriety stacks the deck against non-probabilistic credences and hence is

[8] It is worth noting that we can generate a hyperreal scoring rule by using an ultraproduct construction with respect to an ultrafilter on $\mathfrak{A}$ and the above construction.

dialectically inadequate in an argument for probabilism. Moreover, there does not appear to be a good argument for quasi-strict propriety that isn't also an argument for strict propriety. We might try to run an argument for probabilism using non-strict propriety and some additional conditions, but we do not at present know what additional conditions might do the job here—neither additivity nor strict truth-directedness does.

Furthermore, the non-strictly proper scoring rules that exist in the finitely additive case with the probabilities defined on the powerset, and in the uncountably many coin flips case, not only do not support an accuracy argument for Bayesian update, but every such scoring rule has the unhappy feature that in some cases it will assign exactly the same score to a Bayesian update as to an anti-Bayesian update (where one conditionalizes on the complement of the evidence).

If we are willing to restrict to credences defined over countable opinion sets and work with finitely additive probabilities, we can get an argument for finitely additive probabilism based on results of Kelley [9]. However, such a restriction is a serious one and allows the anti-probabilist a way out in infinite cases.

There is a need for future research concerning scoring rules with values that are not extended real numbers. The current state of the art does not appear sufficient to evaluate this possibility for hyperreal values. We can, however, define strict propriety for scoring rules whose values are nets of real numbers, and such strictly proper scoring rules will always exist and yield an expected inaccuracy argument for Bayesian update. We do not know what happens for domination for net-valued scoring rules.[9]

Appendix: Proofs

Proof of Proposition 1. To obtain a contradiction, suppose that s is continuous at a fixed $p \in \mathcal{P}$. For $x \in [M, \infty)$ and $n > 0$, let $B_n(x) = (x - 1/n, x + 1/n)$ and let $B_n(\infty) = (\max(M, n), \infty]$. Then $B_n(x)$ is a neighborhood of x and $\bigcap_n B_n(x) = \{x\}$. Let

$$V_{\omega,n} = \{s' \in [M, \infty]^\Omega : s'(\omega) \in B_n(s(p)(\omega))\}.$$

This is a neighborhood of $s(p)$ in in the product topology on $[M, \infty]^\Omega$. Note that

$$\bigcap_{(\omega,n)\in\Omega\times\mathbb{N}} V_{\omega,n} = \{s(p)\}.$$

For a finite subset F of $\mathscr{P}\Omega$, a credence c and an $r > 0$, let

$$B(c, F, r) = \{c' \in \mathcal{C} : \forall A \in F(|c(A) - c'(A)| < r)\}.$$

The collection of the $B(c, F, r)$ forms a neighborhood basis for $\mathcal{C}$ with the product topology.

[9]I would like to thank [anonymized] for much extended correspondence on these topics and [anonymized] for help with cardinal exponentiation. I am grateful to two anonymous readers whose comments have helped improve the clarity of the text and arguments.

Since s is continuous at $p \in \mathcal{P}$, for each $(\omega, n) \in \Omega \times \mathbb{N}$, using the Axiom of Choice, select a finite $F_{\omega,n} \subseteq \mathscr{P}\Omega$ and an $r_{\omega,n} > 0$ such that if $W_{\omega,n} = B(p, F_{\omega,n}, r_{\omega,n})$, then $s[W_{\omega,n}] \subseteq V_{\omega,n}$. Observe that:

$$s\left[\bigcap_{(\omega,n)\in\Omega\times\mathbb{N}} W_{\omega,n}\right] \subseteq \bigcap_{(\omega,n)\in\Omega\times\mathbb{N}} s[W_{\omega,n}] \subseteq \bigcap_{(\omega,n)\in\Omega\times\mathbb{N}} V_{\omega,n} = \{s(p)\}.$$

Since s is $\mathcal{P}$-distinguishing, nothing in $\mathcal{C} - \mathcal{P}$ can have the same score as p. Hence, $W = \bigcap_{\omega,n} W_{\omega,n}$ must be a subset of $\mathcal{P} \subseteq \mathcal{P}_{\mathrm{f}}$. But we shall see this is impossible. For let $F = \bigcup_{\omega,n} F_{\omega,n}$. This is a union of $|\Omega \times \mathbb{N}| = |\Omega|$ many finite sets and hence $|F| \leq |\Omega|$ (here we used the Axiom of Choice twice). Thus by Cantor's Theorem, $|F| < |\mathscr{P}\Omega|$ and since F is a subset of $\mathscr{P}\Omega$, it must be a proper subset of it. Now, $p \in W$, and any credence c such that $c(A) = p(A)$ for all $A \in F$ will be a member of each of the $W_{\omega,n}$ and hence of W. Fix any $B \notin F$. Let c be any credence function such that $c(A) = p(A)$ for all $A \in F$, but such that $c(B)$ is chosen so as to be different from $1 - c(\Omega - B)$ (if $\Omega - B$ is in F, we have no choice as to the value of $c(\Omega - B)$, and if $\Omega - B \notin F$, we can let $c(\Omega - B)$ be whatever we like). Then c is not finitely additive, but is in W, a contradiction. □

