Physicalism without the idols of mathematics

László E. Szabó
Department of Logic, Institute of Philosophy
Eötvös Loránd Uiversity Budapest

Email: laszlo.e.szabo@gmail.com
ORCID: 0000-0001-7454-8421

Abstract

I will argue that the ontological doctrine of physicalism inevitably entails the denial that there is anything conceptual in logic and mathematics. The elements of a formal system, even if they are tagged by suggestive names, are merely meaningless parts of a physically existing machinery, which have nothing to do with concepts, because they have nothing to do with the actual things. The only situation in which they can become meaning-carriers is when they are involved in a physical theory. But in this role they refer to elements of the physical reality, i.e. they represent a physical concept. "Mathematical concepts" are just idols, that philosophy can completely deny and physics can completely ignore.

There remains one simple way of getting our teaching across, namely to introduce men to actual particulars and their sequences and orders, and for men in their turn to pledge to abstain for a while from notions, and begin to get used to actual things.

Francis Bacon[1]

I. Introduction

1. In this article I will not make a single argument for physicalism. The ontological doctrine of physicalism, i.e. the thesis that nothing exists in the world

[1] *The New Organon*, Aphorisms – On the Interpretation of Nature and the Kingdom of Man [Book I], XXXVI. (Bacon, 2000, 40).

but particular physical objects, will be taken as a starting point, and all further arguments will be based on this starting thesis. In a certain sense, of course, the whole article can be taken as an argument for physicalism, but only insofar as the fact that a number of things can be accounted for within the framework of a given philosophical assumption can be taken as an argument for that assumption.

I will argue that the ontological doctrine of physicalism inevitably entails the denial that there is anything conceptual in logic and mathematics. Since physicalism denies the existence of mental and abstract things, there is nothing surprising in this conclusion, at least if mathematical concepts are taken as abstract or mental entities in the literal sense. As we shall see, the situation is more complicated than this, and there are two basic reasons for this. On the one hand, those strands of the philosophy of mathematics that deny in principle the existence of mental or abstract mathematical entities—essentially for epistemological reasons—either leave the ontological questions about the fundamental nature of mathematics completely open, or suggest an ontological picture that retains traces of certain elements of Platonism or mentalism. On the other hand, those schools of philosophy which are in principle committed to physicalism are not willing to get rid of the traditional view that logic and mathematics are about some conceptual abstractions, or that they are about anything at all in the ordinary sense of the language. However, both problems can be resolved if we hold coherently to our basic physicalist ontological assumption. To show this, I would like to sketch from the ground up a broader framework for how physicalism can account for semantics, logic and mathematics, the basic structure of physical theory, and the role of logic and mathematics in physical theories.

First, in section II, I give what I believe to be the only workable construction of the notion of partial semantics; that is, an interpretation of what it means that certain well-formed formulas of an otherwise meaningless formal system are endowed with meaning pointing to the world outside the formal system. On the basis of the suggested definition of semantics, we will make a few observations which will play crucial roles in the further considerations. An immediate consequence of those observations will be discussed in section III. Namely, that there is no free semantics; that is, to state that a formal system is equipped with semantics involves ontological commitment with respect to the realm referred by the semantics. Combining this with the physicalist presupposition, we shall conclude that there are two possibilities: the formulas are not endowed with meanings at all, or, if they are, then they refer to something physical—trough the semantics of a physical theory.

In section IV we discuss the first case, in which, I will argue, the logical and mathematical statements are, just as the formalist philosophy of mathematics claims, statements about meaningless formal systems. This doesn't mean, however, that formalism can avoid ontological commitment: the formalist claim that mathematics is the science of formal systems involves ontological commitment with respect to the realm of formal systems.

In section V, under the title *Physico-formalism*, I will discuss how formal systems can be accounted for in a physicalist ontology. My main thesis will be that a formal system is a physical system consisting of signs and derivation patterns embodied in concrete physical objects, concrete physical configurations, and concrete physical processes. The logical and mathematical facts are physical facts: facts of the physically existing formal system. Therefore, mathematical statements have exactly the same epistemological status as any other statements about the physical world. I will also argue that the meta-theories about formal systems do not involve different ontology than the formal systems themselves; there is nothing in the ontology of logic and mathematics beyond the physically existing formal systems.

Section VI discusses the situation when the formulas of a formal system do carry meaning indeed; within a physical theory. Our analysis will lead to the conclusion that meaning-carrying cannot be established without a certain kind of underlying causal process in the physical world, which can be identified as the process of learning from experience. On the other hand, it will be also shown that the formal systems—studied in mathematics—play a constitutive role in our empirical knowledge of the physical reality. But, even this constitutive role of a formal system in a physical theory does not imply that there is an *a priori* conceptual schema by which we grasp the physical reality we experience. For what exists has nothing conceptual about it; it is merely a physically embodied formal system.

II. Gödelian view on semantics

2. A scientific theory formulated with sufficient mathematical precision is nothing more than a formal system with partial semantics. In this section I want to consider the question of how such a semantics should be conceived. What does it mean to endow certain well-formed formulas of a formal system, otherwise meaningless, with meanings pointing to the world outside the formal system?

Of course, even a brief overview of the various theories of meaning would be beyond the scope of this article. Acknowledging that almost all of the various approaches—notably Tarski (1933), Carnap (1942), modern model-theoretic interpretation, truth-conditional theory, verificationism, Quinean holism, late Wittgenstein, etc.—have helped us to understand some aspect of the concept of meaning, we must nevertheless conclude that the century-long discourse on the problem of meaning offers no satisfactory answer to the above question. At least, no such answer is to be found in the literature in any well-known explicit form.

However, I would like to draw attention to a construction which, in my view, works perfectly well and in some sense embraces the various aspects mentioned above. And this concept of semantics is not new at all. We find it implicitly in Gödel's 1931 paper, in the proof of the incompleteness theorems. More precisely, in the method of "arithmetization" which prepares the proof. Undoubtedly, the reading of Gödel's construction that we use here became really clear in later commentaries and explanations of the proof (see e.g. Nagel and Newman 1958).

