“Is spacetime locally flat?”: a note

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Abstract

I review three senses in which the slogans, ‘spacetime is locally flat’, and-or ‘spacetime is locally approximately flat’ can be justified. The background is a recent paper of Fletcher and Weatherall, that focuses on two of these senses: (i) that each tangent space is ‘like Minkowski’, and (ii) that around any curve we can construct Fermi-coordinates in which Christoffel symbols vanish. They argue, against the orthodoxy, that these senses cannot be given a substantive content. I will here, if not entirely disagree, attempt to qualify their verdicts. I also introduce a third sense, based on geodesic deviation: that tidal effects can be ignored, in the sense that deviation becomes linear (i.e. like Minkowski), when geodesics are very close to each other.

1 Introduction

1.1 Interpretations of ‘local flatness’

I shall begin by issuing a note to the reader: this paper does not aim to review all the ways in which one could construe spacetime as being locally flat. I will focus on three such ways, two of which were discussed in Fletcher & Weatherall (2022a), and a third that was not included in that work.[1]

The first two senses are:

(i) **Tangent Space Interpretation:** “The tangent space at a point of spacetime is, or is somehow equivalent to, Minkowski spacetime.” (Fletcher & Weatherall, 2022a, p. 7).

(ii) **Coordinate Chart Interpretation:** “At any point of any relativistic spacetime (or along certain curves), local coordinates may be chosen so that, at that point (or along that curve), (a) the components of the metric agree with the Minkowski metric in standard coordinates and (b) all Christoffel symbols vanish.”(Fletcher & Weatherall, 2022a, p. 10)

And the third, which is not included in Fletcher & Weatherall (2022a) is:

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[1]Neither is this paper meant as a review of Lorentzian or Riemannian geometry. I will assume basic acquaintance with some of the concepts, and will at most sketch the idea behind proofs of theorems. For the particular issues discussed here, I suggest the textbooks (Hawking & Ellis, 1975) and (Poisson, 2004).

(iii) **Negligible geodesic deviation (tidal effects):** Given neighboring geodesics of any congruence of time-like geodesics on any relativistic spacetime, the acceleration of their deviation vector approximates—linearly, for small distances—its Minkowski behavior.

I will make each one of these three senses more precise, as I proceed.

Of the three, Fletcher & Weatherall (2022a) are dismissive of (i), agree with (ii) but argue that it needs clarification (which they seek to provide), and omit (iii). In contrast, I will: defend (i), agree that (ii) needs clarification and that they provide it, and champion (iii) as the most important sense of local flatness.

In my view, Fletcher & Weatherall (2022a) provide a very important clarification that cannot be found in the extant literature: a reconstrual of (ii) in purely geometric terms, i.e. without the explicit use of coordinates.

Before we begin, I will briefly review the necessary mathematical and physical facts about Lorentzian manifolds and their interpretation in general relativity.

1.2 Mathematical background

General relativity models vacuum spacetime by Lorentzian manifolds, that is, a doublet $\langle M, g_{ab}\rangle$, where M is a smooth differentiable manifold and g_{ab} is a symmetric, bilinear, non-degenerate tensor, with signature $(1,3)$. I will use abstract index notation with Roman letters and coordinate components will be denoted with Greek letters.

Signature.

The first mathematical object that is important for us here is the *signature*: it is the number (counted with multiplicity) of positive and negative eigenvalues of the real symmetric matrix g_{ab} of the metric tensor *at each point*, with respect to a basis.[2]

Two comments are in order: first, that the concept of signature is applied pointwise, and thus one could consider manifolds that have a metric of varying signature.[3] Second, by 'Sylvester's law of inertia', the signature does not depend on the basis, and, moreover, at each point one can find a basis in which g_{ab} is diagonal, with all elements being -1 and 1.[4]

Curvature

Here I take parallel transport in M to be defined by a Levi-Civita connection, Γ, which at any point is a function of the metric and its first derivatives there. The curvature is a tensor that encodes the rotation due to parallel transport around an infinitesimal parallelogram, and it is at any point a function of the Levi-Civita connection and its first derivatives. Here is a reminder of these relations :

$$\nabla_{[b}\nabla_{c]}X_d = R^a{}_{bcd}X_a; \tag{1.1}$$

$$R^a{}_{bcd} = \partial_c\Gamma^a{}_{db} - \partial_d\Gamma^a{}_{cb} + \Gamma^a{}_{cf}\Gamma^f{}_{db} - \Gamma^a{}_{df}\Gamma^f{}_{cb}. \tag{1.2}$$

[2] I am here focussing on non-degenerate metrics, and so ignore the possibility of zero eigenvalues.

[3] If spacetime is allowed to change signature, there are possible physical differences between signature (1,3) and (3,1); see Gibbons (1994).

[4] It is this fact that is behind the tetrad formalism for general relativity.

