

# Born rule: quantum probability as classical probability

Ovidiu Cristinel Stoica

*Dept. of Theoretical Physics, NIPNE—HH, Bucharest, Romania.*

*Email: [crisi.stoica@theory.nipne.ro](mailto:crisi.stoica@theory.nipne.ro), [holotronix@gmail.com](mailto:holotronix@gmail.com)*

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I provide a simple derivation of the Born rule as giving a classical probability, that is, the ratio of the measure of favorable states of the system to the measure of its total possible states.

In classical systems, the probability is due to the fact that the same macrostate can be realized in different ways as a microstate. Despite the radical differences between quantum and classical systems, I show that the same can be applied to quantum systems, and the result is the Born rule.

This works only if the basis is continuous, but all known physically realistic measurements involve a continuous basis, because they are based eventually on distinguishing positions.

The continuous basis is not unique, and for subsystems it depends on the observable.

But for the entire universe, there are continuous bases that give the Born rule for all measurements, because all measurements reduce to distinguishing macroscopic pointer states, and macroscopic observations commute. This allows for the possibility of an ontic basis for the entire universe.

In the wavefunctional formulation, the basis can be chosen to consist of classical field configurations, and the coefficients  $\Psi[\phi]$  can be made real by absorbing them into a global U(1) gauge.

For the many-worlds interpretation, this result gives the Born rule from micro-branch counting.

Keywords: Born rule; state counting; Everett’s interpretation; many-worlds interpretation; branch counting.

## I. INTRODUCTION

In quantum mechanics, the Born rule prescribes that the probability that the result of a quantum measurement is the eigenvalue  $\lambda_j$  of the observable is

$$\text{Prob}(\lambda_j) = \langle \psi | \hat{P}_j | \psi \rangle, \quad (1)$$

where the unit vector  $|\psi\rangle$  represents the state of the observed system right before the measurement, and  $\hat{P}_j$  is the projector on the eigenspace corresponding to  $\lambda_j$ .

The *Projection Postulate* states that  $|\psi\rangle$  projects onto one of the eigenspaces  $\hat{P}_j$  with a probability given by (1).

von Neumann expressed already in 1927 the desirability of having a derivation of the Born rule “from empirical facts or fundamental probability-theoretic assumptions, *i.e.*, an inductive justification” [31]. Gleason’s theorem shows that any countably additive probability measure on closed subspaces of a Hilbert space  $\mathcal{H}$ ,  $\dim \mathcal{H} > 2$ , has the form  $\text{tr}(\hat{P}\hat{\rho})$ , where  $\hat{P}$  is the projector on the subspace and  $\hat{\rho}$  is a density operator [16]. If the state is represented by  $\hat{\rho}$ , this can be interpreted as the Born rule. Gleason’s theorem is very important, in showing that if there is a probability rule, it should have the form of the Born rule. But it does not say that the density operator of the observed system is the same  $\hat{\rho}$ , how the probabilities arise in the first place, and what they are about [11]. For example, it is unable to convert the amplitudes of the branches in the many-worlds interpretation (MWI) [9, 12, 29, 33] into actual probabilities. For this reason, the search for a proof of the Born rule continues.

There are numerous proposals to derive the Born rule. Earlier attempts to derive it from more basic principles include [14], [17], [13] *etc.* Such approaches based on a frequency operator were accused of circularity [7, 8]. Other proposals, in relation to the MWI, are based on im-

posing conditions like additivity to the probability measure [12], many-minds [1], decision theory [10, 22, 32] (accused of circularity in [2, 3]), invariance [34] (accused of circularity in [23]), measure of existence [28], branch counting based on refinements of the branching structure [21] *etc.* For a review see [30].

In this article, I investigate the possibility of obtaining probabilities that are very similar to the classical ones. As in classical physics, what we observe are not actually the states (also named *microstates* in this context), but the macrostates. If each macrostate can be realized in different ways as a microstate, probabilities can arise from the relative count, or rather the relative measure (since the basis is uncountable), of the states underlying each macrostate, just like in the standard understanding of probabilities.

In Section §II I argue that, contrary to the common view on quantum mechanics, an “ontic” or “classical” basis for the entire universe is possible, allowing for classical-like probabilities in quantum mechanics.

In Sec. §III I prove the main result, that probability density can be understood as a distribution of “ontic” or “classical” states.

In Sec. §IV I discuss the physical interpretation of this derivation of the Born rule, how it makes possible the existence of a “classical” or ontic basis for the entire universe, how complex numbers appear, and how this yields probabilities in the many-worlds interpretation.

## II. CLASSICAL VS. QUANTUM PROBABILITIES

In this Section we investigate the differences and similarities between classical probabilities and probabilities in quantum mechanics. This will provide the physical

justification to interpret probabilities in a quantum universe, based on the proof given in Sec. §III.

Let us recall what classical probabilities are.

**Definition 1** (Classical probabilities). A *probability space*  $(\Omega, \mathcal{F}, P)$  consists of a *sample space*  $\Omega$  (the set of possible *outcomes*), an *event space*  $\mathcal{F}$  (a  $\sigma$ -algebra of subsets of  $\Omega$ ), and a *probability function*  $P : \mathcal{F} \rightarrow [0, 1]$ .

Even in a classical and deterministic universe, where a ‘‘Laplace’s demon’’ who knows the physical state in full detail should be able to assign to each event a probability equal to either 0 or 1, the evolution of a system can appear to be unpredictable at the macro-level. The reason is that the state cannot be known exactly. We can only access the macrostates. The macrostates correspond to events, and the microstates to outcomes, *cf.* Definition 1.

**Observation 1.** Nontrivial probabilities exist for agents who lack complete information about the microstate.

For example, the classical probability that throwing a pair of dice results in the events

$$\begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array} + \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array} \quad (2)$$

is given by the probability measure of the set of microstates (outcomes) that realize the macrostate in which the event is (2) divided by the total probability measure.

