Born rule from counting states

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I give a very simple derivation of the Born rule by counting states from a continuous basis.

More precisely, I show that in a continuous basis, the contributing basis vectors are present in a state vector with real and equal coefficients, but they are distributed with variable density among the eigenspaces of the observable. Counting the contributing basis vectors while taking their density into account gives the Born rule without making other assumptions. State counting yields the Born rule only if the basis is continuous, but all known physically realistic observables admit such bases.

The continuous basis is not unique, and for subsystems it depends on the observable.

But for the entire universe, there are continuous bases that give the Born rule for all measurements, because all measurements reduce to distinguishing macroscopic pointer states, and macroscopic observations commute. This allows for the possibility of an ontic basis for the entire universe.

In the wavefunctional formulation, the basis can be chosen to consist of classical field configurations, and the coefficients $\Psi[\phi]$ can be made real by absorbing them into a global U(1) gauge.

For the many-worlds interpretation, this result gives the Born rule from micro-branch counting.

Keywords: Born rule; state counting; Everett's interpretation; many-worlds interpretation; branch counting.

I. INTRODUCTION

In quantum mechanics, the Born rule prescribes the probability that the outcome of a quantum measurement is the eigenvalue λ_j of the observable is

$$\operatorname{Prob}(\lambda_j) = \langle\psi|\widehat{\mathsf{P}}_j|\psi\rangle, \tag{1}$$

where the unit vector $|\psi\rangle$ represents the state of the observed system right before the measurement, and $\widehat{\mathsf{P}}_j$ is the projector on the eigenspace corresponding to λ_j.

The *projection postulate* states that $|\psi\rangle$ projects onto one of the eigenspaces $\widehat{\mathsf{P}}_j$ with the probability from (1).

von Neumann expressed already in 1927 the desirability of having a derivation of the Born rule "from empirical facts or fundamental probability-theoretic assumptions, *i.e.*, an inductive justification" [22]. Gleason's theorem shows that any countably additive probability measure on closed subspaces of a Hilbert space $\mathcal{H}$, $\dim\mathcal{H} > 2$, has the form $\operatorname{tr}(\widehat{\mathsf{P}}\widehat{\rho})$, where $\widehat{\mathsf{P}}$ is the projector on the subspace and $\widehat{\rho}$ is a density operator [13]. If the state is represented by $\widehat{\rho}$, this can be interpreted as the Born rule. Gleason's theorem is very important, in showing that if there is a probability rule, it should have the form of the Born rule. But it does not say that the density operator of the observed system is the same $\widehat{\rho}$, how the probabilities arise in the first place, and what they are about [9]. For example, it is unable to convert the amplitudes of the branches in the many-worlds interpretation (MWI) [7, 10, 20, 24] into actual probabilities. For this reason, the search for a proof of the Born rule continues.

There are numerous proposals to derive the Born rule. Earlier attempts to derive it from more basic principles include [12], [14], and others [11]. Such approaches based on a frequency operator were accused of circularity [5, 6]. Other proposals, in relation to MWI, are based on many-minds [1], decision theory [8, 17, 23] (also accused of circularity in [2, 3]), envariance [25] (accused of circularity in [18]), measure of existence [19] *etc.* For a review see [21]. The necessity to obtain the Born rule in MWI by branch counting was advocated in [16].

In this article I follow this guideline:

Goal 1. Ideally, the Born rule should be obtained in the old-fashioned way, as *the ratio of the number of favorable outcomes to the total number of possible outcomes.*

I show that, in a continuous basis, it is possible to express the state vector as a linear combination of basis vectors of equal norm, but distributed unevenly. Then the probability density can be understood as a distribution of "classical" states (Fig. 1).

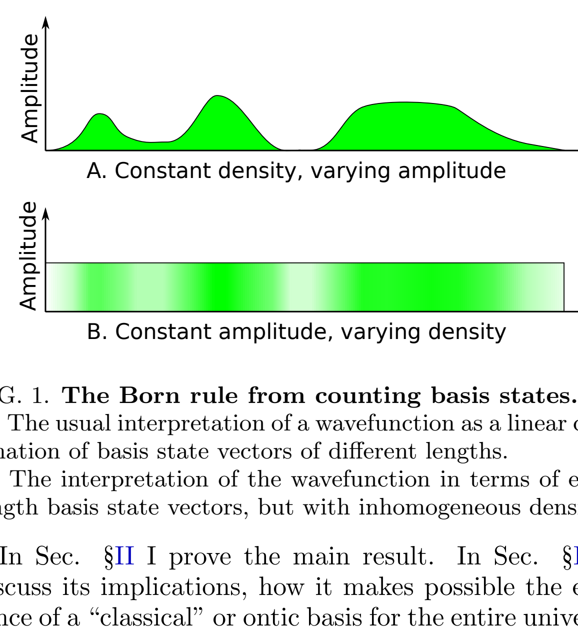

FIG. 1. **The Born rule from counting basis states.**
A. The usual interpretation of a wavefunction as a linear combination of basis state vectors of different lengths.
B. The interpretation of the wavefunction in terms of equal length basis state vectors, but with inhomogeneous density.

