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On Certainty, Change, and "Mathematical Hinges"

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Abstract Coliva (2020a) asks, "Are there mathematical hinges?" I argue here, against Coliva's own conclusion, that there are. I further claim that this affirmative answer allows a case to be made for taking the concept of a hinge to be a useful and general-purpose tool for studying mathematical practice in its real complexity. Seeing how Wittgenstein can, and why he would, countenance mathematical hinges additionally gives us a deeper understanding of some of his latest thoughts on mathematics. For example, a view of how mathematical hinges relate to Wittgenstein's well-known riverbed analogy enables us to see how his way of thinking about mathematics can account nicely for a "dynamics of change" within mathematical research—something his philosophy of mathematics has been accused of missing (e.g., by Ackermann (1988) and Wilson (2006)). Finally, the perspective on mathematical hinges ultimately arrived at will be seen to provide us with illuminating examples of how our conceptual choices and theories can be ungrounded but nevertheless the right ones (in a sense to be explained).

Keywords Wittgenstein · *On Certainty* · mathematical innovation · hinge epistemology · philosophy of mathematics

1 Introduction

The peculiar status and function of mathematical propositions is never long out of mind for the later Wittgenstein.[1] This preoccupation persists through the notebooks containing his final writings more directly concerned with the peculiar status and function of certain propositions having an "empirical form." In particular, *On Certainty*, a selection of remarks taken from these last notebooks, is sprinkled with mathematical examples throughout.

Buoyed by the surge of interest in so-called "hinge epistemologies" taking inspiration from *On Certainty*,[2] interest in a deeper understanding of the precise role played by mathematical propositions in the work has also recently emerged: are the statements of mathematics themselves hinges? if not *all* of them are, are *some* of them? are mathematical propositions simply useful objects of comparison for Wittgenstein's main object of investigation? etc. In what follows, I'd like to float some answers to these questions with the aim of showing that the debate over them to date has missed some important alternatives and that by taking some of these overlooked alternatives seriously we can arrive at the beginnings of a broadly attractive Wittgensteinian "hinge philosophy of mathematics."[3,4] By placing the *shifting* aspect of Wittgen-

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[1] Given his understanding of mathematics as "twisting and turning about within" and creating rules (Wittgenstein 1956/1983: I.§§165-168), discussion of mathematical *propositions* is sometimes deemed to be inappropriate in relation to Wittgenstein's thought. See especially (Moyal-Sharrock 2004: Ch. 2) for this judgment in the present context. As a number of commentators have pointed out however (see, e.g., (Baker and Hacker 2009: Ch. VII.4-6) and (Coliva 2010: 80-82)), by the time of the *Investigations* Wittgenstein seems to view *proposition* itself as a family-resemblance concept (*cf.* (Wittgenstein 1953/2009: §136) and (Wittgenstein 1969: §320)). His adoption of this perspective makes the objections to proposition-talk raised by Moyal-Sharrock and others less pressing, at least for present purposes. See Travis (2012) for a helpful discussion of the different understandings of 'proposition' Wittgenstein endorsed over the course of his career.

[2] See Moyal-Sharrock (2004), Coliva (2015), and Pritchard (2016) for examples of the hinge approach to epistemological problems in action. Coliva and Moyal-Sharrock (2017) is a recent collection of essays on the subject.

[3] This phrase comes from (Coliva 2020a: 365), which will be discussed in some detail presently.

[4] For the purposes of this paper, I'll be consistently using the now-firmly-established language of "hinges" and "hinge propositions." Whether or not the general hinge approach to *On Certainty* is the most effective way to get a handle of this swath of Wittgenstein's thinking is a larger question not taken up here.

stein's well-known river-bed analogy at the center of the discussion, it's possible to see how Wittgenstein can distinguish between hinges and non-hinges even if mathematical propositions are all essentially rule-like—apparently a necessary first step if anything like a Wittgensteinian hinge philosophy of mathematics is to get underway. This shift in emphasis also allows for a reasonable Wittgensteinian account of the "dynamics of change" in mathematical research—a dynamics some, e.g., (Ackermann 1988: 216-217), suggest Wittgenstein cannot account for.[5,6] The view achieved here, then, can be used to illustrate the richness and applicability—especially in its very latest form—of Wittgenstein's thought about mathematics and its practice. It also allows us to see Wittgenstein's claims in *On Certainty* about the ungroundedness of some of our most important commitments at work in some particularly nice cases where an idea of correctness nevertheless still finds a place.[7]

The plan to show all of this is as follows. In Section 2, I'll set out the current state of the debate over the role of the mathematical examples in *On Certainty* and address the question of whether or not (some) mathematical propositions are hinges. I'll aim to show that the arguments of Coliva (2020a) don't establish that hinges have no place in Wittgenstein's philosophy of mathematics (while his understanding of mathematical statements as being rule-like is held fixed).[8] Having made the case that Wittgenstein can distinguish between hinges and non-hinges in a rule-based conception of mathematics, Section 3 will look for some examples of mathematical hinges that go beyond the very simple propositions like $2 \times 2 = 4$ and $12 \times 12 = 144$ that play a prominent role in *On Certainty* as well as aim to make room for the concept of a hinge to play a role in not-strictly-Wittgensteinian visions of mathematical practice. The existence of more substantive and general examples of mathematical hinges will bolster the case for taking the hinge concept to be a useful tool for studying mathematical practice in its real complexity. Next, using an understanding of the river-bed analogy elaborated in Section 4, Section 5 will put forward a general characterization of hinges in mathematics and an account of the special role they can play in guiding research and change in the field. Finally, in Section 6, I'll provide an illustration of how Wittgenstein's understanding of the motivation for change works in a particular case before closing with a comment about the pertinence of Wittgenstein's *On Certainty*-thinking to contemporary and actively-investigated questions in the philosophy of mathematics.

2 Are There Mathematical Hinges?

2.1 Characterizing Hinges in General

In order to answer the question posed in the section heading, we first need to settle on a general characterization of the "hinges" that play a central role in *On Certainty*. A further necessity is that this characterization be broadly acceptable to the main participants in the debate over the question so far: primarily Moyal-Sharrock (2004), Kusch (2016), and Coliva (2020a). The brief exposition of hinges given in (Coliva 2020a: 348-350), which glosses over some disagreements that aren't particularly relevant to the question at hand (e.g., whether hinges are *ineffable*),[9] fits the bill as well as any could, so I'll take it on board without significant change.[10]

For present purposes, the following are the four most important characteristics that make a proposition count as a hinge.[11]

1. The proposition plays a normative role (as opposed to a descriptive one). It does this either by contributing to the constitution of the meaning of (some of the) terms in the proposition (e.g., "*This* is a tree") or by establishing a framework within which evidence can have bearing and significance—hinges playing this role "mediate the most basic evidential connections" (Wright 1985: 463).[12]

[5] (Wilson 2006: 279, 566) also raises some worries about Wittgenstein's ability to understand the complexities of conceptual change in real-world mathematical cases. Wilson (2020) is more circumspect about this judgment.

[6] See Eriksen (2020) for another recent discussion of Wittgenstein on the "dynamics of change." Understanding the dynamics of change is of course also central to the large, on-going project of making sense of mathematical progress more generally. For a recent discussion of the state and goals of this project see Weisgerber (2022).

[7] See, e.g., (Wittgenstein 1969: §§110, 166) for the grounding metaphor.

[8] I add this qualification because Coliva considers how hinges might find a place in Wittgenstein's thought about mathematics both "on the *vulgata*," where he takes mathematical statements to behave like rules, and according to a view where he gives them a more traditional understanding (Coliva 2020a: 347). I take Shanker (1987) and (Baker and Hacker 2009: Ch. VII), among others, to establish that a rule-like/normative role for mathematical statements must be taken on board for a view to count as being Wittgensteinian, so aside from parts of Section 3 I'll only be speaking with the vulgar in what follows.

[9] See (Moyal-Sharrock 2004: 94-97) and (Kusch 2016: §6) for discussion of this question.

