Directed Temporal Asymmetry from Scale Invariant Dynamics: Is the Problem of Time's Arrow Solved?

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Abstract

The scale invariant model of Newtonian gravity by Barbour, Koslowski, and Mercati (2014, 2013, 2015) purports to solve the problem of the arrow of time, but has received minimal philosophical analysis. This omission is amended in the present work, in which I describe how the model manages to derive asymmetric behaviour from symmetric physics. The Janus point structure of the proposed solution holds significant preliminary promise for deriving asymmetry from symmetry. However, the proposal does not recover sufficient supervenience relationships between various other arrows of time to regard the problem as being solved. I propose a line of research which defines statistical mechanics within the BKM model, which would be a significant advance toward a satisfactory solution.

1 Introduction

Research programmes can receive support for a variety of reasons, but two of the most obvious are attractive philosophical principles and explanatory power. The relational research programme commenced by Julian Barbour with Tim Koslowski, Flavio Mercati (BKM), Henrique Gomes, Sean Gryb and David Sloan (Barbour et al., 2014, 2013, 2015; Gomes, Gryb, & Koslowski, 2011; Gomes & Koslowski, 2012; Koslowski, Mercati, & Sloan, 2018; Sloan, 2018, 2019; Sloan & Gryb, 2021) claims both these merits. Among other things, this research programme proposes to: I) provide the correct metaphysical framework for physics, II) offer a solution to the problem of the arrow of time, and III) offer singularity resolution in general relativity. In the present work, I will discuss the second claim.

The problem of the arrow of time concerns the fact the we can identify directed temporal asymmetries, but the physics we take these asymmetric phenomena to emerge from is time symmetric. This is in fact two questions; i) how does asymmetry emerge from symmetry? and, ii) why are our phenomena directed one way along the asymmetry rather than the other? This latter question will importantly shape what we regard as a solution to the problem of the arrow of time.

I will present the usual programme to solve the problem of the arrow of time: by appeal to statistical mechanics (SM) (section 2.1) with a past hypothesis (PH) (section 2.2). I will argue that this programme fails, and for reasons which encourage pessimism about the prospects for the future, thus motivating the investigation of alternatives (section 2.4). The *BKM model* is such an alternative.

The BKM model is an application of relational principles, specifically *dynamical similarity*, to N-body Newtonian gravity. Dynamical similarity (section 3.1) is similar to scale invariance, except times are scaled in addition to length; systems are then describe by only dimensionless variables invariant under such transformations. This administers the relational principle that the Universe has no external measures: length and durations are determined as ratios.

The BKM model (section 3.2) shows that a dimensionless measure of homogeneity, which they term *complexity*, displays secularly increasing behaviour on either side of a shared minimum. Identifying arrows of time with complexity, they offer new horizons in the old debate regarding the arrow of time.

This dynamically similar model emerges from the contemporary *shape dynamics* (Mercati, 2018) theory, which offers an alternative to general relativity. However, the BKM model does not require shape dynamics. Instead, I will discuss it as an application of dynamical similarity to standard Newtonian physics. Similarly, the real cosmological models presented in section 5 will be applications of dynamical similarity to general relativity, rather than full shape dynamics.

In my analysis of the BKM solution, I will distinguish two projects concerning the problem of the arrow of time, a modest project and an ambitious project (section 3.3). The modest project is specifically concerned with deriving an observed temporal asymmetry from time symmetric physics. The ambitious project asks for explanations of supervenience relationships between temporal asymmetries. Proposals in support of the ambitious project will necessarily have various degrees of success, according to the number of temporal asymmetries explained. I will argue that the BKM model suggests an attractive solution to the modest project which avoids the failures of SM-with-PH (section 3.4), but has limited success in the ambitious project.

A natural view is that the particular difficulty with the problem of the arrow of time is to derive temporal asymmetries from symmetric physics, and so the modest project represents the core of the problem. It has been suggested on these grounds that the BKM project can claim solving the problem of the arrow of time into its list of accolades. *Contra* this view I argue that, in order for the BKM project to claim to have solved the problem, a greater degree of success than it attains in the ambitious project is necessary (section 3.5).

Building on this conclusion, I propose a line of research which would advance the solution to the ambitious project considerably, and as such, advance BKM's claim to a solution of the problem of the arrow of time. This proposal combines the merits of both the BKM model and SM to offer a complete solution to the problem. However it faces difficulties, and requires further mathematical development to be realised (section 4.3).

Finally, I present work which provides a scale invariant analysis of real cosmological models (section 5). This work consolidates the modest solution, but does not help the ambitious solution.

2 The Statistical Mechanical Arrow of Time

2.1 Statistical Mechanics

I will formalise SM according to formulation of Gryb (2021). The macrostates of a thermodynamic system (e.g. temperature, entropy) are defining by beginning with the phase space of states of a large collection of degrees of freedom, such as particles described in terms of their positions and momenta, each point in which represents a microstate. A macrostate of a system is then defined to be a particular set of microstates, for example corresponding to those states that are assigned the same value by some observable. Supposing the system consists of N particles, the phase space will be a $6N$ dimensional symplectic manifold, X. A macrostate α can then be defined as a (Liouville) measurable subset $A \subseteq X$ such that A picks out microstates which produce a macrostate with properties α.

The power of SM comes from our ability to quantify the probabilistic weight of regions of X. The Liouville measure $\mu_L(A)$ is used on this phase space to assign a weight to any region of the space. One can theoretically use this weight as a probability of finding the state in this region - however, as will be shown, both the use of this measure and its interpretation as a probability are not obviously possible for the infinite phase space of the Universe. For finite systems one can use Birkhoff's ergodic theorem, which states that for ergodic systems the time average of a macro-variable is proportional to the phase space volume given by the Liouville measure, on timescales larger than the Poincaré recurrence time. Alternatively, one can make the stronger assumption that a system is mixing, which implies that in the long run the measure becomes homogeneous and proportionally represents all coarse-grained regions in the phase space, and is thus Liouvillian (see also Arnold and Avez (1968)).

A typical state can now be defined as a region with a comparatively large "volume" of phase space, given by $\mu_L(A)$. Typicality is a fairly slippery property, but the useful definition given by Gryb is that a state A is typical when

$$\frac{\mu_L(X) - \mu_L(A)}{\mu_L(X)} \ll 1. \tag{1}$$

If we keep our assumption that the measure weights define the probability of occupation, then a state that occupies a large proportion of phase space as defined in (1) will be highly likely to be realised, and hence typical. The equilibrium state will then be the state that is vastly more typical than any other state, if it exists, which it will for finite thermodynamic systems.

To initially see a SM arrow of time, we can consider a system in an atypical state. If we time evolve this state, and if it is ergodic, or satisfies some condition that does a similar job like Boltzmann's Stoßzahlansatz, then if given enough time it will almost certainly evolve into a more typical state due to the much larger region of phase space occupied. Therefore, states out of equilibrium evolve towards equilibrium, atypicality towards typicality (see Callender (2010) for an alternative formulation). The problem is that this equilibrating law for SM does not dictate whether the evolution is in the forwards or backwards time direction. This is known as the Loschmidt reversibility paradox: entropy increase is both a future prediction, and a past-oriented retrodiction.

The retrodiction is thoroughly falsified (the previous state of a melting ice cube was not a more melted ice cube) and as such we have not solved the problem of the arrow of time.

The power of a PH is to override the falsified retrodiction. That is, when one asks: why do states only become more typical into the future and not the past, despite SM predicting increasing typicality into the past? One can answer that in the past a very atypical state occured, and so backwards time evolution must evolve to this atypical state, rather than towards typicality.

2.2 The Past Hypothesis

One can approach PH in two ways. First, if we take SM to provide a correct description of reality at all times, for all macro-systems, then something like a PH (or an alternative that does a similar job) is needed to avoid the conclusion that our experience is an unlucky accident. This is because SM without a PH leads to the unfortunate conclusion that all of our perception is an illusion due to a fluctuation from a more typical state. Therefore, if SM is an adequate theory at all times, for all macro-systems, then we have good evidence for PH (Albert, 2000). However, the claim that SM-with-PH provides an adequate description of reality with a time asymmetry at all times, for all macro-systems, is not safe.

For a similar claim consider Price's argument that only a PH can solve the problem of time's arrow. Price (2002) distinguishes two approaches to the problem. One may attempt to find a dynamical mechanism which makes it a general law that entropy will increase, this is the causal-general strategy. Alternatively one uses a PH to argue for a particular low entropy initial condition of the universe, the acausal-particular strategy. Price argues that one must pursue the latter, because the causal-general approach has two-fold time-asymmetric behaviour. If the equilibration driving mechanism was removed, Price supposes equilibrium would still be reached due to Boltzmann's SM. North (2002) correctly comments that this assumes Boltzmannian SM is correct, and its probabilities over phase space knowable *a priori*, both of which we cannot assume. The dynamical mechanism would *replace* SM as the best description of thermodynamics and it's equilibration law, and therefore the causal-general strategy only requires one type of asymmetry. Hence both categories in Price's taxonomy are permissible.

To the same effect our knowledge of reality is only *a priori* dependent on a PH if SM is the true physical description of the thermodynamic behaviour of the Universe. This is not necessarily true (see also Gryb (2021), discussed below). SM certainly correctly describes familiar thermodynamic systems, such as heat engines, up to statements about what preceded an imposed initial condition. But taken beyond this SM can fail, and so this approach to the PH is to be abandoned.