In the preparatory part of his paper, Gödel shows how certain meta-arithmetic facts can be represented in the Peano arithmetic itself. For the present purposes, I only recall the final result of the construction (see e.g. Crossley 1990, 52–54; Hamilton 1988, 145-146). Let $Pr(x,y)$ symbolize the meta-arithmetic fact that 'the formula–sequence of Gödel number x constitutes a proof of the formula of Gödel number y'; and let $\{Pr(x,y)\}_{x,y}$ denote the family of similar meta-arithmetic facts, where x and y are arbitrary Gödel numbers. For a given pair of Gödel numbers x and y, $Pr(x,y)$ is either the case, that is the formula–sequence of Gödel number x is indeed a proof of the formula of Gödel number y, or not. In Gödel's construction, the facts belonging to the family $\{Pr(x,y)\}_{x,y}$ are represented by a suitable family of Peano arithmetic formulas, $\{R(x,y)\}_{x,y}$.[2] Representation means that the following condition is met: for all Gödel numbers x and y, that is, for all paired $Pr(x,y)$ and $R(x,y)$ from the two families,

$$\begin{array}{cccc} & Pr(x,y) \text{ denotes a} & & \text{the corresponding} \\ \text{if} & \text{meta-arithmetic fact} & \text{then} & R(x,y) \text{ is such that} \\ & \text{which is the case} & & \Sigma_{\mathrm{PA}} \vdash R(x,y) \\ & & & \\ & Pr(x,y) \text{ denotes a} & & \text{the corresponding} \\ \text{if} & \text{meta-arithmetic fact} & \text{then} & R(x,y) \text{ is such that} \\ & \text{which is not the case} & & \Sigma_{\mathrm{PA}} \vdash \neg R(x,y) \end{array} \tag{1}$$

where Σ_{PA} denotes the axioms of Peano arithmetic.

Notice that the regularity consisting in the fact that the above condition holds for a whole family $\{Pr(x,y)\}_{x,y}$ of meta-arithmetic facts of a certain type is an essential ingredient of Gödel's conception of representation. This fact also plays an important role in the proof of the incompleteness theorem (e.g. Crossley 1990, 55–56).

3. Now let us turn to the general situation. How can formulas of a formal system become meaning-carrier within a theory describing a certain realm? Symbolically I shall denote such a theory by (L,S,U), where L denotes the formal system in question, U symbolizes the realm to be described, and S stands for the semantics of the theory, which we are going to define now. The formal system L is meant to include the language of the theory, the derivation patterns, and the axiom system which will be denoted below by Σ_L.

Adopting the essence of Gödel's conception of representation, to equip a formal system L with a (partial) semantics pointing to a certain realm U means to give

[2]In accordance with the rest of the paper, I do not make distinction between "numbers" and "numerals". In fact, everywhere I mean what the standard terminology calls "numerals"; but I do not see any reason to call them anything other than numbers.

(A_0) a family $\{a_\lambda\}_\lambda$ of states of affairs in U and a family $\{A_\lambda\}_\lambda$ of formulas in L, such that

(B_0) for all λ, that is for all paired a_λ and A_λ,

if	a_λ denotes a state of affairs in U which is the case	then	the corresponding A_λ is such that $\Sigma_L \vdash A_\lambda$
if	a_λ denotes a state of affairs in U which is not the case	then	the corresponding A_λ is such that $\Sigma_L \vdash \neg A_\lambda$

Immediately notice that the G-sentences[3] of the formal system L cannot participate in such a semantic construction; in the sense that they cannot belong to a suitable family $\{A_\lambda\}_\lambda$. Because, for all A_λ in $\{A_\lambda\}_\lambda$, it is required in (B_0) that either $\Sigma_L \vdash A_\lambda$ or $\Sigma_L \vdash \neg A_\lambda$ is the case.[4] Taking this into account, we define the notion of semantics in the following final form:

To equip a formal system L with a (partial) semantics, denoted by S, pointing to a certain realm U means to give

(A) a family $\{a_\lambda\}_\lambda$ of states of affairs in U and a family $\{A_\lambda\}_\lambda$ of formulas in L, such that for all λ either $\Sigma_L \vdash A_\lambda$ or $\Sigma_L \vdash \neg A_\lambda$ is the case, and

(B) for all λ, that is for all paired a_λ and A_λ,

a_λ denotes a state of affairs in U which is the case	if and only if	the corresponding A_λ is such that $\Sigma_L \vdash A_\lambda$

We usually say that the formulas belonging to $\{A_\lambda\}_\lambda$ are interpreted, are endowed with meaning: namely, A_λ means or refers to a_λ according to the semantics in question; and accordingly it expresses a factual truth or falsity about U, depending on whether $\Sigma_L \vdash A_\lambda$ or not. Although, as we will see in point 4/(e) below, this picture needs further refinement.

4. On the basis of the above construal of semantics, some immediate observations are in order here.

(a) Meaning of a formula of a formal system is relative to the whole semantic construction; namely, it is relative to the whole formal system L and to the choice of the two families, the family of formulas $\{A_\lambda\}_\lambda$ in L and

[3] A sentence X is called G-sentence if neither $\Sigma_L \vdash X$ nor $\Sigma_L \vdash \neg X$ holds.

[4] It is therefore debatable in what sense the notable Gödel sentence can carry any meaning at all; for example, the meaning usually ascribed to it, 'I am not provable in Peano arithmetic'—as Wittgenstein has already pointed out (see Rodych 1999).

the family of state of affairs $\{a_\lambda\}_\lambda$ in U, and the paring between the elements of the two families. For instance, imagine two different semantics for the same formal system L, one with some families $\{A_\lambda\}_\lambda$ and $\{a_\lambda\}_\lambda$, the other with some $\{\bar{A}_{\bar{\lambda}}\}_{\bar{\lambda}}$ and $\{\bar{a}_{\bar{\lambda}}\}_{\bar{\lambda}}$, then it is entirely possible that a formula of L, contained in both families of formulas $\{A_\lambda\}_\lambda$ and $\{\bar{A}_{\bar{\lambda}}\}_{\bar{\lambda}}$, carries completely different meanings according to the two different semantics.

(b) Thus, meaning is not determined by the facts of the formal system L. In other words, the knowledge of all the facts of the formal system L is not enough to "understand" the meaning either of the separate formulas of L or the meaning of the entire formal system L as a structure.