In Sec. §II I prove the main result. In Sec. §III I discuss its implications, how it makes possible the existence of a "classical" or ontic basis for the entire universe, how the wavefunction becomes real, and how this yields probabilities in the many-worlds interpretation.

II. PROBABILITIES FROM COUNTING

Before proving the main result, let us motivate it.

Consider a state vector of the form

$$|\psi\rangle = \frac{1}{\sqrt{n}}\sum_{k=1}^{n}|\phi_k\rangle. \tag{2}$$

where $(|\phi_k\rangle)_{k\in\{1,\dots,n\}}$ are orthonormal vectors from $\mathcal{H}$. Then, if every $|\phi_k\rangle$ is an eigenvector of the operator $\widehat{\mathsf{A}}$ representing the observable, the Born rule simply coincides with the counting of basis states:

$$\begin{aligned}\langle\psi|\widehat{\mathsf{P}}_j|\psi\rangle &= \frac{1}{n}\left(\sum_{k=1}^{n}\langle\phi_k|\right)\left(\widehat{\mathsf{P}}_j\sum_{k=1}^{n}|\phi_k\rangle\right)\\ &= \frac{1}{n}\sum_{|\phi_k\rangle\in\widehat{\mathsf{P}}_j\mathcal{H}}\langle\phi_k|\phi_k\rangle = \frac{n_j}{n},\end{aligned} \tag{3}$$

where $\widehat{\mathsf{P}}_j$ is the projector of the eigenspace corresponding to the eigenvalue λ_j, and n_j is the number of basis vectors $|\phi_k\rangle$ that are eigenvectors for λ_j. This would satisfy Goal 1, but the state vectors of the form (2) are very special.

Interestingly, in the continuous case, the basis vectors can be distributed with nonuniform density, making it possible for the continuous version of eq. (2) to apply to any state vector. This motivates the following results.

Theorem 1. *Let $(|\phi\rangle)_{\phi\in\mathcal{C}}$ be an orthonormal basis indexed continuously by the points of a topological manifold $\mathcal{C}$ with a measure μ on its σ-algebra. Then, any state vector $|\psi\rangle$ so that $|\langle\phi|\psi\rangle|$ is μ-measurable has the form*

$$|\psi\rangle = \int_{\phi\in\mathcal{C}} e^{i\alpha(\phi)}|\phi\rangle d\widetilde{\mu}(\phi), \tag{4}$$

where $\alpha:\mathcal{C}\to\mathbb{R}$, and $\widetilde{\mu}$ is a measure on $\mathcal{C}$ specifying the density of the basis vectors $(e^{i\alpha(\phi)}|\phi\rangle)_{\phi\in\mathcal{C}}$.

If the eigenspace $\mathcal{H}_\lambda$ of an eigenvalue λ of an observable $\widehat{\mathsf{A}}$ is spanned by $(|\phi\rangle)_{\phi\in\mathcal{C}_\lambda}$, where $\mathcal{C}_\lambda$ is μ-measurable,

$$\operatorname{Prob}(\lambda) = \int_{\phi\in\mathcal{C}_\lambda} e^{i\alpha(\phi)}|\phi\rangle d\widetilde{\mu}(\phi). \tag{5}$$

Proof. In full generality, we can assume that $\langle\phi|\psi\rangle\in\mathbb{R}$ for all ϕ. If not, substitute $|\phi\rangle\mapsto e^{i\alpha(\phi)}|\phi\rangle$, where $\alpha(\phi)$ is the phase in the polar form of $\langle\phi|\psi\rangle$, for all $\phi\in\mathcal{C}$. Then,

$$|\psi\rangle = \int_{\phi\in\mathcal{C}} r(\phi)|\phi\rangle d\mu(\phi), \tag{6}$$

where $r(\phi) := |\langle\phi|\psi\rangle|$ and $r\in L^2(\mathcal{C},\mu,\mathbb{R})$ is a real non-negative square-integrable function.

The measure $d\widetilde{\mu}(\phi) := r(\phi)d\mu(\phi)$ satisfies eq. (4):

$$|\psi\rangle = \int_{\phi\in\mathcal{C}} |\phi\rangle d\widetilde{\mu}(\phi). \tag{7}$$

Since $r(\phi)$ is μ-measurable, the measure $\widetilde{\mu}$ is absolutely continuous with respect to μ.