The alternative to this approach is to argue that the PH is evidenced. This is the most common approach, adopted for example by Price (2002). There isn't a single agreed upon PH in the literature, but a sketch will be instrumental (see (Penrose, 1979) for a classic account and (Wallace, 2010) for an analysis and further references).

Cosmological evidence suggests that the very early Universe was very homogeneous. For a gas in a box this is highly typical, but the early Universe is dominated by gravity and therefore what counts as typical is different. Supposedly, homogeneity is highly atypical of gravitating systems, and so the phase space region of this highly homogeneous macrostate of the Universe is very small. The Universe then evolves toward inhomogeneity and, through gravitation, forms clusters, which form star-systems, the heating of planets, and eventually systems which are dominated by pressure and radiation. In non-gravitating systems homogeneity is expected and so clumpiness is atypical; however with gravity, clumpiness is now prolific, and so the usual second law is recovered.

Although this may seem intuitive, it is tenuous to make such claims, because it is unclear how Boltzmann's entropy would apply to the early moments of the Universe. Before discussing these difficulties facing a PH, it is necessary to develop a deeper view of time and its arrows.

2.3 Mapping Gradients and Arrows

Price (1996) has given a book-length argument that time in our universe is fundamentally directionless, and that this claim is compatible with our best evidence from physics, including thermodynamics. A version of this view has more recently been characterised by what (Farr, 2020) calls the C-theory of time, that temporal facts are best characterised by the directionless "betweenness" relation, or C-series of McTaggart (1908).[1] In contrast, in the A-series temporal facts are characterised by past, present and future, while in the B-series temporal facts are characterised by the properties of being "earlier than" or "later than". On the C-series, the only kind of temporal fact one can state about time is that it is a directionless sequence, with no fact as to which way one must go along that sequence.[2]

What then constitutes a B-theorist explanation of the arrow of time? Our problem is to ask how temporal asymmetry that distinguishes macrostates like equilibrium as B-series "later than" macrostates out of equilibrium emerges from temporally symmetric microphysical theories that seem to only state C-series facts. As Zermelo and his contemporaries noted in response to Boltzmann, temporally symmetric microphysics by itself leads to temporally symmetric macrophysics. Let us consider a famous example, the Boltzmann fluctuation model, in which the macrostate of the Universe is a highly unlikely fluctuation out of equilibrium. The problems with this explanation (Boltzmann brains etc.) have been well versed, but let us suspend these criticisms so we can unpack the B-theory. On the B-theory time has an earlier than relation, so we can say that on the "earlier" side of the fluctuation agents experience decreasing entropy, and on the "later" side they experience increasing entropy. This is displayed in Figure 1.

This sort of B-series explanation give no reason as to why we experience increasing entropy other than the coin-flip chance of being on one side of the asymmetry rather than the other. Consequently, no explanation of the arrow of time will ever be accessible. Time becomes something that is pre-science, pre-empirical knowledge, and pre-investigation and we must throw ourselves upon this coin-flip chance. What is left for the B-theorist? Only to derive how the time symmetric physics produces models

[1] See (Roberts, 2022, ch. 5) for a chapter-length critique of the Price-Farr directionless view.

[2] In this work I will refer to the view that time is directed the B-theory. The reader should keep in mind that the A-theory also has a direction, but which will not be relevant for this discussion.

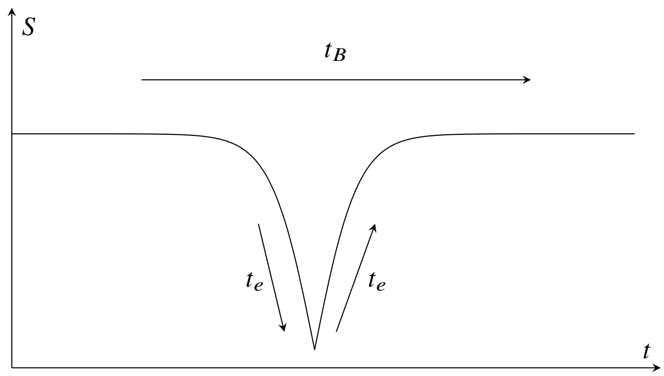

Figure 1: A Boltzmann fluctuation model of the Universe, plotted according to a B-theoretic notion of time. The time axis is used for calculation, and the ontic, B-theoretic direction of time is labelled t_B. It could of course have reasonably been in the opposite direction. Experienced time is then labelled t_e, and experienced up the entropy gradient on one side, and down it on the other.

upon which there are locally temporally asymmetric structures, and to explain how they arise from the symmetric physics.[3]

The alternative C-theory adopts the *atemporal* perspective, advocated by Price (1996, p. 35), that "the direction of time is not an objective matter, but an appearance, a product of our own orientation in time." This view takes temporally asymmetric structures as undirected and complete evolutions, and directionality is mapped onto the structure in accordance with internal features of the structure.

According to Price, the arrow of time is most immediately defined by psychological arrows, such as our asymmetric impression that our conscious experience moves from the past into the future. After this are epistemtic and causal arrows, defined by our knowledge of an unchangeable past, and our ignorance of a influencable future. After this comes thermodynamics and equilibration phenomena, along with radation. Finally, we have the apparent collapse of the wavefunction and cosmological phenomena such as the expansion of the Universe. On the directionless view, to explain arrows of time (once gradients have been derived) is to explain the supervenience relationships of these arrows, and to do no more.

For the rest of this essay I will assume that there is no preferred direction of time; arrows are explained by alignment with each other, rather than with respect to a preferred direction. That is to say, I will assume no more structure than is found in Price's atemporal perspective or Farr's C-theory of time.[4] This is also the theory of time

[3]It is also sensible that the B-theorist would be asked to describe what the Universe would be like on the other side of the symmetry, but as this is empirically inaccessible, it would also not be falsifiable and thus a scientific form of story-telling more than science.

[4]*Pace* Price-Farr, Roberts (2022) defends the thesis that the weak interaction can establish a true direction of time. The assumptions spelled out are unaffected, however, because there is no apparent causal link between the weak interaction arrow, and those arrows that concern us most, such as the cosmological and

adopted by Barbour, "as merely the successive order of things" (Barbour & Bertotti, 1982, p.296), although he doesn't call it the C-theory, and he is motivated by a Machian conception of nature. It would be interesting to consider to what extent the findings of the scale invariant research programme are retained on an A- or B-theoretic approach to the problem of the arrow of time, but it is not within the scope of this paper.

I can now address the reasons the PH fails.

2.4 Problems for the Past Hypothesis

The PH faces difficulties on two accounts: firstly, a well evidenced PH would not solve the problem of time's arrow, only move it, and secondly there is no evidence for it.

As to the first problem, assume that there is a empirically successful and mathematically precise PH for SM, describing an early Universe state which was highly atypical. The highly atypical initial condition is an unexplained, temporally asymmetric structure upon which our explanation depends. It is unclear why the Universe should have this atypical moment from which time's arrow can emerge and thus it requires explanation: but none is forthcoming.

However, we can challenge this: do we in fact need to explain a past hypothesis? Price (2002, 2004) argues that such a explanation is required. On the other hand, Callender (2004a, 2004b) argues that the past hypothesis is not a premise needing to be explained, but rather a contingent fact that, once observed, must be accepted without further challenge.[5] Some amount of scientific practice is in agreement with Price. In cosmology inflation is argued for by appeal to a no fine-tuning argument, and the PH is a very fine-tuned theoretical construct. Consequently, if we have it in our theory it must be explained. Price mentions some potential ways this could be done but none seem to currently have any strong support. Moreover, Price (1996, p. 42) defends the necessity of explanation because he argues if we do not require an explanation we violate the atemporal perspective, as we would require such an explanation of the future. I find it clearest to put the argument in the following way, which makes use of what I call False Statistical Mechanics (FSM): a theory which almost identically resembles SM except that low entropy states in SM are now highly typical and high entropy states in SM are now highly atypical. Suppose, due to some mistake in our theorising which went unchecked, FSM was posited as a theory of our Universe. FSM evolves states, without any boundary conditions, toward lower entropy. We could then conditionalise on a boundary condition on one end of time and create the entropy gradient we are familiar with, high entropy on one end and low entropy on the other.

We then define our past to be the low entropy part of the gradient. If we could truly conceive of ourselves, and formulate our scientific beliefs arising from empirical enquiry, in an atemporal manner, this would be the sensible thing to do given the supposed discovery of FSM, and would produce just as good results as SM. Applying the atemporal perspective, a Future Hypothesis (FH) is no different to a PH, so generating the gradient in this manner is as permissible and effective as the PH.

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thermodynamic arrows.

[5]See Callender (1997) for an argument that even evidence is unnecessary. However, Callender's solution is a non-starter. Callender argues that SM is not fundamental on the grounds of empirical evidence of the time-asymmetry, and then uses this fact to justify the time-asymmetry itself.

Now suppose the scientific community believed that one didn't have to explain the FH, it was a brute contingent fact. In this case we would have recovered and explained the entropy gradient using an incorrect theory, FSM. On the other hand, if one insisted upon explaining the FH, then we would search for a novel explanation. No such explanation would be forthcoming, because the high entropy future is correctly explained by typicality arguments within SM. The failure to adequately explain FH would hopefully spark a research programme in which Boltzmann's mistake was found and SM was discovered.

In conclusion, if one does not have to explain PH (or FH), then the content of the scientific theories becomes inconsequential, and a theory's negation can do the same explanatory work. Therefore, it is necessary to explain PH. Because no good explanation has been offered, even if one had evidence of PH, the problem of time's arrow is simply shifted onto another unexplained asymmetry.