(c) Meaning is not simply a matter of arbitrary assignment between state of affairs in U and the formulas of the formal system, as the "assignment" has to satisfy condition (B) too. But, the satisfaction of condition (B) is a *joint* result of L and U; the formal system L must be such and the object of the theory, U, must be such that condition (B) is satisfied.

(d) Notice that condition (B) is exactly the same as the necessary and sufficient condition for the theory (L, S, U) to be *true*. For, what condition (B) actually says is that the statements of the theory are correct, that is, the interpreted theorems of L are in accurate correspondence with the sate of affairs in U to be described by the theory. Since condition (B) is a part and parcel of the very construal of semantics, that means that the two conceptions *meaning and truth are inextricably intertwined*.

(e) But, this interrelation between meaning and truth is obviously not as simple as assumed in the traditional verifiability or truth-conditional theory of meaning—which was famously criticized by Quine in the *Two Dogmas* or in the *Epistemology Naturalized* (1951; 1969). For, as it follows from the very construction, both meaning and truth are essentially *holistic* conceptions. It makes no sense to talk about the meaning and truth of a single isolated formula. Not only because a whole family of formulas, "as a corporate body", are endowed with meaning—and, simultaneously, with truth or falsity—but also because the fact $\Sigma_L \vdash A_\lambda$ in condition (B) can involve an arbitrarily large part of the formal system L. This practically means that while the meaning carriers, in some literary sense, are the elements of $\{A_\lambda\}_\lambda$, the actual meaning carrying is coming about by the formal system L as a whole.

III. No free lunch: semantics cannot be detached from ontology

5. As we have seen, especially from remarks (b)–(d), there is no meaning without the existence of the realm U, the facts of which correlate with the facts of the formal system in the way of the regularity required in condition (B). Consequently, to state that a formal system is equipped with semantics involves ontological commitment with respect to the realm U, no matter what kind of ontological realm U is. For a physical theory—as well as for a John Stuart Mill style account of logic and mathematics—the realm U is obviously a part of the physical world. For mathematical intuitionism, the realm U is meant to be the contents of the intuitionists' "universal human intuition". In mathematical Platonism the realm U is the realm of abstract mathematical entities and structures:

> For the platonist, mathematical statements are true or false independently of our knowledge of their truth-values: they are rendered true or false by how things are in the mathematical realm. And this can be so only because, in turn, their *meanings* are not given by reference to our knowledge of mathematical truth, but to how things are in the realm of mathematical entities. Thus a statement of the form, 'For some natural number n, $A(n)$', platonistically interpreted, makes no reference to whether or not we are able to cite some numeral ν such that $\ulcorner A(\nu) \urcorner$ is true—or even to whether we can disprove the statement, 'For all n, not $A(n)$'. It relates to whether there is or not a member of the objective, though abstract, domain of natural numbers satisfying the predicate '$A(x)$': and we understand the statement just because we grasp that that is what determines it as true or as false. The mathematician is, therefore, concerned, on this view, with the correct description of a special realm of reality, comparable to the physical realms described by the geographer and the astronomer. (Dummett 1978, 202)

All this means that meaning cannot be detached from ontology. No free lunch. It is not possible to accept Platonic semantics and at the same time to deny Platonic ontology, as in Carnap's 'Empiricism, Semantics, and Ontology' (1950), or in contemporary mathematical fictionalism (Balaguer 2018)—at least, in some understanding of mathematical fictionalism.[5]

And, of course, the ontological *commitment* with respect to the realm U is

[5]"To take seriously the mathematical fictionalist's insistence on a standard semantics, then, it is perhaps better to view mathematical fictionalists as implicitly or explicitly preceding their mathematical utterances with a disavowing preface which excuses them from the business of making assertions when they utter sentences whose literal truth would require the existence of mathematical objects. But this, of course, raises the question of what they think they are doing when they engage, as fictionalists, in mathematical theorizing (both in the context of pure mathematics and in the context of empirical science). Pure mathematics does not present a major difficulty here – fictionalists may, for example, view the purpose of speaking *as if* the assumptions of our mathematical theories as true to be to enable us easily to consider what follows from those assumptions. Pure mathematical inquiry can then be considered as speculative inquiry into what would be true if our mathematical assumptions were true, without concern about the question of whether those assumptions are in fact true, and it is perfectly reasonable to carry out such inquiry as one would a conditional proof, taking mathematical axioms as undischarged assumptions." (Leng 2020) Accordingly, fictionalism in best case is a counterfactual version of "if-thenism", to which we will return in point 7.

only enough to coherently *claim* that the formal system carries meaning. For this claim to be correct, the realm U must exist.

Combining this with our pre-assumed physicalist ontological framework, there are two possibilities:

1. the formulas of the formal system are not endowed with meanings at all, or,
2. if they are, then they refer to something physical, within the framework of a physical theory.

We will return to the discussion of physical theory in section VI. In the next section, we focus on the first case when the formulas of the formal system L have no meanings. At least, we focus on the nature of "pure" mathematics, independently of whether the formal system in question is used in a physical theory or not.

IV. Logical and mathematical facts

6. What is the subject matter of pure mathematics then? What kind of facts are investigated by mathematics? Due to what was said above, there remains only one possible answer to this question; the one given by the most hard-headed formalist: the statements of mathematics are statements about the formal system L itself. For example, the most typical such statement is of the following form: formula A entails from the axioms of the formal system Σ_L, that is, $\Sigma_L \vdash A$, where $\vdash$ is the syntactic entailment. It must be emphasized that neither A nor the elements of Σ_L are statements, which could be true or false. They are just meaningless strings of marks; the logical signs, the connectives, $\neg$ and $\rightarrow$, and the quantifiers, $\exists$, $\forall$, included. The entailment relation $\vdash$, too, has nothing to do with "rational" or "proper" or "truth preserving" reasoning; it simply stands for the fact that there is a finite sequence of meaningless strings of symbols, such that each element of the sequence is either an axiom or fits into one of the derivation patterns with some earlier formulas.