We now show that Sph^* from Section 2 is strictly proper. Suppose first that p and q are probabilities. Restricted to the probabilities, Sph^* is just the spherical scoring rule, which is well-known to be strictly proper [6, Section 4.1], but a proof is given for completeness. We have:

$$E_p \mathrm{Sph}^*(q) = -\frac{\|\hat{p}\hat{q}\|_1}{\|\hat{q}\|_2}$$

and

$$E_p \mathrm{Sph}^*(p) = -\frac{\sum_n \hat{p}_n^2}{\|\hat{p}\|_2} = -\frac{\|\hat{p}\|_2^2}{\|\hat{p}\|_2} = -\|\hat{p}\|_2.$$

The Cauchy-Schwarz inequality says that $\|\hat{p}\hat{q}\|_1 \leq \|\hat{p}\|_2\|\hat{q}\|_2$, with equality if and only if one of $\hat{p}$ and $\hat{q}$ is a scalar multiple of the other. For probabilities p and q, the latter condition is met only if $p = q$. Thus:

$$E_p \mathrm{Sph}^*(q) \geq -\frac{\|\hat{p}\|_2\|\hat{q}\|_2}{\|\hat{q}\|_2} = -\|\hat{p}\|_2 = E_p \mathrm{Sph}^*(p),$$

with equality only when $p = q$.

Now suppose $c \notin \mathcal{P}_{\mathrm{c}}$ and $p \in \mathcal{P}_{\mathrm{c}}$. We need to show that $E_p \mathrm{Sph}^*(p) < E_p \mathrm{Sph}^*(c)$. Now

$$\|\mathrm{Sph}^*(c)\|_2^2 = \sum_{n=0}^{\infty} \frac{1}{(2(n+1))^2} = \frac{\pi}{24},$$

because of the Euler series formula $\sum_{n=1}^{\infty} n^{-2} = \pi/6$. By Cauchy-Schwarz again and as $(\pi/24)^{1/2} < 1$,

$$\begin{aligned}-E_p \operatorname{Sph}^*(c) &= \sum_{n=0}^{\infty} \hat{p}(n)\operatorname{Sph}^*(c)(n) \\ &\leq \|\operatorname{Sph}^*(c)\|_2 \|\hat{p}\|_2 < \|\hat{p}\|_2 = -E_p \operatorname{Sph}^*(p),\end{aligned}$$

and so $E_p \operatorname{Sph}^*(p) < E_p \operatorname{Sph}^*(c)$.

Now for $u \in \ell^1(\Omega)$, let

$$\|u\|_* = \sup_{A \in \mathscr{P}\Omega} \left| \sum_{\omega \in \Omega} u(\omega) \right|.$$

Two norms $\|\cdot\|_A$ and $\|\cdot\|_B$ on a vector space are said to be equivalent if there is a positive real constant c such that for every u in the space $\|u\|_A \leq c\|u\|_B$ and $\|u\|_B \leq c\|u\|_A$. Credences are members of $\ell^\infty(\mathscr{P}\Omega)$, the space of bounded real-valued functions on $\mathscr{P}\Omega$ with norm given by the supremum of the absolute value.

Lemma 1. *The function $\|\cdot\|_*$ on $\ell^1(\Omega)$ is a vector space norm such that $\|u\|_* \leq \|u\|_1 \leq 2\|u\|_*$ for all u. Moreover, for any $p \in \mathcal{P}_{\mathrm{c}}$, we have $\|\hat{p}\|_* = \|p\|_{\ell^\infty(\mathscr{P}\Omega)}$.*

Proof. That $\|\cdot\|_*$ is a vector space norm is easy to check using the triangle inequality and the fact that $\sup_x(f(x)+g(x)) \leq \sup_x f(x) + \sup_x g(x)$.