There is a large literature of well-known criticisms of PH.[6] However, a few of the most serious concerns, undermining the mathematical and empirical basis for PH, have been neatly summarised and extended in a recent paper by Gryb (2021), and so it will be helpful to recapitulate how he frames them.

Gryb (2021) points out that for a successful PH, a number of conditions must be satisfied beyond common sense intuition, and none of which are safe. 1) A measure on the phase space of the Universe must exist that is precise, unambiguous and invariant under gauge symmetries, all in line with usual standards in physics. 2) We must then be able to use the weights of this measure to give the relative likelihood of a phase space region being occupied. 3) Macrostates must then be definable on the microscopic phase space and one of these macrostates must be an equilibrium state which dominates the phase space, in order for a Boltzmannian conception of typical time-evolution to succeed. 4) Finally, given all this, arguments from typicality must be applicable to the Universe and there must be cosmological evidence.[7]

The first difficulty that Gryb raises is that a measure can't be defined on the infinite interval and infinite dimensional phase space of the Universe for SM. In order to do so one must truncate the space: but this is highly subjective and leads to significant inconsistencies. Defining any measure on the phase space of general relativity is a problem that hasn't been solved. This is an issue mostly relating to Gryb's first condition.

Secondly, a global maximum entropy can't exist for a Newtonian self gravitating system unless two poorly motivated "cut-offs" are used (Padmanabhan, 2008): confining the system to a finite radius, and giving a minimum radius of each body. Consequently equilibrium can't be defined - which in turn entails macrostates can't be defined, an issue for Gryb's third, and to a lesser extent fourth, condition.

Thirdly Gryb argues that ergodicity and mixing can't be used to attribute measure weights to probabilities for the Universe (see also Schiffrin and Wald (2012)). The

[6]The PH nomenclature was introduced in Albert (2000, ch. 4), but the idea is discussed and criticised in Price (1996, ch. 2). Critiques since then include (Price, 2002; Earman, 2006; Schiffrin & Wald, 2012; Winsberg, 2004; Davey, 2008; Wallace, 2017; Barbour, 2020). Engage defences of the PH (which nonetheless do not overcome the concerns I discuss here) include (Callender, 2017; North, 2011)

[7]I have condensed Gryb's six points into four, all content is retained.

only other option is Laplace's principle of indifference[8], which is weakly motivated and criticised in both Uffink (1995) and Norton (2006). This is an issue for Gryb's second condition. None of these are insurmountable objections but they indicate the significant difficulties faced by a PH.

Finally, Gryb presents a dilemma. Either the advocate of the PH can use the uniquely time-symmetric Liouville measure but this leads to a breaking of gauge invariance and the introduction of a distinction without difference. Or the advocate can use a different measure but it will necessarily be time-asymmetric and then there is no reason to think the PH is doing explanatory work in the argument for emergent asymmetry. In section 4.2, I will present BKM's argument for a measure which throws itself on this second horn: but this is no defect because they don't need a PH to do explanatory work.

For a thorough exposition the reader is referred to Gryb (2021), but it should be clear that arguing for a PH is not a simple matter. In our philosophising, we must be sure to be good scientists at all times since our intuitions lead us astray; and so the concerns raised above motivate us to consider an alternative approach. The BKM model of a scale invariant Universe provides this alternative and it deserves more attention. I now turn to this model.

3 The BKM Model

I will sketch the BKM ontology, before turning to the philosophical analysis of the BKM model for the arrow of time (the reader is referred to (Barbour, 2020) for further discussion). BKM assume, due to a broadly Machian perspective: i) the Universe is a closed, unbounded dynamical system, (ii) there exists a notion of universal simultaneity.

Regarding the latter condition, this is unproblematic in Newtonian gravity but in general relativity many solutions do not admit a foliation into spacelike hypersurfaces, let alone a preferred one (see Earman (1995, sec 6.3); section 5 of this paper discusses some exceptions). Dynamical similarity requires a this preferred time slicing in order to globally scale time. Hence one cannot make GR generically dynamically similar. Shape dynamics, BKM's preferred rival to general relativity, does has a preferred notion simultaneity.

The former condition is also reasonable, and the expansion of the Universe gives it an empirical character. Three assumptions that BKM make in light of (i) are: 1) The Universe does not have a bounded volume measure (on its representation with scale), 2) all measurements are ratios, and 3) the total energy, angular momentum and linear momentum of the Universe are vanishing, (Barbour et al., 2013).

The unbounded volume measure captures the idea that particles can escape to infinity. The claim that all measurements are ratios invokes a relational perspective on spacetime, debarring external substantivalist characterisations of space and time. This leads us to postulate scale invariance (more precisely, dynamical similarity), and describe the Universe in *shape space*.

By far the least intuitive assumption is that for the universe as a whole, $E = \mathbf{J} = \mathbf{L} = 0$. All the above assumptions will be necessary for what follows, but this one

[8]Which Gryb calls a principle of insufficient reason.

will appear most like an contingent fact, rather than philosophical principle. A defence of this claim will be essential to BKM's proposed solution, and is complex enough to deserve at least one paper devoted to it, and so I shall take it as assumed in this work.[9]

The extent to which the above is derived from a Machian philosophy, and the extent to which it has been taken beyond the work of Mach, and thus is better called Barbourian is an interesting question. Certainly, there is much said here which is not in Mach, but whether it is naturally implied by Mach's writings, or philosophically novel, is open.

3.1 Dynamical Similarity and Shape Space

Consider two triangles with distances between points $(a, a, 2a)$ and $(2a, 2a, 4a)$. BKM argue that only relational degrees of freedom are physically significant, and absolute values are eliminable, and so these two triangles are identical (if they are each the only object in a global system). This simple-minded scale invariance doesn't work in generic dynamical cases, however, because one also has to consider how durations are affected by such a scaling.

To be more precise, let S be an action functional on a space of curves, and γ be a curve through the state space of a theory. Then $S[\gamma]$ denotes the value that the action assigns to the curve, and dynamically possible models are defined as those curves such that $\delta S[\gamma] = 0$. A dynamical similarity, $\mathcal{D}$, is a transformation which rescales the action by a constant, c (for $c \neq 0$); that is

$$\mathcal{D}^* S \to cS + \Phi \tag{2}$$

where Φ is a constant corresponding to some canonical transformation which we are free to include in our transformation. Equation (2) preserves the dynamically possible models of the theory, and yet acts very differently to canonical transformations for which $c = 1$. (Sloan & Gryb, 2021)

As an example, let us analyse Kepler's third law (following Sloan and Gryb (2021)). Consider a two body orbital system in an idealised Universe in which a distant static collection of stars can serve as a fixed reference frame. The distance between the bodies is r, and the line connecting the bodies makes an angle θ with some fixed axes in the reference frame. Absorbing masses and constants into a single value C, the action is

$$S = \int \mathcal{L} dt = \int \frac{1}{2}\dot{r}^2 + \frac{r^2}{2}\dot{\theta}^2 + \frac{C}{r} dt \tag{3}$$

Now suppose we scale the distance between the bodies by β; $\mathcal{D} : r \longrightarrow \beta r$. If we also scale durations by $\beta^{3/2}$; $\mathcal{D} : t \longrightarrow \beta^{3/2} t$, then the Lagrangian, scales by $\mathcal{D} : \mathcal{L} \rightarrow \beta^{-1}\mathcal{L}$. Hence, $\mathcal{D} : S \longrightarrow \beta^{1/2} S$ and we get a dynamical similarity transformation. This preserves the solutions of the equations of motion and such is a symmetry of the theory; the solutions are similar up to their absolute scale.

[9]A sketch of the defence comes from Barbour's theory of 'best-matching' in which space and time emerge by minimising the inter-particle distance between successive shapes. Best-matching leaves the centre-of mass constant, and minimises rotation, which suggests the momenta vanish. One then claims energy vanishes as there is no time variable with which to generate energy (see Gryb (2012, sec 2.3) for a description of best matching, (Barbour, 2020) for further defence of the vanishing constants, and (Barbour & Bertotti, 1982) for a complete account of this formulation of Machian dynamics).

Generalising, if all the coordinates in a system are scaled by α, then the time coordinate should be scaled as $\alpha^{1-\frac{k}{2}}$ where k is the degree of homogeneity of the systems potential $U(\alpha\mathbf{x}) = \alpha^k U(\mathbf{x})$ (e.g. $k = -1$ for Newtonian gravity). Even though the absolute values change under $\mathcal{D}$, this has no real meaning because all values are measured relationally. Thus, we seek to describe the system with quantities invariant under dynamical similarity (cf. Sloan, 2018).

Dynamical similarity transformations will be parameterised by a real number c, thus defining one-dimensional orbits in phase space. Therefore we require an odd-dimensional space of invariants, with one dimension less than the phase space. These odd-dimensional spaces are described by contact geometries. A canonical choice of local coordinates (Q, P, A) can always be found and the geometry of the contact manifold is specified by a contact 1-form, $\eta = PdQ - dA$.

Scale invariant evolution is described in *shape space*. The name shape space is used multifariously in the literature: i) it describes the degrees of freedom of a dynamical system which are invariant under dynamical similarity (e.g. for a Newtonian N-body system, it is the quotient of normal Euclidean space which removes the degrees of freedom relating to rotations, translations and scaling, such that the only degrees of freedom are ratios and angles); and ii) the space of conformal classes of metrics (i.e. how one restricts to relational degrees of freedom in shape dynamics). In this paper, shape space refers to the former. Paths through shape space thus track the evolution of different dynamical similarity invariants.

