Born rule: quantum probability as classical probability

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(Dated: November 2, 2022)

I provide a simple derivation of the Born rule as giving a classical probability, that is, the ratio of the measure of favorable states of the system to the measure of its total possible states.

In classical systems, the probability is due to the fact that the same macro state can be realized in different ways as a micro state. Despite the radical differences between quantum and classical systems, the same can be applied to quantum systems. More precisely, I show that in a continuous basis, the contributing basis vectors are present in a state vector with real and equal coefficients, but they are distributed with variable density among the eigenspaces of the observable. The measure of the contributing basis vectors gives the Born rule without making other assumptions.

This works only if the basis is continuous, but all known physically realistic measurements involve a continuous basis, because they involve the positions of the particles.

The continuous basis is not unique, and for subsystems it depends on the observable.

But for the entire universe, there are continuous bases that give the Born rule for all measurements, because all measurements reduce to distinguishing macroscopic pointer states, and macroscopic observations commute. This allows for the possibility of an ontic basis for the entire universe.

In the wavefunctional formulation, the basis can be chosen to consist of classical field configurations, and the coefficients $\Psi[\phi]$ can be made real by absorbing them into a global U(1) gauge.

This suggests an interpretation of the wavefunction as a nonuniform distribution of classical states. For the many-worlds interpretation, this result gives the Born rule from micro-branch counting.

Keywords: Born rule; state counting; Everett's interpretation; many-worlds interpretation; branch counting.

I. INTRODUCTION

In quantum mechanics, the Born rule prescribes that the probability that the outcome of a quantum measurement is the eigenvalue λ_j of the observable is

$$\mathrm{Prob}(\lambda_j) = \langle\psi|\widehat{\mathsf{P}}_j|\psi\rangle, \tag{1}$$

where the unit vector $|\psi\rangle$ represents the state of the observed system right before the measurement, and $\widehat{\mathsf{P}}_j$ is the projector on the eigenspace corresponding to λ_j.

The *projection postulate* states that $|\psi\rangle$ projects onto one of the eigenspaces $\widehat{\mathsf{P}}_j$ with the probability from (1).

von Neumann expressed already in 1927 the desirability of having a derivation of the Born rule "from empirical facts or fundamental probability-theoretic assumptions, *i.e.*, an inductive justification" [26]. Gleason's theorem shows that any countably additive probability measure on closed subspaces of a Hilbert space $\mathcal{H}$, $\dim\mathcal{H} > 2$, has the form $\mathrm{tr}(\widehat{\mathsf{P}}\widehat{\rho})$, where $\widehat{\mathsf{P}}$ is the projector on the subspace and $\widehat{\rho}$ is a density operator [14]. If the state is represented by $\widehat{\rho}$, this can be interpreted as the Born rule. Gleason's theorem is very important, in showing that if there is a probability rule, it should have the form of the Born rule. But it does not say that the density operator of the observed system is the same $\widehat{\rho}$, how the probabilities arise in the first place, and what they are about [9]. For example, it is unable to convert the amplitudes of the branches in the many-worlds interpretation (MWI) [7, 10, 24, 28] into actual probabilities. For this reason, the search for a proof of the Born rule continues.

There are numerous proposals to derive the Born rule. Earlier attempts to derive it from more basic principles include [12], [15], [11] *etc.* Such approaches based on a frequency operator were accused of circularity [5, 6]. Other proposals, in relation to MWI, are based on many-minds [1], decision theory [8, 19, 27] (accused of circularity in [2, 3]), envariance [29] (accused of circularity in [20]), measure of existence [23] *etc.* For a review see [25]. The necessity to obtain the Born rule in MWI by branch counting was advocated in [18], where Saunders proposed the existence of consistent histories that are more refined than the ones that give the branching structure and have equal amplitude branches.

In this article, I investigate the possibility of obtaining probabilities that are very similar to the classical ones. As in classical physics, what we observe are macro-states. If each macro state can be realized in different ways as a micro state, probabilities can arise from the relative count, or rather the relative measure[1], of the micro-states underlying each macro state, just like in the standard understanding of probabilities.

In Section §II I argue that, contrary to the common view on quantum mechanics, an "ontic" or "classical" basis for the entire universe is possible, allowing for classical-like probabilities in quantum mechanics.

In Sec. §III I prove the main result, that in a continuous basis it is possible to express the state vector as a linear combination of basis vectors of equal norm, but distributed unevenly. Then the probability density can be understood as a distribution of "classical" states

[1] The term "counting" is a misnomer if the basis is continuous, and therefore uncountable, but it may be intuitive.

relatively to that basis (Fig. 1).