Therefore, classical probabilities satisfy

**Condition 1** (Probability). The probability is *the ratio of the measure of favorable outcomes to the total measure of possible outcomes*.

In classical physics, Condition 1 makes sense because the universe is in a unique state at any time. But in quantum mechanics, a system can be in a superposition of multiple states that coexist in parallel.

**Difficulty 1.** Unlike classical states, quantum states seem to be able to coexist in parallel, in a superposition, as shown by interference experiments.

From Difficulty 1, another difficulty follows. In a quantum universe, the central difference seems to be that there are multiple ways in which the macrostate of a subsystem can be realized as microstates, each depending on the experimental settings. For example, the spin of a particle can be interpreted as consisting of definite possible spins  $|\uparrow\rangle_z$  and  $|\downarrow\rangle_z$  if the spin is measured along the axis  $z$ , but not if it is measured along another axis. In classical mechanics, the possible results of the measurement are considered to be independent of the measurement settings, provided that the effect of the observation on the observed system can be made arbitrarily small.

The main difficulty in the applicability of classical probabilities in quantum mechanics is therefore

**Difficulty 2.** Unlike classical systems, the probability space (the set of microstates) seems to depend on the measurement settings.

But in reality the situation is not necessarily as described in Difficulty 2. To see this, consider an example.

The measurement of the spin of a spin-1/2 particle along the axis  $z$  results in the possible states

$$\begin{cases} |\uparrow\rangle_z |\text{up}\rangle_z \\ |\downarrow\rangle_z |\text{down}\rangle_z \end{cases}, \quad (3)$$

where  $|\uparrow\rangle_z$  and  $|\downarrow\rangle_z$  are the spin states of the particle along the axis  $z$ , and  $|\text{up}\rangle_z$  and  $|\text{down}\rangle_z$  are the corresponding states of the pointer of the measuring device. A spin measurement along the axis  $x$  leads to a different decomposition,

$$\begin{cases} \frac{1}{\sqrt{2}} (|\uparrow\rangle_z + |\downarrow\rangle_z) |\text{up}\rangle_x \\ \frac{1}{\sqrt{2}} (|\uparrow\rangle_z - |\downarrow\rangle_z) |\text{down}\rangle_x \end{cases}. \quad (4)$$

The macrostates corresponding to the results of the spin measurements, including those along distinct axes, are orthogonal, because we can tell by examining the experimental setup along what direction in space the spin is measured, and what result is obtained. Therefore, the microstates from eq. (3) are orthogonal to those from eq. (4), even though the states of the observed system  $|\uparrow\rangle_z$  and  $|\downarrow\rangle_z$  are not orthogonal to  $1/\sqrt{2}(|\uparrow\rangle_z \pm |\downarrow\rangle_z)$ .

In general, every quantum measurement ultimately becomes a direct observation of the macrostate of the measuring device. So every measurement reduces to distinguishing macrostates. Macrostates are distinguished by macro-observables, and all macro-observables commute.

*Macrostates* are represented by subspaces of the form  $\hat{P}_\alpha \mathcal{H}$ , where  $(\hat{P}_\alpha)_{\alpha \in \mathcal{A}}$  is a set of commuting projectors on  $\mathcal{H}$ , so that  $[\hat{P}_\alpha, \hat{P}_\beta] = 0$  for any  $\alpha \neq \beta \in \mathcal{A}$ , and  $\bigoplus_{\alpha \in \mathcal{A}} \hat{P}_\alpha \mathcal{H} = \mathcal{H}$ . This claim is empirically adequate, as illustrated by the example of spin measurements. This position is adopted for example in decohering histories approaches [15] and in the MWI [33].

**Observation 2.** We never observe the microstate, only the macrostates.

Since ultimately every measurement translates into an observation represented by the macro projectors, there is a universal basis for all measurements, which diagonalizes all macro projectors. This is not true for subsystems, but this is irrelevant, since any measurement is ultimately one of macrostates.

**Observation 3.** For the entire universe, whose states are represented by vectors in a Hilbert space  $\mathcal{H}$ , there is a universal basis

$$(|\phi\rangle)_{\phi \in \mathcal{C}} \quad (5)$$

compatible with the macrostates. In general, more such bases exist, otherwise the macrostates would coincide with the microstates.

It may seem too much to account for states of the entire universe just to explain the probabilities of the measurement of a single particle. But in fact we always do this, because the observed particle can be entangled with any other system in the universe, so when we measure it, we measure in fact the state of the universe. The usual separation between the observed system and the rest of the universe that enters in our theoretical description is an idealization that may make us not see the forest for the trees. Therefore,

**Observation 4.** The state of the universe is a single quantum state, not the composition of the states of the subsystems.

Now let us return to Difficulty 1. Despite mentioning a superposition of “parallel microstates”, Difficulty 1 does not apply exclusively to the MWI, but also to standard quantum mechanics, because any state can be (and in general it is) in a superposition before the Projection Postulate is invoked. And even after that it remains in a superposition of microstates, as a succession of compatible measurements confirm it.

Consider now a classical universe, in which we assume in addition that multiple classical microstates exist in parallel, distributed according to a measure. Would Condition 1 be invalidated in such a universe? We can conceive a classical universe in which there are more microstates, and the agent does not know in which of them it exists. If an agent knows the macrostate, her ignorance of the microstate is the same in such a universe as in a standard classical universe, in which there is only one microstate. This leads to the following:

**Observation 5 (Equivalence).** Probabilities for a given macrostate are independent of whether there is a unique microstate and the agent does not know it (*epistemic probability*), or if it describes more coexisting microstates, and the agent does not know in which of these microstates it exists (*self-location probability*).