Furthermore,

$$\|u\|_* = \sup_{A \in \mathscr{P}\Omega} \left| \sum_{\omega \in A} u(\omega) \right| \leq \sup_{A \in \mathscr{P}\Omega} \sum_{\omega \in A} |u(\omega)| = \sum_{\omega \in \Omega} |u(\omega)| = \|u\|_1.$$

Next, given $u \in \ell^1(\Omega)$, let $B = \{\omega : u(\omega) \geq 0\}$ and $C = \Omega - B$. Then

$$\begin{aligned}\|u\|_1 &= \sum_{\omega \in B} u(\omega) - \sum_{\omega \in C} u(\omega) = \left| \sum_{\omega \in B} u(\omega) \right| + \left| \sum_{\omega \in C} u(\omega) \right| \\ &\leq 2 \max_{A \in \{B,C\}} \left| \sum_{\omega \in A} u(\omega) \right| \leq 2\|u\|_*.\end{aligned}$$

Finally, if $p \in \mathcal{P}_{\mathrm{c}}$:

$$\|p\|_{\ell^\infty(\mathscr{P}\Omega)} = \sup_{A \subseteq \Omega} p(A) = \sup_{A \subseteq \Omega} \left| \sum_{\omega \in A} \hat{p}(\omega) \right| = \|\hat{p}\|_*.$$

□

Our observations related to the continuity of Sph^* on $\mathcal{P}_{\mathrm{c}}$ in Section 2.1 together with Lemma 1 yield:

Lemma 2. *The scoring rule Sph^* is continuous on $\mathcal{P}_{\mathrm{c}}$ in the $\ell^\infty(\mathscr{P}\Omega)$ topology with respect to the $\mathcal{P}_{\mathrm{c}}$-weak topology on $[M,\infty]^\Omega$.*

To prove Proposition 2 we need one more crucial result.

Lemma 3. *The sets $\mathcal{P}_c$ and $\mathcal{P}_f$ are closed subsets of $\ell^\infty(\mathscr{P}\Omega)$.*

Proof of Lemma 3. We need to show that every member u of $\ell^\infty(\mathscr{P}\Omega) - \mathcal{P}_c$ has a neighborhood that doesn't intersect $\mathcal{P}_c$ and every member u of $\ell^\infty(\mathscr{P}\Omega) - \mathcal{P}_f$ has a neighborhood that doesn't intersect $\mathcal{P}_f$.

If $u \notin \mathcal{C}$, then there is an $A \subseteq \Omega$ such that $u(A) < 0$ or $u(A) > 1$. Then

$$\|u - p\|_{\ell^\infty(\mathscr{P}\Omega)} \geq |u(A) - p(A)|$$

for any $p \in \mathcal{P}_f$. Moreover, we have $p(A) \in [0,1]$, so if $u(A) < 0$, the right hand side is at least $-u(A)$ and if $u(A) > 1$, the right hand side is at least $u(A) - 1$. In the former case, a ball of radius $-u(A)$ around u does not intersect $\mathcal{P}_f$ and in the latter, one of radius $u(A) - 1$ does not intersect $\mathcal{P}_f$.

Next suppose that $u \in \mathcal{C} - \mathcal{P}_f$. Since we've assumed credences take values between 0 and 1, there are two possible ways for u not to be a f.a. probability: either the total probability or the finite additivity axiom fails for u.

If the total probability axiom fails, then $u(\Omega) < 1$. But for any $p \in \mathcal{P}_f$ we have $p(\Omega) = 1$, so $\|u - p\|_{\ell^\infty(\mathscr{P}\Omega)} \geq |p(\Omega) - u(\Omega)| = 1 - u(\Omega)$, and so a ball of radius $1 - u(\Omega)$ around u does not intersect $\mathcal{P}_f$.

Suppose now that finite additivity fails, so $u(A \cup B) \neq u(A) + u(B)$ for some disjoint A and B. Let $\varepsilon = |u(A) + u(B) - u(A \cup B)| > 0$. Suppose that $p \in \mathcal{P}_f$. I will show that $\|u - p\|_{\ell^\infty(\mathscr{P}\Omega)} \geq \varepsilon/3$ for any $p \in \mathcal{P}_f$, from which it follows that a ball of radius $\varepsilon/3$ around u does not intersect $\mathcal{P}_f$. To obtain a contradiction, suppose $\|u - p\|_{\ell^\infty(\mathscr{P}\Omega)} < \varepsilon/3$. Then $|u(D) - p(D)| < \varepsilon/3$ for every event D. Thus,

$$\varepsilon = |u(A) + u(B) - u(A \cup B)| < 3(\varepsilon/3) + |p(A) + p(B) - p(A \cup B)| = \varepsilon,$$

where the inequality follows from the triangle inequality and the last equality from the finite additivity of p. And that's a contradiction.