[10] For a more detailed account of hinges as understood by advocates of the so-called framework view of *On Certainty*, see Coliva (2015). Coliva (2010), McGinn (1989), Moyal-Sharrock (2004), Schönbaumsfeld (2016), and Wright (1985) all advocate versions of the framework reading of the text.

[11] Given that the project of the paper is simply to consider the prospects for a hinge-based philosophy of mathematics deriving from *On Certainty*, I'll not be trying to justify Wittgenstein's general approach to certainty or our special commitment to hinges in what follows.

[12] See, for example, (Wittgenstein 1969: §§57, 268, 308). *Cf.* (Moyal-Sharrock 2004: 85-87). (Coliva 2020a: 439n6) suggests that Kusch doesn't accept this normative role as being essential to hinges, but I take the following passage to put that conclusion in ques-

2. The proposition cannot be epistemically justified. It can't be epistemically justified either because the giving and weighing of evidence presupposes the acceptance of the proposition (e.g., the practice of giving historical evidence may presuppose the acceptance of "The world has existed for more than 10 minutes") or because the certainty we have in the proposition is already maximal in some sense (e.g., "Why shouldn't I test my *eyes* by checking whether they can see that I have a hand?").[13]
3. The proposition cannot be doubted. It may be indubitable either because there is nothing that does or could be taken to count against it ("everything speaks in its favor, nothing against") or because what we are able to regard as evidence depends on taking it for granted in the first place. A hinge's immunity to doubt apparently flows from the same source as its unjustifiability, so the remarks elaborating on (2) will apply here as well.[14]
4. The proposition is acquired through engagement with a community or practice of some kind and it generally comes packaged together with other propositions into an image of the world—or part of the world—we inhabit, in part, via the practice. The examples Wittgenstein provides of hinges upon which our justificatory practices turn don't seem to be propositions we have consciously chosen, but are rather ones we acquire through our upbringings and may eventually realize we can't do without in some sense.[15]

With this brief sketch of an account of hinges in place, the question of whether (some) mathematical statements are hinges can now be approached.

2.2 Coliva on Mathematical Hinges

The primary import of the question asking whether the statements of mathematics are hinges to date has been its bearing on the debated between "epistemic" vs. "non-epistemic" understandings of hinges.[16] But it becomes clear in Coliva (2020a) that the question has significant independent interest. By closely considering Coliva's argument that Wittgenstein has no room for hinges in his thinking about mathematics, I'll now aim to show that hinges actually can and do find an important place within Wittgenstein's work, even while we stick to his view of mathematical propositions as being rule-like. I'll also begin elaborating on the Wittgensteinian hinge philosophy of mathematics that suggests itself once these hinges are in place. This account will be expanded considerably by the work of Sections 3 and 5.

Since the main examples of mathematical propositions in *On Certainty* are simple statements of arithmetic, if these aren't—or if none of them are—hinges, the prospects for a *Wittgensteinian* hinge-based philosophy of mathematics would appear to be dim. With that in mind, I'll begin by considering whether Wittgenstein can reasonably consider examples like $2 \times 2 = 4$, $12 \times 12 = 144$, or $235 + 532 = 767$ to be "mathematical hinges."[17]

Coliva herself agrees that these sorts of arithmetical propositions do all have characteristics (1) and (4) from the list above: $2 \times 2 = 4$ is used normatively, e.g., when I count two apples in each of two boxes and conclude that I must now have four apples—if I count five all together, I'll judge that I've made a mistake; and $2 \times 2 = 4$ is something that I, along with my classmates, was repeatedly drilled on until it, along with its mates like $3 \times 3 = 9$ and $8 \times 8 = 64$, eventually stuck. The debate, therefore, turns on the status of the justifiability and doubtability of these sorts of mathematical propositions; i.e., on characteristics (2) and (3) from our list.

Although Wittgenstein doesn't believe that the statements of arithmetic are justified by empirical evidence, it does seem clear enough that he believes these mathematical statements are justifiable nonetheless.[18]

> [...] If I say "I know" in mathematics, then the justification for this is a proof.[19]

And if this is right, then *any* proven mathematical statement seems to be equally justified and "objectively" certain.[20] This conclusion poses two problems for the claim

tion: "McGinn and Moyal-Sharrock are right to stress this grammatical/linguistic role of certainties. But they pay too little attention to the various epistemic roles that certainties do also play" (Kusch 2016: 138).

[13] See, e.g., (Wittgenstein 1969: §§56, 231, 359, 454). *Cf.* (Moyal-Sharrock 2004: 75-80). (Kusch 2016: §9) argues that this characteristic isn't shared by all hinges. One of the reasons he makes this claim, however, is that he holds mathematical propositions to be hinges and, citing (Wittgenstein 1969: §563), counts proofs as evidence for them. Since the status of mathematical propositions as hinges is in question here, it seems worth keeping this item as part of our characterization of hinges at least for now. *Cf.* (Coliva 2020a: 439n6).

[14] See, for example, (Wittgenstein 1969: §§4, 87, 117). *Cf.* (Moyal-Sharrock 2004: 72-74). Kusch (2016) doesn't raise any special objections to this characteristic, but once one puts hinges in the realm of the justifiable, it seems unlikely that they can be spared from being doubtable as well.

[15] See, for example, (Wittgenstein 1969: §§93-97, 138, 162, 233). *Cf.* (Moyal-Sharrock 2004: 80-85). (Kusch 2016: 135) shows Kusch to be in agreement that this is a characteristic of hinges at least some of the time.

[16] *Cf.* (Kusch 2016: 121-122). See (Pritchard 2016: Chs. 3 and 4) for a thorough consideration of the merits of each of the two readings.

[17] *Cf.* (Moyal-Sharrock 2004: 119).

[18] There are subtle questions about what exactly justification is supposed to come to here raised by Wittgenstein's repeated characterization of mathematics as "akin both to what is arbitrary and to what is non-arbitrary" (Wittgenstein 1967: §358). (See, e.g., (Baker and Hacker 2009: Ch. VII.11-12) for discussion.) The main question under discussion here is addressable without fully engaging these other difficult questions though, so I'll set them aside wherever possible.

[19] (Wittgenstein 1969: §563)

[20] See (Coliva 2020a: 351) and (Kusch 2016: 128): "[A]ll mathematical sentences are certainties." ('Certainties' is Kusch's way of referring to hinges.) On the difference between 'subjective' and 'objective' certainty, see, e.g., (Wittgenstein 1969: 194).

that there are mathematical hinges: first, being justified apparently makes a statement not a hinge; and, second, if all mathematical statements are equally certain, there is no subclass of them to single out as especially "hinge-like."[21] It additionally looks unlikely that the statements of arithmetic could function as hinges given the characterization above since, in Coliva's estimation, we don't need to presuppose the truth of, say, $2 \times 2 = 4$ or $1,057 \times 216 = 228,312$ in order for the project of providing mathematical evidence and proof to get off the ground. If we were forced to accept such statements for that reason, we might be led to the conclusion that they are immune to doubt, but it seems as if we're not so forced and that these statements needn't be taken to be indubitable in the way hinges are meant to be as a result.[22]

The above considerations are at the heart of Coliva's argument that there aren't mathematical hinges.[23] And on one way of looking at proof in mathematics, perhaps a way held by Frege or Russell, Coliva's points about the justifiability or indubitability of $2 \times 2 = 4$ and $1,057 \times 216 = 228,312$ look to be correct. But given what we know about the later Wittgenstein's views about proof, there is room for doubt. Consider especially the following point Wittgenstein makes in Part III of his *Remarks on the Foundations of Mathematics*: "A shortened procedure tells me what *ought* to come out with the unshortened one. (Instead of the other way round.)"[24] The kind of case Wittgenstein is asking us to consider is one in which there is a purported proof in, say, the system of *Principia Mathematica* that is supposed to prove the translation there of $10^{100} = 10^{100} + 1$. To the question, "Does this proof show that $10^{100} = 10^{100} + 1$?" Wittgenstein offers up the natural reaction, "Of course not—$10^{100} \neq 10^{100} + 1$." That is, even if painstaking checks of the very long Russellian proof, repeatedly revealed no errors, the response of everyone considering the matter would be expected to be, "Well, there must be a mistake *somewhere*."[25] This is because we can see immediately that these are different numbers.[26] It's in this sense that the "shortened" procedure, can tell us what ought to come out from the longer one: facts like $10^{100} \neq 10^{100} + 1$ appear to play a foundational role with respect to the evaluation of certain complicated (purported) proofs.