We can modify the example of a classical universe with many microstates evolving in parallel, by allowing the evolution to “scramble” the microstates, in the sense that the dynamics cannot be applied to individual microstates, but only to sets of such states, so that the measure describing how are they distributed is conserved. For example, a single microstate may evolve into multiple microstates, so that the density is preserved. If we compare such a classical universe with one with the same state space, but so that only one microstate exists, and it evolves indeterministically according to the same measure, Observation 5 would still apply.

An implicit assumption underlying Observation 5 is that an agent or observer *supervenies* (in the sense that its states depends) on a single microstate, even if there are more microstates coexisting in parallel. But if there can be more parallel microstates, an additional condition is needed:

**Condition 2 (Correspondence).** If there are more parallel microstates, and at a given time different instances of an agent exist in more of them, each instance of the agent supervenes on only one of these microstates.

In other words, the physical microstates should be able to support ontologically the existence of agents or observers, so that their experience of probabilities depends on their ignorance of the microstate.

Despite Difficulty 2, the existence of a basis as in Observation 3 will turn out to make it possible for quantum mechanics to satisfy Condition 2. This requires

**Principle 1.** In the quantum universe, there is a basis as in Observation 3, so that all instances of an agent can be realized only in microstates from that basis. We will call it *ontic basis* and its elements *ontic states*.

This, as we will see, allows probabilities in quantum mechanics to be just like classical probabilities, provided that we apply them to the entire universe.

Another difficulty in quantum mechanics is that the state is found, after wavefunction collapse or decoherence, to be  $\widehat{P}_\alpha|\psi\rangle/|\widehat{P}_\alpha|\psi\rangle|$ . The Born rule prescribes the probability that the state vector becomes  $\widehat{P}_\alpha|\psi\rangle/|\widehat{P}_\alpha|\psi\rangle|$  is given by  $\langle\psi|\widehat{P}_\alpha|\psi\rangle$ , as in (1). In the ontic basis (5),  $|\psi\rangle$  is decomposed as a linear combination with distinct coefficients (amplitudes)  $\langle\phi|\psi\rangle$ , so

$$\text{Prob}(\alpha) = \langle\psi|\widehat{P}_\alpha|\psi\rangle = \int_{\mathcal{C}_\alpha} |\langle\phi|\psi\rangle|^2 d\mu(\phi), \quad (6)$$

where  $\mathcal{C}_\alpha = \{\phi \in \mathcal{C} | \langle\phi|\psi\rangle \in \widehat{P}_\alpha\mathcal{H}\}$  and  $\mu$  is the measure on  $\mathcal{C}$ . While  $|\langle\phi|\psi\rangle|^2$  gives a measure, this is not sufficient to interpret it probabilistically:

**Difficulty 3.** Just because it is a measure, it does not mean that it is a probability.

In Sec. §III we will see that Principle 1 allows Condition 2 to be satisfied in quantum mechanics, and Difficulties 1, 2, and 3 to be avoided, so by Observation 5, Condition 1 is satisfied too.

### III. DERIVATION OF THE BORN RULE

Before proving the main result, let us motivate it.

Consider a state vector of the form

$$|\psi\rangle = \frac{1}{\sqrt{n}} \sum_{k=1}^n |\phi_k\rangle. \quad (7)$$

where  $(|\phi_k\rangle)_{k \in \{1, \dots, n\}}$  are orthonormal vectors from  $\mathcal{H}$ . Then, if every  $|\phi_k\rangle$  is an eigenvector of the operator  $\widehat{A}$  representing the observable, the Born rule simply coin-

cides with counting basis states:

$$\begin{aligned} \langle \psi | \widehat{P}_j | \psi \rangle &= \frac{1}{n} \left( \sum_{k=1}^n \langle \phi_k | \right) \left( \widehat{P}_j \sum_{k=1}^n | \phi_k \rangle \right) \\ &= \frac{1}{n} \sum_{|\phi_k\rangle \in \widehat{P}_j \mathcal{H}} \langle \phi_k | \phi_k \rangle = \frac{n_j}{n}, \end{aligned} \quad (8)$$

where  $\widehat{P}_j$  is the projector of the eigenspace corresponding to the eigenvalue  $\lambda_j$ , and  $n_j$  is the number of basis vectors  $|\phi_k\rangle$  that are eigenvectors for  $\lambda_j$ .

In the MWI, this seems to be the only situation when “naive branch counting” coincides with the Born rule. It is used as a starting point for arguments that the Born rule is present in the MWI for “less naive” counting rules such as [10, 21, 28, 32].

Eq. (8) satisfies Condition 1, but unfortunately it seems to work only for very special state vectors. We cannot make it work for all vectors in a finite-dimensional Hilbert space, since the eigenbasis vectors either have to contribute to eq. (7) with the same absolute value  $1/\sqrt{n}$  of the coefficient  $\langle \phi_k | \psi \rangle$ , or to be absent.

Interestingly, as if by magic, the idea works in the continuous case without problems, because the basis vectors can be distributed with nonuniform density, making it possible for the continuous version of eq. (7) to apply to any state vector.

To see this, we have to show that we can take the continuous limit of eq. (7) while keeping the vectors from (7) distinct. It is important to keep them distinct, because the finite case proof from eq. (8) only works when they are distinct. While this is a very severe limitation in finite-dimensional Hilbert spaces, it works in the infinite-dimensional case, in a continuous basis.

Let  $\mathcal{C}$  be a topological manifold with a measure  $\mu$  on its  $\sigma$ -algebra, and  $\mathcal{H} := L^2(\mathcal{C}, \mu, \mathbb{C})$  the Hilbert space of square-integrable complex functions on  $\mathcal{C}$ . Let  $(|\phi\rangle)_{\phi \in \mathcal{C}}$  be an orthogonal basis of  $\mathcal{H}$ , so that

$$\int_{\mathcal{C}} \langle \phi | \phi' \rangle \psi(\phi') d\mu(\phi') = \psi(\phi) \quad (9)$$

for any compact-supported continuous function  $\psi \in \mathcal{H}$ .

