Spacetime Conventionalism Revisited

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Abstract. We characterise and critically evaluate five formulations of the thesis that the structure of spacetime is conventional, rather than empirically determined. The proliferation of formulations comes from considering generalisation first, via a more liberal Carnapian understanding of the conventionalism argument in terms of universal effects, rather than Reichenbachian universal forces, and second, via a more liberal understanding of the modal context of the empirical underdetermination, that is as underdetermination between observers rather than models. Whilst three of the five formulations of conventionalism will be found to fail, two are found to open up new interesting problems for researchers in the foundations of general relativity. In all five cases, our analysis will explore interplay between geometric identities, symmetry, conformal structure, and the dynamical content of physical theories. The conventionalism dialectic is thus deployed towards as a tool of explication, clarification, and exploration.

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1. Conventionalism About What?

Conventionalism about geometry is the view that statements about the geometry of space or spacetime are not empirical facts, but rather depend on certain freely stipulated conventions. Thus: the straightness of a line drawn on a page is only defined with respect to a standard ruler; and, the isochrony of a ticking watch is only defined with respect to a standard clock. This perspective was proposed by Poincaré (1902), following in the tradition of Riemann (1854), and developed in a variety of forms by Dingler (1911), Carnap (1922), and Reichenbach (1928).[1]

For our purposes, it will prove highly instructive to demarcate specific varieties of conventionalism which we can understand to arise from how one answers the following two questions:

(1) Which *geometric statements* are not *empirically-determined* truths?
(2) Which physical *differences* arise in the comparison of different *conventions* about such statements?

Reichenbach (1928) famously responded to the first question that statements regarding the spatial or spacetime *metric tensor* of any given model are not empirically determined; and, to the second question that the physical difference between two metrics (and thus two models) would manifest as an apparent *universal force*. In particular, Reichenbach (1928), like Poincaré in the disc experiment, compares the measuring rod of observers living in two different worlds having flat spacetime. In this thought experiment, inhabitants of one of the world have rulers under the effect of heat and perceive the spacetime as curved. However, "(h)eat as a force can thus be demonstrated directly" (p. 13) because heat affects different materials differently. As a result, the geometry of spacetime is underdetermined if the heat is taken as a universal force:

> "Thus the only distinguishing characteristic of a field of heat is the fact that it causes different effects on different materials. But we could very well imagine that the coefficients of heat expansion of all materials might be equal then no difference would exist between a field of heat and the geometry of space." (Reichenbach 1928, p. 26)

A great deal of modern philosophy of science regarding the conventionality of geometry has been directed to the evaluation the conventionalist thesis as considered in Reichenbachian form (Grünbaum 1968; Sklar 1974, 1985; Glymour 1977; Friedman

[1]See Ben-Menahem (2006) for an overview. Ivanova (2015a,b) gives a helpful analysis of Poincaré's conventionalism, and Torretti (1978) of Dingler's. A translation of the 1919-1921 doctoral dissertation of Carnap (1922) can be found in Carnap (2019).

1983; Malament 1985; Norton 1994; Dewar et al. 2022). Our discussion will consider one such treatment due to Weatherall and Manchak (2014) in terms of the view we demarcate as *Spacetime Conventionalism 1* in the following Section 2.

What forms of conventionalism might we consider beyond the specific Reichenbachian form? As just noted, in his original argument Reichenbach took metrical relations of spacetime to be conventional and made the specific assumption that the choice in convention could be underdetermined by introducing universal forces. Significantly, Dieks (1987) argues that it is in the Reichenbachian spirit to interpret gravitation as a universal 'force', since any effect of the curvature of spacetime can be explained by a suitable gravitational effect. Indeed, in Reichenbach (1928) we find the following text:

> [The] universal effect of gravitation on all kinds of measuring instruments defines therefore a single geometry. In this respect we may say that gravitation is *geometrized*. (Reichenbach 1928, p. 256) (italics in original)

An explicit defence of this more liberal 'universal effect' reading can be found in the later work of Carnap (1966):

> Reichenbach called them 'forces', but it is preferable here to speak [...] in a more general way, as two kinds of 'effects'. (Forces can be introduced later to explain the effects.) [p. 169]

The first dimension of novelty in our analysis is the re-framing of the debate via the Carnapian form of the conventionalist thesis. In particular, we will consider different formulations of the conventionalist thesis in terms of different proposals for the *spacetime structure* that could be understood to be empirically underdetermined and the *physical differences* that are taken to result from different proposals for the relevant *universal effect*.