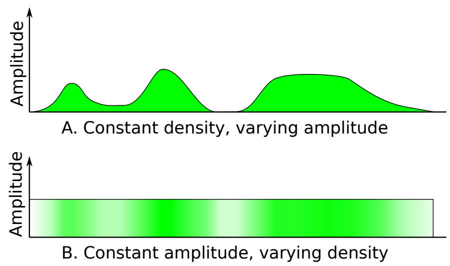

FIG. 1. **The Born rule from "counting" basis states.** **A.** The usual interpretation of a wavefunction as a linear combination of basis state vectors of different amplitudes. **B.** The interpretation of the wavefunction in terms of basis vectors with uniform amplitude but inhomogeneous density.

In Sec. §IV I discuss the physical interpretation of this derivation of the Born rule, how it makes possible the existence of a "classical" or ontic basis for the entire universe, how complex numbers appear, and how this yields probabilities in the many-worlds interpretation.

II. CLASSICAL *VS.* QUANTUM PROBABILITIES

In this Section we look at classical probabilities to identify what conditions they require to be satisfied, and whether they can be satisfied in quantum mechanics. This will provide the physical justification to interpret probabilities in a quantum world, based on the proof given in Sec. §III.

If the world would be classical and deterministic, the result of a process can appear to be impredictable at the macro-level, even if "Laplace's demon", who knows the physical state in full detail, should be able to assign to each event a probability equal to either 0 or 1.

Observation 1. Nontrivial probabilities exist for agents that lack complete information about the micro-state of the system.

For example, the classical probability that throwing a pair of dice results in the outcome

$$\text{⚅} + \text{⚅} \tag{2}$$

is given by the measure of the set of micro-states that realize the macro-state in which the outcome is (2) divided by the total measure. Let us summarize this:

Condition 1 (Probability)**.** The probability is *the ratio of the measure of favorable outcomes to the total measure of possible outcomes.*

In classical physics, Condition 1 makes sense because the universe is in a unique state at any time. But in quantum mechanics, an observed system can be in a superposition of multiple states that coexist in parallel.

Difficulty 1. Unlike classical systems, quantum micro-states seem to be able to coexist in parallel, in superposition, as shown by interference experiments.

But would Condition 1 be invalidated if multiple classical worlds would exist in parallel? In a classical universe where there are more worlds and the agent doesn't know in which of them it exists, the ignorance of the micro-state assuming the knowledge of the macro-state is the same as in a universe where there is only one world whose state is incompletely known to the agent. This leads to the following:

Observation 2 (Equivalence)**.** Probabilities for a given macro-state are independent on whether the distribution describes the probability that the agent exists in a unique world and ignores its micro-state, or if it describes more worlds, and the agent does not know in which of these worlds it exists.

An implicit assumption underlying Observation 2 is that an agent or observer *supervenes* (in the sense that its states depends) on a single world, even if there are more parallel worlds. But if there can be more parallel worlds, this condition is additionally needed:

Condition 2 (Correspondence)**.** If there are more parallel worlds, and at a given time different instances of an agent exist in more of them, each instance of the agent supervenes on only one of these worlds.

In other words, the physical state of the world should be able to support ontologically the existence of agents or observers, so that their experience of probabilities depends on their ignorance of the micro-state.

In a quantum world, the central difference is that there are multiple ways in which the macro-state of a subsystem can be realized as micro-states, each depending on the experimental settings. For example, the spin of a particle can be interpreted as consisting of definite possible spins $|\uparrow\rangle_z$ and $|\downarrow\rangle_z$ if the spin is measured along the axis z, but not if it is measured along another axis. In classical mechanics, the possible outcomes are considered to be independent of the measuring settings, provided that the observation's effect on the observed system can be made arbitrarily small.

The main difficulty in the applicability of classical probabilities in quantum mechanics is therefore

Difficulty 2. The possible observed quantum states of a subsystem depend on the settings of the measuring device performing the observation of that subsystem.