These considerations relate very naturally to the examples considered earlier; i.e., $2 \times 2 = 4$ and $1,057 \times 216 = 228,312$. Although each proposition can be proved, just as we use $1 + 1 = 2$ to check the correctness of the *Principia* proof of $1 + 1 = 2$, the "occasionally useful" fact finally proved on page 83 of the second volume of the work,[27] we can immediately use $2 \times 2 = 4$ to check the outcome of any purported proof of the proposition or its negation we are presented with. $1,057 \times 216 = 228,312$ doesn't have this same status, however. Consider also the following familiar way of demonstrating that $1,057 \times 216 = 228,312$.[28]

$$\begin{array}{r} 1\,0\,5\,7 \\ \times \quad 2\,1\,6 \\ \hline 6\,3\,4\,2 \\ 1\,0\,5\,7 \\ 2\,1\,1\,4 \\ \hline 2\,2\,8\,3\,1\,2 \end{array}$$

Simple arithmetic propositions involving the addition of digits between 0 and 9 as well as the basic entries in a multiplication table are the tools used to justify and check the correctness of such a proof. That being the case, it looks like the means we have for justifying and checking proofs in our arithmetical practices in these slightly less simple cases—for giving and evaluating this kind of "mathematical evidence"—does depend on taking the collection of basic additions and multiplications as both unjustifiable (because completely certain) and indubitable. They are the ungrounded grounds upon which the rest of the practice rests.

Coliva does note the somewhat different roles played by $2 \times 2 = 4$ and $1,057 \times 216 = 228,312$, but doesn't think the difference is great enough to matter. That is, she roughly echoes Alonzo Church's sentiment that definitions are nothing but "concessions in practice to the shortness of human life and patience"[29] by suggesting that the rote memorization of $2 \times 2 = 4$ is simply a way of speeding up the process of our longer calculations.[30] And she further argues that the evident difference between the way we work with the two equations is nothing more than a reflection of the contingent fact that we're more familiar with the simpler one of the pair.[31] However, I take it that the comments in the previous two paragraphs show why Wittgenstein would disagree with the first claim given the functioning of our actual mathematical practices—we use, e.g., $6 \times 7 = 42$, to carry out and confirm computations like $1,057 \times 216 = 228,312$ and not the other way around, and that fact is built into

[21] See (Coliva 2020a: 347-348) for the later worry.

[22] See (Coliva 2020a: 352).

[23] Another important part of Coliva's case is her explanation of why the following passage doesn't immediately settle the question: "The mathematical proposition has, as it were officially, been given the stamp of incontestability. I.e.: "Dispute about other things; *this* is immovable—it is a hinge on which your dispute can turn" (Wittgenstein 1969: §655). (See (Coliva 2020a: 359-360).) I think she's correct that this passage on its own is not enough to put the debate to rest, so I don't rely on it here.

[24] (Wittgenstein 1956/1983: III.§18)

[25] That is, if the system's consistency isn't called into question in the process.

[26] Given that our system of naming numbers is set up appropriately. See Kim (2021) for a discussion of what an appropriate setup might look like.

[27] (Whitehead and Russell 1927: ∗110.643)

[28] *Cf.* (Mühlhölzer 2020: 193-195).

[29] (Church 1956: 76)

[30] See (Coliva 2020a: 352).

[31] See (Coliva 2020a: 352-354).

the particularities of this practice. Further, the second claim about familiarity seems to be largely beside the point in the context of *On Certainty*. Of course, in some sense, we could have learned $1,057 \times 216 = 228,312$ first and it could have been as familiar to us as any arithmetical statement,[32] but Wittgenstein's concerns in *On Certainty* focus on those hinges that we do happen to have, explicitly acknowledging that there might be other people with other sorts of fundamental commitments.[33] So, the fact that we could have taken $1,057 \times 216 = 228,312$ to be basic to our justificatory practices in the same way we do take $2 \times 2 = 4$ to be doesn't seem to tell us anything important here. As a matter of fact though, it's not even clear how a proposition like $1,057 \times 216 = 228,312$ could play anything like the role $2 \times 2 = 4$ does for us. If we switched to, say, a base-1111 number system, such a role would be conceivable, but prior to such a drastic change a foundational role for this equation is difficult to envision. That fact alone shows there to be a large gulf between the roles played by $2 \times 2 = 4$ and $1,057 \times 216 = 228,312$ in our mathematical lives—a gulf large enough to make room for the former's counting as a mathematical hinge and the latter's not.

It's worth emphasizing that what distinguishes $2 \times 2 = 4$ from $1,057 \times 216 = 228,312$ is not the fact that the former is more familiar or simpler than the latter, but that the former plays a different role in our arithmetical practices than the latter. In order to make this point a bit clearer, consider a different arithmetical practice that doesn't rely on either addition or memorization of multiplication tables to calculate.[34]

What has recently been called the "Japanese way to multiply"[35]—although this suggested origin is doubtful—asks the calculator to first draw from left to right groups of n vertical lines for each digit n in the first multiplicand (using a single dashed vertical line if the digit is zero) and then to similarly draw n horizontal lines from top to bottom for each digit n in the other (using the same convention for zeros). The setup for the calculation $1,057 \times 216$ is illustrated in Fig. 1. The product of the two numbers being multiplied is then computed simply by counting the number of intersections of lines starting in the bottom right corner of the grid for the ones-digit and then moving to the left in diagonals as indicated by the arrows for the subsequent digits—ignoring any intersections involving a dotted line. E.g., there are 42 intersections in the bottom right corner, so the ones-digit in the product is 2. The leftover 4 is carried to the tally that computes the tens-digit: there are, therefore, $(30+7)+4$ intersections on this diagonal, so the tens-digit is 1 and another 4 is carried to the hundreds-digit count $(0+5+14+4)$. Etc.

Note that although I've written the required counting of the tens-digit place as $(30+7)+4$, if this practice is to really be one where our ordinary addition and multiplication tables don't play any logical role, the procedure must be thought of as simply counting the 30 lowest intersections first, then counting seven further to 37, and then counting the four extra intersections carried from the initial ones count. Someone familiar with our ordinary means of adding and multiplying would certainly stop counting all the intersections almost immediately (relying on multiplication to find the number of intersections instead) and could use our multiplication tables to check the results of her counting while she was still otherwise applying the technique faithfully (e.g., "The ones-digit should be 2 because $6 \times 7 = 42$").[36] These different possibilities for someone who is part of our arithmetical practice and for someone imagined to only have access to the intersection-counting technique, however, are grounded in the special logical roles played by equations like $2 \times 2 = 4$ in our practice. These same roles are played by *no* equations in the intersection-counting practice—this technique may require hinges like, "Lines don't cross or uncross during the count," instead—and are also not played by equations like $1,057 \times 216 = 228,312$ in our familiar practices. As a result, it's reasonable to ascribe different logical statuses to these differing sorts of equations.[37]

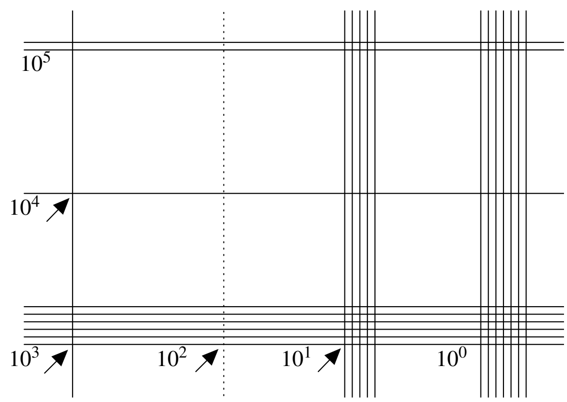


Fig. 1: "Japanese" calculation of 1057×216

[32] "In some sense" because it's not clear how the $1,057 \times 216 = 228,312$ learned prior to, say, $2 \times 2 = 4$ would relate to the equation $1,057 \times 216 = 228,312$ as we know and relate to it now.