Without discussing the technology for finding this full set of invariants, (Q, P, A), one can pick an invariant contact Hamiltonian, $\mathcal{H}^c$, which generates equations of motion equivalent to the Hamiltonian equations of motion of the full phase space, which take the form:

$$\dot{Q} = \frac{\partial \mathcal{H}^c}{\partial P} \tag{4}$$

$$\dot{P} = -\frac{\partial \mathcal{H}^c}{\partial Q} - P\frac{\partial \mathcal{H}^c}{\partial A} \tag{5}$$

$$\dot{A} = P\frac{\partial \mathcal{H}^c}{\partial P} - \mathcal{H}^c \tag{6}$$

These equations are distinct from Hamilton's equations due to the additional terms. The equations for $\dot{A}$ and $\dot{P}$ contain terms linear in P often associated with 'friction-like' behaviour.

The natural measure on the original phase space, the Liouville measure, is not invariant under dynamical similarity, and so is inappropriate to characterise volumes on the contact geometry. Using η, a contact manifold instead admits a natural the volume-form $\mu_c = \eta \bigwedge (d\eta)^{N-1}$, where N is the dimension of the original configuration space.[10] μ_c is not preserved by the evolution equations. It's differential is given by

$$\dot{\mu}_c(R) = -N\frac{\partial \mathcal{H}^c}{\partial A}\mu_c(R) \tag{7}$$

[10]The non-degeneracy of this differential form is a necessary and sufficient condition for η to be a contact 1-form.

Therefore, when $\frac{\partial \mathcal{H}^c}{\partial A} \neq 0$, the equation for $\dot{P}$ takes a non-symplectic form and volumes are *not* preserved on the contact manifold. The measure can shrink or grow, depending on the sign of $\frac{\partial \mathcal{H}^c}{\partial A}$, corresponding to the focusing and defocusing of the dynamical trajectories in shape space. This measure focusing is, again, often characterised as *friction*, or friction-like.

We are now able to define a *Janus point*, $\mathcal{J}_P$: a point on a solution in shape space where $\frac{\partial \mathcal{H}^c}{\partial A}$, and thus μ_c, changes sign. These come in two types: *type 1* occurs when $\frac{\partial \mathcal{H}^c}{\partial A} = 0$ but $\ddot{\mu_c} \neq 0$; *type 2* occurs at a point x when $s\frac{\partial \mathcal{H}^c}{\partial A}(x^-) \to \infty$ and $-s\frac{\partial \mathcal{H}^c}{\partial A}(x^+) \to \infty$ for $s \in \{-1, 1\}$ and x^- approaches x from below and x^+ exits x from above.

3.2 The BKM Model; or Dynamical Similarity Applied to the N-body Model

Before returning to the arrow of time it will be helpful to have a concrete example of the structures described above: let us look at dynamical similarity for N-body Newtonian dynamics, which characterises the BKM model. The first job is to remove external Newtonian space and time. We then construct a function that characterises the state of the relational degrees of freedom. Finally we construct dimensionless scale invariant variables, and inspect the dynamics of the theory.

3.2.1 N-body Shape Degrees-of-Freedom

In order to describe the global system's scale without absolute space, BKM use the quantity

$$I_{cm} = \frac{1}{m_{tot}} \sum_{i<j} m_i m_j r_{ij}^2 \tag{8}$$

which is a parameter dependent upon the relative shape and density of the system, expressed in terms of a system of point masses m_i with relative distances r_{ij} and a total mass of m_{tot}. BKM call this the 'centre-of-mass moment of inertia' (hence the suggestive notation). However, this is a misnomer as no axis of rotation is defined, and such one cannot define a moment of inertia. The object is in fact the trace of the moment of an inertia tensor calculated about the centre-of-mass, divided by two. Rather than re-name the object, I will opt to simple refer to it as I_{cm} throughout.[11]

A potential V with total energy E and which is homogenous of degree k obeys the Lagrange-Jacobi relation

$$\ddot{I}_{cm} = 4E - 2(k+2)V. \tag{9}$$

For Newtonian gravity, $k = -1$ and $E_{cm} \geq 0$, which implies

$$\ddot{I}_{cm} = 4E - 2V_{New} \geq 0 \tag{10}$$

[11] I am grateful to an anonymous referee for noting this subtlety.

Consequently, I_{cm} is a U-shaped function with a minimum at the point where $D = 0$, defining $D = \dot{I}_{cm}$ as the dilatational momentum (see Barbour (2003)). The point at which $D = 0$ is a Janus point of type 1.

The type of evolution that I_{cm} displays shall henceforth be referred to as a 'Janus curve', $\mathcal{J}_C$, defined as a curve that exhibits locally asymmetric behaviour on either side of a shared 'past' or 'central point'.[12] Typically this central point will be at, or very near, a Janus point. $\mathcal{J}_C$ will have different types: secular, monotonic, linear and so on.

Because D is monotonic it satisfies a minimal standard of being interpretable as a time variable. Labelling it τ, we can now construct the τ-dependent Hamiltonian for the system, removing the external Newtonian time t. This Hamiltonian will encapsulate the dynamics of the scale invariant evolution of the system.

This can be done as follows: Barbour et al. (2014) adopt 'pre-shape space', the quotient of configuration space by only global translations and dilatations, rather than shape space. Rotations commute with the Hamiltonian and dilatations, so the implications of scale invariance are investigable without including them. Pre-shape space is parameterised by shape coordinates $\sigma_a = \sqrt{\frac{m_a}{I_{cm}}}\mathbf{r}_a^{cm}$ which generate the canonical momenta, called shape momenta $\pi_a = \sqrt{\frac{I_{cm}}{m_a}}\mathbf{p}^a - D\sigma_a$.

The Hamiltonian is

$$\mathcal{H}(\tau) = \ln(\sum_{a=1}^{N} \pi^a \cdot \pi^a + \tau^2) - \ln(I_{cm}^{1/2}|V_{New}|) \tag{11}$$

The result is a fully relational expression of the dynamics of an N-body Newtonian system. Our next step is to describe a property of the system that will help us characterise its temporal symmetries.

3.2.2 Complexity

BKM require a dimensionless variable invariant under dynamical similarity to characterise the state of the system. For this they use a ratio of lengths which they call *complexity*,

$$C_S := \frac{l_{rms}}{l_{mhl}} \tag{12}$$

where l_{rms} is the root-mean-squared length of the system

$$l_{rms} := \frac{1}{m_{tot}}\sqrt{\sum_{a<b} m_a m_b r_{ab}^2} = \frac{1}{m_{tot}} I_{cm}^{1/2} \tag{13}$$

and l_{mhl} is the mean harmonic length of the system

$$\frac{1}{l_{mhl}} := \frac{1}{m_{tot}^2}\sum_{r<a}\frac{m_a m_b}{r_{ab}} = \frac{1}{m_{tot}^2}|V_{New}| \tag{14}$$

[12]This Janus *curve* terminology is my own, not introduced by BKM who simply refer to Janus points.

The first term, l_{rms}, is dominated by the largest r_{ab} so as to be larger for a system which has particles much further away from others than the average. In contrast, $1/l_{mhl}$ is dominated by the relatively closest together particles, so if a system has particles much closer together than the average this term is higher. Complexity, defined in (12), is maximised for high clustering and minimised for low clustering, or in other words it measures the inhomogeneity of a system. Since complexity is a ratio, it is observable, and moreover it is the central observable in the study of modern cosmology (Kolb & Turner, 1990).

BKM plot complexity against Newtonian time and generate a secular-linear $\mathcal{J}_C$, displayed in Figure 2. Specifically, this $\mathcal{J}_C$ has no upper bound, but one clear minimum. Complexity fluctuates as it increases on either side of the minimum, and BKM show that as the number of particles increases the fluctuations become increasingly small, and a clear linear growth develops. This seems to agree with our knowledge of cosmological growth: an initial highly homogeneous state under gravity evolves into a highly clustering state, with solar systems, galaxies and galaxy clusters.

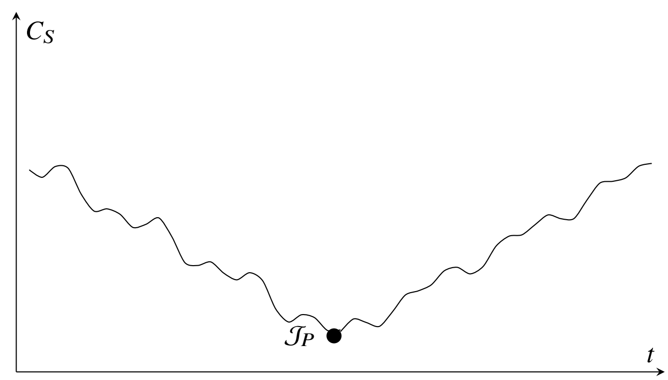

Figure 2: This figure is indicative of the Janus curve for complexity with Newtonian time. The increase is secular-linear, as one might expect with temporal orientation, away from the Janus point, indicated by a circle.

BKM propose to solve the problem of the arrow of time by identifying time with complexity. BKM postulate there are two worlds, one on either side of the complexity minimum, with temporal asymmetry supervening upon the complexity asymmetry. The complexity minimum will be very close to, but not necessarily at, $\mathcal{J}_P$ defined by $D = 0$.