7. Notice that formalism in the above sense essentially differs from all versions of if-thenism. The main difference is that logic is not in a distinguished position in the formalist approach. The logical signs, the logical axioms, and the derivation patterns are on a par with the rest of the formal system. In contrast:

> To articulate [if-thenism] rigorously, one would distinguish the logical terms like 'and', 'if. . .then', 'there exists', and 'for all' from the non-logical, or specifically mathematical, terminology such as 'number', 'point, 'set', and 'line'. The logical terminology is understood with its normal meaning, while the non-logical terminology is left uninterpreted, or is treated as if it were uninterpreted. (Shapiro 2000, 149–150)

The other differences may vary with the different versions of if-thenism. One thing is for sure: in the formalist philosophy of mathematics, neither the logical nor the non-logical objects in the formal system have meanings whatsoever; and a mathematical assertion like '$\Sigma_L \vdash A$' is not a (counterfactual) conditional statement (with some meaningful antecedent and consequent) but a simple unconditional statement of a fact about the formal system L.

8. According to a widespread opinion, it is precisely the merit of the formalist account that it implies no ontological commitment whatsoever. At first sight this seems to be correct, since, according to the formalist claims, the formulas of the formal system carry no meanings at all, they do not refer to anything. This doesn't mean however that the formalist mathematics can avoid ontological commitment.

For, it is agreed that "mathematics is the science of formal systems" (Curry 1951, 56). The assertions of mathematics are *meaningful* assertions; namely, a statement like '$\Sigma_L \vdash A$' refers to a fact *of* the formal system L. So to speak, all mathematics is meta-mathematics. Euclidean geometry is a sience *of* a formal system called *Euclidean Geometry.* Peano arithmetic is a since *of* a formal system called *Peano Arithmetic*, and so on. At first glance, these are more descriptive sciences. Peano arithmetic's claim that '$\Sigma_{PA} \vdash 2 + 2 = 4$' is just like geography's claim that 'Vienna and Budapest are connected by a river'. However, even such simple descriptive narratives must be understood as scientific theories in the sense that we have generally given in point 3; which contain such terms as 'proof' or 'river'.

Accordingly, a mathematical theory is a normal scientific theory of the form (M, S, L), where M is some formal system (the "language" of the meta-theory), S is a semantics in the sense of point 3, and the formal system L is the realm whose facts constitute the subject matter of the mathematical theory.

Now, as was said in point 5, there is no free semantics; the formalist claim that a mathematical theory is about a formal system L involves ontological commitment with respect to the realm of L; and, this claim can be true only if L exists.

What kind of ontological entity is a formal system? One thing is for certain: if, as we assumed, the doctrine of physicalism is true, then formal systems must be physical objects, constituting particular parts of the physical reality.

V. Physico-formalism

9. How can a formal system be accounted for in a physicalist ontology? In earlier papers (Szabó 2003; 2012; 2017) I sketched the outlines of such an account, which I call physico-formalist philosophy of mathematics. The basic idea is that the logical and mathematical facts, that is, the formal facts, are *physical* facts. A formal system is nothing but a physical system; the marks and derivations are *concrete physical objects, concrete physical configurations, and* *concrete physical processes*; such as, for example, ink-configurations on a paper, neural configurations and processes of a brain, electronic configurations and processes in a computer, etc., or their various combinations. A formal fact, like '$\Sigma_L \vdash A$', is a fact of such a physically existing formal system.

Formalism, in the sense I use the term, is often called "game formalism". Adopting this phraseology, my claim is that the "game" itself is a part of the physical reality. A chess game—to stick to the standard analogy—is a part of the physical reality. The chess pieces are physical objects. Their moves are physical changes. The rules, constraining these moves, are real physical constraints; the fulfillment of which is not a "metaphysical necessity" but a causal consequence of certain physical conditions. The way in which the knight can jump is determined by the physical state of the surface of the CD[6] from which the chess game is installed and other similar conditions, and the contingent laws of physics governing the computer's behavior. So, by formal system I mean something similar to what Haugeland (1985, 76) calls (particular material embodiment of) "automatic" formal system, "a formal system that 'works' (or 'plays') by itself".[7] An iconic example is a computer that, in a certain order, prints out the theorems derivable from a given system of axioms, according to some derivation patterns. Though, a physically existing formal system, in my understanding, does not necessarily mean a *temporal* progression; or at least temporality is inessential. There is nothing temporal in the fact, for example, that a formula sequence on the paper constitutes a proof.

Highlighting the fact that the marks are physical objects and that the whole game is in the physical realm—in a "finite", empirically available part of the physical realm—has been quite common in philosophy of mathematics since the first occurrence of formalism. As common as the belief that there must be something more to mathematics than such a finite game with physically existing marks. Interestingly enough, this belief, in some form, is still present in the various branches of formalism, too. As Alan Weir (2010, 68) puts it: "Now it would certainly be stupid to hold that mathematics is an empty manipulation of meaningless symbols." In contrast, the physico-formalist approach implies

[6] That is, "*Computer Science Without Programs*" is entirely possible (cf. Boolos 1998, 129).

[7] It is interesting to observe how computerization has changed our natural intuitions. In 1903, Frege writes in the *Grundgesetze der Arithmetik, Vol. II*:

> "[T]o speak of the behaviour of the signs with respect to the rules seems to me unfortunate. I do not behave with respect to the civil laws simply by being subject to them, but only in obeying or disobeying them. Since neither the chess pieces nor the numerical figures have a will of their own, it is the player or the calculator—and not the pieces or figures—who, by obeying or disobeying the rules, behaves with respect to them." (Frege 1960, 190)

In 1985, Haugeland writes:

> [A]n automatic formal system is like a set of chess pieces that hop around the board, abiding by the rules, all by themselves, or like a magical pencil that writes out formally correct mathematical derivations without the guidance of any mathematician. These bizarre, fanciful images are worth a moment's reflection, lest we forget the marvel that such systems (or equivalent ones) can now be constructed. (Haugeland 1985, 76)

the complete denial of the existence of anything in logic and mathematics over and above the particular physically existing formal systems. This is not some additional nominalism, but, as we will see, a straightforward consequence of the physicalist ontology.