Suppose that the projectors  $(\widehat{P}_\alpha)_{\alpha \in \mathcal{A}}$  associated to the possible results of the measurements are *compatible* with the basis  $(|\phi\rangle)_{\phi \in \mathcal{C}}$ , that is, each subspace  $\widehat{P}_\alpha \mathcal{H}$  is spanned by the vectors  $|\phi\rangle$ ,  $\phi \in \mathcal{C}_\alpha$ , for some  $\mu$ -measurable set  $\mathcal{C}_\alpha \subseteq \mathcal{C}$ .

**Theorem 1.** *Let  $(|\phi\rangle)_{\phi \in \mathcal{C}}$  be a continuous basis compatible with the projectors associated to the possible results. Then, for any state vector  $|\psi\rangle$ , the continuous limit of counting basis vectors gives the Born rule as the ratio of the measure of favorable states of the system to the measure of its total possible states.*

*Proof.* Let  $|\psi\rangle \in \mathcal{H}$  be a unit vector. The wavefunction  $\psi$  defined by  $\psi(\phi) = \langle \phi | \psi \rangle$  is  $\mu$ -measurable, so it has measurable support  $D \subseteq \mathcal{C}$ . Since the function

$\varrho : \mathcal{C} \rightarrow \mathbb{R}$ ,  $\varrho(\phi) := \psi^*(\phi)\psi(\phi)$  is measurable and positive, it defines a measure  $\mu' := \varrho\mu$  on  $D$ , and since  $\langle \psi | \psi \rangle = 1$ ,  $\mu'(D) = 1$ . For any  $n \in \mathbb{N}$ , we choose a partition  $\mathcal{D}_n = \{D_{n,1}, \dots, D_{n,2^n}\}$  of  $D$  into  $2^n$  measurable sets of equal measure  $\mu'(D_{n,k}) = \mu'(D)/2^n = 1/2^n$  for all  $k \in \{1, \dots, 2^n\}$ , so that  $\mathcal{D}_{n+1}$  is a *refinement* of  $\mathcal{D}_n$  (that is, every set from  $\mathcal{D}_n$  is the union of some sets from  $\mathcal{D}_{n+1}$ ). Then, for any  $n$ , the unit vectors

$$|n, k\rangle := \sqrt{2^n} \int_{D_{n,k}} \psi(\phi) d\mu(\phi), \quad (10)$$

where  $k \in \{1, \dots, 2^n\}$ , are mutually orthogonal, and

$$|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{k=1}^{2^n} |n, k\rangle. \quad (11)$$

The first thing to notice is that we can refine  $|\psi\rangle$  from eq. (11) as much as we want. This is not possible in the finite-dimensional case, because  $2^n$  cannot exceed the dimension of the Hilbert space.

We define, for each  $n$  and  $\alpha$ , the following set indexing the sets consistent with  $\mathcal{C}_\alpha$  from each partition  $\mathcal{D}_n$ ,

$$M_{n,\alpha} := \{k \in \{1, \dots, 2^n\} | \mu'(D_{n,k} \setminus \mathcal{C}_\alpha) = 0\}. \quad (12)$$

In particular,  $\mu'(D_{n,k} \setminus \mathcal{C}_\alpha) = 0$  is ensured when  $D_{n,k} \subseteq \mathcal{C}_\alpha$ ; in general,  $D_{n,k}$  should be “almost” included in  $\mathcal{C}_\alpha$ , except for a zero-measure remaining subset of  $D_{n,k}$ .

The partitions  $\mathcal{D}_n$  can be chosen so that, as  $n \rightarrow \infty$ , the sets consistent with  $\mathcal{C}_\alpha$  approximate  $\mathcal{C}_\alpha \cap D$  in the measure  $\mu'$ , *i.e.* for any  $\alpha \in \mathcal{A}$ ,

$$\lim_{n \rightarrow \infty} \sum_{k \in M_{n,\alpha}} \mu'(D_{n,k}) = \mu'(\mathcal{C}_\alpha \cap D). \quad (13)$$

Then, for any  $\alpha \in \mathcal{A}$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{2^n}} \sum_{k \in M_{n,\alpha}} |n, k\rangle = \widehat{P}_\alpha |\psi\rangle. \quad (14)$$

Each partition  $\mathcal{D}_n$  is equivalent to a set of projectors  $\mathcal{P}_n$  that form a partition of the identity and are compatible with the ontic basis, so that each  $\mathcal{P}_{n+1}$  refines  $\mathcal{P}_n$ ,  $\mathcal{P}_n$  determines a decomposition (10), and  $\mathcal{P}_n$  converges to a refinement of  $(\widehat{P}_\alpha)_{\alpha \in \mathcal{A}}$ .

From equations (13) and (14) it follows that counting the unit vectors consistent with each macro-projector converges to the Born rule. Therefore, in the continuous limit, the Born rule results from “counting” the basis vectors  $(|\phi\rangle)_{\phi \in \mathcal{C}}$ , as the ratio of the measure of favorable states of the system to the measure of its total possible states, where the density is  $\psi^*(\phi)\psi(\phi)d\mu(\phi)$ .  $\square$

**Observation 6** (Probability). If Principle 1 is assumed in quantum mechanics, Theorem 1 shows that the density of the ontic states satisfies the Born rule for the macro-observables  $\widehat{P}_\alpha$ ,  $\alpha \in \mathcal{A}$ . This allows Condition 2 to be satisfied, and by Observation 5, Condition 1 is satisfied too, despite Difficulties 1–3, according to the Born rule.



For any physically realistic quantum measurement there is a continuous basis in which the observable is diagonal, as required by Theorem 1. Even for a single particle in nonrelativistic quantum mechanics, the Hilbert space is infinite-dimensional, and admits continuous bases, *e.g.* the position basis.

**Observation 7.** All measurements satisfy, in practice, the continuity condition required for Theorem 1, because the actual observation is that of the pointer.