However, quantum mechanics can satisfy Condition 2. For example, a measurement of the spin of a spin-1/2 particle along the axis z results in the possible states

$$\begin{cases} |\uparrow\rangle_z |\text{up}\rangle_z \\ |\downarrow\rangle_z |\text{down}\rangle_z, \end{cases} \tag{3}$$

where $|\uparrow\rangle_z$ and $|\downarrow\rangle_z$ are the spin states of the particle along the axis z, and $|\text{up}\rangle_z$ and $|\text{down}\rangle_z$ are the corresponding states of the pointer. A spin measurement along the axis x leads to a different decomposition,

$$\begin{cases} 1/\sqrt{2}\,(|\uparrow\rangle_z + |\downarrow\rangle_z)\,|\text{up}\rangle_x \\ 1/\sqrt{2}\,(|\uparrow\rangle_z - |\downarrow\rangle_z)\,|\text{down}\rangle_x. \end{cases} \tag{4}$$

The macro-states corresponding to spin measurements along distinct axes are orthogonal, so the micro-states from eq. (3) are orthogonal to those from eq. (4), even though the observed system's states $|\uparrow\rangle_z$ and $|\downarrow\rangle_z$ are not orthogonal to $1/\sqrt{2}\,(|\uparrow\rangle_z \pm |\downarrow\rangle_z)$. Since all four states $|\text{up}\rangle_z$, $|\text{down}\rangle_z$,$|\text{up}\rangle_x$, and $|\text{down}\rangle_x$ are macroscopically distinct, they are orthogonal, so the four states from equations (3) and (4) are also orthogonal.

In general, every quantum measurement ultimately becomes a direct observation of the macro-state of the measuring device. So every measurement reduces to distinguishing macro-states. Macro-states are distinguished by macro-observables, and all macro-observables commute.

Macro-states are represented by subspaces of the form $\widehat{\mathsf{P}}_\alpha\mathcal{H}$, where $(\widehat{\mathsf{P}}_\alpha)_{\alpha\in\mathcal{A}}$ is a set of commuting projectors on $\mathcal{H}$, so that $[\widehat{\mathsf{P}}_\alpha, \widehat{\mathsf{P}}_\beta] = 0$ for any $\alpha \neq \beta \in \mathcal{A}$, and $\bigoplus_{\alpha\in\mathcal{A}}\widehat{\mathsf{P}}_\alpha\mathcal{H} = \mathcal{H}$. This claim is empirically adequate, as illustrated by the example of spin measurements. This position is adopted for example in decohering histories approaches [13] and in MWI [28].

Observation 3. We never observe the micro-state, only the macro-states.

Since ultimately every measurement translates into an observation represented by the macro projectors, there is a universal basis for all measurements, which diagonalizes all macro projectors.

Observation 4. For the entire universe, whose states are represented by vectors in a Hilbert space $\mathcal{H}$, there is a universal basis

$$(|\phi\rangle)_{\phi\in\mathcal{C}} \tag{5}$$

compatible with the macro-states. In general, more such bases exist.

It may seem too much to account for states of the entire universe just to explain the probabilities of the measurement of a single particle. But in fact we always do this, because the observed particle can be entangled with any other system in the universe. The usual separation between the observed system and the rest of the universe that enters in our theoretical description is an idealization that may make us not the forest for the trees. Then,

Observation 5. The state of the universe is not a set of independent states of subsystems, but a single state.

Despite Difficulty 2, the existence of a basis as in Observation 4 will turn out to make it possible for quantum mechanics to satisfy Condition 2. This requires

Principle 1. In quantum mechanics, there is a basis as in Observation 4, so that all instances of an agent can be realized only in worlds whose states are from that basis. We will call it *ontic basis* and its elements *ontic states*.

But another difficulty in quantum mechanics is that the state is found, after wavefunction collapse or decoherence, to be $\widehat{\mathsf{P}}_\alpha|\psi\rangle/|\widehat{\mathsf{P}}_\alpha|\psi\rangle|$. The Born rule prescribes the probability that the state becomes $\widehat{\mathsf{P}}_\alpha|\psi\rangle/|\widehat{\mathsf{P}}_\alpha|\psi\rangle|$ is given by $\langle\psi|\widehat{\mathsf{P}}_\alpha|\psi\rangle$, as in (1). In the ontic basis (5), $|\psi\rangle$ is decomposed as a linear combination with distinct coefficients (amplitudes) $\langle\phi|\psi\rangle$, so

$$\text{Prob}(\alpha) = \langle\psi|\widehat{\mathsf{P}}_\alpha|\psi\rangle = \int_{\mathcal{C}_\alpha} |\langle\phi|\psi\rangle|^2\,(\phi)d\mu(\phi), \tag{6}$$

where $\mathcal{C}_\alpha = \{\phi \in \mathcal{C} | |\phi\rangle \in \widehat{\mathsf{P}}_\alpha\mathcal{H}\}$ and μ is the measure on $\mathcal{C}$. While $|\langle\phi|\psi\rangle|^2$ gives a measure, this is not sufficient to interpret it probabilistically:

Difficulty 3. Measure alone is not probability.