10. In my view, the belief that mathematics must be something more than the investigation of the particular, physically existing, formal systems is rooted in the misinterpretation of the fact that mathematics is a collection of claims *about* formal systems. It is assumed that "the theory of the game" involves different ontology than "the game itself" (Frege 1960, 203). Alan Weir writes:

> The metatheory is itself a substantial piece of mathematics, ostensibly committed to an infinite realm of objects which are not, on the face of it, concrete. Tokens of the expressions of the object language game calculus may be finite—ink marks and the like; but since there are infinitely many expressions, theorems and proofs, these themselves must be taken to be abstract types. (Weir 2015)

Of course, a metatheory—that is, in our terminology, a mathematical theory—(M, S, L) describing the facts of a physically existing formal system L must be distinguished from L itself; for, a theory, as such, is certainly not identical with its subject matter. But, this difference does not mean that (M, S, L) represents some *a priori* knowledge of the facts of L, prior to or independent of the physically existing L. As it is clear from the physico-formalist approach, (M, S, L) is an ordinary physical theory about L, and has the same epistemological status as any other physical theory about any other part of the physical world (see point 15).[8]

The existence of (M, S, L) does not imply commitment to abstract entities whatsoever. It involves ontological commitment only with respect to the physically existing formal system L, *and another physically existing formal system* M, and the physical correlation[9] between them required in S; and these all are in the physical realm.

For example, it can be an entirely valid claim that a certain object in the physically existing formal system L is a token of a certain type introduced in the mathematical theory (M, S, L). But, it doesn't mean that the token and the type belong to different ontological realms. The type, from ontological point of view, is just the same kind of object in the physically existing formal system

[8] This challenges the widespread view that

> "the formal languages and deductive systems were formulated with sufficient clarity and rigour for them to be studied as mathematical objects in their own right. That is, the mathematician can prove things *about* formal systems." (Shapiro 2000, 153)

For, such a "proof" would mean the possibility of synthetic *a priori* proposition about a part of the physical world. We only can make (hypothetical) *predictions* about a (physically existing) formal system, by means of a mathematical theory (or meta-mathematical theory, depending on the terminology) which is as much fallible as any other physical theory.

[9] For more about this see point 15.

M as its token in L, or any other objects in L or in M. The fact that they constitute a type–token pair consists in a special relationship through the regularity described in point 3/(B).

Similarly, it can be entirely possible that two or more concrete formal systems have similarities, just as any contingently existing physical objects may have similarities in some features. There is a strong temptation to interpret the similar physically existing formal systems as particular physical "representations" of one and the same "formal system", something that is obtained by "abstraction". Haskell Curry writes:

> [A]lthough a formal system may be represented in various ways, yet the theorems derived according to the specifications of the primitive frame remain true without regard to changes in representation. There is, therefore, a sense in which the primitive frame defines a formal system as a unique object of thought. This does not mean that there is a hypostatized entity called a formal system which exists independently of any representation. On the contrary, in order to think of a formal system at all we must think of it as represented somehow. But when we think of it *as* formal system we abstract from all properties peculiar to the representation. (Curry 1951, 30)

In line with what was said about the type–token relationship, it must be clear that abstraction does not produce an *abstract* formal system over and above the physically existing ones. To abstract from some peculiar properties of, say, two physically existing formal systems, L_1 and L_2, and to isolate the common essential features of them, first of all requires knowledge of the properties of the physical objects in question. That is, we must have a mathematical theory $(M, S, \{L_1, L_2\})$ jointly describing L_1 and L_2. Only in such a theory it is possible to formulate the steps of abstraction, to isolate the common essential features of L_1 and L_2, to talk about similarity or isomorphism between the objects representing them, and, for example, to introduce some equivalence classes, which could be regarded as the end product of the abstraction procedure. But, the formal system M in the mathematical theory $(M, S, \{L_1, L_2\})$ itself is a physically existing formal system. The whole abstraction procedure is just an object in the physically existing formal system M, including the result of the abstraction. Thus, *abstraction does not lead out of the realm of the physically existing formal systems*.[10] In other words, the physically existing formal systems can be rep-

[10]The above reasoning is also true in general. To abstract from some peculiar properties of two arbitrary physical objects O_1 and O_2, and to isolate the common essential features of them, requires a physical theory $(L, S, \{O_1, O_2\})$ jointly describing objects O_1 and O_2. Only in such a theory it is possible to formulate the steps of abstraction, to isolate the common essential features of O_1 and O_2, to talk about their similarity, etc. The whole abstraction procedure is just an object in the physically existing formal system L. So, abstracting from particular properties of physical objects does not lead us out of the physical realm, the realm of the objects O_1 and O_2, the physically existing formal system L, and the physical process producing the correlation required in the semantics S. And, above all, mathematics, investigating the facts of formal system L, does not in itself say anything about some "idealized" features, which are common to physical objects O_1 and O_2.

resented in each other, through the semantics of a suitable mathematical theory; that's all. (Accordingly, taking into account that a mathematical theory is a physical theory—of one or more physically existing formal systems—, the faithfulness of the representation is limited, uncertain, approximate, and based on *a posteriori* means.)

Notice that, from the point of view of the above considerations, it is irrelevant whether the formal systems in question, L, L_1, L_2 and M, are finite or infinite physical objects—though, assumably, they are finite.

Thus, our conclusion is that "the theory of the game" does not involve different ontology than "the game itself"; there is nothing in the ontology of logic and mathematics over and above the physically existing formal systems.

11. This means that the logical and mathematical statements are statements of physical facts; therefore, they have exactly the same epistemological status as any other statements about the physical world.[11] Consequently, taking into account what will be said about physical theory in the next section, especially in point 15, the logical and mathematical truths—"truths of reason" *par excellence*—are synthetic, *a posteriori*, and fallible. ("Deduction" itself is nothing more than experiencing the physical existence of a series of formulas that form a proof.) Therefore, logical and mathematical truths do not deliver us absolute certainty. On the other hand, however, they do have factual content; accordingly, they "can be true and useful and surprising" (Ayer 1952, 72). They express objective, mind independent, facts; which can be "discovered" (cf. Hardy 1929)—just like we can discover an objective feature of a plastic molecule or the characteristics of a transistor. From metaphysical point of view, the logical and mathematical facts are contingent facts. Not only because a formal system is an artifact, but also because the laws of nature predetermining its features, themselves, are contingent.