One of the few engagements with this idea in the literature is Zeh (2016). Many of Zeh's concerns are ontological and thus do not concern us here. What is most relevant to us is Zeh's argument that the 'initial condition' which BKM evolve from, which is in the region of the Janus point, is an initial condition assumption which injects an improbable past hypothesis into the model. However, this seems to be a misunderstanding. Barbour et al. (2013) use 'mid-point' data, from which one can calculate time evolution in either direction. It is a starting point for *calculation* and is not necessarily the initial point of experienced time. As discussed in section 2.3, $\mathcal{J}_C$ exists as an atemporal path in its entirety, with temporal structure then assigning according to internal structures. Furthermore, Janus points are generic features of solutions. Consequently, Zeh's argument against BKM misses the mark. This is the reply adopted by Barbour, Koslowski, and Mercati (2016).

Before discussing complexity as a solution to the problem of the arrow of time further, I will complete the dynamical similarity analysis of the N-body model.

3.2.3 Dimensionless N-body Model

In order to understand the scale invariant dynamics, one must construct a dimensionless Hamiltonian, which requires the removal of the scalable time parameter. It is possible to construct a time-independent, autonomous Hamiltonian by making the parameterisation dimensionless with parameter $\lambda = \log(\tau)$, and defining the dimensionless shape momenta to be $\omega^a = \pi^a/D$. The fully scale invariant Hamiltonian is then

$$H_0 = \log(\sum_{a=1}^{N} \omega^a \cdot \omega^a + 1) - \log(C_S) \tag{15}$$

where one has noted that $I_{cm}^{1/2}|V_{New}| = C_S$. This Hamiltonian generates equations of motion given by

$$\frac{d\sigma_a}{d\lambda} = \frac{dH_0}{d\omega^a} \tag{16}$$

$$\frac{d\omega^a}{d\lambda} = -\frac{dH_0}{d\sigma_a} - \omega^a \tag{17}$$

We see here again the 'friction-like' term characteristic of the non-symplectic equations of motion generated by the contact Hamiltonian, as in Equation (5)

3.3 The Modest and Ambitious Projects for the Arrow of Time

The BKM model provides a definition of time as determined by complexity which gives rise to locally asymmetric temporal behaviour, on either side of the $\mathcal{J}_P$, from the time symmetric laws of Newtonian mechanics. However, not only does one need to confirm the asymmetry matches observation, one should give reasons as to why any other arrows of time should align with the derived asymmetry. Importantly, there is no immediately obvious theoretical (non-observational) reason why we should define the arrow of temporal experience of one of these half solutions toward or away from $\mathcal{J}_P$ as indicated in Figure 3.[13]

Explaining the arrow of time has two components: one must explain how local temporal asymmetry arises from symmetric physics, and one must explain how all other arrows of time map to this arrow. This delineates a modest project and an ambitious project.

[13]This critique could be made stronger if one accepted a philosophy of time with more structure. Suppose that rather than finding relationships between distinct arrows of time, one gave time an ontological direction which supervened upon some internal structure. For example, the direction of time itself may be attached to the direction of increasing typicality of states. I take this as a defensible view, but one which violates a key assumption of the BKM model. Whether the BKM proposal could be successful given such a philosophy, which is neither clearly, A-, B- or C- theoretic, is an open question.

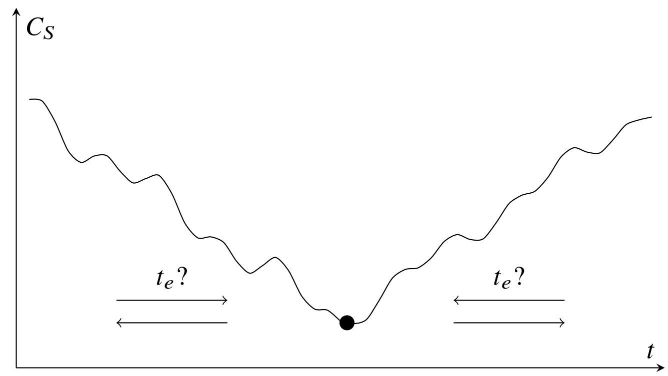

Figure 3: This figure displays a heavily simplified complexity curve with a Janus point, indicated by a circle. The pairs of arrows indicating possible directions of temporal experience. The top pair indicates the temporal direction we would consider contrary to observation, the bottom pair is what our observation indicates. Why we should choose the bottom pair rather than the top pair is the problem of mapping discussed in section 2.3.

The Modest Project To fully explain the emergence of an observed temporal gradient.

The Ambitious Project To explain the supervenience relationships between the asymmetry derived in the modest project, and asymmetries pertinent to our experience of time.

I will argue that the BKM model has good success with the modest project, but falls problematically short in the latter project.

3.4 BKM Proposal for the Modest Project

In the literature on the BKM approach, there are two distinct explanations of the creation of local temporal asymmetry. The first (presented in Barbour et al. (2014, 2013)) describes the behaviour of a single path through shape space, which is subject to dissipative equations of motion and attractors. The second (presented in Barbour (2020), Sloan (2018), Sloan and Gryb (2021) and in personal communications) describes the evolution behaviour of a measure on shape space. I will now clarify the structure of the former, in response to some criticism in the literature, and conclude that on clarification these are equally successful, although some difficulties are raised for each.

3.4.1 Dissipation

Shape space contains attractors: regions in the invariant space to which the dynamical flow converges. Attractors are characteristic of stable structures. In Equation (11), the potential term depends on the complexity, and shapes of greater complexity act as

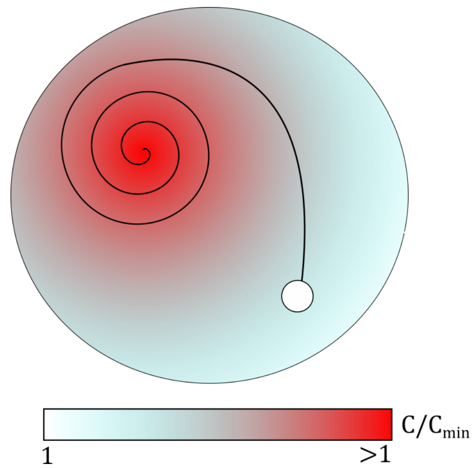

Figure 4: An emblematic, heavily simplified representation of half a Janus curve through shape space, beginning with a Janus point, and being drawn into higher and higher potentials by the asymmetry of the potential space. There will be many more dimensions, with multiple maxima and minima, and the Janus curve will of course go in two opposite directions. Low complexity shapes are lighter shades of blue, high complexity shapes are darker shades of red.

attractors in shape space. Moreover, as noted above, the equations of motion are non-conservative: Equation (17) contains a friction-like term. Therefore, BKM characterise the system as controlled by a potential $-\log(C_S)$ and subject to friction.

What is the structure of the potential on shape space? Taking the negative complexity, $-C_S$, gives rise to a potential which BKM call the 'shape potential'. The topology is characterised by two types of stationary points, maxima and saddle points, which form an undulating plateau, intersperse with sharp, infinitely deep potential wells. This structure exhibits the fact that the complexity of an N-body system generally has a minima but no upper bound.

Equipped with this topology, we can choose any point in shape space and run the equations of motion in both directions. The dissipative structure of the laws means that shape momentum is depleted. The motion will evolve into a falling orbit towards the potential wells, and the system cannot escape this well due to the lost momentum. (Barbour et al., 2013, p.22)

This behaviour of inevitably 'falling' into a potential well of higher complexity due to dissipation is roughly demonstrated in Figure 4.

Against Zeh-like concerns about dependence on initial conditions, recall Janus curves are generic; the set of solutions which do not have Janus points is zero.

A more prescient concern is that of Gibbons and Ellis (2014, p.8), who argue that the system being dissipative is a product of a gauge choice, and not fundamental to the dynamics. They write:

> One can make the system appear to be dissipative by a choice of time

> parameter non-linearly related to t, for example $\tau = \log t$. However this is an artefact of an unphysical choice of the time parameter (on the same basis, the simple harmonic oscillator will also appear to be dissipative), and one would make the system appear to be dissipative in the opposite direction of time if one instead chose $\tau = -\log t$[14]

Of course, a typical Newtonian system's behaviour is locally deterministic, thus $\mathcal{J}_C$ is certainly not a result of a gauge choice. What the Ellis and Gibbons criticism undermines is our ability to explain $\mathcal{J}_C$ by appeal to dissipation, i.e. the target is the proposed solution to the modest problem.

BKM respond by noting that adherence to their philosophical principles, the removal of external space and time, and any dimensionful quantities, requires the time transformations carried out. The use of dilitational momentum as a clock removes external time, and taking the logarithm makes the dynamics autonomous.

The argument for the sign of the time parameter is then simply that "a 'mere' reversal of its direction would make the Universe become less complex with time." (Barbour et al., 2013, p.24) This response is confusing as it would seem that BKM choose the sign in order to recover the empirical fact of complexity increases with time. I don't believe this is the correct interpretation.

Rather, one should note that, although anti-dissipative dynamics invert the evolution of complexity when *computed*, our philosophical stance implies that $\mathcal{J}_C$ exist as entire atemporal paths. Hence, this makes absolutely no difference to the internal temporal mapping structure of the Universe. This is a promising defence, however the proposal for the modest project is still undermined. If the explanans contain a statement about the system being controlled by a potential and dissipative dynamics, if anti-dissipative dynamics produce an equally valid account for the explanandum, then the explanans don't seem to be doing the intended job.

This problem is easily resolved: it is convenient but imprecise to speak of dissipative dynamics as dissipation presupposes a temporal direction. Rather, the explanan should be the non-conservative structure of equations of motion. Thus formulated, it becomes irrelavant what sign the time parameter has, and thus whether the dynamics are dissipative or anti-dissipative, as the internal relational structure is unchanged, and the gradient is explained by the non-conservative laws.

3.4.2 Measure Focusing and Scale

In chapter 11 of (Barbour, 2020), Barbour uses Liouville's theorem and scale in order to explain $\mathcal{J}_C$.