[33] See, e.g., (Wittgenstein 1969: §§106, 264).

[34] This feature of the method to be discussed is noted, but not commented on at (Coliva 2020b: 352n10).

[35] See, e.g., https://www.popularmechanics.com/science/a32131826/ancient-multiplication-method/ (accessed 13 September 2022).

[36] This sort of check is especially valuable because all the counting involved makes the intersection-counting technique more liable to error than our normal method of multiplying.

[37] A practice that used Roman numerals instead of Arabic ones for calculation would similarly have a different collection of equations with a special logical status supporting justification within the practice. E.g., to multiply MLVII by CCXVI one would need to know that V times V is XXV, that L times V is CCL, and that five Cs add up to one D, and so on. These considerations are perhaps mundane, but they are part of a realistic investigation of this mathematical practice nonetheless. See Detlefsen et al. (1976) for more on techniques of Roman numeral arithmetic.

Taking those mathematical propositions like $2 \times 2 = 4$ that play a foundational role with respect to proofs of other propositions in a practice to be mathematical hinges already gives us a preliminary way of thinking about what a Wittgensteinian hinge philosophy of mathematics might look like. Yet, even supposing that the basic additions of the digits 0 through 9 and the multiplication table, say, up to 12 play distinct foundational, hinge-like roles in our arithmetical practices, it looks like we might still be left with a serious problem. If Wittgenstein takes all mathematical propositions to be akin to expressions of rules, they must all be equally objectively certain and not subject to doubt. But the basic picture of hinges and the role they play as laid out in *On Certainty* seems to depend on the thought that hinges are somehow special and *more* certain than the propositions that depend on them for their evidential support. How can we account for this "extra" degree of certainty?[38]

An analogous problem arises in the discussion of so-called mathematical coincidences.[39] In mathematics, everything is necessary, so the concept of a mathematical *coincidence* seems almost to be an oxymoron. The way forward in the coincidence case was to come up with a new understanding of 'coincidence' that didn't rely on the existence of contingencies happening to align in surprising ways for the term to be applicable. Here, the way forward must be to formulate a new notion of degrees of certainty where the certainty of the propositions weighed by these degrees is already maximal. I'll put forward a formulation of this kind in Section 4 using Wittgenstein's river-bed analogy as a guide, but first I'd like to briefly consider some other sorts of mathematical hinges to flesh out the picture obtained so far with some more substantive examples.

3 Other Potential Mathematical Hinges

Although the mathematical examples in *On Certainty* are uniformly fairly dull, Wittgenstein did have other pieces of mathematics on his mind around the time he was working on the manuscripts that would eventually make up the work.[40] So, for example, he discusses the complex numbers occasionally in "Part II" of the *Investigations* and geometric constructions come up in his *Remarks on Colour*.[41] These particular examples will turn out to be good ones for further illustrating the variety of roles that hinges might play in the practice of mathematics in Section 5. The current section will canvas a few of these possible roles and hinges on the way to eventually making that case. Because part of the point of this section is to indicate the applicability of the hinge concept to mathematical practice in general, I won't confine myself here strictly to examples Wittgenstein himself would have likely accepted as hinges.

3.1 Identity and Non-Identity Statements

The discussion in Section 2.2 suggested that basic additions and the entries in a multiplication table were mathematical hinges, but already some other candidates for propositions with a hinge-like status emerged there: obvious non-identity statements and perhaps with them trivial identity statements. Consider, in this vein, the following passage from Frege's *Grundgesetze* where the indubitability and unjustifiability of this type of statement is claimed.

> [W]e may say: we are forced to make judgments by our own nature and external circumstances; and if we do so, we cannot reject this law – of identity for example; we must acknowledge it unless we wish to reduce our thought to confusion and finally renounce all judgment whatever. I shall neither dispute nor support this view; I shall merely remark that what we have here is not a logical consequence. What is given is not a reason for something's being true but for our taking it to be true.[42]

In addition to the law's being indubitable and unjustifiable, Frege seems to be suggesting that accepting the law of identity is necessary for us to engage in the practice of judging at all and that this necessity stems from our nature and circumstances. If he is correct, the law and, perhaps, its instances meet the criteria for being hinges. (There do seem to be ways of viewing hinges as "following" in some sense from other hinges while still maintaining their hinge status. See, for example, Duncan Pritchard's account of particular examples of hinges as being spellings out of one fundamental and general "über hinge commitment."[43] So even if (proposed) hinges like $2 = 2$ originate in a more general law of identity, their status as hinges isn't by that fact threatened.)

There are two worries worth raising about this potential form of mathematical hinge however. First, statements like $2 = 2$ (and $10^{100} \neq 10^{100} + 1$ to be discussed further next) might be thought to really be *logical* propositions and so not especially mathematical.[44] This way of drawing the distinction between the logical and the mathematical may very well be a good one, but someone interested in studying the particularities of a practice already deemed to be mathematical

[38] Something like this worry seems to lie behind the general thrust of Coliva's arguments. See, e.g., (Coliva 2020a: 347-348).

[39] See, e.g., Baker (2009), Davis and Hersh (1981), (Lange 2017: Ch. 8), and Martin (2022) for discussion of the phenomenon.

[40] See van Gennip (2003) for discussion of the timeline of the manuscripts mined for the content of *On Certainty*.

[41] See, e.g, (Wittgenstein 1953/2009: Part II.xi §165) and (Wittgenstein 1977: §10).

[42] (Frege 1893/1964: xvii)

[43] Pritchard (2021) provides a recent overview of this account.

[44] This is likely the line Frege, for example, would take given the rest of his discussion in the *Grundgesetze*.

needn't distinguish so sharply between the logical and the mathematical in their investigation of the hinges grounding aspects of the practice. For some purposes, thinking of $2 = 2$ as mathematical may be more appropriate than grouping it in with the logical. Second, despite some brief indications to the contrary,[45] for someone taking Wittgenstein's perspective, $2 = 2$ may be indubitable and unjustifiable, but only because it's essentially meaningless.[46] As I noted in the introduction to this section, however, the potential hinges under discussion here aren't meant only to be palatable to those taking Wittgenstein's view.

Similar questions can be raised about non-identities like $10^{100} \neq 10^{100} + 1$ that I've suggested can play a hinge-like role in mathematical practice in Section 2.2. Again, even if these are hinges, they are perhaps better thought of as logical propositions and so are not especially appropriately counted as mathematical hinges. In response, a case for considering $10^{100} \neq 10^{100} + 1$ to be mathematical is easy enough to construct given the mathematical operations involved in the proposition, but even, e.g., $100 \neq 101$, can reasonably be seen as being mathematical despite its having roughly the same form as the (presumably) non-mathematical Venus $\neq$ Mars. Given that a hinge of any sort—by characteristic (4)—is found embedded within a practice, the status of any proposition as a hinge or not should only be queried in the context of the practice in which its role being investigated. In some settings, $100 \neq 101$ and Venus $\neq$ Mars may be equally logical or equally non-mathematical, but within the practice of arithmetic, $100 \neq 101$ can be seen to be essentially equivalent to $100 \neq 100 + 1$, which allows any participant in the practice to immediately grasp the correctness of the proposition in a way that isn't available for a superficially similar proposition like Zoe $\neq$ Zoey embedded in a practice using ordinary names.[47] So, although there is room for debate here, obvious non-identities of this sort are arguably also potentially mathematical hinges, and in this case they may even be acceptable as such by Wittgenstein himself.[48]

3.2 Consistency

Mathematical research is often thought of as involving the free investigation of any system of axioms that is free from contradiction. In this spirit, Poincaré, e.g., suggests that "[i]n defining an object, we assert that it involves no contradiction."[49] The implication being that if the definition of the object did involve a contradiction, it wouldn't exist and so couldn't be investigated. But how exactly does the "assertion" that the defined object involves no contradiction get made in practice? It's not through the actual assertion of consistency as one axiom among others in most cases.[50]

The commitment to the consistency of a mathematical system under investigation arguably has all the characteristics of a hinge commitment.[51,52] If the Poincaré thought is right, an inconsistent set of axioms doesn't even constitute an object of investigation, so the consistency commitment does seem to be normative by playing a role in constituting the relevant object(s) and meaning(s); given the limitative results of Gödel, there is no evidence for the consistency of a system of axioms that can be provided while we confine ourselves to the provability strength of those axioms; a proof within an inconsistent system of axioms couldn't be taken to provide evidence for the proposition proved, so the giving and weighing of evidence requires taking consistency for granted; and the commitment to the consistency of the system appears to be something that simply comes to be accepted amongst investigators of those axioms through the process of working with them. All of these facts suggest that consistency claims may rightly be understood on the model of hinges in *On Certainty*.