The above conclusions obviously contradict to the traditional beliefs about logic and mathematics. The universal *illusion* that logical and mathematical truths are necessary, certain and *a priori* can be explained by two practical facts:

1. The illusion of necessity is rooted in the fact that formal systems are simple physical systems of relatively stable, predictable behavior, like a clockwork.

2. The illusion of aprioricity is rooted in the fact that knowledge of logical and mathematical facts does not need experience of the "external" physical world—external relative to the physically embodied formal system in question.

[11] To avoid any possible misunderstanding, it is worth emphasizing that physico-formalism is in no sense the same as John Stuart Mill's philosophy of mathematics. According to physico-formalism, the statement referring to the physical world is '$\Sigma_L \vdash A$', and it is about the formal system L as a special part of the physical world. According to Mill, the statement is 'A', and this would say something about the things in the physical world, like apples, circles drawn on paper, etc. In other words, the physico-formalist would not suggest to Mill, as Balaguer (2014) does, that he should become a mathematical fictionalist, but that he should first become a true formalist, and then—in line with his naturalism, anti-platonism, and empiricism—adopt physico-formalism.

12. All this applies to all mathematical facts, involving all kinds of mathematical concepts, simple or sophisticated, finite or infinite. To be more precise, the usage of the word "concept" is inadequate; after all, a meaningless mark is not a concept. And, from this point of view, it makes no difference whether the mark in question is termed by "two", or "triangle", or "probability", or by any other name that suggests any other meaning; it is not a concept that expresses something of the world. This is precisely what Carnap, for example, objected to in the formalist construction of mathematics. To shed more light on my point, it is worth taking a closer look at a longer passage from Carnap's *Logical Syntax*:

> The formalist view is right in holding that the construction of the system can be effected purely formally, that is to say, without reference to the meaning of the symbols; that it is sufficient to lay down rules of transformation, from which the validity of certain sentences and the consequence relations between sentences follow; and that it is not necessary either to ask or to answer any questions of a material nature which go beyond the formal structure. But the task which is thus outlined is certainly not fulfilled by the construction of a logico-mathematical calculus alone. For this calculus does not contain all the sentences which contain mathematical symbols and which are relevant for science, namely those sentences which are concerned with the *application of mathematics*, i.e. synthetic descriptive sentences with mathematical symbols. For instance, the sentence "In this room there are now two people present" cannot be derived from the sentence "Charles and Peter are in this room now and no one else" with the help of the logico-mathematical calculus alone, as it is usually constructed by the formalists; but it can be derived with the help of the logicist system, namely on the basis of Frege's definition of '2'. A logical foundation of mathematics is only given when a system is built up which enables derivations of this kind to be made. The system must contain general rules of formation concerning the occurrence of the mathematical symbols in synthetic descriptive sentences also, together with consequence-rules for such sentences. Only in this way is the application of mathematics, i.e. calculation with numbers of empirical objects and with measures of empirical magnitudes, rendered possible and systematized. *A structure of this kind fulfills, simultaneously, the demands of both formalism and logicism.* For, on the one hand, the procedure is a purely formal one, and on the other, the meaning of the mathematical symbols is established and thereby the application of mathematics in actual science is made possible, namely, by *the inclusion of the mathematical calculus in the total language*. The logicist requirement only appears to be in contradiction with the formalist one; this apparent antithesis arises as a result of the ordinary formulation in the material mode of speech, namely, "an interpretation for mathe-

> matics must be given in order that it maybe applied to reality". By translation into the formal mode of speech this relation is reversed: the interpretation of mathematics is effected by means of the rules of application. The *requirement of logicism* is then formulated in this way: *the task of the logical foundation of mathematics is not fulfilled by a metamathematics (that is, by a syntax of mathematics) alone, but only by a syntax of the total language, which contains both logico-mathematical and synthetic sentences.* (Carnap 1937, 326–327)

Notice that the "syntax of a total language, which contains both logico-mathematical and synthetic sentences" is nothing but the language of a *theory* (L, S, U), more precisely, the formal system L that serves as a constituent of the theory. This is even more obvious in the light of Carnap's *Theories as Partially Interpreted Formal Systems*, written a few years later (to which we will return in point 13). Now, Carnap's two example sentences about Charles and Peter belong to such a theory (L, S, U), which is capable of describing all the things in the world that are involved in the situation mentioned. The language of the theory, formal system L, must include such terms as "Charles", "Peter", "room", "in the room", etc. The axiom system of L, presumably in addition to some logico-mathematical axioms, also contains non-mathematical axioms. Taking the whole axiom system into account, it is quite possible that Carnap's two sentences have the relation of consequence he requires. However, whether this is the case or not is not a matter of "logical necessity", as the logicism maintains, but depends on how the things like 'Charles', 'Peter', 'room', 'being in the room' are in reality. Carnap's "syntax of the total langugae", that is the formal system L, as a component of a correct theory merely reflects this fact. (Note that the theory must be correct in order to have a semantics at all.) But, and this is the most important thing for us now, even if that is the case within L, the two sentences in question—as well as the non-mathematical axioms themselves—remain meaningless formulas which in themselves, without the semantics, that is without the whole theory (L, S, U), do not express anything of the world, even though they contain non-mathematical terms (as I shall argue more fully in point 14).

Thus, no matter how the objects of a formal system are termed, they are just meaningless bricks in a physically existing formal system. They have nothing to do with concepts, according to any ordinary interpretation of the term (e.g. Margolis and Laurence 2007): they do not represent abstract objects (the existence of which we have rejected anyway), and they cannot be regarded as elements of mental representation of the world (even under some physicalist interpretation of mental representation). *They simply do not express anything.* Consequently, they do not generate and do not solve conceptual problems. For example, just as a formal construction called "two" has nothing to do with the state of affairs in the room when only Charles and Peter are present, a formal construction called "infinity" has nothing to do with the metaphysical debate about the actual vs. potential infinity, as it has nothing to do with neither the state of affairs along a timeslice of the physical world nor a long/endless physical process (nor, of course, some "mental" or "abstract" procedure).