A second dimension in which one can generalise the form of the conventionalist thesis is in terms of the *modal context* of empirical underdetermination thesis. The idea is that the context in which one is answering the questions (1) and (2) is itself underdetermined to the degree that it might be specified as obtaining between two observers within a single model of a theory rather than two models within the same theory, or two models of different theories. Careful consideration of this aspect will be combined with the Carnapian generalisation to effects as the two principal dimensions of analysis framing of our discussion.

A third issue that connects the different parts of our project is the formal interplay between geometric identifies, symmetry, conformal structure, and dynamical equations within the foundations of general relativity. Fascinatingly, each of the forms of conventionalism we will consider, to at least some degree, trades on a package of re-reading the interconnection between these aspects of the formalism. Of particular significance will be the role of the Bianchi identities of general relativity, variously understood as formal geometric identities, conservation equations, and dynamical field equations, and of decomposition of tensor fields into conformally invariant and non-conformally invariant parts. Outlining clearly the interplay between various of these formal relations will be a subsidiary major goal in what follows.

The fourth aspect of our project concerns dialectical context in which the conventionalism thesis is being proposed and analysed. Our goal in what follows is neither bury nor resurrect the view. Rather, the revisitation of spacetime conventionalism is deployed as an tool of foundational analysis towards both clarification of old conceptual problems, or indeed pseudo-problems, and the identification of new problems. Articulation of new conceptual problems is, in turn, understood in a Laudanian spirit as a constitutive element of scientific progress (Laudan 1977). Thus, our ultimate aim is to apply conventionalism as a lense to advance scientific understanding of spacetime theory.

In Section 2 our analysis commences with consideration of a recent formulation of the classical Reichenbachian form of conventionalism. In particular, we will briefly review a precise formulation of the thesis as articulated and refuted by Weatherall and Manchak (2014). We will conclude that as powerful and convincing as this analysis is within the relevant domain, the field is still open for consideration of varieties of the more generalised forms of conventionalism.

In Section 3 we present our first attempt at formulating a Carnapian form of conventionalism with reference to inertial structure and putative universal effects that result from differing representations local stress-energy 4-current by coincident observers following different world-lines. Ultimately, the view is found to be unconvincing, since there are good reasons to reject the physical salience of the energy 4-current and thus this prima facie plausible form a conventionalism can be dissolved.

In Section 4 we then consider conformal structure and the universal effect of tidal deformation. We argue that by considering the invariant decomposition of the Riemann tensor into Weyl and Ricci parts we can dissolve a further potential source of conventionalism with regard to the deforming and non-deforming aspects of tidal effects. In each case, although the new forms of conventionalism are found to fail, our discussion serves to elucidate new perspectives on the representational roles of different objects within the foundations of general relativity.

In Section 5, we pull our focus back to a wider form of interpretational question regarding the dynamical understanding of the Weyl tensor fields and the role of the Bianchi identities. This analysis serves to open up the possibility of a very different form of conventionalism in terms of how to isolate the dynamical content of general relativity.

Finally, in Section 6, we consider a proposal based upon the decomposition of the Einstein tensor into a conformally invariant and non-conformally invariant parts. The potential for underdetermination of this decomposition is then taken to be the hallmark of a novel form conventionalism whose validity is expressible in terms of (the failure of) a precise mathematical 'Bach conjecture', whose truth is as yet unknown.

2. Metric Structure and Universal Forces

Spacetime Conventionalism 1

(1) The *metric structure* of *relativistic spacetime* is not an empirically-determined truth.

(2) Physical differences *between conformally equivalent models of relativistic spacetime* with regard to the existence of *universal forces* arise in the comparison of different conventions about *affine structure*.

Any pair of conformal equivalent relativistic spacetimes are, by definition, such that $(M, g_{\mu\nu})$ and $(M, \tilde{g}_{\mu\nu})$ where $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$. Each of these spacetimes is accompanied by a metric compatible, torsion-free, Levi-Civita derivative operator, ∇ and $\tilde{\nabla}$ respectively. Relativistic spacetime conventionalism can then be formulated as the thesis that the *same curve*, γ, can be a geodesic relative to ∇ but be associated with an acceleration relative to $\tilde{\nabla}$ given by a 'universal force' induced by a tensor field $G_{\mu\nu}$ via the equation $G^{\nu}_{\mu}\tilde{\xi}^{\mu}$ where $\tilde{\xi}$ is the tangent field to γ with unit length relative to $\tilde{g}_{\mu\nu}$. So formulated a relativistic conventionalist thesis, equivalent to Spacetime Conventionalism 1, is provably false (Weatherall and Manchak 2014, Proposition 2).