**Example 1.** Consider a measurement of the spin of a particle, whose spin state is initially  $|\psi\rangle_s = a|\uparrow\rangle_z + b|\downarrow\rangle_z$ , where  $|a|^2 + |b|^2 = 1$ . The particle also has position degrees of freedom, so its state vector is in fact

$$\psi(\mathbf{x}, t) = \psi_u(\mathbf{x}, t)|\uparrow\rangle_z + \psi_d(\mathbf{x}, t)|\downarrow\rangle_z. \quad (15)$$

At the initial time  $t_0$ ,  $\psi_u(\mathbf{x}, t_0) = a\psi_0(\mathbf{x})$ ,  $\psi_d(\mathbf{x}, t_0) = b\psi_0(\mathbf{x})$ , and  $\langle\psi_0|\psi_0\rangle = 1$ . The measurement process consists of using a magnetic field to entangle the spin and the position of the particle, then the position where the deflected particle hits a screen or photographic plate is observed. After passing through the magnetic field, at time  $t_1$ ,  $\psi_u(\mathbf{x}, t_1)$  becomes restricted to a region  $U$  of the screen, and  $\psi_d(\mathbf{x}, t_1)$  to a region  $D$ . From the resulting position, the spin is inferred to be either “up” or “down”. The regions  $U$  and  $D$  of the screen are almost identical, but the norms of  $\psi_u(\mathbf{x}, t_1)$  and  $\psi_d(\mathbf{x}, t_1)$  are proportional to  $|a|^2$  and respectively  $|b|^2$ . We obtain, for the two possible regions  $U$  and  $D$ ,

$$\begin{cases} |\psi_u, t_1\rangle|\uparrow\rangle_z &= \int_U |\psi_u(\mathbf{x}, t_1)| e^{i\theta_u(\mathbf{x})} |\mathbf{x}\rangle d\mathbf{x} \\ |\psi_d, t_1\rangle|\downarrow\rangle_z &= \int_D |\psi_d(\mathbf{x}, t_1)| e^{i\theta_d(\mathbf{x})} |\mathbf{x}\rangle d\mathbf{x}. \end{cases} \quad (16)$$

Then, we invoke collapse or decoherence to explain why only one result is observed. This illustrates how, despite apparently making a binary measurement of a qubit, the actual basis is continuous, as required by Theorem 1.  $\square$

**Observation 8.** In the proof of Theorem 1, the projectors used to construct the unit vectors (10) are defined to be compatible with the ontic basis, in order to satisfy Principle 1. Principle 1 is essential for Theorem 1.

Any argument based on counting worlds or branch refinements while using the self-locations of the agents should count only the worlds defined by a fixed basis, as Principle 1 states. Otherwise, we would be forced to admit that any unit vector from any macrostate supports agents, and to count all such vectors as worlds. This overcounting is inconsistent with the Born rule.

To see this, let  $\alpha \in \mathcal{A}$  be so that  $\widehat{P}_\alpha|\psi\rangle \neq 0$ . Let  $\widehat{P}_\alpha|\psi\rangle = 1/\sqrt{n} \sum_{k=1}^n |\phi_{\alpha,k}\rangle$  be an expansion of  $\widehat{P}_\alpha|\psi\rangle$  as a sum of equal-norm orthogonal vectors from  $\widehat{P}_\alpha\mathcal{H}$ , as in eq. (7). Then, any unitary transformation of  $\widehat{P}_\alpha\mathcal{H}$  that preserves  $\widehat{P}_\alpha|\psi\rangle$  would transform any of these vectors into another one from an equally valid expansion, and the world supported by the resulting vector should be valid as well. To “count” these worlds, we actually need to

compare the measures induced by  $\mu$  on the subsets of these vectors. Due to unitary symmetry, this measure depends only on the norm of  $\widehat{P}_\alpha|\psi\rangle$  and the dimension of  $\widehat{P}_\alpha\mathcal{H}$ . If  $\dim \widehat{P}_\alpha\mathcal{H} > \dim \widehat{P}_\beta\mathcal{H}$  for some  $\beta \in \mathcal{A}$  for which  $\widehat{P}_\beta|\psi\rangle \neq 0$ , the relative measure of the vectors to be counted from  $\widehat{P}_\beta\mathcal{H}$  compared to that of those from  $\widehat{P}_\alpha\mathcal{H}$  is zero, independently of the norms of the projections  $\widehat{P}_\alpha|\psi\rangle$  and  $\widehat{P}_\beta|\psi\rangle$ . If, on the other hand,  $\dim \widehat{P}_\alpha\mathcal{H} = d$  is the same for all  $\alpha \in \mathcal{A}$  for which  $\widehat{P}_\alpha|\psi\rangle \neq 0$ , the relative measures would depend only on the norm, but not as  $|\widehat{P}_\alpha|\psi\rangle|^2$ , but as  $|\widehat{P}_\alpha|\psi\rangle|^d$ . Therefore, self-location counting arguments that do not assume that the only state vectors that support worlds with agents form a basis fail to recover the Born rule.  $\square$

Theorem 1 can be seen as the continuous limit of the refined branch-counting method proposed in [21], amended with Principle 1.

## IV. INTERPRETATION OF THE WAVEFUNCTION

### A. Uniformization of the ontic states

The continuous limit from the proof of Theorem 1 keeps the unit vectors orthogonal, so that we can interpret any state vector as the limit of a sum of the form (7), and by this, the basis vectors as distinct outcomes in the sample space (*cf.* Definition 1).

Without loss of generality, for any given state vector  $|\psi\rangle$  so that  $|\psi(\phi)|$  is a  $\mu$ -measurable function of  $\phi$ , we can assume that  $\psi(\phi) \in \mathbb{R}$  for all  $\phi$ . If not, substitute the basis by  $|\phi\rangle \mapsto e^{i\theta(\phi)}|\phi\rangle$ , where  $\theta(\phi)$  is the phase appearing in the polar form of  $\langle\phi|\psi\rangle$ , for all  $\phi \in \mathcal{C}$ .