The curvature is a tensor, and so transforms linearly under coordinate transformations: if it vanishes at a point in one coordinate system, it will vanish at that point for all coordinate systems. On the other hand, ${\Gamma^a}_{cb}$ is a pseudo-tensor, and thus, at any given point, it can vanish in one coordinate system without vanishing in all. Differences between Christoffel symbols will transform as tensors; this is what allows the Riemann curvature as defined in (1.2) to transform as a tensor, and the reason we slightly abused notation and wrote Christoffel symbols in the abstract index notation. More strictly, in a given coordinate system x^α, Chistoffel symbols are given by:

$$ {\Gamma^\alpha}_{\beta\gamma} = g^{\alpha\sigma}(\partial_\beta g_{\sigma\gamma} + \partial_\gamma g_{\sigma\beta} - \partial_\sigma g_{\beta\gamma}) \tag{1.3} $$

Fermi coordinates

In order to describe Fermi coordinates, we first go back to the context of flat spacetime. Let a particle trajectory in Minkowski spacetime be given, in inertial coordinates, by $x^\alpha(\lambda)$.[5] In these coordinates, the equations of motion for a unit-parametrized geodesic are:

$$ \frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\lambda^2} = 0. \tag{1.4} $$

More generally, in these coordinates, a test-particle under forced motion will follow Newton's second law, according to

$$ \frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\lambda^2} = F^\alpha, \tag{1.5} $$

for some source F.

But even in the absence of external forces, under more general coordinate systems, the unit-parametrized geodesic (1.4) of flat spacetime becomes:

$$ \frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\lambda^2} + {\Gamma^\alpha}_{\beta\gamma} \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\gamma}{\mathrm{d}\lambda} = 0. \tag{1.6} $$

The difference between (1.6) and (1.4) is the presence of the Christoffel symbols. Their presence is necessary because, while equation (1.4) is invariant under Poincaré transformations, it is not invariant under general coordinate transformations. But (1.6) *is* invariant under general coordinate transformations; the presence of Γ and their transformation properties is what guarantees general coordinate-invariance, even in flat spacetime.

Comparing (1.5) with (1.6), we get to the idea of 'fictional forces' in a flat spacetime: (1.6) gives differences in linear motion that could, naively, be interpreted as a force—as in (1.5)), albeit a sourceless one—but which actually vanish in an appropriate choice of coordinates (namely, inertial). Again, this is possible because Γ is not a tensor, and so it can vanish in one coordinate system but not in the other.

The question is: does the same argument apply to curved spacetime, thereby supporting a construal of geodesic motion as 'inertial'? That is, if the Riemann curvature is non-vanishing, can we similarly find coordinates around a geodesic curve in which the terms encoding deviation from linear behavior (for the coordinates of a test-particle travelling a geodesic) vanish?

[5]Here, since we are discussing particular coordinate components, we have dropped the abstract index notation.

The answer is a qualified 'yes', and it is a corollary of the existence of Fermi coordinates.[6] So, I will take the Fermi coordinates associated to the time-like curve γ, to be a choice of coordinates for which

$$g_{\mu\nu}|_\gamma = \eta_{\mu\nu}|_\gamma, \text{ and } \Gamma^\alpha{}_{\beta\gamma}|_\gamma = 0. \quad \text{(Fermi coordinates)} \tag{1.7}$$

But unlike the flat case, Fermi coordinates only guarantee that the Christoffel symbols vanish on the geodesic itself; they do not vanish in a neighborhood of the geodesic (otherwise their derivatives, and thus the Riemann curvature, would vanish there as well); they only 'approximately vanish' in small neighborhoods of the geodesic.

Geodesic deviation

Suppose we are given a two-parameter family of non-intersecting unit-parametrized geodesics, $\gamma(s,t)$, where t is the parameter along each geodesic, and the geodesics are labelled by (ie distinguished form each other by) the parameter s, i.e. at fixed s the curve $\gamma(s,t)$ is a geodesic. (e.g. if they are part of a geodesic congruence.) We then have a two dimensional surface that is parametrized by $\gamma(s,t)$. Using bold-face to denote vector fields, we define two families of vector fields on this surface:

$$\frac{\mathrm{D}\gamma}{\mathrm{d}t}|_{(t',s')} := \mathbf{v}(t',s'), \text{ which satisfies } v^a\nabla_a v^b = 0, \; v^a v_a = -1; \tag{1.8}$$

and

$$\frac{\mathrm{D}\gamma}{\mathrm{d}s}|_{(t',s')} := \mathbf{r}(t',s') \quad \text{which is called the } \textit{geodesic deviation vector}. \tag{1.9}$$

Since these are tangent vectors to a surface, $[\mathbf{r},\mathbf{v}] = 0$, and since we have zero torsion,

$$v^a\nabla_a r^b = r^a\nabla_a v^b. \tag{1.10}$$

Using (1.10) and (1.8), we can set $r_a v^a = 0$.[7]