In Sec. §III we will see that Principle 1 allows Condition 2 to be satisfied in quantum mechanics, and Difficulties 1, 2, and 3 to be avoided, so by Observation 2, Condition 1 is satisfied too.

III. DERIVATION OF THE BORN RULE

Before proving the main result, let us motivate it. Consider a state vector of the form

$$|\psi\rangle = \frac{1}{\sqrt{n}}\sum_{k=1}^{n}|\phi_k\rangle. \tag{7}$$

where $(|\phi_k\rangle)_{k\in\{1,\dots,n\}}$ are orthonormal vectors from $\mathcal{H}$. Then, if every $|\phi_k\rangle$ is an eigenvector of the operator $\widehat{\mathsf{A}}$ representing the observable, the Born rule simply coincides with counting basis states:

$$\begin{aligned} \langle\psi|\widehat{\mathsf{P}}_j|\psi\rangle &= \frac{1}{n}\left(\sum_{k=1}^{n}\langle\phi_k|\right)\left(\widehat{\mathsf{P}}_j\sum_{k=1}^{n}|\phi_k\rangle\right) \\ &= \frac{1}{n}\sum_{|\phi_k\rangle\in\widehat{\mathsf{P}}_j\mathcal{H}}\langle\phi_k|\phi_k\rangle = \frac{n_j}{n}, \end{aligned} \tag{8}$$

where $\widehat{\mathsf{P}}_j$ is the projector of the eigenspace corresponding to the eigenvalue λ_j, and n_j is the number of basis vectors $|\phi_k\rangle$ that are eigenvectors for λ_j.

In MWI, this is the only situation when "naive branch counting" coincides with the Born rule. It is used as a starting point for arguments that the Born rule is present in MWI for "less naive" counting rules [8, 18, 23, 27].

Eq. (8) would satisfy Condition 1, but it seems to work only for very special state vectors. We cannot even make it work for all vectors in a finite-dimensional Hilbert space, since the basis vectors either have to contribute to eq. (7) with the same absolute value $1/\sqrt{n}$ of the coefficient $\langle\phi_k|\psi\rangle$, or to be absent.

Interestingly, as if by magic, the idea works in the continuous case without problems, because the basis vectors can be distributed with nonuniform density, making it possible for the continuous version of eq. (7) to apply to any state vector. Let $\mathcal{C}$ be a topological manifold with a measure μ on its σ-algebra, and $\mathcal{H} := L^2(\mathcal{C},\mu,\mathbb{C})$ the Hilbert space of square-integrable complex functions on $\mathcal{C}$. Let $(|\phi\rangle)_{\phi\in\mathcal{C}}$ an orthogonal basis of $\mathcal{H}$, so that

$$\int_{\mathcal{C}} \langle\phi|\phi'\rangle\psi(\phi')d\widetilde{\mu}(\phi') = \psi(\phi) \tag{9}$$

for any square-integrable function $\psi \in \mathcal{H}$.

Without loss of generality, for any given state vector $|\psi\rangle$ so that $|\langle\phi|\psi\rangle|$ is μ-measurable, we can assume that $\langle\phi|\psi\rangle \in \mathbb{R}$ for all ϕ. If not, substitute the basis by $|\phi\rangle \mapsto e^{i\theta(\phi)}|\phi\rangle$, where $\theta(\phi)$ is the phase appearing in the polar form of $\langle\phi|\psi\rangle$, for all $\phi \in \mathcal{C}$.

Theorem 1. *The state vector $|\psi\rangle$ has the form*

$$|\psi\rangle = \int_{\mathcal{C}} |\phi\rangle d\widetilde{\mu}(\phi), \tag{10}$$

where $\theta : \mathcal{C} \to \mathbb{R}$, and $\widetilde{\mu}$ is a measure on $\mathcal{C}$ specifying the density of the basis vectors $(|\phi\rangle)_{\phi\in\mathcal{C}}$.

Any projector $\widehat{\mathsf{P}}_\alpha$ diagonal in the basis $(|\phi\rangle)_{\phi\in\mathcal{C}}$ corresponds to a subset $\mathcal{C}_\alpha \subseteq \mathcal{C}$. If $\mathcal{C}_\alpha$ is μ-measurable,

$$\left|\int_{\mathcal{C}_\alpha} |\phi\rangle d\widetilde{\mu}(\phi)\right|^2 = \int_{\mathcal{C}_\alpha} r^2(\phi)d\mu(\phi). \tag{11}$$