So, far we have shown that any $u \notin \mathcal{P}_f$ has a neighborhood that doesn't intersect $\mathcal{P}_f$, and hence doesn't intersect $\mathcal{P}_c$. In particular, we've shown that $\mathcal{P}_f$ is closed.

Finally, suppose that $u \in \mathcal{P}_f - \mathcal{P}_c$ so that we can show that there is a neighborhood of u that doesn't intersect $\mathcal{P}_c$. Finite but not countable additivity must hold for u. Thus we have disjoint A_i such that

$$\varepsilon = \left| u\left(\bigcup_{i=1}^\infty A_i\right) - \sum_{i=1}^\infty u(A_i) \right| > 0.$$

But we have:

$$\sum_{i=1}^\infty u(A_i) = \lim_{n\to\infty} \sum_{i=1}^n u(A_i) = \lim_{n\to\infty} u\left(\bigcup_{i=1}^n A_i\right)$$

by finite additivity. Thus:

$$\varepsilon = \left| u\left(\bigcup_{i=1}^\infty A_i\right) - \lim_{n\to\infty} u\left(\bigcup_{i=1}^n A_i\right) \right|.$$

To obtain a contradiction, suppose $\|u - p\|_{\ell^\infty(\mathscr{P}\Omega)} < \varepsilon/2$ for some $p \in \mathcal{P}_{\rm c}$. Then $|u(D) - p(D)| < \varepsilon/3$ for any D and hence by the triangle inequality again:

$$\varepsilon \leq \left| p\left(\bigcup_{i=1}^{\infty} A_i\right) - \lim_{n\to\infty} p\left(\bigcup_{i=1}^{n} A_i\right)\right| + 2\varepsilon/3 = 2\varepsilon/3,$$

by the countable additivity of p, and contradicting the positivity of ε. Hence, again, a ball of radius $\varepsilon/3$ around u does not intersect $\mathcal{P}_{\rm c}$, and our proof of the closedness of $\mathcal{P}_{\rm c}$ is complete. □

Proof of Proposition 2. By the Dugundji extension theorem [4, Theorem 4.1], together with the closedness (Lemma 3) and convexity of $\mathcal{P}$, and the local convexity of $\mathcal{P}$-weak topology, there is an $\ell^\infty(\mathscr{P}\Omega)$-continuous function $\psi : \mathcal{C} \to \mathcal{P}$ whose restriction to $\mathcal{P}$ is the identity function.

Define $s_1(c) = s(\psi(c))$. Since this agrees with s on $\mathcal{P}$ and has the same range as s while s is proper, this is a proper scoring rule on $\mathcal{C}$.

Now, let $\delta(c) = \inf_{p\in\mathcal{P}} \|c - p\|_{\ell^\infty(\mathscr{P}\Omega)}$ be the distance from c to $\mathcal{P}$. By the triangle inequality, this is a continuous function, and since $\mathcal{P}$ is closed (Lemma 3), $\delta(c_0) > 0$ for our $c_0 \in \mathcal{C} - \mathcal{P}$. Let $\phi(c) = \max(1, \delta(c)/\delta(c_0))$ and define

$$\bar{s}(c)(\omega) = \phi(c) f_0(\omega) + (1 - \phi(c)) s_1(c)(\omega)$$

for any credence c. Note that s, s_1 and $\bar{s}$ all agree on $\mathcal{P}$ (since $\phi(p) = 0$ for $p \in \mathcal{P}$). Then if $c \notin \mathcal{P}$, we have $\phi(c) > 0$ and so for any $p \in \mathcal{P}$:

$$\begin{aligned} E_p\bar{s}(c) &= \phi(c)E_p f_0 + (1 - \phi(c))E_p s_1(c) \\ &> \phi(c)E_p s(p) + (1 - \phi(c))E_p s_1(c) \\ &\geq \phi(c)E_p s(p) + (1 - \phi(c))E_p s_1(p) \\ &= \phi(c)E_p s_1(p) + (1 - \phi(c))E_p s_1(p) \\ &= E_p s_1(p) = E_p\bar{s}(p) \end{aligned}$$

by our assumption on f_0 and the propriety of s_1. Hence E_p is $\mathcal{P}$-strictly proper. Moreover, it is continuous because δ is continuous. And $\phi(c_0) = 1$ so $\bar{s}(c_0) = f_0$. □

For the proof of Proposition 3, we will need the following.