Then, using (1.8) and (1.10), it is easy to show, by commuting derivatives, that the *acceleration of the geodesic deviation* is given by:

$$\frac{\mathrm{D}^2 r^a}{\mathrm{d}t^2} := v^c\nabla_c(v^d\nabla_d r^a) = R^a{}_{bcd}v^b v^d r^c \tag{1.11}$$

2 In what sense is a generic spacetime locally flat?

At several places, Fletcher & Weatherall (2022a) criticize notions of local flatness for "telling us nothing about the curvature tensor" at that point. In discussing interpretation (i), that 'the tangent space at a point of spacetime is, or is somehow equivalent to, Minkowski spacetime', they say:

[6] I include a quick sketch of what is involved in the proof of this result in the appendix. There are different nomenclatures involved, depending on whether one is in a Riemannian or Lorentzian manifold, and chooses these coordinates around a point or a curve. Here I am not interested in these matters, and so will call any family of coordinates that are chosen so as to have vanishing Christoffel symbols 'Fermi'.

[7] We obtain that it is constant along the geodesics, and so we we set it to zero. That is:

$$v^a\nabla_a(r_b v^b) = v^b v^a\nabla_a r_b = v^b r^a\nabla_a v_b = r^a\nabla_a(v^b v_b) = 0.$$

> An advocate for this interpretation might reply that "local flatness" means that infinitesimal neighborhoods of each point—that is, the tangent space—should be thought of as equivalent to Minkowski spacetimes with a distinguished point, since after all we are rep resenting a neighborhood of a particular point. Fine. But even if we set aside the structural differences between Minkowski spacetime and the tangent space at a point of a relativistic spacetime, if the tangent space interpretation is all that is meant by "local flatness", it is strikingly weak. This is because Riemann curvature is a tensor field, and so it determines a tensor acting on the tangent space at each point. Thus, there is a sense in which, even from the perspective of the tangent space at a point, one can "see" the curvature of spacetime near that point, by considering the curvature tensor there.[8]

Of course, all hands agree that the curvature tensor is, indeed, a tensor, and, in general, it is, well, arbitrary; thus there is clearly no sense in which it could be (generically) "approximately zero" at any given point. But being locally flat can be given different senses, in particular the senses (i-iii) I listed in Section 1.1; and so 'local flatness' need not just mean 'approximately zero curvature'. Indeed, each of (i), (ii), and (iii) provide one way to construe *an analogy or relationship* between curved spacetime and Minkowski spacetime, and none of these ways should be contingent on the values of the curvature.

In Sections 2.1 and 2.2 I will report and assess Fletcher & Weatherall (2022a)'s critiques of the two interpretations of local flatness listed as (i) and (ii) in Section 1.1. Then, in Section 2.3 I will assess the interpretation (iii), of flatness via geodesic deviation; and in 2.4 I will show that the effects of curvature could, in principle be witnessed even over a single geodesic, but, in fact, no classical matter could realize these effects for arbitrarily short distances.

2.1 According to the Tangent Space Intepretation

In particular, the 'tangent space interpretation' provides a non-trivial sense in which a general spacetime is like Minkowski spacetime. In the quote above, Fletcher & Weatherall (2022a) describe Minkowski space as an affine metric space. But affine metric spaces are modeled, *uniquely*, on vector spaces with an inner product; and each such inner product has a signature. The noteworthy sense in which Lorentzian manifolds are 'locally like Minkowski', according to (i), is the following: *for any* p, T_pM has an inner product, $g_{ab}(p)$, and Minkowski space is the *unique* affine space modeled on (any of these) T_pM, with an inner product that has the same signature as $g_{ab}(p)$. And to reiterate: this is true for any choice of $p \in M$. In this sense—of being linear, inner product spaces of the same signature—tangent spaces of M are, mathematically, a 'microcosm' of Minkowski space. Thus sense (i) serves as a robust interpretation of spacetime being 'locally like Minkowski', at the mathematical level.

Though at one level I agree with Fletcher & Weatherall (2022a) that one could sell this 'unique' property of Minkowski, among all of the other Lorentzian metrics, as merely one pertaining to a vague notio of 'simplicity', I believe in this case the price is not quite right: after all affine spaces are, in a strong mathematical sense, *like* vector spaces.

[8] And in (p. 10, Ibid), they apply a similar criterion to the coordinate chart intepretation, (ii): "What does it tell us about local curvature?"

2.2 According to the Coordinate Chart Interpretation

In this Section, I will first, in Section 2.2.a, criticize the standard interpretation of Fermi coordinates, agreeing with Fletcher & Weatherall (2022a); then, in Section 2.2.b I will summarize what I take to be the main result of Fletcher & Weatherall (2022a), assessing the geometric significance of Fermi coordinates; and in Section 2.2.c I will provide a mild criticism of Fletcher & Weatherall (2022a)'s deflationary gloss on local flatness according to (ii).

2.2.a In agreement with Fletcher & Weatherall (2022a)

As indicated above, I agree with Fletcher & Weatherall (2022a), that there are many ways to interpret the phrase 'spacetime is locally flat'. Of these, I think (ii), the coordinate chart intepretation, is the least contentful.