Proof. Let $r(\phi) := |\langle\phi|\psi\rangle|$. Then, $r \in L^2(\mathcal{C},\mu,\mathbb{R})$ is a real non-negative square-integrable function, and

$$|\psi\rangle = \int_{\mathcal{C}} r(\phi)|\phi\rangle d\mu(\phi). \tag{12}$$

The following measure satisfies eq. (10),

$$d\widetilde{\mu}(\phi) := r(\phi)d\mu(\phi). \tag{13}$$

Then, evidently $\widehat{\mathsf{P}}_\alpha|\psi\rangle = \int_{\mathcal{C}_\alpha} |\phi\rangle d\widetilde{\mu}(\phi)$, and

$$\left|\int_{\mathcal{C}_\alpha} |\phi\rangle d\widetilde{\mu}(\phi)\right|^2 = \langle\psi|\widehat{\mathsf{P}}_\alpha|\psi\rangle = \int_{\mathcal{C}_\alpha} r^2(\phi)d\mu(\phi). \tag{14}$$

The reader may think that $\int_{\mathcal{C}_\alpha} |\phi\rangle d\widetilde{\mu}(\phi)$ cannot have finite norm, or that, in any case, it has to be larger than 1. So let us check this more explicitly:

$$\begin{aligned}\left|\int_{\mathcal{C}_\alpha} |\phi\rangle d\widetilde{\mu}(\phi)\right|^2 &= \left(\int_{\mathcal{C}_\alpha} \langle\phi| d\widetilde{\mu}(\phi)\right)\left(\int_{\mathcal{C}_\alpha} |\phi'\rangle d\widetilde{\mu}(\phi')\right) \\ &= \int_{\mathcal{C}_\alpha}\left(\int_{\mathcal{C}_\alpha} \langle\phi|\phi'\rangle d\widetilde{\mu}(\phi')\right)d\widetilde{\mu}(\phi) \\ &= \int_{\mathcal{C}_\alpha}\left(\int_{\mathcal{C}_\alpha} \langle\phi|\phi'\rangle r(\phi')d\mu(\phi')\right)d\widetilde{\mu}(\phi) \\ &= \int_{\mathcal{C}_\alpha} r(\phi)d\widetilde{\mu}(\phi) = \int_{\mathcal{C}_\alpha} r^2(\phi)d\mu(\phi).\end{aligned} \tag{15}$$

This double-checks eq. (11). □

Observation 6 (Probability)**.** If Principle 1 is assumed in quantum mechanics, Theorem 1 shows that the density of the ontic states satisfies the Born rule for the macro-observables $\widehat{\mathsf{P}}_\alpha$, $\alpha \in \mathcal{A}$. This allows Condition 2 to be satisfied, and by Observation 2, Condition 1 is satisfied too, despite Difficulties 1–3, according to the Born rule.

But this requires the following:

Condition 3 (Continuity)**.** The ontic basis has to be continuous, in the sense that $\mathcal{C}$ is a topological manifold and $d\mu$ is continuous.

For any physically realistic quantum measurement there is a continuous basis in which the observable is diagonal, as required by Theorem 1. Even for a single particle in nonrelativistic quantum mechanics, the Hilbert space is infinite-dimensional, and admits continuous bases, *e.g.* the position basis.

Observation 7. All measurements satisfy, in practice, Condition 3 required for Theorem 1.

Example 1. Consider a measurement of the spin of a particle, whose spin state is initially $|\psi\rangle_s = a|\uparrow\rangle_z + b|\downarrow\rangle_z$, where $|a|^2 + |b|^2 = 1$. The particle also has position degrees of freedom, so its state is in fact $\psi(\mathbf{x},t) = \psi_u(\mathbf{x},t)|\uparrow\rangle_z + \psi_d(\mathbf{x},t)|\downarrow\rangle_z$. At the initial time t_0, $\psi_u(\mathbf{x},t_0) = a\psi_0(\mathbf{x})$, $\psi_d(\mathbf{x},t_0) = b\psi_0(\mathbf{x})$, and $\langle\psi_0|\psi_0\rangle = 1$. The measuring device also has position degrees of freedom. The measurement process consists of using a magnetic field to entangle the spin and the position of the particle, then it detects the position. After passing through the magnetic field, at time t_1, $\psi_u(\mathbf{x},t_1)$ becomes restricted to the "up" region of a screen or photographic plate, and $\psi_d(\mathbf{x},t_1)$ to the "down" region. From the position, the spin is inferred to be either "up" or "down". The regions "up" and "down" of the screen are almost identical, but the densities of $\psi_u(\mathbf{x},t_1)$ and $\psi_d(\mathbf{x},t_1)$ are proportional to $|a|^2$ and respectively $|b|^2$. Eq. (10) becomes, for the two possible outcomes,