Even if this is true, however, it's not clear that a commitment to consistency would be a part of a *Wittgensteinian* hinge-based philosophy of mathematics. Wittgenstein's heterodox views on the significance of coming across an inconsistency in a mathematical system changed between his so-called middle and late periods,[53] but he never seems to have held the view that a contradiction could vitiate what was done in the system before noticing the trouble, so it's

[45] I'm thinking especially of Juliet Floyd's recent work on Wittgenstein's philosophy of mathematics that appears to suggest that Wittgenstein may have once had some sympathy for Frege's line of thought in at least some form. (See (Floyd 2021: §3.4) and (Wittgenstein 1930/1975: §163).)

[46] See, e.g., (Wittgenstein 1953/2009: §216), where Wittgenstein suggest that there is "no finer example of a useless proposition" that the law of identity.

[47] *Cf.* The discussion in (Wittgenstein 1939/1989: 41-42) considering the ways that even $25 \times 25 = 625$ might be deemed mathematical or not depending on usage.

[48] See, e.g., (Wittgenstein 1930/1975: §163): "'Every symbol is what it is and not another symbol.'"

[49] See (Poincaré 1914/2001: 466-467). *Cf.* (Hilbert 1899/1980: 39-40) and (Russell 1903/2010: §1). (Wilson 2020: 39) suggests, however, that this "if-thenist" picture of mathematical research is "deeply insincere." For more on that line of thought, see Section 7.

[50] Sometimes logicians are interested in studying systems that do assert their own consistency though. E.g., Peano Arithmetic (PA) + "PA is consistent" is a so-called consistency extension of Peano Arithmetic that has been investigated. See, e.g., (Franzén 2004: Ch. 13). The consistency of this larger system must still be asserted somehow too though if Poincaré's idea is on the right track.

[51] Pedersen (2021) argues that this type of consistency claim is in fact a hinge (or a "cornerstone proposition" has he puts it).

[52] In order to make this commitment into a more evidently mathematical claim, it could be reformulated as a commitment to the claim that the axioms have a model.

[53] See, e.g., (Floyd 2021: §3.6) for discussion.

not clear that he would insist on committing oneself to consistency before working with a system of axioms.[54] Nevertheless, noting the similarity between commitments to consistency claims and hinge propositions shows that mathematical hinges might not always be found amongst the "ordinary" statements of mathematics. And the implications of treating consistency as a hinge seem on their own to warrant investigation whether or not Wittgenstein would've been inclined to carry out such an inquiry.[55]

3.3 Axioms

Finally, the best and most obvious place to locate various mathematical hinges of real interest is among the axioms that constitute an area of inquiry within the subject.[56]

It's easy enough to see how to argue that a set of axioms has characteristics (1)-(4) from our list in Section 2.2 as well as how Wittgenstein himself might make such an argument. First, if all mathematical statements are normative, then the axioms must be too. A set of axioms can also be seen as constituting the meanings of key terms involved in them: e.g., the axioms of geometries of various forms are often taken to provide us with all the (mathematical) meaning there is for terms like 'point', 'line', and 'intersection'.[57] Second, we don't and can't give evidence in the form of a proof for axioms, so they are unjustifiable. Third, axioms can't be doubted if the whole edifice of giving proof and evidence for statements formulable within the system is to remain standing.[58] And, finally, someone working with a set of axioms for any period of time is essentially trained to take them on board through that training. So, conditions (1)-(4) are apparently met.

In further support of *Wittgenstein* taking axioms to be hinges, consider the following passage where he describes axioms in terms very similar to those he uses to talk about other hinge-propositions in *On Certainty*.

> "If the proof convinces us, then we must also be convinced of the axioms." Not as by empirical propositions, that is not their role. In the language-game of verification by experience they are excluded. Are, not empirical propositions, but principles of judgment.[59]

This, in combination with the earlier remarks, suggests that giving axioms a hinge or hinge-like status in some area of mathematical research is something Wittgenstein is amenable to at the very least. And once axioms count among the mathematical hinges, mathematical hinges have all the richness, complexity, and interest had by the "MULTICOLOURED *mixture*" of mathematics as a whole.[60] Yet, that Wittgenstein is still struggling with how exactly to make this hinge-like perspective work within the general framework of his thinking about mathematics at the time is indicated by the following passage from the same manuscript as the quotation above.

> A language-game: How have I to imagine one in which axioms, proofs and proved propositions occur?[61]

What is Wittgenstein worrying about here? And how could such a basic question still be bothering him at this late stage in his thinking about mathematics? I don't think it's a stretch to suppose that Wittgenstein's worry is related to Coliva's previously-discussed concern about distinguishing axioms/hinges from ordinary, proved mathematical propositions in the context of a rule-like treatment of both. Wittgenstein may feel as if he's lacking the tools to comprehend both axioms and proved propositions within this 1944 framework, but I'll suggest in the next section that by the time of *On Certainty* the river-bed analogy has provided him with the means necessary to draw all the required lines.

4 The River-bed Analogy and Degrees of Certainty

Having now solidified a place for the notion of mathematical hinges within Wittgenstein's later thought, I return to the problem of distinguishing mathematical hinges from other rule-like statements of mathematics that are "merely" objectively certain. *On Certainty*'s river-bed analogy points the way.

Images similar to *On Certainty*'s river-bed appear early on in Wittgenstein's thinking about certainty. Consider, for instance, the following example from the 1937 notes that came to be published as "Cause and Effect: Intuitive Awareness," which contain reflections on related topics.[62]

54 See, e.g., (Wittgenstein 1939/1989: 210).

55 Again, see Pedersen (2021) here.

56 This is where Coliva thinks it's wisest for us to look for mathematical hinges as well. (See (Coliva 2020a: §6).) The main differences between the account on offer here and Coliva's own suggestions are that the present account is able to incorporate Wittgenstein's insight that mathematical statements seem to behave uniformly in a rule-like fashion (see Friederich (2011) for persuasive way of making this case); it posits more hinges than just axioms; and it attempts to give a more detailed explanation of the difference between a proposition's being listed as an axiom when stating a theory and its playing a truly axiomatic role within the practice. (Coliva briefly touches on this distinction at (Coliva 2020a: 363).)

57 *Cf.* (Hilbert 1899/1980: 40-41). See also Resnik (1974).

58 *Cf.* (Coliva 2020a: 363).

59 (Wittgenstein 1956/1983: VII.§73). This passage from *Remarks on the Foundations of Mathematics* is from the latest manuscript year to be represented in the work, 1944: it appears at MS 124, 197.

60 (Wittgenstein 1956/1983: III.§46, emphasis in the original) (in Felix Mühlhölzer's amended translation).

61 (Wittgenstein 1956/1983: VII.§73)

62 *Cf.* van Gennip (2003) for the relation between "Cause and Effect" and the *On Certainty* manuscripts.

> First there must be firm, hard stone for building, and the blocks are laid rough-hewn one on another. *Afterwards* it's certainly important that the stone can be trimmed, that it's not *too* hard.[63]

The stone in this picture plays the role of the grammatical framework within which our linguistic practices play out, and the fact that these firm guideposts may be *consciously* changeable makes an appearance in the image. However, the natural, "imperceptible" shifts involved in the *On Certainty* river-bed analogy are missing. Any variety among the stones that make up these firm foundations also goes unremarked upon.