**Proposition 1.** *The state vector  $|\psi\rangle$  has the form*

$$|\psi\rangle = \int_{\mathcal{C}} |\phi\rangle d\tilde{\mu}(\phi), \quad (17)$$

where  $\theta : \mathcal{C} \rightarrow \mathbb{R}$ , and  $\tilde{\mu}$  is a measure on  $\mathcal{C}$ .

Any projector  $\widehat{P}_\alpha$  diagonal in the basis  $(|\phi\rangle)_{\phi \in \mathcal{C}}$  corresponds to a subset  $\mathcal{C}_\alpha \subseteq \mathcal{C}$ . If  $\mathcal{C}_\alpha$  is  $\mu$ -measurable,

$$\left| \int_{\mathcal{C}_\alpha} |\phi\rangle d\tilde{\mu}(\phi) \right|^2 = \int_{\mathcal{C}_\alpha} r^2(\phi) d\mu(\phi). \quad (18)$$

*Proof.* Let  $r(\phi) := |\langle\phi|\psi\rangle|$ . Then,  $r \in L^2(\mathcal{C}, \mu, \mathbb{R})$  is a real non-negative square-integrable function, and

$$|\psi\rangle = \int_{\mathcal{C}} r(\phi) |\phi\rangle d\mu(\phi). \quad (19)$$

The following measure satisfies eq. (17),

$$d\tilde{\mu}(\phi) := r(\phi) d\mu(\phi). \quad (20)$$

Then, evidently  $\widehat{P}_\alpha|\psi\rangle = \int_{\mathcal{C}_\alpha} |\phi\rangle d\tilde{\mu}(\phi)$ , and

$$\left| \int_{\mathcal{C}_\alpha} |\phi\rangle d\tilde{\mu}(\phi) \right|^2 = \langle \psi | \widehat{P}_\alpha | \psi \rangle = \int_{\mathcal{C}_\alpha} r^2(\phi) d\mu(\phi). \quad (21)$$

It may seem that the vector  $\int_{\mathcal{C}_\alpha} |\phi\rangle d\tilde{\mu}(\phi)$  cannot have finite norm, or that its norm may be larger than 1. So let us check this more explicitly:

$$\begin{aligned} \left| \int_{\mathcal{C}_\alpha} |\phi\rangle d\tilde{\mu}(\phi) \right|^2 &= \left( \int_{\mathcal{C}_\alpha} \langle \phi | d\tilde{\mu}(\phi) \right) \left( \int_{\mathcal{C}_\alpha} |\phi'\rangle d\tilde{\mu}(\phi') \right) \\ &= \int_{\mathcal{C}_\alpha} \left( \int_{\mathcal{C}_\alpha} \langle \phi | \phi'\rangle d\tilde{\mu}(\phi') \right) d\tilde{\mu}(\phi) \\ &= \int_{\mathcal{C}_\alpha} \left( \int_{\mathcal{C}_\alpha} \langle \phi | \phi'\rangle r(\phi') d\mu(\phi') \right) d\tilde{\mu}(\phi) \\ &= \int_{\mathcal{C}_\alpha} r(\phi) d\tilde{\mu}(\phi) = \int_{\mathcal{C}_\alpha} r^2(\phi) d\mu(\phi). \end{aligned} \quad (22)$$

This double-checks eq. (18).  $\square$

Figure 1 illustrates the uniform-amplitude representation of the wavefunction from eq. (17).

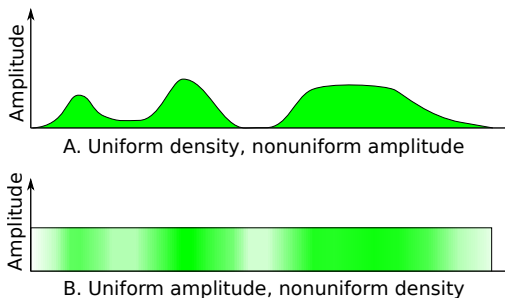


FIG. 1. **The Born rule from “counting” basis states.**

**A.** The usual interpretation of a wavefunction as a linear combination of basis state vectors of different amplitudes.

**B.** The interpretation of the wavefunction in terms of basis vectors representing ontic states.

**Observation 9.** We notice the existence of three densities. The first one is  $d\mu$ , given by the measure on  $\mathcal{C}$ , and it is independent of states. The second density is  $d\tilde{\mu} = r(\phi)d\mu$ , which describes how the ontic states contribute to the state vector  $|\psi\rangle$  in eq. (17). The third density,  $d\mu'(\phi) = r^2(\phi)d\mu$  is the probability density corresponding to the Born rule, as in eq. (18).

This may seem strange, despite the explicit calculation from (22), so let us try to understand the interplay between these densities.

*Remark 1* (“Magic” accident). One may expect that we have to define the measure  $\tilde{\mu}$  so that  $d\tilde{\mu}(\phi)$  is  $r^2(\phi)d\mu(\phi)$ , rather than as in eq. (20). But, interestingly, eq. (18) follows without this, simply by choosing the measure  $\tilde{\mu}$  so that the amplitudes become uniformly equal to 1. Moreover, it does not even work otherwise, because  $|\psi\rangle \neq \int_{\mathcal{C}} r^2(\phi)|\phi\rangle d\mu(\phi)$ .  $\square$

*Remark 2* (Why does it work?). Naively, it may seem that the norm of  $\int_{\mathcal{C}_\alpha} |\phi\rangle d\tilde{\mu}(\phi)$  cannot be finite, or at least that it is equal to  $\int_{\mathcal{C}_\alpha} d\tilde{\mu}(\phi)$  and it can be larger than 1, but this is incorrect. Eq. (18) is correct, as checked in (21) and double-checked in (22), because  $r(\phi)$  is square-integrable, and since it is  $\mu$ -measurable, the measure  $\tilde{\mu}$  is *absolutely continuous* with respect to  $\mu$ .