As a brief sketch, consider shape space with scale re-introduced, the *extended representation*: each point is characterised by shape coordinates and a scale parameter. A volume in shape space, evolved according to Liouville's theorem in both temporal directions, will spread out increasingly over the extended representation's scale degree of freedom. Liouville's theorem ensures that the volume of the phase space region

[14]The reader should note that the τ here is not the same as that used as the internal clock above, given by the dilitational momentum.

remains constant, and thus the area taken up by the shape degree's of freedom must reduce.

Observers only have access to the *reduced representation*, shape space without scale re-added, on which the projected area will be shrinking. Moreover, the shape potential attracts the system toward higher complexity shapes. BKM interpret this as a statement that the law of the Universe necessarily attracts all points in shape space to higher complexity shapes, and measures focus as shape degrees of freedom are lost to scale, which locks the system into the high complexity regions of shape space.

This argument is troubling because the scale degree of freedom has become explanatorily relevant, which runs hard against the Machian philosophy the BKM model is built upon. One should not be concerned by this, however, because the same measure focusing argument gets a technical treatment in Sloan (2018), in which the measure focusing result arises without any reference to scale. That is, an objective scale is not necessary for the derivation of measure focusing, it arises purely from the non-conservative part of the equations of motion.

Therefore, the two different explanations, dissipative structure and measuring focusing, are equivalent and both a direct product of the non-symplectic form of the dynamical similarity symmetry, and the fact that $\frac{\partial \mathcal{H}^c}{\partial A}$ is non-zero for this particular system. For the N-body model $\frac{\partial \mathcal{H}^c}{\partial A}$ is a dimensionless constant proportional to the dilatational momentum.[15]

3.4.3 Difficulties for the Modest Project

The most prescient difficulty facing the modest project is to show that the local temporal asymmetry derived is an observed temporal gradient.

The complexity gradient doesn't seem well equipped to handle the varying levels of coarse-graining used in our temporal history. The early Universe was homogeneous of radiation and fundamental particles whilst the later Universe's structure consists of stars, galaxies and galaxies clusters. There is a clear change of level of the description at different moments in history. In the Newtonian model we can think of this as the fact that the sort of particles existent near the big bang would not be those we describe in Kepler pairs in the late stage evolution on either side of $\mathcal{J}_C$. Therefore, the description of complexity in terms of large cosmic structures is appealing but may not make sense near $\mathcal{J}_P$. Conversely, although the homogeneity of the early Universe may make sense at $\mathcal{J}_P$, that these early Universe entities are best described by Kepler pairs in the late stage evolution on either side of $\mathcal{J}_C$ is doubtful.

Sharpening this, the near the big bang behaviour of the Universe is well described by classical gravitational field theory coupled to scalar field perturbations. It is not clearly how complexity would apply to this sort of physical system at all, let alone persist into coarse grained systems in later time evolution. These concerns then are intensified when the quantum nature of a system is incorporated. The complexity gradient therefore only seems to have limited empirical support.

Also necessary for BKM is to defend the claim that $E = \mathbf{J} = \mathbf{L} = 0$. If a system

[15]It is noteworthy that measure focusing has the same implicit directionality as dissipation, which must be treated carefully in explanations of temporal and atemporal phenomena.

in the extended representation has a constant of motion that is scaled by the dynamical similarity transformation, then there will be an internal clock with respect to which the dynamics in the reduced representation are conservative, and thus no measure focusing occurs. An example of this is given in Sloan and Gryb (2021), in a 2-body Kepler model, in which one body is idealised as a perfect sphere of finite radius. This radius is modelled as dynamically isolated from the rest of the system and so is a constant of motion, but it is a length so scales under dynamical similarity. This example leads to the usual equations of motion for the Kepler system, but with lengths given in terms of the body radius, and durations given in terms of the sidereal day.

Therefore, BKM require there to be no constants of motion which scale with dynamical similarity. To achieve this one must first note the standard result that there is no gravitational screening, hence no potential length or time measure can ever be perfectly dynamically isolated from the rest of the Universe. In the Keplerian case, the radius of the sphere and its spin angular velocity will both vary under gravitational forces.

However, no gravitational screening results do not effect the fact that energy, translational momentum and angular momentum are conserved, so these will be constants of motion. Consequently, the BKM model requires $E = \mathbf{J} = \mathbf{L} = 0$, so that they do not scale with dynamical similarity transformations, and are therefore invariants.

The arguments for this condition are crucial, and should be independently investigated in the literature, as they are decisively non-trivial.

A less technical concern is regarding what counts as an explanation. When asked: what explains the Janus curve behaviour in the N-body model? the above discussion above shows that BKM can, and must, point to the equations of motion and indicate the non-conservative structure of the dynamics. However, let us compare this explanation with that of other dissipative phenomena. Suppose we are asked to explain why a ball rolling on a table slows down. We could point to the equations of motion and, indicating their non-conservative structure, subsume the phenomena under the laws dictating their behaviour.

I take it as plausible, however, that this strategy misses something crucially important in the explanation. What one seeks to do in explaining a phenomena is to explain, using terminology from Railton (1978), the “mechanism(s) at work”.[16] For a slowing ball, this would amount to an explanation of friction, of the electromagnetic interaction between the ball and the table, which cause the ball’s kinetic energy to be transformed into thermal energy, slowing it down.

This sort of mechanistic explanation resembles what is offered in Barbour (2020), in the sketched account offered above, where shape degrees of freedom are lost to scale, inducing measure focusing, equivalent to dissipation. However, scale cannot be explanatorily powerful if it is banned from our ontology. What then explains the non-conservative dynamics, beyond simply pointing the the equations of motion?

I believe this difficulty is likely surmountable, but I cannot see a clear response to it in the current literature.

[16] I do not endorse Railton’s account of explanation wholesale, but believe this appeal to mechanism is fruitful.

3.5 Difficulties for the Ambitious Project

The ambitious project is, in its very nature, gradiential. A particular research programme may have little success, or some success, or significant success in answering its challenge. I wish to argue for two claims; I) in order for BKM to claim success in tackling the problem of the arrow of time, they require a good degree of success in the ambitious project, and II) they do not attain this good degree of success. I unpack a 'good level of success' below.

In response to claim I, a natural position for BKM to take is that the fundamental challenge of the problem of the arrow of time is to derive temporally asymmetric phenomena from symmetric physics. Thus BKM, *contra* claim I, in deriving their complexity gradient have put the core of the problem of the arrow of time behind them.

However, in addition to the derivation of an empirically instantiated temporal asymmetry, another requirement must be satisfied: the derivation must emerge from a research programme which has enough strength for the derivation to be the chosen explanans for the empirical explanandum.

Unpacking this, consider some phenomena which requires explanation. Two research programmes both offer explanations of the phenomena, but only one is true. We distinguish the true explanation, not by which explanation is best, but rather which research programme is preferred. Of course, the individual explanations themselves will factor into the process of theory choice, but so do other considerations.

In this case, the phenomena that is derived is the cosmological homogeneity gradient, and two research programmes which purport to explain this are the BKM model and standard inflationary cosmology evolving into a late stage universe in which gravity dominates. It is indubitable that in a Newtonian N-body model, the behaviour described by the BKM model is confirmed. But that does not imply that the target explanandum, the cosmological homogeneity gradient, is to be explained by the results of the BKM model. Of course, the current mainstream explanation has serious problems, the problem of the arrow of time being a prime example. However, BKM need a sufficiently strong set of explananda to be accepted as *the* explanation of the local temporal asymmetry in question and thus solve the problem of the arrow of time. Thus, I claim, a good degree of success in the ambitious project is needed for the BKM model, and this is not achieved.

For a good degree of success, one would require that out of the BKM model emerges explanations of how and why other arrows of time, at least (some of) those which trouble us most, align with the complexity arrow. The BKM model is simply not furnished with such explanations.

Moreover, there are distinct problems for such explanations. In Barbour (2020, pg 153n), a Michael Berry comment is included, to the effect that complexity in a cosmological setting does not imply complexity in a planetary or anthropological sense. Within the complexity gradient it seems perfectly coherent that in isolated solar systems, on very large timescales, life would form in one of these systems which could experience time in the direction of decreasing global complexity. Locally the behaviour of the system is very similar to that in our solar system and galaxy, with stable, long lasting orbits; changing the orientation on $\mathcal{J}_C$ would seem to amount, over reasonable timescales, to simply reversing the direction of orbit of the Kepler pair.

It is important to stress that the cosmological clustering arrow is very far removed from our usual experience of time. Psychological, causal, epistemic, thermodynamic, radiative and quantum collapse arrows may all be considered prior in our experience of temporal asymmetry to the cosmological arrow. Thus, if one seeks to solve the problem of the arrow of time with reference to the cosmological arrow, then one is required to show how at least a fair portion of these supervene on the cosmological arrow. Otherwise, this more prescient phenomena will have a different explanation, and the problem has not be felled.

Is the SM-with-PH model not also require success in the ambitious project? Yes, indeed. This is why, for example, Albert (2000) not only attempts to recover the thermodynamic and cosmological arrows (chapter 4) but also the epistemic and causal arrows (chapter 6) (and SM arrow is largely thought to also underwrite the radiative arrow). BKM can call to their defence an intuition pump. The clustering of particles in astrophysical systems gives rise to stellar births. New stars emit radiation, and thus create the conditions on orbiting planets for low entropy states, and thus a local past hypothesis is therefore implemented and a local 2nd law recovered. Thus the homogeneity gradient produces these other arrows. However, this is an intuition pump, and not a derivation within the BKM model. Some of the difficulties faced by an attempt to derived the 2nd law within self-gravitating systems will be discussed in section 4.3.