Another example of a pre-*On-Certainty*-instance of the river-bed analogy comes from a discussion of William James's conception of the "stream of thought" in MSS 165 and 129 of the early 1940s. (These discussions seem to be the immediate precursors to the river-bed analogy as we have come to know it.[64]) In these brief discussions, Wittgenstein objects to James's use of a stream metaphor because James doesn't include a distinction between the grammatical and the empirical, the a priori and the a posteriori as part of his conception of thought. As a result, in Wittgenstein's view, James's thinking about thought would be more accurately captured by a metaphor such as a "space of thought" rather than a stream, where the bed of the stream provides the grammatical channels over which ordinary thought can flow.[65] Again, the emphasis of this discussion is on the important distinction Wittgenstein notices between between what we might now want to call hinges and the other propositions whose justifications turn on these hinges staying fixed. But, again, the shifts that might occur among what we take to be hinges doesn't play a central role here nor does the hinges' variety. In his discussion of both images, Wittgenstein simply "want[s] to say: it is characteristic of our language that the foundation on which it grows consists in *steady* ways of living, *regular* ways of acting."[66]

The river-bed analogy's full import isn't realized, however, until the bed's moving and meandering variety is explicitly noted. Wittgenstein does this noting in *On Certainty*. Here are the relevant passages.

> It might be imagined that some propositions of the form of empirical propositions, were hardened and functioned as channels for such empirical propositions as were not hardened but fluid; and that this relation altered with time, in that fluid propositions hardened, and hard ones became fluid.[67]

Wittgenstein then goes on to describe the "state of flux" amongst these types of propositions as follows.[68]

> And the bank of that river consists partly of hard rock, subject to no alteration or only to an imperceptible one, partly of sand, which now in one place now in another gets washed away, or deposited.[69]

Both the changing nature of our hinges and the variety of statuses they may have—at last clearly in view—will play an important role in what follows.

In particular, once we distinguish between the hard rock and the mud and the sand of the river-bed, we at once have the means to distinguish between different degrees of certainty among our hinge propositions. And while the distinction between the bank and the stream is the most important distinction to note in the context of *On Certainty*, where the relation of hinges to empirical claims is the primary focus, in the context of mathematics, where everything is riverbank, it's useful to be able to distinguish between mathematical hinges and the ordinary proved propositions of the subject by taking the hinges to be the hard rock of the bed and the proved propositions to shade out from there into the sand. By doing so we can make sense of degrees of certainty among already-objectively-certain propositions. In the next section, I'll aim to characterize which propositions are to count as those that are rock and which are the sand in more detail, and I'll also discuss why it's essential to the prospects for a Wittgensteinian hinge-based philosophy of mathematics to allow for there to be shifts between the rocks and sand at that point. But first, I'll briefly consider another historically important analogy similar to Wittgenstein's river-bed in order to clarify what I take Wittgenstein's view to be.

The view of the river-bed analogy presented so far may make it sound as if Wittgenstein wants to commit to something like Quine's "web of belief" view, where some of our beliefs, like $2+2=4$, appear to be fixed and necessary because they are so central to other beliefs in the web.[70] Although comparing Wittgenstein to Quine is a now familiar project,[71] The comparison isn't always the most helpful one, and that holds true here.[72] This is for two primary reasons. First, the fundamental distinction between grammatical and empirical propositions, so central to Wittgenstein's thinking, is famously absent in Quine. Second, the idea of there being a single web of belief is anathema to Wittgenstein's thinking about mathematics (and language more generally) as a patchwork of language-games and practices. If anything

[63] (Wittgenstein 1993: 397, emphasis in the original)
[64] See (Bocompagni 2012: §2). The metaphor of a stream of thought is from (James 1890: Ch. IX).
[65] See MS 165, 24-25 and (Bocompagni 2012: 3).
[66] (Wittgenstein 1993: 397, emphases added)
[67] (Wittgenstein 1969: §96)
[68] (Wittgenstein 1969: §97)
[69] (Wittgenstein 1969: §99). See also (Wittgenstein 1969: §63).
[70] See, e.g., (Quine 1951: 41).
[71] See, e.g., the essays in Arrington and Glock (1996).
[72] I essentially agree with Hacker (1996) when he suggest that there is "proximity at great distance" between the two. See also Moyal-Sharrock (2000) especially on the danger of bringing a Quinian perspective to bear on the river-bed analogy.

like his view is right, the idea of one web of belief weaving through all these distinct pieces of inquiry and activity is chimerical.[73] So, although there are some similarities between the images of a web and a river-bed, the river-bed analogy serves present purposes more effectively and should be our guiding picture.[74]

5 Characterizing Mathematical Hinges

I've so far claimed that if we take seriously all aspects of the river-bed analogy, we can distinguish mathematical hinges from other proved propositions while keeping to a rule-based picture of mathematical statements. In the metaphor, mathematical hinges are the hard rock of the river-bed, the theorems proved by means of these hinges are the sand gathered around the stone but still involved in channeling empirical statements over them via their objective certainty. The goal of the present section is to stop the metaphors and show how a Wittgensteinian hinge-based philosophy of mathematics can be applied to real mathematical practice.

The most important question left to settle is, "What makes something a mathematical hinge? (I.e., which propositions constitute our hard rock?)" Wittgenstein himself doubts that there is a common characteristic that sets hinges apart from other propositions in general,[75] and I share that doubt, but in the more specific realm of mathematics, there is perhaps still something that can be said. The basic thought I'll defend is that mathematical hinges are those propositions that must be held fixed for an inquiry to remain the one that it is. This is intended to be vague enough to capture the flexibility of the practice as well as accommodate the following recognizable precept attributed to J. J. Sylvester by V. I. Arnold.

> "[A] mathematical idea should not be petrified in a formalised axiomatic setting, but should be considered instead as flowing as a river." One should always be ready to change the axioms, preserving the informal idea.[76]

When will a change in axioms preserve the informal idea? When will a subject stay the same despite reformulations or shifts in focus?

Wittgenstein's idea, which I take on board here, is that the answers to these questions depend on what mathematical practice is willing to recognize as being "the same."[77] As an illustration, consider the different axiomatizations of *ring* given in two well-known algebra textbooks: in Herstein's *Topics in Algebra* a ring doesn't have to have multiplicative identity, in Lang's *Algebra* it does.[78] A ring without a multiplicative identity is now sometimes referred to as a "rng" to avoid ambiguity, but it appears as if the mathematical community at large is prepared to recognize the underlying ring-idea to be the same with or without the axiomatic addition of a multiplicative identity.[79] That being the case, it looks like this axiom could be taken or left while the inquiry into rings still remains the inquiry that it is. And, so, I claim that the existence of a multiplicative identity isn't a hinge in the study of rings—the fact, e.g., that a ring's addition-operation is characterized by the group axioms on the other hand *is* a non-negotiable hinge. Although there may be no *general* explanation for why mathematical practice judges a commitment to be a mathematical hinge—i.e., why the commitment must be held fixed if the subject is to remain the one it is—at least the following two types of reasons can be expected to play key roles: historical reasons and functional reasons.

Given the history of the introduction of a concept or the inauguration of an inquiry, there will be characteristics of the object or inquiry that are more central than others. E.g., the origins of graph theory in Euler's Königsberg bridges problem has made the notions of "eulerian" and "hamiltonian" paths more central to the subject than the possible existence of Conway's "99-graph."[80] Similarly, the historical importance of Euclid's *Elements* made the reach of ruler-and-compass constructions of great significance in the study of geometry for centuries. For some purposes, geometric investigation that didn't proceed with these basic tools would no longer be considered geometry, and so in this context restriction to their use counts as a mathematical hinge. For others purposes, for example, in the context of the investigation in Hilbert (1902), one can still do geometry without ever drawing a line. A proposition's status as a hinge will depend on how mathematical practice decides to proceed from case to case, but because the history of the tradition guides the questions of central interest, the modification of hinges from case to case can mostly be expected to be slow, like the shifting of a river-bed.[81]

[73] See (Wilson 2020: 54) for a similar point about all "the mathematics science needs."