A few remarks are in order here. First, it is noteworthy that Reichenbach himself does not, in fact, enforce a requirement of conformal equivalence in his formulation of the thesis. Rather, this is added by Weatherall and Manchak (2014) as a further interpretative step following the observation of Malament (1977) that Reichenbach himself argued in other writings that the causal structure of spacetime was non-conventional. Since two spacetimes are conformally equivalent just in case they agree on casual structure the assumption that conformal equivalence is well justified as a Reichenbachian one. What is of particular interest for our project is the role that we already find an interplay between the putative conventionality of spacetime and conformal structure. This intersection of topics will be found to persist as a theme throughout our analysis.

Second, as Weatherall and Manchak (2014) accept, their formulation of the conventionalist thesis takes place 'closer to the ground floor of spacetime physics' (p. 234) than a maximally general formulation in that it is assumed that forces are represented by rank 2 tensor fields associated with the acceleration of a test particle and spacetimes are represented by Lorentzian manifolds with torsion free derivative operators. To see why such an assumption might be questioned, let us suppose a term K was added to the Einstein-Hilbert. We would then have that:

$$S = \int dx^4 \sqrt{-g}(R + K) \tag{1}$$

In this context, we might then ask why the resulting 'force' from the action would have to require a rank-2 contribution from K, since the rank of K can take any value. In fact, Weatherall and Manchak (2014) note precisely such possibility:

> This option may even be compatible with our description of force fields above, although much more would need to be said about how such a field would give rise to forces and what properties it would have. (p. 245)

Following our Carnapian outlook, one might argue that one need not consider 'forces' at all, rather one need only consider the 'effects' of the term K.

In sum, the argument of Weatherall and Manchak (2014) is self-consciously targeted at a fairly limited and conservative notion of relativistic spacetime conventionalism within the Reichenbachian form. As powerful and convincing as this analysis is within this domain, the field is still open for consideration of varieties of the more generalised Carnapian form of conventionalism. It is to the different possibly further articulations of that project that we turn in the following sections.

We arrive at the form of conventionalism most closely resembling a generalised version of the Weatherall and Manchak (2014) in Section 6. At this initial stage it will prove most profitable first consider the idea of re-embedding the conventionalist dialectic in the context of underdetermination between observers in a single model.

3. Inertial Structure and Local Stress-Energy Flux

Spacetime Conventionalism 2

(1) The *inertial structure* of a *particular relativistic spacetime* is not an empirically-determined truth.

(2) Physical differences between *coincident observers* with regard to *local stress-energy flux* arise in the comparison of different conventions about which *time-like vector field* is chosen as an observer's inertial frame.

The second form of conventionality of spacetime conventionality we will consider relates to the interpretation of *different* observers of the *same model*, and thus the *same geometric facts*, in terms of *different local dynamical facts*. The idea is that within a spacetime there can be observers that are coincident and yet will disagree about the dynamical interpretation of the same local geometric facts. The formal property that we will isolate in the privileged class of observers who have the analogue of a vanishing universal force is that they are *Killing observers* and thus the answer to the first of our two conventionalism questions is *inertial structure*. The physical differences that feature in the second question are then with regard to local stress-energy flux.

We can better understand the motivation for Conventionalism 2 by briefly considering the problem of the local definition of conservation of energy in general relativity.[2] The problem can be straightforwardly stated as follows. For a vanishing cosmological constant the Einstein equations take the familiar form:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu} \tag{2}$$

[2]See Pitts (2010, 2016, 2021); Lam (2011); Read (2020); Dewar and Weatherall (2018); Duerr (2019b,a) for discussions of the problem of defining gravitational energy. A further noteworthy paper, along lightly different lines, is Lehmkuhl (2011). There it is argued that in the context of general relativity energy-momentum density as expressed by $T_{\mu\nu}$, cannot, in fact, be regarded as an intrinsic property of matter, but rather should be understood as a relational property that matter possesses only in virtue of its relation to spacetime structure. For an outstanding general discussion of energy conditions in general relativity see Curiel (2014).

where $T_{\mu\nu}$ is energy-momentum tensor associated with all matter fields and their interactions, and $G_{\mu\nu}$ is the Einstein tensor which is determined by the metric $g_{\mu\nu}$ via the Ricci curvature $R_{\mu\nu}$ and Ricci scalar R.