$$\begin{cases}|\psi_u,t_1\rangle|\uparrow\rangle_z &= \int_{\text{up}} e^{i\theta_u(\mathbf{x})}|\mathbf{x}\rangle d\widetilde{\mu}_u(\mathbf{x}) \\ |\psi_d,t_1\rangle|\downarrow\rangle_z &= \int_{\text{down}} e^{i\theta_d(\mathbf{x})}|\mathbf{x}\rangle d\widetilde{\mu}_d(\mathbf{x})\end{cases} \tag{16}$$

where

$$\begin{cases}d\widetilde{\mu}_u(\mathbf{x}) &= |\psi_u(\mathbf{x},t_1)|\,d\mathbf{x} \\ d\widetilde{\mu}_d(\mathbf{x}) &= |\psi_d(\mathbf{x},t_1)|\,d\mathbf{x}.\end{cases} \tag{17}$$

Now, we invoke collapse or decoherence to explain why only one outcome is observed, and the probability is obtained by applying eq. (11). This illustrates how, despite apparently making a binary measurement of a qubit, the actual basis is continuous, as required by Theorem 1. □

Observation 8. We notice the existence of three densities. The first one is $d\mu$, given by the measure on $\mathcal{C}$, and it is independent of states. The second density is $d\widetilde{\mu} = r(\phi)d\mu$, which describes how the ontic states contribute to the state vector $|\psi\rangle$ in eq. (10). The third density, $r^2(\phi)d\mu$ is the probability density corresponding to the Born rule, as in eq. (11).

This may seem strange, despite the explicit calculation from (15), so let us try to understand the interplay between these densities.

Note 1 ("Magic" accident)**.** One may expect that we have to define the measure $\widetilde{\mu}$ so that $d\widetilde{\mu}(\phi)$ is $r^2(\phi)d\mu(\phi)$, rather than as in eq. (13). But, interestingly, eq. (11) follows without this, simply by choosing the measure $\widetilde{\mu}$ so that the amplitudes become uniformly equal to 1. Moreover, it does not even work otherwise, because $|\psi\rangle \neq \int_{\mathcal{C}} r^2(\phi)|\phi\rangle d\mu(\phi)$. □

Note 2 (Why it works?)**.** Naively, it may seem that the norm of $\int_{\mathcal{C}_\alpha} |\phi\rangle d\widetilde{\mu}(\phi)$ cannot be finite, or at least that it is equal to $\int_{\mathcal{C}_\alpha} d\widetilde{\mu}(\phi)$ and it can be larger than 1, but this is incorrect. Eq. (11) is correct, as checked in (14) and double-checked in (15), because $r(\phi)$ is square-integrable, and since it is μ-measurable, $\widetilde{\mu} \ll \mu$, *i.e.* the measure $\widetilde{\mu}$ is *absolutely continuous* with respect to μ.

There is a reason why, in eq. (15),

$$\int_{\mathcal{C}_\alpha} \langle\phi|\phi'\rangle d\widetilde{\mu}(\phi') = r(\phi) \tag{18}$$

rather than 1. A perhaps more revealing way of understanding this involves the *scaling property* of the Dirac distribution $\delta(x)$ with $a > 1$,

$$\delta(ax) = a^{-1}\delta(x). \tag{19}$$

To see how this works, consider the Hilbert space $L^2(\mathbb{R}^{\mathbf{n}}, \mu, \mathbb{C})$ with the basis $(|\mathbf{x}\rangle)_{\mathbf{x}\in\mathbb{R}^{\mathbf{n}}}$. If $\mathbf{f} : \mathbb{R}^{\mathbf{n}} \to \mathbb{R}^{\mathbf{n}}$ is an invertible reparametrization of $\mathbb{R}^{\mathbf{n}}$, by making a change of variables $\widetilde{\mathbf{y}} = \mathbf{f}(\mathbf{y})$ we obtain the following generalization of eq. (19),

$$\int_{\mathbb{R}^{\mathbf{n}}} \langle\mathbf{x}|\mathbf{y}\rangle \,\mathrm{d}\,\widetilde{\mathbf{y}} = \int_{\mathbb{R}^{\mathbf{n}}} \langle\mathbf{x}|\mathbf{y}\rangle \left|\frac{\partial\mathbf{f}}{\partial\mathbf{y}}\right| \mathrm{d}\,\mathbf{y} = \left|\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right|, \tag{20}$$

where $|\partial\mathbf{f}/\partial\mathbf{x}|$ is the modulus of the determinant of the Jacobian matrix of $\mathbf{f}$ at $\mathbf{x}$. With the notation from Theorem 1 but ϕ replaced by $\mathbf{x}$, μ is the Lebesgue measure on $\mathbb{R}^{\mathbf{n}}$, $d\mu(\mathbf{x}) = \mathrm{d}\,\mathbf{x}$, $d\widetilde{\mu}(\mathbf{y}) = \mathrm{d}\,\widetilde{\mathbf{y}}$, and

$$r(\mathbf{x}) = \frac{d\widetilde{\mu}(\mathbf{x})}{d\mu(\mathbf{x})} = \left|\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right|. \tag{21}$$