As described in Section 1.2, the existence of Fermi coordinates provides another analogy between 'inertial motion' in generally curved spacetimes and in flat spacetimes. But I take this to be a very limited sense in which a general spacetime is, around a 'small neighborhood' of a geodesic, approximately 'like' flat spacetime.[9]

The sense is limited because all it says is that we are able to find coordinates in the neighborhood of a spacetime geodesic such that the equations of motion of a test-particle are approximately of a specific form, of vanishing *coordinate* acceleration (as in (1.4)). It does *not* say that the curvature vanishes along the geodesic: once more, the curvature in (1.2) is a tensor, and it depends not only on the Christoffel symbols, but also on their derivatives. It follows trivially that if and only if the curvature vanishes along a curve will Fermi coordinates—coordinates in which the Christoffel symbols vanish along the curve in question—also eliminate derivatives of Christoffel symbols. Derivatives of Christoffel symbols, as per (1.3), are second order in derivatives of the metric, and thus the existence of Fermi coordinates gives a sense in which metrics on the neighborhood of any curve are 'close to flat' up to its first derivatives.[10]

Fleshing out this rough statement is the great merit of Fletcher & Weatherall (2022a). As (Fletcher & Weatherall, 2022a, p. 10) say:

> it is true that on this interpretation one invokes special coordinates—viz., [Fermi] ones—but the significance of those coordinates requires further commentary. What does the "form" of the metric in some coordinate system tell us about the metric or its derivatives, all of which are coordinate independent structures?

As I will now discuss, Fletcher & Weatherall (2022a) provide a more geometric understanding of Fermi coordinates—that does not mention coordinates—and construe this understanding as 'approximate flatness' in a very literal sense.[11]

[9]Though this existence proof seems to realize Einstein's "happiest thought"—the equivalence principle—the relationship between Fermi coordinates and the weak equivalence principle is not direct: see (Fletcher & Weatherall, 2022a, Footnotes 2 and 6).

[10]I would like to note that the rough statement does not contradict our intuitions about semi-Riemannian geometry: we attribute geometric differences to different curvatures, which require the second derivatives of the metric.

[11]I believe Fletcher & Weatherall (2022a) at one point slightly under-sell the significance of Fermi coordinates; namely, when they say "If it is not clear what features of these coordinates are supposed to be salient, it is hopeless to try to establish the general existence or uniqueness conditions for such coordinates." This is under-

2.2.b 'Geometrizing' Fermi coordinates

Fletcher & Weatherall (2022a)'s main result is that a spacetime is locally flat in the sense that an arbitrary metric locally—i.e. for some small neighborhood of a geodesic—approximates a flat metric, where 'approximate' is understood as arbitrarily close according to a distance function between metrics that only takes into account contributions up to first-order derivatives of the metrics. I will give the technical statement below, in Theorem 1.

Crucially, the result relies on the standard proof of existence for Fermi coordinates. Namely, as they discuss at the end of (p. 14, ibid), that same proof can be used to find a flat metric on γ whose derivative operator $\overline{\nabla}$ agrees with ∇ on γ.[12] The idea here (see (Iliev, 2006, Theorem II.3.2)) is to first find coordinates in which the Christoffel symbols vanish along the curve, then use those coordinates to pull-back the metric on a neighborhood of the geodesic.[13] That pull-back will produce a metric whose Levi-Civita derivative $\overline{\nabla}$ is flat everywhere but nonetheless agrees with the original Levi-Civita derivative of g_{ab}, ∇, on the curve itself. Thus there is a sense in which the standard proof of existence of Fermi coordinates works 'under the hood' of Fletcher & Weatherall (2022a)'s geometric construction.

Now, to discuss approximations of metrics one to another, one first needs to introduce the idea of an auxiliiary positive definite metric, h_{ab}. This auxiliary metric is then used to measure distances between metrics, order by order in derivatives, and it is applied to measure the distance between the original metric and a flat metric on a neighborhood of the geodesic.

In more detail:— given some compact set U, they define a k-distance between two tensors, T, T', as

$$d_{U,h,k}(T,T') := \max_{j\in\{0,\cdots,k\}} \sup_{x\in U} |(\nabla)^j_h(T-T')|_h, \tag{2.1}$$

where here the subscript h denotes that both the covariant derivative and the norm are taken with respect to h_{ab}, and j denotes the number of covariant derivatives taken of the difference between tensors. Then, given metrics g_{ab} in U and g'_{ab} in U', a diffeomorphism $f : U \to U'$ is defined to be an (h, k, δ)-isometry if $d_{U,h,k}(g_{ab}, f^*g'_{ab}) < \delta$.