The Levi-Civita derivative, ∇_μ, can be explicitly constructed from the metric and an arbitrarily chosen derivative operator. Given such a derivative operator on a manifold M it can be proved that there exists a unique smooth tensor field on M such that for all smooth fields ξ^σ:

$$R^{\rho}_{\sigma\mu\nu}\xi^\sigma = -2\nabla_{[\mu}\nabla_{\nu]}\xi^\rho \tag{3}$$

(Malament 2012, Lemma 1.8.1). $R^{\rho}_{\sigma\mu\nu}$ is then the unique Riemann curvature tensor field associated with ∇_μ. The Ricci curvature tensor is defined via the contraction of the Riemann curvature as:

$$R_{\sigma\mu} = R^{\rho}_{\sigma\mu\rho} \tag{4}$$

An important identity that holds for any Riemann tensor is the second Bianchi identity. For the torsion free case this can be expressed in terms of the Levi-Civita derivative as:

$$\nabla_{[\alpha}R^{\rho}_{|\sigma|\mu\nu]} = 0 \tag{5}$$

This equation follows from the basic mathematical properties of the Riemann tensor and is thus naturally understood in this context as a formal identity without physical content.[3] Interestingly, however, when combined with the definition of the Einstein tensor the Equation (5) can be shown to imply:[4]

$$\nabla_\mu G^{\mu\nu} = \nabla_\mu T^{\mu\nu} = 0 \tag{6}$$

At this point, it is very tempting to read the equation $\nabla_\mu T^{\mu\nu} = 0$ as an expression of the local conservation of energy (or stress-energy) and it seems like we have derived a dynamical conservation equation from a formal identity. However, such a naïve way of thinking about energy conservation in GR immediately runs into trouble.

Consider an observer moving along the integral curve of an arbitrary time-like vector field ξ^μ. Let us define the *stress-energy 4-current* along the direction defined by this vector as $j[\xi^\mu] = T^{\mu}_{\nu}\xi^\nu$. This 4-current *does not* in general vanish, so we cannot be guaranteed that $\nabla_\mu j[\xi^\mu] = 0$ by the fact that $\nabla_\mu T^{\mu\nu} = 0$. This might be interpreted to mean that we no longer have 'strict' conservation of energy according to the curved spacetime version of Gauss' law (Wald 1984, pp. 69-70).

[3] See (Malament 2012, p.69-79) and (Landsman 2021, p.61) for proofs.
[4] See (Malament 2012, p.322). N.b. one also uses further symmetry properties of the Riemann tensor in this short derivation.

Moreover, we might plausibly interpret $j[\xi^\mu]$ as the conserved current associated with the observers time translations and thus understand $\nabla_\mu j[\xi^\mu] = 0$ to imply a failure of conservation of locally measured energy. Next, recall that the class of Killing vector fields are given by the satisfaction of the Killing equation:

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \tag{7}$$

This is equivalent to the statement that the metric is invariant along the integral curves of ξ_ν or, in terms of the Lie derivative, that we have that $\mathfrak{L}_\xi g_{\mu\nu} = 0$. The existence of a time-like Killing vector field is then (in this sense) equivalent to the symmetry of time translation invariance holding along the relevant time-like curves. The existence of a time-like Killing vector field is then necessary and sufficient for the vanishing of the stress-energy 4-current. This, in turn, implies that for the class of *Killing observers* whose, worldlines are integral curves of the vector fields solving (7), the vanishing of the stress-energy 4-current and thus 'strict' conservation of energy *is* guaranteed.

We can then formulate Spacetime Conventionalism 2 by interpreting the stress-energy flux for the non-Killing observers as playing the functional role of a Carnapian universal effect. Killing and non-Killing observers can be coincident within the same spacetime patch but will attribute different interpretations to the same geometric facts in terms of the presence or absence of this additional associated flux. Putative physical differences with regard to local stress-energy flux thus can be understood to arise in the comparison of different conventions about which timelike vector field is chosen as an observers rest frame. The existence or not of such effects depends upon a convention as to choice of timelike vector field as a rest frame.

Three clear lines of response to this argument for Spacetime Conventionalism 2 are available. The first is simply to note that in the general class of Einstein spacetimes Killing observers are not guaranteed to exist at all. The existence or not of Killing vector fields depends upon there being non-trivial continuous isometries of the metric, and there are good reasons to believe that such transformations do not exist in general for physically realistic spacetimes.[5] In a generic spacetime

[5]Following the results of Fischer (1970), what can be proved explicitly is that for globally hyperbolic Einstein spacetimes which admit an initial value formulation, the topology of the space of physically distinguishable initial states (i.e. the 'superspace' composed of distinct Riemannian three-geometries) is such that the geometries admitting continuous isometries are singular points. Furthermore, initial data without non-trivial Killing vector fields is generic in the sense that 'geometries of high symmetry are completely contained in the boundary of geometries of lower symmetry' (p. 303). There is thus a good formal basis to expect that a generic (globally hyperbolic) Einstein spacetime will lack Killing vector fields. At a simpler, more intuitive, level it is not hard to convince oneself that the any physically realistic spacetime will be too 'messy' in terms of inhomogeneities and anisotropies to contain Killing vector fields.

we can therefore expect that there will not be any set of 'freely falling' observers who measure locally vanishing stress-energy 4-current. Spacetime Conventionalism 2 would then be best understood as a viable but highly limited conventionalist position.[6]