This explains once more how the homogenization of the amplitude from eq. (10), despite not involving $r^2(\phi)$, leads to its appearance in eq. (15), by using the generalized scaling property of the Dirac delta distribution. □

IV. INTERPRETATION OF THE WAVEFUNCTION

A. Wavefunction or wavefunctional?

Subsystems admit observables that cannot be diagonalized simultaneously, so their continuous bases depend on the observable. But since different measurement settings ultimately translate into distinguishing macrostates defined by the same set of macro projectors, the ontic basis from Observation 4 and Principle 1 is consistent with any observables we measure for the subsystems [21]. This universal basis can be taken as representing "classical states", which may be called *ontic states*. Theorem 1 allows us to interpret the Born rule for any measurement as "counting" such ontic states.

But what are these ontic states? Since each particle is represented on a Hilbert space of wavefunctions that have, among their degrees of freedom, the positions, which play a role in any measurement, and also form a continuous basis, it may be tempting to interpret the ontic states as position eigenstates, as in Example 1. But we know that in fact the world is not described by nonrelativistic quantum mechanics, but by quantum field theory.

A unique basis $(|\phi\rangle)_{\phi\in\mathcal{C}}$ that really is ontic or classical is possible in quantum field theory. In the Schrödinger wavefunctional formulation of quantum field theory [16, 17], $\mathcal{C}$ becomes the configuration space of classical fields, and the Schrödinger *wavefunctional*

$$\Psi[\phi] := \langle\phi|\Psi\rangle \tag{22}$$

replaces the nonrelativistic wavefunction. Here, ϕ stands for a collection of classical fields, $\phi = (\phi_1, \ldots, \phi_n)$.

B. Macro-classicality

The wavefunctional formulation represents quantum states in terms of classical field states, in the sense that the wavefunctional is a complex functional defined on the configuration space of classical fields. The usual Fock representation can be obtained from the basis $(|\phi\rangle)_{\phi\in\mathcal{C}}$ [16]. The Fock representation can then be used to interpret the quantum fields in terms of more commonly used nonrelativistic quantum mechanical wavefunctions and operators. But this is a departure from the more foundational description provided by quantum fields.

We never observe individual particles directly, but only macro-states. Macro-states are imported from the classical theory, and they are adequate, because at the macro level the world looks classical. Therefore, it makes sense to assume that states of the form $|\phi\rangle$ belong to macro-states, *i.e.* for every $|\phi\rangle$ there is a macro-state $\widehat{\mathsf{P}}_\alpha\mathcal{H}$ so that $|\phi\rangle \in \widehat{\mathsf{P}}_\alpha\mathcal{H}$, as in Observation 4 and Principle 1.

Principle 2. At any instant, at the macro level, a classical world in the classical state ϕ looks the same as a quantum world in the quantum state $|\phi\rangle$ or linear combinations of such states from the same macro-state.

And indeed, it took us a very long time to realize that our world is not classical, but quantum.

C. Interpretation of complex numbers

Recall that eq. (10) is based on absorbing the phase factor in the vector by substituting $|\phi\rangle \mapsto e^{i\theta[\phi]}|\phi\rangle$, done just before stating Theorem 1. This substitution depends on the state $|\Psi\rangle$, in particular $\theta[\phi]$ changes in time. So we cannot simply interpret $|\Psi\rangle$ directly as a set of classical states distributed according to the density from eq. (10).

But the phase change $|\phi\rangle \mapsto e^{i\theta[\phi]}|\phi\rangle$ can be identified with an U(1) gauge transformation of the classical field, denoted $\phi \mapsto e^{i\theta[\phi]}\phi$ (in fact U(1) acts differently on different fields, but I will use a uniform notation for its action), so that

$$e^{i\theta[\phi]}|\phi\rangle \equiv |e^{i\theta[\phi]}\phi\rangle. \tag{23}$$

This makes sense because (1) multiplying a state vector with a phase factor changes the vector, but not the physical (quantum) state it represents, and (2) an U(1) gauge transformation of a classical field represents the same physical (classical) state.