Now they show:

Theorem 1 *Given any spacetime (M, g_{ab}), embedded curve $\gamma : I \to M$, point $p \in \gamma[I]$, compact neighborhood U of p, Riemannian metric h_{ab} on U, real $\delta > 0$, and point p' in Minkowski spacetime $(\mathbb{R}^4, \eta_{ab})$, there exist neighborhoods $O \ni p$ and $O' \ni p'$, an embedded curve $\gamma' : I \to \mathbb{R}^4$ with $p' \in \gamma'[I]$, and an $(h, 1, \delta)$-isometry $f : O \to O'$ between (O, g_{ab}) and (O', η_{ab}) satisfying $f \circ \gamma = \gamma'$ on I and $f^*(\eta_{ab}|_{\gamma'}) = g_{ab}|_\gamma$.*

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selling because it is clear that existence is supposed to be the relevant fact, not uniqueness, and *that* is well established; moreover, the salient features of these coordinates are also clear. But I think this is a momentary lapse, for they get to the heart of the matter when they go on to say that: "it is unclear whether spatial flatness is meant to be a claim about the existence of certain coordinate systems, *which in turn expresses something about the local geometry of relativistic spacetimes, or if it is supposed to include a further interpretive claim about idealized measurement apparatuses of natural motion, which, of course, would go far beyond any facts about curvature, local or otherwise*" (my emphasis). I will return to this point about idealized measurements in Section 2.2.c.

[12]Indeed, a crucial ingredient of their Theorem 1, which plays an importart role in all subsequent results, is a corollary of the usual proof of existence of Fermi coordinates: see (Iliev, 2006, Theo. II.3.2).

[13]Here they say "parallel transport" the metric along those coordinates: but I believe that is just typo (the only metric at one's disposal thus far in the argument is the original g_{ab}, and it is already parallel transported along those curves, since its covariant derivative vanishes).

Moreover, they show that the corollary extends to $k > 1$ (i.e. to derivative higher than the first) if and only if the curvature of g_{ab} vanishes everywhere.

Their upshot is that:

> while one can isolate a precise and accurate statement to the effect that spacetime is locally approximately Minkowskian, this statement is misleadingly specific given that local approximation is pervasive. Perhaps a better way of characterizing the situation is that, to first order, all spacetimes with the same metric signature have a universal character, in the sense that they all locally approximate one another. It is only at second order and higher that differences in structure between different spacetimes can be seen in arbitrarily small neighborhoods of a point or curve. That spacetimes cannot approximate one another arbitrarily well to second order is, of course, closely related to the fact that curvature is a tensor. (Fletcher & Weatherall, 2022a, p. 27,28)

Which is in line with the proof of existence of Fermi coordinates.

This all seems like good news for local flatness. But Fletcher & Weatherall (2022a) quickly curb the initial enthusiasm by showing that η_{ab} plays no special role here: the same theorem can be proven using any other metric of the same signature in place of η_{ab}. The idea is merely to use the triangle inequality with respect to (2.1), with the Minkowski metric as the intermediate vertex of the triangle: if g_{ab} and g'_{ab} are each a certain distance from η_{ab}, $d(g, \eta)$ and $d(g', \eta)$, then the distance between them, $d(g, g')$, is bounded by the sum $d(g, \eta) + d(g', \eta)$, which, in the context of the proof, is arbitrarily small.[14]

In sum, the original existence proof of Fermi coordinates says, between the lines, that all metrics are similar to Minkowski along a curve, up to zero derivatives of the Christoffel symbols (i.e. up to first order in derivatives of the metric). And since all metrics are similar to Minkowski, they are all similar to each other in this way. Now, it takes real ingenuity to translate that general idea into a precise mathematical statement, as Fletcher & Weatherall (2022a) have ably done. They accomplished that by using two auxiliary metrics: one flat metric, constructed via the existence of Fermi coordinates, and one arbitrary auxiliary *Riemannian* metric, used to measure distances and compute derivatives of the metric in a covariant manner.

2.2.c A concern

While Fletcher & Weatherall (2022a)'s main result, described in the previous Section, is technically correct, I take its deflationary gloss to be a consequence of the triangle inequality; there

[14]But so far as I know, there is no direct proof of Theorem 1 for g'_{ab} in place of η_{ab}; i.e. no proof that does not invoke η_{ab} at any step. The fact that the construction of Fermi coordinates is operating under the hood of the results—it is necessary to construct the flat metric in the first place—should give one pause about how such a direct proof would go for different metrics, e.g. for a general, non-vanishing Christoffel symbols along γ.

Here is a sketch of how such a proof would go: choose some other set of Chistoffel symbols Γ', and find the coordinates that pick it out. But now, following the analogous step in the construction of the local flat metric, how could the pull-back of the metric on the curve along these coordinates give rise to a general, *non-homogeneous* metric? It is not clear to me.

Of course, this may be simply a practical matter of computation: by the triangle inequality we know that any two Lorentzian metrics will approximate each other, and so it is just a matter of finding a direct relation that leap-frogs over any mention of Minkowski. Nonetheless, I think this practical matter is significant, as I discuss in the next Section.

is still a sense in which it is the Minkowiski metric that anchors those relations.