The second response relies upon the introduction of further structure into the theory. The first step is to note that the *magnitude* of the stress-energy 4-current depends upon the affine properties of the world-line of the observer. This means that the *quantity* given the the flux into a region is still conventional in the sense that it will be different for different observers moving within the same spacetime patch irrespective of the existence of not of Killing vector fields. In this context, one could consider introduction of a gravitational-energy-momentum tensor $t^{\mu\nu}[g_{\alpha\beta}, \nabla_{\alpha\beta}]$ which is such that a total energy-momentum complex is conserved with respect to the the flat metric's torsion free derivative operator (Pitts 2010, 2016, 2021). This would eliminate any potential for conventionalism with regard to total energy conservation but at the cost of introducing extra background or auxiliary structures into the theory.

This brings us to our third ultimately more satisfactory response to Killing conventionalism: rejection of physical salience of the energy 4-current. The key observation is that it is far from clear that focusing on localised energy fluxes is germane to the context of relativistic physics; arguably the ambiguities we have encountered are a product of focusing on concepts and quantities that are a relic of the bygone, pre-relativistic era. Consider in particular the fact that *the failure of the vanishing of the stress-energy 4-current can occur for non-Killing observers even in flat spacetimes.* This surely indicates that we should not think of it as encoding the results of additional universal effect resulting in the failure of energy conservation. Rather it motivates us to look for a completely general characterisation of the *causal basis* of genuine universal gravitational effects due to curvature without the need for background structure: this is to re-frame the discussion to focus on tidal forces and their explicit geometric representation, rather than energy conservation. To do this it will prove essential to consider the physical significance of conformal structure within general relativity.

[6]It is true that, in general, local conservation of stress-energy will approximately hold for 'small enough' patches of spacetime, with how small depending on the relevant curvature (Wald 1984, pp. 70). Thus, there will, in general, exist 'approximate Killing observers' for whom within some 'small enough' region there are no tidal forces and gravitational stress-energy 4-current vanishes. Operationally, however, *observers cannot be arbitrarily small*, and thus making such a move would violate the spirit of the Reichenbachian argument.

4. Conformal Structure and Tidal Deformation

Spacetime Conventionalism 3

(1) The *local conformal structure* of a *particular relativistic spacetime* is not an empirically-determined truth.

(2) Physical differences between *coincident observers* regard to *geodesic deformation* arise in the comparison of different conventions about the decomposition of the Riemann tensor into deforming and non-deforming parts.

Consider a smooth one-parameter family of geodesics. Define two vector fields: a timelike vector field ξ^μ tangent to the family of geodesics and second vector field χ^ν that represents the infinitesimal displacement to an infinitesimally nearby geodesic. For a given Riemann curvature tensor, the acceleration due to *geodesic deviation* is given by:

$$a^\mu = -R^\mu_{\beta\nu\alpha}\xi^\alpha\xi^\beta\chi^\nu \tag{8}$$

In general terms, tidal 'forces' should be understood as the effects of geodesic deviation as induced by Riemann curvature. There are two physically distinct senses in which a geodesic can undergo deviation. The first is through conformal re-scaling, which would change local spatial scales but preserve local shape. The second is through deformation which would preserve local spatial scales but change local shape. There are good physical reasons for us to consider only the second as an effect that observers would attribute to a genuine physical force since deformation of a body is unambiguously associated with stress.

Spacetime Conventionalism 3 is then the view that seeks to exploit underdetermination in the representation of tidal deformation. The proposal would be that different coincident observers may decompose the consequences of the geodesic deviation given by Equation (8) differently, and so adopt different conventions with regard to degree of tidal deformation. This would not only underdetermine the tidal 'forces' in a physically significant sense but was also underdetermine the local conformal structure of spacetime.

The proposed new formulation of spacetime conventionalism has, however, been set up to fail. It is in precisely in the relevant respect that the Riemann curvature tensor has a unique decomposition in terms of its Ricci and Weyl parts. This decomposition equates to exactly the decomposition of geodesic deviation into deforming and non-deforming parts. We can see this as follows. The Weyl curvature tensor, $C_{\rho\sigma\mu\nu}$ can be defined via the Riemann and Ricci curvature tensors as the traceless tensor:

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{1}{2}\left(g_{\rho[\nu}R_{\mu]\sigma} + g_{\sigma[\mu}R_{\nu]\rho}\right) - \frac{R}{6}\left(g_{\rho[\mu}g_{\mu]\sigma}\right) \tag{9}$$