One may claim that such questions should not be levied at the physicist. They likely require advances in our understanding of human psychology and neuroscience, in order to understand the physical mechanisms which are part of the composition of our mental temporal phenomena. But even if this is so, the SM-with-PH model describes clear asymmetric phenomena at a scale pertinent to us as agents. That is to say, SM reasonably purports to be theoretically relevant to our human psychology and physiology, and therefore the supervenience relationship between the asymmetry structure in SM, typicality, and our experience of time is well motivated. These claims are weaker for the BKM model.

It is tempting to go beyond what BKM actually achieve (roughly, the demonstration of the emergence of clustering, with Kepler pairs, in an gravitational N-body model) and assume that this will imply the emergence of all other local arrows of time such as the thermodynamic arrows and the experiential arrows, but this has not been proven.

BKM acknowledge the need to show how all other arrows map to their complexity arrow (pg 161 Barbour, 2020), but what I wish to emphasise is that without this ambitious proposal, the research programme is weakened, and if the research programme is not sufficiently strong, then the problem of the arrow of time is not solved.

4 SM-BKM Proposal for the Ambitious Project

BKM have argue that their model exhibits a type of record formation and a measure of typicality on Universes. I sketch these proposals in section 4.1 and section 4.2 respectively, and argue that although they improve the standing of the research programme, they do not contribute to an ambitious solution, so don't help solve the problem of the arrow of time. I will then suggest a line a research which will be very helpful to the proposal, and discuss the challenges it faces.

4.1 Branching and Emergent Structure

Away from $\mathcal{J}_P$, the path through shape space moves towards 'branching' structures, which maximise complexity by creating tightly bound subsystems far from other matter. Barbour et al. (2014) claim these resemble our familiar cosmic sub-system structures of star systems, galaxies and galaxy clusters.

Although we assumed $E = \mathbf{J} = \mathbf{L} = 0$, subsystems that are distilled can have non-zero values of these conserved quantities. As the branching occurs, the systems become increasingly stable and 'locked into' these values, with fluctuations from the conserved total becoming smaller and smaller. Similarly, the Kepler pairs that form become increasingly stable rods and clocks, with orbital period and axes length fluctuating less and less. For Barbour et al. (2013) this describes the information stored in a subsystem increasing: more 'data' is 'stored'.[17] Using these stable orbital systems as physical measuring devices, Newtonian time and space can emerge, with scale defined as ratio to these functional absolutes.

In Barbour et al. (2014), the authors state "our evidence for the passage of time is in locally stored records (including memory), which by agreeing with each other, lead us to believe in a dynamical law that has generated them from a unique past". This suggestably is an approach to the ambitious project. Our mental states are (very roughly) characterised by records being made and stored as memories. Moving away from mental states, BKM's subsystems become the memory devices; as their inherent structure increases due to the increasingly stable isolation, the past is the direction of decreasing information structure, the future is the direction of increasing information structure.

This argument, however, is not fully worked through. What exactly is the causal link between information being stored in an orbiting system and complex life having internal memory storage? Of course it can't be a direct supervenience relationship as then a astronomical event will disturb our experience of the direction of time, which is absurd. If the stabalisation of Kepler pairs is indeed meant to imply a direction of record formation within that Kepler pair, then such a causal relationship must be defended explicitly.

This will be further complicated by the need to recover the second law of thermodynamics within these subsystems. In non-gravitationally dominated systems, an early time ordered state needs to evolve into more disordered states. This is superficially in tension with a proposal to have the arrow of information storage in a Kepler pair supervene on some more anthropological notion of information. Such suggestions are too speculative to ascertain whether an arrow of cosmic information storage could help solve the ambitious project; it is an interesting proposal, but with little meat on the bone. Regardless, the proposal I offer in section 4.3 is more promising.

4.2 Typicality and Entaxy

Entropy is not well defined over a universe, and as such SM statements of the typicality of universal states break down. Barbour et al. (2015) offer an alternative description of

[17] Our ability to define these structures as isolated, and such subject to entropic description is a debatable point which Albert (2000) challenges. We accept it unchallenged for the present discussion.

typicality. Rather than defining a measure over states, BKM follow Gibbons, Hawking, and Stewart (1987) in defining a measure over models. The shape coordinate and momentum at $\mathcal{J}_P$ is the 'mid-point data' and defines $\mathcal{J}_C$ entirely and uniquely. Therefore the typicality of a universe is a measure of the typicality of $\mathcal{J}_P$.

The measure in Gibbons et al. (1987) is infinite, which of course was a problem raised for SM measures in section 2.4. However, BKM produce a finite measure by using the dynamical similarity of the mid-point data. To use the measure as a quantification of typicality, BKM use the principle of indifference. The principle of indifference received attacks from Gryb in section 2.4 and such it is not immediately clear to me why BKM's measure is immune from these same complaints.

Using the BKM measure, the complexity of a shape defines its typicality, with homogeneous shapes being more vastly more typical (see Figure 4 in Barbour et al. (2015, pg. 21)). Therefore, the considerable majority of Janus curves, will have a $\mathcal{J}_P$ very close to the maximally homogeneous state, and will decrease in inhomogeneity away from $\mathcal{J}_P$. This movement away from typicality is, as described above, driven by the the structure of the dynamics, unlike Boltzmann entropy which statistics drive toward typicality (see Frigg (2009) for criticism of such claims). BKM term this entropy-like measure of typicality *entaxy*. Entaxy's typical states are maximally homogeneous, and such entaxy decreases away from $\mathcal{J}_P$.[18]

Supposing the BKM measure holds up to rigour, it adds further appeal to the research programme, but it does not help offer a solution to the ambitious project. Janus curves exist as entire atemporal paths in correct accordance with the C-theory of time. There is no *a priori* reason to suppose that experienced time has to be aligned in any way with typicality given our philosophical assumptions. One would need to argue that the typicality of shapes along the Janus curve in some way relate to other arrows of time, and there is no good reason to think that any such argument is forthcoming. Typicality may well play a role in explanations in statistical mechanics, because it has explanatory power (if we have the entropy gradient because of the typicality gradient). But typicality is not causal in the BKM model (except, perhaps, in selecting an entire universe).

Given the modest solution, entaxy measures are a conceptually and empirically useful description. However, entaxy alone is impotent for solving the ambitious problem.

4.3 SM and BKM

In this section I state my proposal which identifies SM within the BKM model, and using this internal structure, suggests progress on the ambitious project. This proposal is anticipated in Barbour et al. (2015), however I here advance the work by contextualising it within the problem of the arrow of time, emphasising its significance, and highlighting the challenges faced. This is not a fully developed mathematical model, rather it is a suggestion for how the BKM model could develop to make real progress in the problem of the arrow of time.

First, let us consider how this would work in principle. Away from $\mathcal{J}_P$, in both temporal directions, clusters form as subsystems. Within these subsystems it becomes

[18]Barbour et al. (2015) claim that, through no-fine tuning arguments, this provides falsifiable predictions.

possible to define an entropy. The subsystem would need to be sufficiently stable to be treated as an isolated, entropic sub-system, but also needs to be in a state that is out of statistical equilibrium. The sub-system would then have to evolve monotonically (or very approximately so) towards higher entropy states.

Putatively, this avoids the difficulties presented in 2.4. We don't need to define a measure over the Universe in this case, so we can then have an evidenced PH within the subsystem in which a finite measure is used. In section 2.4 I argued that even an evidenced PH fails as a solution due to the unexplained asymmetric contingent fact: the atypical initial condition. This symmetric explanation however is provided by $\mathcal{J}_C$, and the PH in this case is just the low entropy state of the subsystem emergent on one side of $\mathcal{J}_C$.

Let us turn to the details of this proposal. The results of this section are largely from Padmanabhan (1990, 2008). The emergence of stable subsystems seems to be a reliable predication of the BKM model, as discussed in section 4.1. As these subsystems collapse under their own gravity, potential energy will be converted into kinetic energy, and the system will thermalise, with high energy particles escaping, leaving a stably bound system in virial equilibrium.[19] The subsystems will virialize, because the moment of inertia grows at most like $t^{4/3}$, and a necessary and sufficient condition for a 'sharp' form of the virial theorem (Pollard, 1966) is $\lim_{t\to\infty} t^{-2} I = 0$, which holds for the maximum $I(t)$.

The emergence of Newtonian distance and time by using stable rods and clocks within subsystems will enable us to use units for state functions, which could be E, $\mathbf{J}$ and $\mathbf{L}$.

The first challenge is to define an entropy for the subsystem. This will have to be the microcanonical entropy because self-gravitating systems in virial equilibrium have a negative heat capacity, and the canonical distribution can only have positive specific heat.

The microcanonical entropy for a system of energy E, up to a positive additive constant, is $S = \log g(E)$, where $g(E)$ is the volume of a constant energy surface, $H(\{p_i, q_i\}) = E$, in phase space. In order to define this entropy (i.e. have it not be infinite for all energies), the constant energy surface must be compact, and this requires both a short distance and large distance cutoff. The short distance cutoff prevents divergences due to the $r_{12} \to 0$ behaviour of the potential. The large distance cutoff is necessary because a particle placed at an arbitrary distance with appropriate kinetic energy can leave the energy constant, but increase entropy.