If one is not careful enough, one may think that eq. (7) cannot represent a normalized vector. But it does:

$$\begin{aligned}\langle\psi|\psi\rangle &= \int_{\phi\in\mathcal{C}}\langle\phi|d\widetilde{\mu}(\phi)\int_{\phi'\in\mathcal{C}}|\phi'\rangle d\widetilde{\mu}(\phi')\\ &= \int_{\phi\in\mathcal{C}}\left(\int_{\phi'\in\mathcal{C}}\langle\phi|\phi'\rangle d\widetilde{\mu}(\phi')\right)d\widetilde{\mu}(\phi)\\ &= \int_{\phi\in\mathcal{C}}\left(\int_{\phi'\in\mathcal{C}}\langle\phi|\phi'\rangle r(\phi')d\mu(\phi')\right)d\widetilde{\mu}(\phi)\\ &= \int_{\phi\in\mathcal{C}} r(\phi)d\widetilde{\mu}(\phi) = \int_{\phi\in\mathcal{C}} r^2(\phi)d\mu(\phi) = 1.\end{aligned} \tag{8}$$

Eq. (5) follows directly from eq. (7). □

Therefore, the density $\widetilde{\mu}$ of the basis states corresponds to the Born rule, according to Goal 1.

III. IMPLICATIONS

Remark 1. For any physically realistic quantum measurement there is a continuous basis in which the observable is diagonal, as required by Theorem 1. Even for a single particle in nonrelativistic quantum mechanics, the Hilbert space is infinite-dimensional, and admits continuous bases, *e.g.* the position basis. In general, measurements reduce to position measurements: the pointer indicates the result by its position, for a photographic plate we read the position where the particle hit it *etc.* In practice, these are not points, but regions of space of positive area or volume, so **all measurements satisfy, in practice, the conditions from Theorem 1.** □

Remark 2. Subsystems admit observables that cannot be diagonalized simultaneously, so the continuous basis depends on the observable. Therefore, **for subsystems there are no continuous bases universal for all observables**. □

Remark 3. However, **there is a universal continuous basis for the entire universe**. Every measurement ultimately becomes a direct observation of a macro-state, the state of the pointer of the measuring device. So every measurement reduces to distinguishing macro-states. *Macro-states* are represented by subspaces of the form $\widehat{\mathsf{P}}_\alpha\mathcal{H}$, where $(\widehat{\mathsf{P}}_\alpha)_{\alpha\in\mathcal{A}}$ is a complete set of commuting projectors on $\mathcal{H}$, so that $[\widehat{\mathsf{P}}_\alpha,\widehat{\mathsf{P}}_\beta]=0$ for any $\alpha\neq\beta\in\mathcal{A}$, and $\bigoplus_{\alpha\in\mathcal{A}}\widehat{\mathsf{P}}_\alpha\mathcal{H}=\mathcal{H}$. Since ultimately every measurement translates into an observation represented by the macro projectors, there is a universal continuous basis for all measurements, which diagonalizes all macro projectors. Therefore, this universal basis can be taken as representing "classical states", which may be called *ontic states*. Theorem 1 allows us to interpret the Born rule for any measurement as counting such ontic states. This is consistent with any observable we measure for the subsystem, since different measurement settings ultimately translate to distinguishing macro-states defined by the same set of macro projectors.

It may seem too much to count states of the entire universe just to account for the probabilities of the measurement of a single particle. But in fact we always do this, because the observed particle can be entangled with any other system in the universe. □

Remark 4. A basis $(|\phi\rangle)_{\phi\in\mathcal{C}}$ that really is ontic or classical is possible. In the Schrödinger wavefunctional formulation of quantum field theory [15], $\mathcal{C}$ becomes the configuration space of classical fields, and the wavefunctional $\Psi[\phi] := \langle\phi|\Psi\rangle$ replaces the nonrelativistic wavefunction. **Now our basis literally consists of classical states.** While it may be unusual to interpret quantum mechanics in this way, it makes sense, once we remember that we never observe individual particles, but macro-states, and these are imported from the classical theory. □

Remark 5. The phase change $|\phi\rangle \mapsto e^{i\alpha(\phi)}|\phi\rangle$ from the proof of Theorem 1 can be identified with an U(1) gauge transformation of the classical field, $\phi \mapsto e^{i\alpha(\phi)}\phi$, so that $e^{i\alpha(\phi)}|\phi\rangle = |e^{i\alpha(\phi)}\phi\rangle$, because both are unphysical. Charged and spinor fields admit an U(1) symmetry, but also the photons [4], since classical electromagnetic field admits a complex form. Then, $\Psi[\phi]$ **can be made real by changing the global** U(1) **gauge of the classical states**, and eq. (7) can be interpreted directly as a distribution of classical states. □

Remark 6. In the **many-worlds interpretation**, if we "naively" count the worlds or macro-branches that result after a measurement, the result coincides with the Born rule only if the state has the form (2) in the eigenbasis of the observable. But counting micro-branches that correspond to the basis $(|\phi\rangle)_{\phi\in\mathcal{C}}$ gives the correct probabilities (even if they may interfere in the future, unlike the macro-branches), in accord with Goal 1. □

Remark 7. In the wavefunctional approach each micro-branch consists of classical fields ϕ. **These are the local beables. This justifies counting each micro-branch as a world.** □

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