The tensorial nature of this expression of course means that the decomposition of Riemann curvature into Ricci and Weyl parts is invariant under the push-forward of diffeomorphisms, ϕ of the spacetime manifold M. That is, for any three tensors which are related such that $A = B + C$ we will have that $\phi^{\star}(A) = \phi^{\star}(B + C) = \phi^{\star}B + \phi^{\star}C$. Observers using different coordinate systems in a given spacetime will necessarily agree on the decomposition. Moreover, the decomposition of Riemann curvature tensor into Ricci and Weyl components has its origin in the symmetry transformation properties of the Riemann tensor. In particular, we can explicitly derive the decomposition by considering the symmetric and anti-symmetric parts of a decomposition of the Riemann tensor in terms of SO(3,1) irreducible tensors.[7] We thus find that there is a solid mathematical foundation for taking the distinction as observer independent and intrinsic to the structure of a given spacetime.

A further decomposition of the Weyl tensor then allows an explicit and insightful connection to tidal deformation.[8] Let us first interpret the Weyl tensor as the free gravitational field, and the metric tensor as its (2nd order) potential field and consider a timelike unit vector field $\xi^{\nu}\xi_{\nu} = -1$ representing a family of observers. Next, parallel to the way one can split the electromagnetic field into electric and magnetic parts in the rest frame of ξ^{ν}, we can split the Weyl curvature tensor, $C_{\rho\sigma\mu\nu}$, into electric and magnetic parts constructed as symmetric traceless

[7]Following Ramond (1997) we can decompose $R^{\rho}_{\sigma\mu\nu}$ in terms of $SU(2) \otimes SU(2)$ as it is locally isomorphic to $SO(3,1)$. We decompose the Riemann curvature tensor into symmetric parts

$$(3 \otimes 3) \oplus (5 \otimes 1) \oplus (1 \otimes 5) \oplus (1 \otimes 1) \oplus (1 \otimes 1) \tag{10}$$

and antisymmetric parts

$$(3 \otimes 3) \oplus (3 \otimes 1) \oplus (1 \otimes 3)$$

We then find that the Weyl tensor $C_{\rho\sigma\mu\nu}$ transforms as $(5 \otimes 1) \oplus (1 \otimes 5)$ and it is conformally symmetric. The object that transforms as $(3 \otimes 3)$ in the symmetric part the corresponds to traceless part of Ricci tensor R.

[8]Here we are following Goswami and Ellis (2021). The origin of this decomposition is Matte (1953). Further discussion can be found in Bel (2000). Thanks to Juliusz Doboszewski for help with these historical sources.

tensors orthogonal to ξ^ν. The explicit expressions are:

$$E_{\rho\sigma} = C_{\rho\sigma\mu\nu}\xi^\mu\xi^\nu \tag{11}$$

$$H_{\rho\sigma} = \frac{1}{2}\epsilon_{\rho\nu\alpha}C^{\nu\alpha}_{\sigma\mu}\xi^\mu \tag{12}$$

where $\epsilon_{\rho\sigma\mu}$ is the effective volume element in the rest space of the comoving observer and is explicitly given by:

$$\epsilon_{\rho\sigma\mu} = \sqrt{|\det \mathrm{g}|}\delta^0_{[\rho}\delta^1_\sigma\delta^2_\mu\delta^3_{\nu]}\xi^\nu \tag{13}$$

(Goswami and Ellis 2021, Appendix A). The equation for geodesic deviation due to the Weyl curvature is then simply:

$$a^\mu = -C^\mu_{\beta\nu\alpha}\xi^\alpha\xi^\beta\chi^\nu \tag{14}$$

$$= E^\mu_\nu\chi^\nu \tag{15}$$

We can thus understand the electric part of the Weyl tensor as uniquely responsible for the tidal deformation effect – that is, geodesic deviation that changes the shape of bodies in geodesic motion.[9] Tidal deformation is a non-conventional effect which can be associated with a specific part of the Weyl curvature tensor of any given spacetime. Spacetime Conventionalism 3 is a resounding failure.

5. Coupling and Dynamical Fields

Spacetime Conventionalism 4

(1) The *dynamical or non-dynamical* interpretation of a *particular tensor field* in a relativistic spacetime is not an empirically-determined truth.

(2) Physical differences with regard to *dynamical and non-dynamical* interpretations of field equations arise in the comparison of different conventions about whether certain *fields are understood to couple or interact* within equations.

Whilst the dynamical role of the Weyl tensor forecloses the possibility for a putative conventionalism about tidal deformation, it opens up an opportunity for a more interesting, and ultimately more tenable, form of conventionalism with regard to the dynamical and non-dynamical interpretations of the equations that enforce the connection between Weyl and Ricci curvature. We can articulate the line of thought behind this form of conventionalism as follows.