Lemma 4. *Assume the Axiom of Choice. Let p and q be any f.a. probabilities on $\mathscr{P}\Omega$ where $\kappa = |\Omega|$ is infinite. If p and q differ on any event, they differ on 2^κ events.*

Proof. Suppose A is such that $p(A) \neq q(A)$. Either A or its complement A^c has cardinality κ, and if p and q differ on A, they differ on A^c by finite additivity. Thus, replacing A with A^c if necessary, assume $|A^c| = \kappa$. Let $F = \{B \subseteq A^c : p(B) = q(B)\}$. If $|F| < 2^\kappa$, then $|\mathscr{P}(A^c) - F| = 2^\kappa$ (here we use Choice and the fact that $|\mathscr{P}(A^c)| = 2^\kappa$), and so there are 2^κ many events on which p and q differ. Thus suppose $|F| = 2^\kappa$. But then any $B \in F$ is disjoint from A and so we have

$$p(A \cup B) = p(A) + p(B) = p(A) + q(B) \neq q(A) + q(B) = q(A \cup B)$$

and so p and q differ on the event $A \cup B$, and there are 2^κ such events. □

Proof of Proposition 3. Fix $p \in \mathcal{P}_c$ such that $p(\{\omega\}) > 0$ for all $\omega \in \Omega$. If we have quasi-strict propriety, then $E_p(s(p)) < E_p(s(c))$ for any non-probability c, and hence $\infty > E_p(s(p)) = \sum_\omega p(\{\omega\})s(p)$. Since p is non-zero on every singleton, it follows that $s(p)(\omega) < \infty$ for every ω. Since (inaccuracy) scores cannot equal $-\infty$, we must have $s(p)(\omega)$ finite for every ω.

Now, a sum of uncountably many non-zero values is either undefined or infinite. Thus for any fixed ω, the summand $s_A(p(A), 1_A(\omega))$ is zero except for at most countably many $A \in \mathscr{P}\Omega$. Let

$$N_p = \{A \in \mathscr{P}\Omega : \exists \omega \in \Omega(s_A(p(A), \omega) \neq 0)\}.$$

Let $\kappa = |\Omega|$. Then N_p is the union of κ sets, each of which is countable, and κ is infinite, so $|N_p| \leq \aleph_0 \times \kappa \leq \kappa \times \kappa = \kappa$, where we used the Axiom of Choice in both inequalities.

Let q be any c.a. probability other than p such that $q(\{\omega\}) > 0$ for all ω. Let $N = N_p \cup N_q$. Then for any $A \notin N$ and $\omega \in \Omega$, we have $s_A(p, 1_A(\omega)) = 0 = s_A(q, 1_A(\omega))$ and $|N| \leq \kappa$, as κ is infinite.

By Lemma 4, p and q differ on 2^κ events. Since $|N| \leq \kappa < 2^\kappa$, by the pigeonhole principle choose a $C \notin N$ for which $p(C) \neq q(C)$. Let $c(A) = p(A)$ if $A \neq C$ and $c(C) = q(C)$. Then finite additivity is violated by c:

$$c(C) + c(C^c) = q(C) + p(C^c) = q(C) + 1 - p(C) \neq q(C) + 1 - q(C) = 1.$$

But

$$s(c)(\omega) = \sum_{A \in \mathscr{P}\Omega} s_A(c(A), 1_A(\omega)) = \sum_{A \in \mathscr{P}\Omega} s_A(p(A), 1_A(\omega)) = s(p)(\omega),$$

for all ω, since $c(A) = p(A)$ except when $A = C$, in which case $s_A(c(A), 1_A(\omega)) = s_A(q(A), 1_A(\omega)) = 0 = s_A(p(A), 1_A(\omega))$ since $C \notin N$. Therefore, $s(c) = s(p)$ everywhere, and hence we cannot have $E_p s(c) > E_p s(p)$ as would be necessary for either f.a. or c.a. quasi-propriety since c is not a f.a. probability. □

To prove Proposition 4, we need the following improvement on Cantor's Theorem, whose proof was essentially given to me by my colleague [name].

Lemma 5. *If X is a set, then there is no function $f : \mathscr{P}X \to X$ such that $f(A) \neq f(B)$ whenever $A \subset B$.*

Proof. Suppose we have such a function f. Let ON be the class of ordinals. Define $F : ON \to X$ by transfinite induction: $F(0) = f(\varnothing)$ and $F(\alpha) = f(\{F(\beta) : \beta < \alpha\})$ whenever α is a successor *or* limit ordinal.

I prove by transfinite induction that F is one-to-one on α for any ordinal α. This is trivially true for $\alpha = 0$.

Suppose F is one-to-one on β for all $\beta < \alpha$. If α is a limit ordinal, then it immediately follows that F is one-to-one on α as well.