[74] See also Friedman (2001) and its notion of a "relativized a priori" for a better object of comparison to Wittgenstein's late views.

[75] "I can enumerate various typical cases, but not give any common characteristic" (Wittgenstein 1969: §674).

[76] (Arnold 2000: 404), cited in (Wilson 2020: 57) where Wilson also connects this idea to Wittgenstein's river-bed analogy.

[77] *Cf.* (Travis 2011: 52) in relation to the question of when a system of concepts would be recognized by us as color-concepts. The idea that "the parochial" must do this kind of work for us in general plays a central role in Travis's reading(s) of Wittgenstein. See also (Thomasson 2020a: 73-76) and Thomasson (2020b) for how these judgments about sameness play a role in other Wittgenstein-inspired theorizing.

[78] See (Herstein 1964: 83) and (Lang 2002: 83).

[79] See Kleiner (1996) for more on the history of the abstract ring concept.

[80] See Gross et al. (2013). For Conway's problem, see https://oeis.org/A248380/a248380.pdf (accessed 13 September 2022).

[81] See, e.g., Corfield (2012), MacIntyre (1988), and Martin (2021) for more on the role of "traditions" in guiding inquiry.

In addition to historical reasons—and often likely deriving from them—are functional reasons for judging a concept or inquiry to be the same through various changes. For example, since groups are used (in part) to study permutations in the abstract, any concept that didn't facilitate such a study couldn't be counted as still being a group-concept. This basic function of the group-concept, therefore, establishes mathematical hinges in the practice of group theory. This particular way of thinking about persistence through meaning-change has become familiar through the work, e.g., of Sally Haslanger and the recent interest in "conceptual engineering."[82] As Haslanger puts it, a "proposed shift in meaning of the term would seem semantically warranted if central functions of the term remain the same, e.g., if it helps organize or explain a core set of phenomena that the ordinary terms are used to identify or describe."[83] The "central functions" of a mathematical concept may be best understood as the hinges around which other functions of the concept can accrue and as determined by the judgments of the practice. Which of these functions are central is something that we can expect to shift over time, but where the central functions are, so will be the subject's hinges.

One way to test the acceptability of this account of mathematical hinges is to check whether the purported examples of hinges considered in Sections 2.2 and 3 all come out as hinges according to it. Unsurprisingly they do. We might imagine an arithmetic that defines 1/0=0.[84] This would have consequences beyond the acceptance of the new definition, but arguably we'd still be doing arithmetic after the adoption of this new rule. But rejecting $2 \times 2 = 4$, which I have argued in Section 2.2 does have a distinct logical status as compared to some other multiplications, like $1,057 \times 216 = 228,312$, and plays a key role in various forms of arithmetical justification, would plausibly take us out of the realm of (our practice of) arithmetic as judged by competent practitioners. That being the case, $2 \times 2 = 4$ can be distinguished from $1,057 \times 216 = 228,312$ as a hinge on these grounds, despite the fact that both propositions are objectively certain.

As for the examples in Section 3, axioms are often such that they can't be changed without changing the subject, but as the ring-theory example showed, this isn't always the case. We need—and are able—to distinguish between a proposition's being listed as an axiom and its playing a hinge-like axiomatic role within a subject, but axioms are nevertheless often hinges given the characterization on offer. Consistency claims (on a non-Wittgensteinian view) arguably also come out as hinges according to this account since inconsistency would destroy the subject's matter.[85] Finally, any inquiry that denied, e.g., that $2 = 2$ could be argued to be one that we couldn't even recognize as an inquiry in the first place.[86] So, again, in order for the subject to stay the one it is, various identity and non-identity claims would have to say fixed and accepted.

This account of mathematical hinges gains some further support by providing an explanation for why mathematical hinges have the four characteristics in the list from Section 2.2. Being necessarily held fixed in order to keep the subject or inquiry the one it is automatically gives mathematical hinges a normative standing. Further, the constitutive role of these propositions makes the giving and evaluating of evidence for other propositions in the area of inquiry dependent on their being accepted, so the unjustifiability and indubitbaility characteristics follow. Finally, the hinges we judge as constitutive of a subject are ones that are gained through practical engagement with particular cases, examples, and problems. It's additionally true that when drastic changes in setting that require major adjustments to hinges occur, e.g., when a function is transported from the real numbers to the complex numbers, the acceptance of the shift can be understood more accurately as being along the lines of a conversion (or a "seeing of the light") than as a reasoned decision as is similarly the case in a number of examples discussed in *On Certainty*.[87] (I'll provide an example to illustrate this "conversion"-claim in the next section.) The relevant reminder Wittgenstein provides us below speaks to a common occurrence of this sort in mathematical research.

> Remember that one is sometimes convinced of the correctness of a view by its simplicity or symmetry, i.e., these are what induce one to go over to this point of view. One then simply says something like: "That's how it must be."[88]

This last passage naturally raises the questions of when and why mathematical practice might decide that "That's how it must be"; what accounts for the dynamics of change here? Earlier versions of Wittgenstein's thinking about mathematics couldn't make sense of the kinds of shifts in meaning involved in such realizations and of the sort discussed more generally in the section. This is because, during his middle period, he was committed to the idea that "if you

[82] See Burgess et al. (2020) for an overview of the conceptual engineering project.

[83] (Haslanger 2000: 35)

[84] (Hodges 1993: 10) suggests one reason that we might want to make this choice.

[85] Although, see Mortensen (1995).

[86] *Cf.* (Frege 1893/1964: xvii). We are here again considering the hinge concept in general and not only as Wittgenstein would want to apply it.

[87] See, e.g., (Wittgenstein 1969: §92). See (Floyd 2021: §1.2) for more discussion of Wittgenstein on coming to see aspects in mathematics.

[88] See (Wittgenstein 1969: §92). It's worth noting that this comment comes just before the river-bed analogy. I'll be suggesting below that it's the realization, "Oh, that's how things must be," that leads to the shifting of the river-bed in this sort of case.

change even one rule it would be a different game."[89] But by the time of *On Certainty*, Wittgenstein is ready to note that "a language-game does change with time."[90] This change in perspective is important because the ability to comprehend change, innovation, and progress in the field is essential for any philosophy of mathematics. I'd like to close this section with a discussion of the view the Wittgensteinian hinge philosophy of mathematics on offer here provides us of these aspects of mathematical practice.

Travis (2009) and more recently Kuusela (2019) have offered an understanding of Wittgenstein's language-games as objects of comparison where the primary point of the comparison is to resolve philosophical puzzlement. A language-game can do this resolving work by shifting our perspective on an old problem and allowing us to see it in a new light where things aren't forced to proceed problematically.[91] For example, the opening section of the *Investigations* shows us a language-game where a shopkeeper acts in response to a slip of paper with "five red apples" written on it. We can compare this language-game to our ordinary linguistic practices and perhaps draw the conclusion that 'five' is not best understood on the model provided by the Augustinian picture of language and thereby avoid some difficulties involved in making sense of reference to abstracta.[92] In this kind of case, the arbiters of whether or not a philosophical problem is resolved or whether things really are seen more clearly in an alternative light can only be those of us who are gripped by the problem or enfolded by the obscurity. Similarly, if we ask why and when a mathematical inquiry adjusts its hinges—or shifts the makeup of its river-bed—we should expect that very often the reason is that the shift allows for a clearer view of parts of the subject (without introducing additional troubles). What exactly a clearer view will come to will be different from case to case, but, nevertheless, through the engagement and judgments of the practitioners, it is possible to arrive at the consensus that, "That's how it must be." When this consensus is reached and when the outstanding anomalies are simultaneously minimized, mathematicians do often judge that the right setting for a problem or inquiry has been found; e.g., most judge that the "native land" for complex functions is the Riemann surface rather than the ordinary complex plane.[93] Through these sorts of judgments, which themselves remain ultimately ungrounded, the appropriateness of our choices of various mathematical settings and commitments are grounded. The dynamics of change, which Wittgenstein can finally make sense of in his latest views on the subject, proceed at least partly through the trying out of different points of view provided by different frameworks of hinges and seeing which ones give us the clearest vision. Here is Wittgenstein making a similar point.