[9]There is also an important connection between the electric part of the Weyl tensor and gravitational waves - see Goswami and Ellis (2021) for more details.

First, recall that solving the Einstein equation (2) *does not* uniquely determine the Riemann curvature. This can be seen most obviously in the example of vacuum Einstein spacetimes which are defined as the class of Ricci flat Lorentzian manifolds – i.e. Einstein spacetimes in which the Ricci curvature tensor $R_{\mu\nu}$ is zero. The Riemann curvature of a Ricci flat Lorentzian manifold is entirely determined by the Weyl curvature tensor $C_{\mu\nu\alpha\beta}$. In contrast, we might also consider Weyl flat spacetimes, in which the Ricci curvature encodes all geometric degrees of freedom. Weyl flat spacetime cannot admit any local deformations of shape and are illustrated most vividly by the FLRW spacetimes that approximately describe our universe. The intersection of the Ricci flat and Weyl flat cases is then the unique Riemann flat spacetime: Minkowski spacetime. In general, a spacetime will have non-zero Ricci and Weyl curvature and the obvious question is then how these two forms of curvature can be related if the Einstein equation only relates to the Ricci part.

The answer is found in the Bianchi identities. Equation (5) can be re-written as:

$$\nabla_{[\alpha} R_{\mu\nu]\rho\sigma} = 0 \tag{16}$$

this can be expanded via the Weyl-Ricci decomposition (9) as:

$$\nabla_\nu C^{\rho\sigma\mu\nu} = \nabla^{[\sigma} R^{\rho]\mu} + \frac{1}{2} g^{\mu[\sigma} \nabla^{\rho]} R \tag{17}$$

In the context of the further decomposition of the Weyl tensor as per Equation (11), the contracted Bianchi identities then lead directly to the temporal and spatial derivatives of the electric and magnetic part of the Weyl tensor, and these equations then lead in vacuo to the propagation equations for gravitational waves (Goswami and Ellis 2021).[10]

The next crucial step towards Spacetime Conventionalism 4 is to consider the form of Maxwell's equations:

$$\nabla_\sigma F^{\rho\sigma} = J^\rho \tag{18}$$

where $F^{\rho\sigma}$ is the electromagnetic field tensor and J^ρ is the source current, and note that we can re-write the equations (17) such that they take an analogous form:

$$\nabla_\nu C^{\rho\sigma\mu\nu} = J^{\rho\sigma\mu} \tag{19}$$

[10]For the original treatments along related lines see Newman and Penrose (1962); Newman and Unti (1962); Hawking (1966).

where $J^{\rho\sigma\mu} = \nabla^{[\sigma} R^{\rho]\mu} + \frac{1}{2} g^{\mu[\sigma} \nabla^{\rho]} R$ (Hawking and Ellis 1973, p.85). This striking analogy suggests we might think of the Equation (17) as *field equations for the Weyl curvature*, just as the Einstein equations provides field equations for the Ricci curvature. Furthermore, we can use the Einstein equation to re-write the source current directly in terms of stress-energy such that we have an expression of the form:

$$\nabla_\nu C^{\rho\sigma\mu\nu} = \nabla^{[\sigma} T^{\rho]\mu} + \frac{1}{2} g^{\mu[\sigma} \nabla^{\rho]} T^\nu_\nu \tag{20}$$

We would then seem to have a dynamical equation for the Weyl curvature as sourced by the stress-energy tensor (Danehkar 2009). In such circumstances the difference between Weyl and Ricci curvature appears to have disappeared in the sense that both can be seen as *dynamical fields* encoding geometric degrees of freedom that are sourced by stress-energy.

Notwithstanding the argument just given, we can also arrive at a non-dynamical interpretation of the Weyl tensor by starting with a different reading of the *modal status* of the Bianchi identities. That is, we understand the identities as encoding restrictions on *kinematical* rather than *dynamical* possibilities. A Bianchi identity is, in the general case, defined as the consequence of the 'gauge' symmetry properties of a theory via Noether's second theorem.[11] The Noether derivation of Bianchi identities does not require stationarity of the relevant action. Then, in the context of general relativity, as was discussed in Section 3, the second Bianchi identity follows from the basic mathematical properties of the Riemann tensor and is thus naturally understood as a *formal identity without physical content*. As such, Equation (17) is derivable completely independently of the action and would hold for any spacetime theory formulated on Riemannian geometries with a torsion-free connection.