If $\alpha = \beta + 1$ for some ordinal β, I claim that it is one-to-one as well. For given that F is one-to-one on β, the only possible failure of injectivity would be if $F(\beta) = F(\gamma)$ for some $\gamma < \beta$. Suppose that happens. Let

$$H_\delta = \{F(\varepsilon) : \varepsilon < \delta\}.$$

Note that $F(\beta) = f(H_\beta)$ and $F(\gamma) = f(H_\gamma)$. Furthermore, $H_\gamma \subset H_\beta$ since F is one-to-one on β. Thus, $F(\gamma) = f(H_\gamma) \neq f(H_\beta) = F(\beta)$, a contradiction.

So, by transfinite induction, F is one-to-one on ON, and hence embeds ON in the set X, which is impossible by Burali-Forti. □

Proof of Proposition 4. Let s be a strictly truth-directed scoring rule defined for all extreme credences. Fix $\omega_0 \in \Omega$. Given a subset U of $\mathscr{P}\Omega$, let c_U be the extreme credence function that is correct at all and only the members of U. Thus, $c_U(A)$ is 1 if both $\omega_0 \in A$ and $A \in U$ or both $\omega_0 \notin A$ and $A \notin U$, and is 0 otherwise. Note that if $U \subset V$, then c_V is strictly truer at ω_0 than c_U, and hence $s(c_U)(\omega_0) > s(c_V)(\omega_0)$. Let $h(U) = s(c_U)(\omega_0)$. Thus, h is a function from $\mathscr{P}\mathscr{P}\Omega$ to $[M, \infty]$ such that $h(A) \neq h(B)$ whenever $A \subset B$. But if Ω is infinite, then by the Axiom of Countable Choice, $\aleph_0 \leq |\Omega|$, and $|[M, \infty]| = 2^{\aleph_0} \leq |\mathscr{P}\Omega|$, and so there is a one-to-one function g from $[M, \infty]$ to $\mathscr{P}\Omega$. Letting $f = g \circ h$ and $X = \mathscr{P}\Omega$, we get a function whose existence contradicts Lemma 5. □

For the proof of Proposition 5, we need this easy fact.

Lemma 6. *Assume the Axiom of Countable Choice. If $\mathcal{F}$ is an infinite σ-algebra on Ω, then there is countably infinite partition of Ω by non-empty members of $\mathcal{F}$.*

Proof. Let $\mathcal{F}_0 = \mathcal{F}$ and $\Omega_0 = \Omega$. Given an infinite σ-algebra $\mathcal{F}_n$ on Ω_n, I claim there is a a non-empty member Ω_{n+1} of $\mathcal{F}_n$ such that $\mathcal{F}_n \cap \mathscr{P}(\Omega_n - \Omega_{n+1})$ is infinite. To see this, given the infinitude of $\mathcal{F}_n$, let B be any member of $\mathcal{F}_n$ such that neither B nor $\Omega_n - B$ is empty. Then every member of $\mathcal{F}_n$ is the union of a member of $\mathcal{F}_n \cap \mathscr{P}B$ and a member of $\mathcal{F}_n \cap \mathscr{P}(\Omega_n - B)$. Hence at least one of these two sets is infinite. If $\mathcal{F}_n \cap \mathscr{P}B$ is infinite, let $\Omega_{n+1} = \Omega_n - B$, and otherwise let $\Omega_{n+1} = B$. Let $\mathcal{F}_{n+1} = \mathcal{F}_n \cap \mathscr{P}(\Omega_n - \Omega_{n+1})$. Note that getting such a sequence of $\mathcal{F}_n$ and Ω_n requires the Axiom of Countable Choice since the choice of B was not determinate.

Let $A_n = \Omega_{n-1} - \Omega_n$ for $n \geq 1$. Then $A_1, A_2, ...$ are non-empty pairwise disjoint members of $\mathcal{F}$. If their union is Ω, we are done, and otherwise just add $\Omega - \bigcup_{n=1}^{\infty} \Omega_n$ to the sequence. □

Sketch of Proof of Proposition 5. Let $A_1, A_2, \ldots$ be a countably infinite partition of Ω by members of $\mathcal{F}$ by Lemma 6. Let $B_1, B_2, \ldots$ be an algebra that σ-generates $\mathcal{F}$ (we assumed $\mathcal{F}$ is countably generated, and then the algebra generated by the generating set will also be countable). Let $C_{mn} = A_m \cap B_n$. Given an event A, let

$$b_A(p)(\omega) = -(1_A(\omega) - p(A))^2$$

be the A-Brier score of a (c.a.) probability p. It is elementary to check that

$$E_p b_A(q) = -p(A)(1 - q(A))^2 - (1 - p(A))(q(A))^2 \geq E_p b_A(p)$$

with equality if and only if $p(A) = q(A)$. In particular, b_A is a proper score.