Recall in section 2.4 it was a problem for SM-with-PH that these cut-offs were needed to define an equilibrium state. In general, self-gravitating systems have no maximum entropy, instead they have increasingly large entropy until quantum mechanical or general relativistic effects become significant. Let us, for now, ignore this problem, assuming that the very late stage evolution of a subsystem reaches some system in thermodynamic equilibrium, such as a black hole, and concentrate on the dynamical evolution toward statistical equilibrium.

$S = \log g(E)$ defines the Boltzmann entropy of an equilibrium state (which only

[19]The virial theorem states that the time average of total kinetic energy is equal to minus half the time average of the potential energy. Understand this here as the system being stable.

exists with the cutoffs). However, it is possible to define an entropy measure out of equilibrium by discarding the full $6N$-dimensional description of the subsystem. This still requires a large length scale cutoff, which BKM plausibly justify by claiming the potential well of the subsystem is an effective box. We also assume this effective box operationally bounds the system. Within this box we take a mean field approximation, starting with the one-particle distribution function

$$f(x_1, p_1, t) = \int f_N(x_1, p_1; ...x_N, p_N) d^3x_1 d^3p_1 ... d^3x_N d^3p_N \tag{18}$$

where f_N is the full distribution function. Functions of f can only describe macrostates accurately if one assumes that the granularity and, more provocatively, the correlations between particles, are not significant. The mean field distribution then takes a coarse-graining by assuming the functions in Equation (18) are smooth. This is not true in an exact description, as the functions would be composed of delta functions. A Gibbs entropy[20] called the mean field entropy can then be given for the system as

$$S = -\int f \ln f d^3x d^3p \tag{19}$$

The entropy will be unbounded above, but one can investigate its temporal evolution and local stationary points. This function does indeed have local maxima of entropy, and so there exist metastable configurations, which could serve as equilibrium states for practical purposes, before UV effects dominate.

It follows from the 2nd law of thermodynamics that the mean field entropy will not decrease, and will generically increase, given a configuration which is not in a local maxima of entropy. However, one would hope to derive this conclusion from the dynamics of the N-body model.

Here further difficulties arise. By observation we know astrophysical systems reach statistical equilibrium (from observation of constant velocity distribution functions). Self-gravitating systems relax into equillibrium through two different processes, collisional relaxation and violent relaxation. Collisional relaxation is better understood and in Padmanabhan (1990) it is shown (with a critical commentary on the assumptions involved, and the conceptual challenges) that, under the Fokker-Planck description of collisional evolution, the 2nd law of thermodynamics for the mean field entropy will be obeyed. However, the collisional relaxation time for several astrophysical systems is longer than the age of the Universe. Hence, violent relaxation will be the dominant process during the dynamical evolution toward equilibrium.

In Barbour et al. (2015) it is claimed that in Padmanabhan (1990) it is proven that Equation (19) will increase. But this is only proven for collisional relaxation. It is reasonable to assume that violent relaxation obeys the 2nd law of thermodynamics, but this is not proven in Padmanabhan (1990). One seeks to recover the 2nd law without assuming its validity, but this has not yet been achieved in the literature on the BKM model.

Finally, let us assume that we can derived the 2nd law within the BKM model; it is necessary to show that all subsystems are in non-equilibrium configurations closer to

[20]Because the one particle distribution function is best understood as an un-normalized probability function for a particle to occupy a particular phase space region.

the Janus point for significant periods of time. Essentially, *we require a derivation of what the past hypothesis assumes*: a low entropy boundary condition on all subsystems, in the temporal region of the subsystem's evolution closest to the Janus point. This is the most challenging condition.

An isolated black hole or neutron star is in thermodynamic equilibrium. Once one has the capacity to define the entropy of an astrophysical system, such an equilibrium state will be overwhelmingly typical of the state. Why, then, do subsystems not emerge as black holes and neutron stars? The proponent of the BKM model could potentially appeal to some properties of $\mathcal{J}_P$, and argue that these properties are generic, upon the assumption of $E = \mathbf{J} = \mathbf{L} = 0$ and using entaxy. However, this has two problems. Firstly, one would need to explain the $E = \mathbf{J} = \mathbf{L} = 0$ condition, which is a non-trivial task. Secondly, one notes that *the Universe is not an N-body model*. The past hypothesis equivalent in the BKM model will depend vitally on near $\mathcal{J}_P$ behaviour. Although the BKM model has been useful for describing late Universe cosmological structure, it is not a realistic cosmological model, and this failure is especially acute in the early Universe. And so the near $\mathcal{J}_P$ behaviour of astronomical systems will depend *vitally* on early universe physics, which requires general relativistic and quantum mechanical treatment. That is, the past hypothesis cannot be derived within the BKM model, as it requires claims about the state of systems as they emerge from the Janus point, which will not be well described by N-body behaviour.[21]

In section 5 I discuss real cosmological models which show that the big bang does plausibly have a Janus point structure, but it is a further significant task to derive the emergence of self-gravitating, out-of-equilibrium sub-systems from these real cosmological models.

The emergence of a supervenience relationship between the complexity arrow and the 2nd law of thermodynamics would be a serious success for the BKM model, but this achievement, as always, must be derived and not assumed. It is, I take it, fertile ground for further study.

5 Real Cosmological Models

Work to apply dynamical similarity to real cosmological models, which could be reasonably claimed to correctly describe our Universe, is presented in two excellent papers, Koslowski et al. (2018) and Sloan (2019). These papers give dynamical similarity treatments of Bianchi cosmologies and FLRW cosmology with scalar fields, and remarkably indicate that the big bang singularity is in fact a type-2 Janus point, and the system of dynamical similarity invariants can be smoothly continued through the big bang, evidencing the Janus point structure of the Universe. Together these two papers are very significant contributions to the research programme, which in turn strengthens the claim for the solution to the problem of the arrow of time, as discussed in the research programme treatment of the problem in section 3.5. However, they do not alleviate the concerns about the ambitious project.

Dynamical similarity requires a global scaling of a time-coordinate, which typically is not available in general relativity due to the relativity of simultaneity. However, if a

[21] I am grateful to an anonymous referee for emphasising the importance of this point.

spacetime is homogeneous, then one will be able to give a global foliation of spacetime, parameterised by 'cosmic time'(see Smeenk (2013) and Read and Qureshi-Hurst (2020) for discussion). Moreover, our Universe is described, up to a good degree of accuracy, by such a symmetric cosmology, called FLRW cosmology, which assumes a flat, spatially homogeneous and isotropic metric, coupled to scalar matter fields and an arbitrary potential. Sloan (2019) applies dynamical similarity to such a cosmology.

The usual singularity reached in the FLRW metric is identifiable when the components of the Hamiltonian vector field are not Lipschitz continuous. This arises in the case of FLRW either because some variable blows up to infinity, or the volume vanishes, $v = 0$. The former problem is solved when the dynamics are expressed in terms of relational quantities in shape space, because it involves a compactification which tames the infinities. The latter problem is solved because, when one moves to the autonomous system of dynamically similar invariants, the contact Hamiltonian has no v dependence. As one would expect from a scale invariant description, the overall scale is redundant to the description, and thus the $v = 0$ singularity does not arise in the relational description.

As in the N-body model, we find that the dynamics are 'frictional'. The Hubble parameter, which measures the rate of 'expansion' of the Universe, plays the role of the dilitational momentum in the FLRW model. Whereas the diliational momentum provides a type-1 Janus point, the Hubble parameter is infinite at $v = 0$ in the extended representation, and decreases monotonically in both directions and is asymptotically constant. This is a type-2 Janus point, where the measure blows up at the Janus point rather than reaches a minimum, but irregardless there exists a focusing gradient either side the point.

More general results are derived in Koslowski et al. (2018). Homogeneous, anisotropic spacetimes are categorised as Bianchi models. Importantly, the BKL conjecture indicates that near singularities, the behaviour of GR solutions is well described by such models. More specifically, Koslowski et al. (2018) argue that there is very good reason to believe that a Bianchi IX model with a massless scalar field is a good description of the near singularity behaviour of GR solutions. Again, the volume v falls out of the description and the dynamical equations are shown to be Lipschitz continuous at $v = 0$.

These results are very impressive, but let us now consider how they interact with what has been said in the rest of this paper. Of course, it does a significant amount of work to show that the Janus curve structure is retained in cosmological models far more realistic than the N-body model. Moreover, the results keep the structure of the modest solution intact; an local temporal asymmetry is derived from the time symmetric models of general relativity.

The real cosmological models presented here do not recover the clustering behaviour of BKM, the complexity measure is never used. Instead, they recover the apparent expansion of the Universe via red-shift (Sloan & Gryb, 2021), and thus there is asymmetric temporal behaviour. This has the clear advantage of strengthening the research programme, which does give more weight to the Janus point solution to the problem of the arrow of time.

On the other hand, these results bring into starker focus the vital importance of the ambitious project. Whereas the complexity gradient resembles astrophysical clustering, which via intuition pump we can believe produces planets, stars, solar systems and temporally directed life as we know it, the red-shifting expansion of our Universe seems even more weakly connected to the other arrows of time. This is a very promising research programme, but the problem of the arrow of time may remain a thorn in the side of the physics community for some years to come.

6 Conclusion

One has good reason to be pessimistic about solving the problem of the arrow of time by appeal to SM-with-PH. Applying dynamical similarity to systems taken to describe the universe offers a promising alternative research programme. The BKM model is suggestive of a solution to the modest project but certain outstanding issues must be dealt with. Moreover, BKM must make progress in the ambitious project, for their derivation of temporal asymmetry to take the mantle of *the* derivation of temporal asymmetry. For now, this is an exciting research programme, and it offers great prospects for a solution to the problem.

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