Here is an analogy (suggested to me by Weatherall, in conversation): the only sense in which we can approximate, to first order, a quintic curve by a parabola is to arrange the two curves so that they have the same tangent, at the point around which the approximation is taking place. That is, the only way to to have the parabola approximate a quintic to first order is that they both approximate the same straight line. We can go probe this idea a bit further, by trying to replace the role of the straight line with yet a third curve. That is, could we say that the only way to have the parabola and quartic approximate each other at a point is that they both approximate a quintic there? No: they could be good approximations beyond first order, and therefore fail to approximate a quintic to that order. It becomes apparent that what is doing the work here is the linearized structure of the curves, and the spacetime. Does this give the straight line a more distinguished status in these approximations than any other curve? I would say 'yes', that it is the straight line that anchors those relations, since those are linearized, or first-order relations; Fletcher & Weatherall (2022a) may disagree.

2.3 Negligible geodesic deviation

In this Section, I want to address a point that Fletcher & Weatherall (2022a) do not consider: namely, local flatness as understood in terms of the geodesic deviation equation, (1.11).

The equation provides a physical interpretation of curvature. The main idea behind its derivation can be applied to model, for example, how a spherical dust of particles is deformed under free-fall, the effects we usually call 'tidal' due to their historical roots.[15]

What does equation (1.11) have to do with spacetime being locally flat? When the curvature vanishes, i.e. in flat spacetime, the rate of change of the deviation stays constant, as expected. For instance, in flat spacetime, if the rate of change is initially zero, it will stay zero: as we expect from an application of Euclid's parallel postulate, the neighboring geodesics stay at constant separation.

The analogy to more generally curved backgrounds is simply that, the smaller the deviation is at some initial time, the closer the evolution of the deviation is to being linear, for any (fixed) Riemann tensor. In the limit, the deviation is linear. The physical interpretation is that sufficiently close test-particles are not accelerated with respect to each other. In plain English: the closer the geodesics are, the less the curvature affects the evolution of the deviation between them.

To get the gist of this result, let us suppose $\mathbf{v}$ is timelike and that the acceleration vector of the spacelike $\mathbf{r}$ is either timelike or spacelike. And write $\mathbf{e}_r := \frac{\mathbf{r}}{|\mathbf{r}|}$, with $\alpha := |\mathbf{r}| > 0$. Then depending on the smallness of α, the acceleration can be arbitarily small, i.e. for any such $\mathbf{r}$ and $\delta > 0$, there is an $\alpha > 0$ such that:

$$|\frac{\mathrm{D}^2 r^a}{\mathrm{d}t^2}| = \alpha |R^a{}_{bcd} v^b v^d e^c_r| < \delta. \tag{2.2}$$

[15] In Newtonian theory, the relative acceleration of two particles separated by r^a under a potential φ is $r^a \nabla_a \nabla_b \varphi$, which is what gives rise to 'tidal forces'. Here the analogous term is $R^a{}_{bcd} v^b v^d$, and it can be interepreted by picturing a spherical dust-cloud, freely-falling towards the Earth, that becomes an ellipsoid because the dust particles that are nearer to the Earth will fall faster than those far away. See (Hawking & Ellis, 1975, Ch. 4), describing more general properties of the deviation vector field, as it gets dragged along time.

To see that this behavior under the shortening of the deviation says something interesting about geometry, take a different example, in which we bring together the worldlines of particles with the same electric charge: there the acceleration is inversely proportional to their separation and we could not obtain an analogue of (2.2): the more we shorten $|\alpha|$, the greater the acceleration.

Having said that, the geodesic deviation equation is an equation about vectors and vectors fields, and thus should be interpreted as being of merely first order in distances. Thus there is a sense in which we are aligned with, if not straight-up vindicating, Fletcher & Weatherall (2022a)'s claims about similarity up to first order in derivatives.

But here, as in the challenge of footnote 14 (about using the same method of proof to construct arbitrary geometries around the geodesic without ever invoking a flat metric at any step), I would issue a second challenge: could we pick out, to first order, another, non-flat, Riemann tensor in a geodesic deviation equation, perhaps by employing a different limiting procedure?

Although I have no mathematical proof, I have a strong intuition that no such alternative can be found. And thus my conclusion is that the geodesic deviation equation *does* provide a geometric sense, more interesting than (ii), in which spacetime is locally flat.

Lest we over-interpret the significance of this geometric fact about Lorentzian manifolds and geodesic congruences, I will now show that there is a sense in which geometric effects of curvature along a single geodesic could be witnessed if (classical) particles with intrinsic spin were admitted in our models. Since there is a good argument against the existence of such particles, I will conclude that this geometric caveat (to local flatness according to negligible geodesic deviation) has no physical correlate.