There is a reason why, in eq. (22),

$$\int_{\mathcal{C}_\alpha} \langle \phi | \phi' \rangle d\tilde{\mu}(\phi') = r(\phi) \quad (23)$$

rather than 1. A perhaps more revealing way of understanding this involves the *scaling property* of the Dirac distribution  $\delta(x)$  with  $a > 1$ ,

$$\delta(ax) = a^{-1}\delta(x). \quad (24)$$

To see how this works, consider the Hilbert space  $L^2(\mathbb{R}^n, \mu, \mathbb{C})$  with the basis  $(|\mathbf{x}\rangle)_{\mathbf{x} \in \mathbb{R}^n}$ . If  $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is an invertible reparametrization of  $\mathbb{R}^n$ , by making a change of variables  $\tilde{\mathbf{y}} = \mathbf{f}(\mathbf{y})$  we obtain the following generalization of eq. (24),

$$\int_{\mathbb{R}^n} \langle \mathbf{x} | \mathbf{y} \rangle d\tilde{\mathbf{y}} = \int_{\mathbb{R}^n} \langle \mathbf{x} | \mathbf{y} \rangle \left| \frac{\partial \mathbf{f}}{\partial \mathbf{y}} \right| d\mathbf{y} = \left| \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right|, \quad (25)$$

where  $|\partial \mathbf{f} / \partial \mathbf{x}|$  is the modulus of the determinant of the Jacobian matrix of  $\mathbf{f}$  at  $\mathbf{x}$ . With the notation from Proposition 1 but  $\phi$  replaced by  $\mathbf{x}$ ,  $\mu$  is the Lebesgue measure on  $\mathbb{R}^n$ ,  $d\mu(\mathbf{x}) = d\mathbf{x}$ ,  $d\tilde{\mu}(\mathbf{y}) = d\tilde{\mathbf{y}}$ , and

$$r(\mathbf{x}) = \frac{d\tilde{\mu}(\mathbf{x})}{d\mu(\mathbf{x})} = \left| \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right|. \quad (26)$$

This explains once more how the homogenization of the amplitude from eq. (17), despite not involving  $r^2(\phi)$ , leads to its appearance in eq. (22), by using the generalized scaling property of the Dirac delta distribution.  $\square$

**Observation 10.** This explains that, while the probability density of the ontic states  $\phi$  is  $r^2(\phi)d\mu$ , the state vector  $|\psi\rangle$  taking part in the Schrödinger equation is composed by using the density  $d\tilde{\mu}$ , as in eq. (17). In fact, this is the continuous limit of eq. (11).

## B. Wavefunction or wavefunctional?

Subsystems admit observables that cannot be diagonalized simultaneously, so their continuous bases depend on the observable. But since different measurement settings ultimately translate into distinguishing macrostates defined by the same set of macro projectors, the ontic basis from Observation 3 and Principle 1 is consistent with any observables we measure for the subsystems [24]. This universal basis can be taken as representing “classical states”, which may be called *ontic states*. Theorem 1 allows us to interpret the Born rule for any measurement as “counting” such ontic states.

But what are these ontic states? Since each particle is represented on a Hilbert space of wavefunctions that have, among their degrees of freedom, the positions, which play a role in any measurement, and also form a continuous basis, it may be tempting to interpret the ontic states as position eigenstates, as in Example 1. But we know that in fact the universe is not described by nonrelativistic quantum mechanics, but by quantum field theory, in which there are no localized particles.

A unique basis  $(|\phi\rangle)_{\phi \in \mathcal{C}}$  that really is ontic or classical is possible in quantum field theory. In the Schrödinger wavefunctional formulation of quantum field theory [18, 19],  $\mathcal{C}$  becomes the configuration space of classical fields, and the Schrödinger *wavefunctional*

$$\Psi[\phi] := \langle \phi | \Psi \rangle \quad (27)$$

replaces the nonrelativistic wavefunction. Here,  $\phi$  stands for a collection of classical fields,  $\phi = (\phi_1, \dots, \phi_n)$ . The configuration space  $\mathcal{C}$  is endowed with a measure  $\mu^1$ .

### C. Macro-classicality

The wavefunctional formulation represents quantum states in terms of classical field states, in the sense that the wavefunctional is a complex functional defined on the configuration space of classical fields. The usual Fock representation can be obtained from the basis  $(|\phi\rangle)_{\phi \in \mathcal{C}}$  [18]. The Fock representation can then be used to interpret the quantum fields in terms of more commonly used nonrelativistic quantum mechanical wavefunctions and operators. But this is a departure from the more foundational description provided by wavefunctionals.

We never observe individual particles directly, but only macrostates. Macrostates are imported from the classical theory, and they are empirically adequate, because at the macro level the universe looks classical. Therefore, Principle 1, which says that states of the form  $|\phi\rangle$  belong to macrostates, *i.e.* for every  $|\phi\rangle$  there is a macrostate  $\hat{P}_\alpha \mathcal{H}$  so that  $|\phi\rangle \in \hat{P}_\alpha \mathcal{H}$ , makes sense.

**Principle 2.** At any instant, at the macro level, a classical universe in the classical state  $\phi$  looks the same as a quantum universe in the quantum state  $|\phi\rangle$  or linear combinations of such states from the same macrostate.

And indeed, it took us a very long time to realize that our universe is not classical, but quantum.

<sup>1</sup> To admit a Lebesgue measure  $\mu$ , the classical configuration space  $\mathcal{C}$  should be finite-dimensional. Maybe it is, because the *entropy bound* [4, 5] requires the Hilbert space of fields defined on compact regions of space to have finite dimension. Also the fields are constrained by equations, the gauge degrees of freedom have to be factored out, and there are severe constraints related to the arrow of time [26]. So we assume that  $\mathcal{C}$  admits a measure  $\mu$ .