VI. Physicalist account for physical theory

13. Let us turn now to the situation where the formulas of a formal system are meaning-carriers indeed. This is the second case in point 5, when the formulas of a formal system L are endowed with meaning within a physical theory (L, S, U)—a theory in the sense of point 3, where U is a certain part of the physical reality. The system of axioms of L are usually divided into the logical axioms, the axioms of some mathematical theories, and some physical axioms. Beyond the traditions and some practical purposes (Szabó 2017), this kind of classification is however inessential—as it will be obvious from the discussions below.

Of course, this view on physical theory goes back to Carnap's *Theories as Partially Interpreted Formal Systems* (1939):

> Any physical theory, and likewise the whole physics, can in this way be presented in the form of an interpreted system, consisting of a specific calculus (axiom system) and a system of semantic rules for its interpretation; the axiom system is, tacitly or explicitly, based upon a logico-mathematical calculus with customary interpretation. (Carnap 1939, Section 23)

There is a debate in the literature as to how exactly Carnap interpreted the concept of calculus, and also how he conceived of semantic rules (e.g. Frost-Arnold 2013; Lavers 2015; Friedman 1988; Murzi 2019). Nevertheless, it is worth clarifying that the physico-formalist approach proposed in this paper has two essential elements in which it seems to differ from Carnap's original conception. First, according to the physico-formalist approach, the "logico-mathematical calculus" is merely a physically existing formal system without any "customary interpretation". Second, according to the notion of semantics we have adopted, the referential relationship between the elements of the formal system L and the states of affairs of the world U described by the theory cannot be given by linguistic means alone; even if by language we mean a meta-language. For there are two possibilities: (1) the "semantic rules" are some meaningless sentences in a meta-language L', or (2) they are meaningful sentences in a meta-theory $(L', S', \{L, S, U\})$ that can describe both the formal system L and the world U, and the semantic relation between them within the theory (L, S, U). In the latter case, however, the problem is simply transposed to the even more complicated question of interpreting the semantics of the meta-theory. In contrast, in our interpretation, as we pointed out in point 4/(c), semantics is something external, or, at least, partly external to the language. In fact, semantics is a *phenomenon* jointly produced by two parts of the physical world, the formal system L and the part of the external reality U to be described. (It is a phenomenon that has a causal explanation, as we shall see in Point 15.)

14. Semantics is a part and parcel of physical theory. (No reason to talk separately about a "physical theory" and its "interpretation". Without semantics, L is just a meaningless physical object.) In case of failure of the theory (L, S, U), any component can be the object of revision; from the language and the derivation patterns, through the logical, mathematical, and physical axioms, up to the semantics. In other words, all these components—including the logical/mathematical axioms and the semantics—are as much hypothetical as the physical axioms.

This also means that the distinction between "pure" and "applied" mathematics is a misconception—except of course the sociological usage of the terms (cf. Pincock 2009). For, there is no such a thing as "applied mathematics". Mathematics is "pure" mathematics: the discipline investigating the facts of a formal system. On the one hand, mathematical facts do not imply how to "apply" mathematics for the description of the physical world; on the other hand, within a physical theory, the facts like $\Sigma_L \vdash A_\lambda$ in condition 3/(B) are "pure" mathematical facts, which are independent of the semantic construction they are involved in—even if L contains so called "physical terms", or contains *only* "physical terms" like in Hartry Field's (1980) nominalization project.[12]

15. a_λ is a symbol in our meta-language standing for a state of affairs in the physical world, specifically in U. According to the physico-formalist approach, $\Sigma_L \vdash A_\lambda$ also stands for a physical fact, namely, for a fact of the physically existing formal system L. That means that condition 3/(B) requires the existence of a regularity between two parts of the physical world; a *correlation* between the state of affairs in L and the state of affairs in U. According to Reichenbach's (1956) *principle of common cause*, there cannot exists correlation in the physical world without causal explanation; more precisely, without the existence of a *causal physical process producing the correlation* in question. This means that there must exist a physical process in the common causal past of the state of affairs in L and the state of affairs in U, which brings about the correlation required in condition 3/(B). Thus, we must conclude, the semantics as well as the truth of a physical theory, that is *knowledge* of the physical world, can be obtained only as a result of a certain causal process in the physical world. After some reflection, it is clear that this physical process is nothing else but what we normally call *learning from experience*. All this means that no genuine knowledge of the physical world—hence knowledge simpliciter—is possible without experience. Due to the fact that meaning and truth are intertwined,

[12] After all, a nominalized version of a physical theory (L, S, U), say (L', S', U), is a normal physical theory in which L' is an ordinary formal system. The facts of L' are ordinary logical and mathematical facts. So, Field's nominalization does not eliminate mathematics from physical theories. This doesn't mean, however, that the Quine–Putnam indispensability argument is a valid argument in favor of Platonism. For, what is indispensable in a physical theory (L, S, U) are the facts of L—physical facts of a physically existing formal system. (L, S, U) involves ontological commitment only with respect to the physical world U, the physically existing formal system L, and the physical process (see point 15) producing the correlation between them, which is required in S. And these all are in the physical realm.

no meaningful talk (or thought) about the physical world is possible without experience. There is no *a priori* meaning and there is no *a priori* truth.[13]

16. Learning from experience, that is the physical process in the common causal pasts of L and U, bringing about the correlation in S, embraces everything in the past history that causally explains the formation of (L, S, U); including the continuous creation, investigation, and selection of formal systems, and the continuous tuning of them—first of all the so called physical axioms—together with the continuous tuning of the semantics. There is therefore nothing "miraculous" about the "applicability of mathematics" in physics.

Notice that the formation of a given theory (L, S, U) can be influenced by various causal factors outside the past and present of part U of the physical reality. That is to say, it can be the case that the theory is not completely determined by that part of the world which is described by the theory. It is therefore entirely possible that there exist different physical processes leading to formation of different physical theories, $(L, S, U), (L', S', U), (L'', S'', U), \ldots$, such that all describe the same part U of the physical reality. This kind of underdetermination[14] of theory also confirms that there is no "pre-established harmony" between the facts of the realm of U and the facts of the realm of the formal system L. The "harmony", consisting in the relationship 3/(B), is established by the contingent causal process leading to the formation of (L, S, U).