2.4 Witnessing curvature on a spacetime geodesic

Of course, the argument of Section 2.3 is not contingent on the values of the curvature, but the relevant scales of separation at which we choose to ignore the remaining curvature effects *are* contingent. In physical terms, what is a relevant scale at which curvature effects are or aren't ignorable will depend on more details about the dynamics: in this sense, the notion of "locally flat spacetime region" is not definable geometrically; it depends on which instrumentation is being used in the region and its sensitivity to curvature. So a given region can be both locally flat and not—as a mere matter of vagueness of 'flat' and so 'locally flat'. Nonetheless, *given* any instrumentation, one can always find a small enough region so that curvature effects are ignorable for the relative distances of a geodesic congruence. Physically, it says that one can always find a small enough freely-falling 'elevator' in which dust particles will behave as if there were no curvature, i.e. as in Minkowski spacetime.

From a more mathematical perspective, why does the curvature appear in the geodesic deviation equation and not in the geodesic equation? First, note that the geodesic equation, as applied to test-particles, is an equation that is second order in covariant derivatives of a scalar quantity (or first-order for vector quantities). Since covariant derivatives applied to a scalar function commute, i.e. $\nabla_{[a}\nabla_{b]}\varphi = 0$, we cannot extract from the evolution of these functions any information about the curvature, which depends on non-commuting covariant derivatives (see (1.2)). To extract information about the curvature from test particles, we thus require neighboring geodesics and their deviation vector.

Of course, we know that some operations on vectors—parallel transporting it around a parallelogram—would result in a local curvature effect. But, from a physical standpoint, we are interested in the evolution of quantities, in equations of motion, and, in particular, in acceleration (or hyperbolic equations, requiring initial data for positions and velocities).[16]

But let us for now ignore the intricacies of the dynamics of matter fields and assess what would be involved in the acceleration of a quantity that has an intrinsic direction, like spin, along a spacetime geodesic. That equation can be found using only parallel transport along geodesic curves, and, in this case, it will also include curvature terms. And if the vectorial character is intrinsic and of fixed length, unlike the geodesic deviation equation (1.11), there will be no parameters that we can take to zero to reveal an approximately flat evolution equation.

In more detail, by supposing we have a time-like geodesic with tangent v^a, an intrinsic vectorial quantity, s^a, with zero initial velocity,[17] with $s^a s_a = 1$,:

$$v^a \nabla_a v^b = 0, \;\; v^a v_a = -1, \;\; v^a \nabla_a s^b = s^a \nabla_a v^b, \tag{2.3}$$

we can recycle the derivation of (1.11), obtaining the same result for the parallel transport acceleration of s^a along the time-like direction determined by the geodesic:

$$\frac{\mathrm{D}^2 s^a}{\mathrm{d}t^2} := v^c \nabla_c (v^d \nabla_d s^a) = R^a{}_{bcd} v^b v^d s^c \tag{2.4}$$

The important difference is that now there is mathematical meaning associated to a limiting procedure in which the 'intrinsic spin' s^a is taken to vanish. Were this true, even if we were in a freely-falling 'arbitrarily small elevator', we should be able to distinguish the evolution of s^a in a curved background from that in a flat background. Indeed, this is essentially the source of the famous frame-dragging effect, recently experimentally verified by the Gravity Probe-B experiment (Everitt et al., 2011).

But of course, the instrumentation of Gravity Probe-B *has* physical extension. So the question arises: is it really possible to give physical meaning to a limiting procedure of a point-like particle with intrinsic spin, *in classical general relativity*? And here the answer is *no*: as discussed in (Weatherall, 2018, Footnote 10), such an approximation would flout all the reasonable energy conditions expected of a classical material source:

> Although it is a side issue for present purposes, observe that this result points to a problem with certain approaches to treating the motion of rotating particles that represent "spin" by higher order distributions supported on a curve (Papapetrou, 1951; Souriau, 1974): such particles are incompatible with the energy condition. There is good physical reason for this. For ever smaller bodies to have large angular momentum (per unit mass), their rotational velocity must increase without bound—leading to superluminal velocities, which are incompatible with the energy condition.

[16] The physical interpretation of the evolution of matter fields is not treated in Fletcher & Weatherall (2022a), it is left for Fletcher & Weatherall (2022b).

[17] This term is here ambiguous: I mean such that the Lie derivative of $\mathbf{s}$ initially vanishes in the direction of $\mathbf{v}$: $\mathcal{L}_{\mathbf{v}}\mathbf{s} = 0$. This assumption is only in place so that we can recycle the proof of (1.11): different assumptions would generically (if no other constraints are imposed) also lead to equations of motion involving the curvature.

3 Conclusion

I analysed three senses in which spacetime could be interpreted as locally like Minkowski. I conclude that sense (i)—the tangent space interpretation—is robust, but not very informative; sense (ii)—the local coordinate chart interpretation—is indeed weak, and the more strictly geometric understanding of Fermi coordinates, mentioned at Section 2.2.b as the 'great merit' of Fletcher & Weatherall (2022a), lays bare that weakness; and (iii)—the geodesic deviation interpretation—is also robust, and it is more informative than the tangent space interpretation, as it describes physical conditions under which the effects of curvature can be ignored.