Next, define

$$b(p) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} 2^{-m-n} b_{C_{mn}}(p)$$

The scoring rule b is continuous on $\mathcal{P}_c(\mathcal{F})$ in our topology. It is proper being the sum of proper scoring rules.

Moreover, b is strictly proper on the probabilities. For suppose that p and q are probabilities that are not identical.

I now claim that p and q differ on B_n for some n. For suppose p and q are equal on all the B_n. Now $\{B_1, B_2, \dots\}$ is an algebra that σ-generates $\mathcal{F}$. If c.a. probabilities agree on $U_1 \subseteq U_2 \subseteq \dots$, they agree on the union, and similarly if they agree on $U_1 \supseteq U_2 \supseteq \dots$, they agree on the intersection. Let $\mathcal{G}$ be the sets on which p and q agree. This is thus a monotone class that contains $\{B_1, B_2, \dots\}$, and hence by the Monotone Class Theorem, $\mathcal{F} \subseteq \mathcal{G}$, and hence they agree on $\mathcal{F}$, a contradiction.

Since B_n is in turn partitioned into $C_{1n}, C_{2n}, \dots$, it follows that p and q differ on C_{mn} for some m. Thus by the propriety of $b_{C_{mn}}$ and the condition for equality in the propriety inequality for A-Brier scores, we have $E_p b(p) < E_p b(q)$ as desired.

Now, define

$$s_1(c)(\omega) = \begin{cases} -\frac{\sum_{n=1}^{\infty} 1_{A_n}(\omega) c(A_n)}{\left(\sum_{n=1}^{\infty} (c(A_n))^2\right)^{1/2}} & \text{if } c \in \mathcal{P}_c(\mathcal{F}) \\ -\sum_{n=1}^{\infty} \frac{1_{A_n}(\omega)}{2(n+1)} & \text{otherwise.} \end{cases}$$

Given a probability $p \in \mathcal{P}_c(\mathcal{F})$, let $\phi(p)$ be the unique c.a. probability on $\mathscr{P}\mathbb{N}$ such that $\phi(p)(\{n\}) = p(A_{n+1})$. Note that ϕ is a continuous function from $\mathcal{P}_c(\mathcal{F})$ to $\mathcal{P}_c(\mathscr{P}\mathbb{N})$ with respect to the ℓ^∞ topologies. Extend ϕ to all of $\mathcal{C}(\mathcal{F})$ by letting $\phi(c) = d_0$ for all $c \in \mathcal{C}(\mathcal{F}) \backslash \mathcal{P}_c(\mathcal{F})$ and any fixed non-probability credence d_0 on $\mathscr{P}\mathbb{N}$. Then it is easy to verify that:

$$E_q s_1(c) = E_{\phi(q)} \operatorname{Sph}^*(\phi(c)).$$

The $\mathcal{P}_c(\mathcal{F})$-weak continuity and quasi-strict propriety of s_1 then follow from those of Sph^* (we do not get strict propriety, because ϕ is not one-to-one on the c.a. probabilities, though we do get quasi-strictness because it maps probabilities to probabilities and non-probabilities to non-probabilities).

Fix some non-probability c_0 in $\mathcal{C}(\mathcal{F})$. Let $f_0 = s_1(c_0)$. Note that $E_p s_1(p) < E_p f_0$ for all c.a. probabilities p.

Let $s_2(p) = b(p) + s_1(p)$ for a probability p. This is the sum of a strictly proper and a proper scoring rule on $\mathcal{P}_c(\mathcal{F})$, so it is strictly proper there. It is continuous and uniformly bounded. Using Proposition 2, extend s_2 to a continuous strictly proper scoring rule on all of $\mathcal{C}(\mathcal{F})$ with $s_2(c_0) = f_0$. It remains to check that we do not have even non-strict domination. To that end, choose $\omega_n \in A_n$ (this uses the Axiom of Countable Choice). Observe that

$$\sum_{n=1}^{\infty} f_0(\omega_n) = -\sum_{n=1}^{\infty} \frac{1}{2(n+1)} = -\infty$$

but

$$\sum_{n=1}^{\infty} s_1(p)(\omega_n) = -\sum_{n=1}^{\infty} \frac{p(A_n)}{\left(\sum_{n=1}^{\infty}(p(A_n))^2\right)^{1/2}} > -\infty$$

for any probability p by countable additivity. Moreover,

$$\sum_{n=1}^{\infty} b(p)(\omega_n) \geq -\sum_{n=1}^{\infty} 2^{-n} > -\infty,$$

since $b(p)(\omega) \in [-2^n, 0]$ for $\omega \in A_n$. It follows that f_0 cannot be even non-strictly dominated by $b(p) + s_1(p)$ for any probability p. □

Proof of Proposition 6. Say that a function f depends only on the coordinates in $C \subseteq \kappa$ if for any ω and ω', if $\omega(x) = \omega'(x)$ whenever $x \in C$, then $f(\omega) = f(\omega')$.