> If someone asks us "but is that true?" we might say "yes" to him; and if he demanded grounds we might say "I can't give you any grounds, but if you learn more you too will think the same."[94]

We'll see an exchange similar to this one in the next section.

6 Illustrating a Change of Hinges

Having now laid out the beginnings of a Wittgensteinian hinge-based philosophy of mathematics, I'd like to illustrate last section's abstract discussion of "providing a clearer view" by presenting a instance of mathematical clarification involving a fairly dramatic shift in mathematical hinges.

According to the account presented in Section 5, one important motivator for innovation and change in mathematical research is that an alteration of our hinges or a restructuring of the river-bed gives can give us a clearer view of the objects or problems of our inquiry. But not only do we sometimes add or remove axioms in order to obtain a clearer view, sometimes our view is improved, surprisingly, by the countenancing of new elements.[95] Consider the following example.[96]

Any real function has a Taylor series expansion about any point at which it has derivatives of all orders. In general, however, this power series will only converge to the original function within some finite interval containing the point of expansion. The real function

$$f(x) = \frac{1}{1+x^2}$$

is everywhere infinitely differentiable, so it has a Taylor series expansion in particular about zero. This series converges only for $|x| < 1$. But many have found this puzzling. Why is there convergence for only these values? After all, f "is a beautiful function of the real variable x" that is everywhere bounded,[97] and the graph of the function below fails to suggest any obvious point at which the series should be expected to diverge. "There appears to be nothing in the nature

[89] (Wittgenstein 1939/1989: 24)

[90] (Wittgenstein 1969: §256). He doesn't always seem to be settled on his views of this matter though. See, e.g., (Wittgenstein 1977: III.§124).

[91] *Cf.* "How must we look at this problem in order for it to become solvable?" (Wittgenstein 1977: II.§11).

[92] (Wittgenstein 1953/2009: §1)

[93] (Weyl 1955/2009: vii). Mark Wilson often appeals to this example as well.

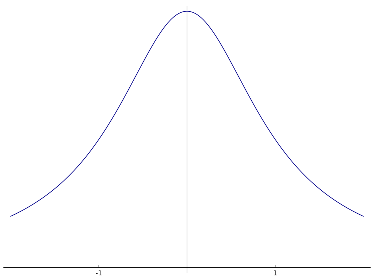
[94] (Wittgenstein 1969: §206)

[95] See Manders (1989) and more recently Bellomo (2021) for further discussion of the advantages of domain extension.

[96] This example has been discussed by philosophers in numerous places: see, e.g., (Lange 2017: 290-292), (Leng 2011: 68), (Steiner 1978: 18-19), (Waismann 1954/1982: 29-30), and (Wilson 2006: 313-314). See (Shanker 1987: 338) for a very different take on the significance of the example.

[97] (Gamelin 2001: 146)

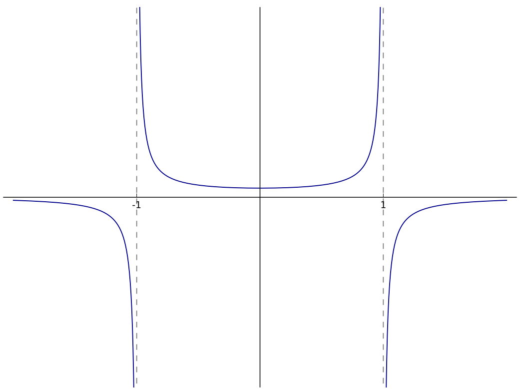
of the function to account for" the convergence behavior of f's Maclaurin series.[98,99]



The Maclaurin series of a function like

$$g(x) = \frac{1}{1-x^2},$$

on the other hand, *is* naturally expected to diverge outside of the interval $(-1,1)$ because the function itself is unbounded as x approaches ± 1, as illustrated below.



The facts presented so far constitute the puzzle Tristan Needham has called, "The Mystery of the Real Power Series."[100]

One can try to explain the convergence behavior of Taylor series expansions of functions like the ones above using only the methods of real analysis, but things get very messy very quickly (in part because techniques like partial fraction decomposition don't always work when we stick only to the real numbers). If, however, instead of considering f as a function of the real variable x, we examine the function

$$f(z) = \frac{1}{1+z^2}$$

with domain and codomain $\mathbb{C}$, the convergence behavior of f's power series expansion seems to start making sense. This is because $f(z)$ has poles at $\pm i$. Just as the Maclaurin series of the function $g(x)$ was expected to diverge unless $|x| < 1$ because of the singularities at ± 1, $f(z)$'s series is now expected to diverge outside the circle $|z| = 1$. This is because convergence for any value of z such that $|z| > 1$ would imply convergence at $\pm i$, which is impossible. So, if the Maclaurin series of our original function had converged to $1/(1+x^2)$ for any x with $|x| > 1$, this fact would have implied an impossible convergence in the complex plane. And this impossibility is thought to explain why the series does not converge to $f(x)$ for any such x. That is, by enlarging our domain we've been given a clearer view of the behavior of certain functions and haven't obviously introduced any additional anomalies. Further, a very general theorem about such convergences—the Cauchy-Hadamard theorem—is straightforward to prove in this new setting. The behavior of the function's Taylor series expansion about zero is now seen to be explained "fully and without residue" by its behavior in the complex plane.[101]

The shift in mathematical hinges between the first presentation of this puzzle and its resolution are striking, as the history of the resistance to accepting the reality of $\sqrt{-1}$ clearly shows.[102] The clearer view reached in this particular example is obviously not the only reason the hinges involved in complex analysis came to be accepted by mathematicians, but the example nevertheless illustrates one way in which "a clearer view" has played out in a real case as well as how the dynamics of change and innovation can be viewed from within Wittgenstein's account. One *could* insist that a real function must be investigated only amongst the real numbers and with real techniques, but the view of mathematical practice has settled on a different "proper setting" and speaks with Wittgenstein as we heard him at the end of the last section: "if you learn more you too will think the same."[103]

7 Conclusion

I hope to have shown that not only can Wittgenstein countenance mathematical hinges, but that by doing so he and we can formulate an attractive Wittgensteinian hinge-based philosophy of mathematics that provides tools and a framework useful for inquiry into the realities of mathematical practice. Mark Wilson is right to suggest that

> the motivational factors that drive mathematics to continually reshape old domains into considerably altered configurations ought to remain a central ingredient within our attempts to gauge the conceptual capabilities of human thought more generally.[104]

Wittgenstein's latest thinking about mathematics seen in the light of the elaboration of the river-bed analogy presented

98 (Ponnusamy and Silverman 2006: 188)
99 "Maclaurin series" is just another name for the Taylor series expansion about zero.
100 (Needham 1997: 64-67)
101 (Waismann 1954/1982: 30)
102 See, e.g., Nahin (1998).
103 *Cf.* (Wittgenstein 1969: §206).
104 (Wilson 2020: 4)

here allows these motivational factors to play a prominent and realistic role in our theorization of the subject. They also provide a useful antidote to the "if-thenism" Wilson laments throughout his recent monograph, *Innovation and Certainty*.[105] By noting the fixity of our mathematical hinges, we can see how it's through the adjustment, refinement, re-statusing, and abandonment of these fundamental commitments that research progresses: just any old set of consistent axioms can hardly be taken to be the primary focus of the practice when seen from this perspective.[106] An investigator into any area of mathematical research would be expected to say, given the Wittgensteinian account expounded here, that our questions are *our* questions because of the make up and content of the mathematical hinges that structure our inquiry. For these two reasons alone, the aspects of Wittgenstein's thought about mathematics in focus in this paper deserve increased attention and elaboration.

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[105] See (Wilson 2020: §5).

[106] Wilson himself thinks that the profession of the "if-thenist" doctrine is primarily a way for mathematicians to avoid pestering questions from philosophers anyway. See (Wilson 2020: 39).

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