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A Sketch of a proof of the existence of Fermi coordinates

Here is a quick sketch of what is involved in the proof of this fact. First, it is important to note that the equations of motion for the test particles in a curved background are identical to (1.6), and so the question amounts to whether we can find coordinates in which the Γ vanish along the curve. I will not get into the details, but the idea for the construction of these coordinates is quite simple: given any point $p \in M$, the *exponential map*: $\mathsf{Exp}_p : T_pM \to M$ is a diffeomorphism from some ball in T_pM onto a convex normal neighborhood of p.[18] Choosing an ordered basis of T_pM, we can then parametrize that normal neighborhood with the corresponding n-tuple. For instance, suppose we choose a unit orthornormal basis, $\{\mathbf{e}_{(\alpha)}\}_{\alpha=1,\cdots n}$, where we put in the Greek indices under parenthesis since they are not the abstract indices standing for tensorial character: they are just the indices of the basis. According to the diffeomorphism above, we identify, within the appropriate normal neighborhoods, a point e.g. $q = z^\alpha$ as $\mathsf{Exp}_p(z^\alpha \mathbf{e}_\alpha)$. Then, $\frac{\partial}{\partial x^\alpha} = \mathbf{e}_{(\alpha)}$ and, by (1.6), ${\Gamma^\alpha}_{\beta\gamma}|_p = 0$. For the proof of existence of Fermi coordinates along a geodesic, see (Poisson, 2004, Sec. 1.11). The main idea there is to restrict the exponential map to a normal neighborhood of the geodesic, but the main points of the proof follow in similar lines.

It is easy to see that the construction is itself coordinate invariant. Start with any metric and any coordinate system, $g_{\mu\nu}$, and the given geodesic γ, and, you will find a metric-dependent coordinate transformation, $f(\gamma, g)$ that takes the original metric $g_{\mu\nu}$ to $\tilde{g}_{\mu\nu}$, to the Fermi-normal one, with the required properties (with vanishing Γ over γ). We can write, for that target metric: $f(\gamma, g)^* g_{\mu\nu} = \tilde{g}_{\mu\nu}$. The proposition that such a functional f exists is

[18] The idea of the exponential map is that it 'shoots out' geodesics in the directions of the unit vectors, with parameter length given by the size of the vectors, so, e.g. the curve $\gamma_u(t) := \mathsf{Exp}_p(tu)$ is a geodesic, such that $\gamma_u'(0) = u$. Since the tangent to the curve $\gamma_u(t)$ at the origin is u, the tangent to the exponential map at the origin is the identity, thus it has injective rank, and therefore, by the implicit function theorem, there are appropriately defined neighborhoods in domain and co-domain in which it is a diffeomorphism.

coordinate-independent. Said in a different way, any two metrics that are related by a coordinate transformation will, under this procedure, be projected to the same $\tilde{g}_{\mu\nu}$.

Indeed, it is possible to endow the construction procedure with specific operational significance (using light-rays, clocks, etc).

References

Everitt, C. W. F., DeBra, D. B., Parkinson, B. W., Turneaure, J. P., Conklin, J. W., Heifetz, M. I., ... Wang, S. (2011, May). Gravity probe b: Final results of a space experiment to test general relativity. *Phys. Rev. Lett.*, *106*, 221101. Retrieved from https://link.aps.org/doi/10.1103/PhysRevLett.106.221101 doi: 10.1103/PhysRevLett.106.221101

Fletcher, S. C., & Weatherall, J. O. (2022a). *The local validity of special relativity, part 1: Geometry.* arXiv. Retrieved from https://arxiv.org/abs/2206.10768 doi: 10.48550/ARXIV.2206.10768

Fletcher, S. C., & Weatherall, J. O. (2022b). *The local validity of special relativity, part 2: Matter dynamics.* arXiv. Retrieved from https://arxiv.org/abs/2206.10774Search... doi: 10.48550/ARXIV.2206.10774Search...

Gibbons, G. W. (1994). Changes of topology and changes of signature. *International Journal of Modern Physics D*, *03*(01), 61-70. Retrieved from https://doi.org/10.1142/S0218271894000071 doi: 10.1142/S0218271894000071

Hawking, S. W., & Ellis, G. F. R. (1975). *The Large Scale Structure of Space-Time (Cambridge Monographs on Mathematical Physics).* Cambridge University Press. Retrieved from http://www.amazon.com/Structure-Space-Time-Cambridge-Monographs-Mathematical/dp/0521099064

Iliev, B. (2006). *Handbook of normal frames and coordinates.* Birkhauser, Basel.

Poisson, E. (2004). *A Relativist's Toolkit.* Cambridge University Press. doi: 10.1017/cbo9780511606601

Weatherall, J. O. (2018). *Geometry and motion in general relativity.* arXiv. Retrieved from https://arxiv.org/abs/1810.09046 doi: 10.48550/ARXIV.1810.09046