17. It is a widespread view that the failure of a theory (L, S, U) means the following situation:

(1) A refers to a according to the semantics of the theory.

(2) A is a prediction of the theory, that is $\Sigma_L \vdash A$.

(3) But, a is not the case in U.

One can easily see however that the three claims cannot hold true at the same time, given that L is consistent. For (1) and (3) are in contradiction with (2), according to conditions (A) and (B) in point 3.

An immediate consequence of this simple meta-theoretical fact is that the state of affairs in U when a is not the case—let us denote this state of affairs

[13] Due to the EPR–Bell problem in quantum mechanics, many question whether the common cause principle deserves to be regarded as universally valid principle. My own view is that it does. In fact, none of the counter-examples provides a compelling reason to deny it. (Cf. Arntzenius 2010; Hofer-Szabó *et al*. 2013) In any event, in the present argument, learning from experience, as an existing and known macroscopic physical process, produces the correlation in question without any problem.

[14] I intentionally don't call it 'empirical underdetermination' or 'underdetermination by evidences'. For it is not the case that the theory is underdetermined by experience but that the experience itself is underdetermined by (the past and present of) the part U of the physical reality—where experience means the causal process in the physical world that produces the correlation described in condition 3/(B). Also, there are no such things as empirical facts or evidences independent of the whole theory, as it will be pointed out in point 19.

by a^*—cannot be expressed as "$\neg A$". Simply because condition 3/(B) fails, therefore not only $\neg A$ does not carry meaning, but there is no semantics at all. Consequently, we are not able to attribute a feature, whatsoever, to the physical reality in the situation a^*.

Of course, one can imagine that the theory formation process leads to a modified (or completely new) theory, (L', S', U'), which is constructed with some new family of state of affairs $\{a'_{\lambda'}\}_{\lambda'}$ and new family of formulas $\{A'_{\lambda'}\}_{\lambda'}$, such that $a^* = a'_{\lambda'_*}$ and conditions (A) and (B) in point 3 are satisfied. Then the corresponding $A'_{\lambda'_*}$ will be attributed to a^*, as a true feature of reality in state a^*—at least, according to the new theory (L', S', U').

18. The above example not only illustrates the thesis of falsification holism mentioned in point 4/(e), but also reveals that semantics plays *constitutive* role in furnishing the world with entities and their attributes. 'Constitutive' in the sense of Reichenbach's conception of constitutive principles of coordination, that is, the way in which physical things are coordinated to mathematical objects (Reichenbach 1965). In fact, due to the holistic nature of semantics and the intertwining of semantics with the truth of the theory as a whole, the constitutive role goes not to some separate operational definitions of isolated notions or quantities—as in traditional operationalism—but to the theory (L, S, U) as a whole; that is, to the whole formal system L and to the whole, empirically established, semantic construction (Szabó 2020). So the formal system L is constitutive, in this partial sense, of our knowledge of the physical world.

However, even this constitutive role of L does not imply that there is some *a priori* conceptual schema, symbolically represented by L, in whose terms we grasp the experienced physical reality. There is no "conceptual side of the coordination" (cf. Reichenbach 1965, p. 53) as a system of analytic truths. For, as it was pointed out in point 12, what does exist is anything but conceptual: we have the physically existing formal system L without any meaning. Once the formulas of L are endowed with meaning, in the sense of point 3, they become true or false in *a posteriori* sense, where experience means the causal process in the physical world that produces the correlation described in condition 3/(B) between the state of affairs in the physically existing L and the physically existing U.

19. One might be inclined to think that this is true only for the "interpreted" part of the language of the theory, that is, only for those formulas of the formal system L that are involved in the family $\{A_\lambda\}_\lambda$ on which the semantic construction is based; while the rest, so called "purely theoretical" part of the formal system possesses some "pure conceptuality" free from the shackles of any reference to the physical world—and, that the various "mathematical concepts" belong to that "purely conceptual" realm.

Beyond the obvious ontological problem that this kind of "pure conceptuality" would smuggle back some sort of mental or abstract entities incompatible with physicalism, notice that the whole idea is rooted in an essential distinction between interpreted and non-interpreted parts of physical theory. Such a distinction is, however, untenable. For, the dichotomy is based on the assumption that the family $\{A_\lambda\}_\lambda$ is a special group of formulas of the formal system which have some distinguished relationship to the physical reality, by being capable of referring to the physical world by themselves, independently of the rest of the theory. This is, however, not the case. In bringing about the meaning-carrying, the "non-interpreted" elements of the formal system are on a par with the ones belonging to $\{A_\lambda\}_\lambda$. The correlation in condition 3/(B) is not between the formulas $\{A_\lambda\}_\lambda$ and the sates of affairs $\{a_\lambda\}_\lambda$ but between the facts $\{\Sigma_L \vdash A_\lambda\}_\lambda$ and $\{a_\lambda\}_\lambda$. And, as it was already mentioned in point 4/(e), the facts $\Sigma_L \vdash A_\lambda$ can involve a large part of the formal system L; practically, they are facts of the formal system as a whole.

However, let me emphasize again, the elements of the formal system, regardless of being involved in the family $\{A_\lambda\}_\lambda$ or not, by themselves, without such meaning-carrying phenomenon, which is jointly produced by the formal system L, as a whole, and the external physical reality U to be described, are merely meaningless marks; no matter if they are called "infinity" or "two", or "straight line", or "probability", "information", "disjunction", "conjunction", "for all", "there exists", "set", "set of", "mapping", "category", etc.; or even "charge", "electric field strength", "temperature", or "spin up".

20. Thus, there is nothing conceptual in logic and mathematics. Mathematical objects are just meaningless elements of a physically existing formal system. They can become meaning-carriers only when they are involved in the semantics of a physical theory. In such roles, however, they refer—holistically, as parts of the whole theory—to elements of physical reality, and in this way they can become physical concepts. The meaning carried by a mathematical object in a physical theory is not *a priori* given; it is not even determined by the formal system in question. It is relative to and determined by the empirically established physical theory.

The task of mathematics is to explore the facts of the physically existing formal systems; but we have to abstain from "mathematical concepts", which are just idols, that philosophy can completely deny and physics can completely ignore.

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