Let Q be the set of all intervals (a, b) with a and b rational numbers. If f is a real-valued measurable function on Ω and $I \in Q$, then the preimage $f^{-1}[I]$ is a member of $\mathcal{F}$, and hence depends only on countably many coordinates. Suppose that $f^{-1}[I]$ depends only on the coordinates in a countable set C_I (use the Axiom of Choice to choose C_I), then let C be the union of C_I as I ranges over Q. Then C is a countable set and $f^{-1}[I]$ depends only on the coordinates in C for any such interval with rational number endpoints. Since $f(x) = y$ if and only if x is a member of $f^{-1}[I]$ for all $I \in Q$ such that $y \in I$, it follows that f depends only on the coordinates in C.

For any countable subset C of κ, the number of measurable functions that depend only on the coordinates in C is the number of measurable functions on $\{0, 1\}^{\omega}$, with respect to the product σ-algebra, and that is just $\mathfrak{c}$, since this product σ-algebra is countably generated. There are at most κ^{α} subsets of κ with cardinality α and so at most κ^{ω} countable subsets of κ (here we use the Axiom of Choice twice). Thus, again using the Axiom of Choice, there are at most $\kappa^{\omega} \times \mathfrak{c} = \kappa^{\omega}$ measurable functions on $\{0, 1\}^{\kappa}$, and given the assumption that $2^{\kappa} > \kappa^{\omega}$, and the fact that there 2^{κ} extreme probabilities concentrated at singletons, the proof is complete. □

Finally, if $\mathcal{Q}$ is a convex subset of a vector space, say that a topology on $\mathcal{Q}$ is *line segment compatible* provided that for any x and y in $\mathcal{Q}$, the function $t \mapsto (1-t)x + ty$ from $[0, 1]$ to $\mathcal{Q}$ is continuous in that topology.

Any topology on $\mathcal{P}_\mathrm{f}$ which is derived from embedding in a topological vector space of real-valued functions on $\mathcal{F}$ is line segment compatible.

Proposition 9. *Let Ω be any non-empty set. Suppose $\mathcal{P}$ is either $\mathcal{P}_\mathrm{f}$ or $\mathcal{P}_\mathrm{c}$ and has a line segment compatible topology. Let s be a proper scoring rule on $\mathcal{P}$ such that $p \mapsto s(p)(\omega)$ is continuous for every fixed $\omega \in \Omega$. Then if $E_p s(p) = E_p s(q)$ and both $E_p s(p)$ and $E_q s(q)$ are finite, we have $s(p) = s(q)$ everywhere on Ω.*

Proof. Fix $\omega \in \Omega$. Let u_ω be the probability measure such that $u_\omega(\{\omega\}) = 1$. Let $p_t = (1-t)p + tu_\omega$ for $t \in [0,1]$. Then:

$$\begin{aligned}(1-t)E_p s(p) + tE_{u_\omega} s(q) &= (1-t)E_p s(q) + tE_{u_\omega} s(q)\\ &= E_{p_t} s(q)\\ &\geq E_{p_t} s(p_t)\\ &= (1-t)E_p s(p_t) + tE_{u_\omega} s(p_t)\\ &\geq (1-t)E_p s(p) + tE_{u_\omega} s(p_t),\end{aligned}$$

where the first equality follows from the assumed equality in the statement of the proposition and the two inequalities follow from propriety. Since $E_p s(p)$ is finite, it follows that

$$E_{u_\omega} s(q) \geq E_{u_\omega} s(p_t)$$

for all $t \in (0,1]$. Since $E_{u_\omega} f = f(\omega)$ for any f, taking the limit as $t \to 0+$, it follows by continuity and line segment compatibility that

$$s(q)(\omega) = E_{u_\omega} s(q) \geq E_{u_\omega} s(p) = s(p)(\omega).$$

We have thus shown that if p and q are such that $E_p s(p) = E_p s(q)$, then $s(q) \geq s(p)$ everywhere. It follows that $E_q s(q) \geq E_q s(p)$ and by propriety that $E_q s(q) = E_q s(p)$. Applying what we have just proved with p and q swapped, we conclude that $s(p) \geq s(q)$ everywhere, and hence that $s(p) = s(q)$ everywhere. □

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