Charged and spinor fields, and electromagnetic potentials, admit a nontrivial U(1) symmetry, but it is sufficient that ϕ includes one such field. The gauge transformation depends on the state $|\Psi\rangle$, so it changes in time.

Observation 9. *$\Psi[\phi]$ can be made real by changing the global* U(1) *gauge of the basis of classical fields.*

Principle 3. The wavefunctional $|\Psi\rangle = \int_{\mathcal{C}} |\phi\rangle d\widetilde{\mu}[\phi]$ can be interpreted as a set of gauged classical fields distributed according to a density functional (Fig. 2).

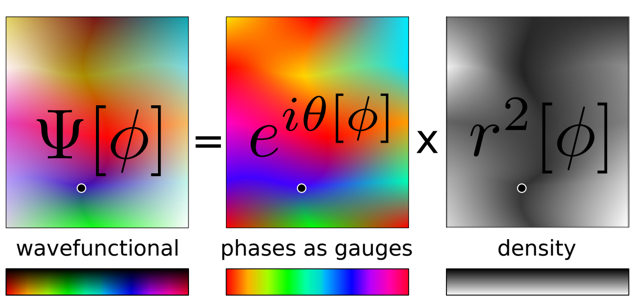

FIG. 2. **Interpretation of the wavefunctional.** The U(1) gauge or phase is represented by the pure color hues in the color wheel. The density is represented as shades of gray. Their combination gives the wavefunctional $|\Psi\rangle = \int_{\mathcal{C}} |\phi\rangle d\widetilde{\mu}[\phi]$ as a set of classical fields with varying densities and gauges.

D. Local beables

There are several benefits in using the interpretation of the wavefunctional from Principle 3 as starting point in the investigations of the foundations of quantum theory. It is more foundational, since quantum field theory is more foundational than nonrelativistic quantum mechanics. It comes with an ontology – each state $|\phi\rangle$ corresponds to a set of fields defined on the 3d-space, not on the configuration space. These fields are the *local beables*, whose necessity was advocated by Bell [4]. The Born rule can be interpreted in terms of such ontic states.

A state does not consist of a single ontic state, but of a set of such states (Principle 3). The projection postulate should not be understood as collapsing the system to a basis state $|\phi\rangle$, no measurement can extract the complete information about the state of the entire universe. Only the ontic states making $\Psi[\phi]$ belonging to the resulting macro-state $\widehat{\mathsf{P}}_\alpha\mathcal{H}$ should remain after the projection.

E. Many worlds

But if decoherence makes the components of $\Psi[\phi]$ corresponding to different macro-states no longer interfere, there is no need to invoke the projection postulate, and we can adopt the *many-worlds interpretation* (MWI). However, "naively" counting the worlds or macro-branches gives the correct probabilities only if the state decomposes into macro-branches as in eq. (7).

Observation 10. "Counting" micro-branches that correspond to the basis $(|\phi\rangle)_{\phi\in\mathcal{C}}$ gives the correct probabilities in MWI, in accord with Condition 1. Even if, unlike the macro-branches, the micro-branches may interfere in the future, they do this within the same macro-branch. Moreover, since each micro-branch consists of classical fields ϕ, and since these are the local beables, it becomes justified to count each micro-branch as a world.

Observation 11. We should also include quantum gravity in our foundational investigations of quantum theory. In background-free approaches to quantum gravity, it becomes impossible to interpret physically all linear combinations as superpositions, because states in which the geometry of space is different cannot be superposed, so the ontic states dissociate automatically [22]. They can reassociate, unless the dissociation becomes irreversible due to decoherence. This provides an additional justification for the many-worlds interpretation (in a revised form [22]).

Observation 12. Solid arguments were made that a rational agent should believe that the probability in MWI is given by the Born rule [8, 19, 27], and also that we should assign to the branches a measure of existence consistent with the Born rule [23, 25], even if "naive" branch counting gives a different answer. Does Theorem 1 contradict these proposals? No, in fact it shows that they are consistent with "naive counting" applied to an ontic basis (Observation 10).

We can make an analogy with the *existence theorems*. There are situations when we can prove mathematically that an equation has a solution, without having the solution itself. And it is possible that later we are able to construct an explicit solution. This doesn't contradict the existence theorem, it confirms it. Similarly, Theorem 1 might just provide an explicit situation in which "naive counting" validates decision-theoretic and other arguments. It can also be seen as the limit of the refined branch-counting method proposed in [18].

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