### D. Interpretation of complex numbers

Recall that eq. (17) is based on absorbing the phase factor in the vector by substituting  $|\phi\rangle \mapsto e^{i\theta[\phi]}|\phi\rangle$ , done just before stating Proposition 1. This substitution depends on the state  $|\Psi\rangle$ , in particular  $\theta[\phi]$  changes in time. So we cannot simply interpret  $|\Psi\rangle$  directly as a probability density over the classical states.

But the phase change  $|\phi\rangle \mapsto e^{i\theta[\phi]}|\phi\rangle$  can be identified with an U(1) gauge transformation of the classical field, denoted  $\phi \mapsto e^{i\theta[\phi]}\phi$  (in fact U(1) acts differently on different fields, but I will use a uniform notation for its action), so that

$$e^{i\theta[\phi]}|\phi\rangle \equiv |e^{i\theta[\phi]}\phi\rangle. \quad (28)$$

This makes sense because (1) multiplying a state vector with a phase factor changes the vector, but not the physical (quantum) state it represents, and (2) an U(1) gauge transformation of a classical field represents the same physical (classical) state.

Charged and spinor fields, and electromagnetic potentials, admit a nontrivial U(1) symmetry, but it is sufficient that  $\phi$  includes one such field. The gauge transformation depends on the state  $|\Psi\rangle$ , so it changes in time.

**Observation 11.**  $\Psi[\phi]$  can be made real by changing the global U(1) gauge of the basis of classical fields.

**Principle 3.** The wavefunctional  $|\Psi\rangle = \int_{\mathcal{C}} |\phi\rangle d\tilde{\mu}[\phi]$  can be interpreted as a set of gauged classical fields distributed according to a density functional (Fig. 2).

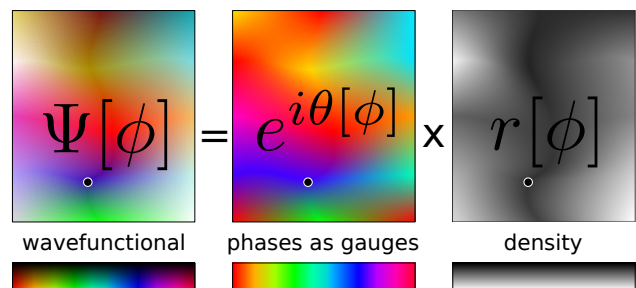


FIG. 2. **Interpretation of the wavefunctional.** The U(1) gauge or phase is represented by the pure color hues in the color wheel. The density  $r[\phi]$ , represented as shades of gray, is the density of state vectors as they combine in  $|\Psi\rangle$  (while the probability density over the sample space  $\mathcal{C}$  is  $r^2[\phi]$ ). Their combination gives the wavefunctional  $|\Psi\rangle = \int_{\mathcal{C}} |\phi\rangle d\tilde{\mu}[\phi]$  as a set of classical fields with varying densities and gauges.

In this way, the probability density  $r^2[\phi]$  is similar to the probability density in a classical system. This relation becomes visible for a quantum system obtained as the Koopman-von Neumann representation of a classical system [20, 27]. In this case, indeed the classical probability density becomes  $r^2[\phi]$ .

### E. Local beables

There are several benefits in using the interpretation of the wavefunctional from Principle 3 as starting point in the investigations of the foundations of quantum theory. It is more foundational, since quantum field theory is more foundational than nonrelativistic quantum mechanics. It comes with an ontology – each state  $|\phi\rangle$  corresponds to a set of fields defined on the 3d-space, not on the configuration space. These fields are the *local beables*. The necessity of local beables was extensively advocated by Bell [6]. The Born rule can be interpreted in terms of such ontic states, based on Principle 1.

A state does not consist of a single ontic state, but of a set of such states (Principle 3). The Projection Postulate should not be understood as collapsing the system to a basis state  $|\phi\rangle$ , no measurement can extract the complete information about the state of the entire universe. Only the ontic states making  $\Psi[\phi]$  belonging to the resulting macrostate  $\hat{P}_\alpha\mathcal{H}$  should remain after the projection.

**Observation 12.** From Difficulty 1 we see that even in standard quantum mechanics the probability density  $r^2[\phi]$  should be understood as self-location probability, not as epistemic probability (see Observation 5). Before the collapse occurs, the state is a superposition of ontic states. But even after the collapse occurs, the state is still a superposition of ontic states, because the macroprojectors are always compatible with more ontic states.

### F. Many-worlds

If decoherence makes the components of  $\Psi[\phi]$  corresponding to different macrostates no longer interfere, there is no need to invoke the Projection Postulate, and we can adopt the *many-worlds interpretation* (MWI).

Observation 12 shows that we should already assume many-worlds even in standard quantum mechanics, particularly if we want it to satisfy Condition 1 and to be able to support the derivation of the Born rule from Theorem 1 interpreted according to Principle 1.

**Observation 13.** “Counting” micro-branches that correspond to the basis  $(|\phi\rangle)_{\phi\in\mathcal{E}}$  gives the correct probabilities in the MWI, in agreement with Condition 1. Even if, unlike the macro-branches, the micro-branches may interfere in the future, they interfere within the same macro-branch. Moreover, since each micro-branch consists of classical fields  $\phi$ , and since these are the local beables, it becomes justified to count each micro-branch as a world, as stated by Principle 1.

**Observation 14.** We should also include quantum gravity in our foundational investigations of quantum theory. In background-free approaches to quantum gravity, it becomes impossible to physically interpret all linear combinations as superpositions, because states in which the

geometry of space is different cannot be superposed unambiguously, so the ontic states dissociate automatically [25]. They can reassociate, unless the dissociation becomes irreversible due to decoherence. This provides an additional justification for the many-worlds interpretation (in the revised form from [25]).

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