These considerations lead us to a second non-dynamical interpretational option in which we treat the Bianchi identities as fundamental *kinematic* or *pre-nomic* restrictions. In the context of general relativity, this means that the space of kinematically possible models of the theory is preselected such that the identities are obeyed. This, in turn, means that the Ricci and Weyl curvatures are kinematically constrained to be coordinated such that the Riemann curvature obeys the relevant identity. Dynamics is then encoded solely within the Einstein equation which then fixes the dynamically allowed Ricci curvature. Residual freedom within the Weyl

[11]For detailed discussion of Noether's second theorem in a historical context see Kosmann-Schwarzbach (2010). For a rigorous formal overview see Olver (1993). For discussion in context of gauge theories and quantization see Henneaux and Teitelboim (1992).

curvature is can be understood as fixed via the choice of initial conditions. Under such an interpretation the relation between the Ricci and Weyl curvature is a product of a pre-established kinematical harmony not a substantive dynamical relationship. The Weyl tensor is a non-dynamical field.

A useful way of understanding the connection between the interpretation of equations and the interpretation of fields has been suggested by Lehmkuhl (2011):

> [A]n interaction demands that all fields present are dynamical fields [...] it seems sensible to make a distinction between speaking of two fields *interacting* and two fields *coupling*. For a dynamical field can couple to a non-dynamical field [...] but we would not speak of an *interaction* if only one of the two fields was dynamical: a non-dynamical field acts without being acted upon if it couples to a dynamical field. Hence, two fields interacting should be seen as sufficient but not necessary for the fields to couple, whereas two fields coupling is necessary but not sufficient for the two fields to interact. (p.469)

Following this formulation, we would then have it that if the Bianchi identities (17) describe an interaction then the Weyl tensor is a dynamical field of the same status as the Ricci tensor; they each act whilst simultaneously are acted upon. However, if the Bianchi identities (17) describe a coupling, then we should offer a non-dynamical interpretation of the Weyl curvature tensor, it merely couples to the Ricci tensor, the two fields do not interact; the Ricci tensor acts upon the Weyl tensor without being acted upon.

What would appear to be the best strategy for breaking this interpretational underdetermination is to consider the initial value problem. In particular, if the initial value problem of general relativity were found to be fully specifiable independently of the Bianchi identities, one could argue that only the Einstein equations encode dynamical degrees of freedom, and the non-dynamical interpretation is supported. Conversely, if the Bianchi identities play an explicit role in encoding interactions between degrees of freedom within the initial value problem then the dynamical interpretation would be supported.

Unfortunately, in practice what is found does not allow for such a simple means of differentiation. First, we may observe that the Einstein equations are a set of second-order quasiliniear partial differential equations for the metric. However, the system is both overdetermined and underdetermined, and the equations do not have the hyperbolic form that would allow access to strong formal results regarding the initial value problem. For this reasons, approaches to the explicit solution of the initial value problem in general relativity proceed via hyperbolic reductions based upon a particular gauge choice.[12]

The most famous of these is the 'harmonic' or 'wave' gauge treatment due to Choqouet-Bruhat. In this approach, it may be observed that although the Bianchi identities do play a substantive role, they do not form part of the basic system of hyperbolic dynamical equations that is obtained as a reduction of the Einstein equations. This would seem to support the non-dynamical interpretation. However, there exists an alternative approach to the hyperbolic reduction of general relativity due to Friedrich (1996) in which the Bianchi identities *are* explicitly understood as hyperbolic propagation equations for the Weyl tensor. Explicitly, and considering the vacuum case, within this approach, the Weyl tensor is treated as one of the *fundamental dynamical variables* in a system of equations given by:

$$R^{\mu}_{\nu\lambda\rho} = C^{\mu}_{\nu\lambda\rho} \tag{21}$$

$$\nabla_{\mu} C^{\mu}_{\nu\lambda\rho} = 0 \tag{22}$$

which are equivalent to the vacuum forms of the the Einstein equation and Bianchi identity respectively. This is precisely to understand the Einstein equations and Bianchi identities as interaction equations of equal status and, moreover, to explicitly treat the Weyl tensor as a dynamical field.

We thus see that it is far from clear whether study of the initial value problem of general relativity should lead us to attribute a dynamical or non-dynamical role to the Bianchi identities and the Weyl tensor. If anything the cause of conventionalism is strengthen by detailed consideration of this context since the the role of the Bianchi identities, and the implied consequences for the Weyl tensor, appears to depend upon a 'convention' as to which approach towards the hyperbolic reduction to an initial value problem is taken.

At this point, it is perhaps insightful here to draw the analogy between the interpretation of the role of the cosmological constant in the Einstein equation. Consider a universe in which the stress-energy tensor is zero but the cosmological constant is non-zero. The resultant field equation can be understood to describe a 'Λ-vacuum' spacetime and take the form:

$$G_{\mu\nu} + g_{\mu\nu}\Lambda = 0 \tag{23}$$

[12]Here we are following (Choquet-Bruhat 2008, §VI) and (Landsman 2021, 7.5).

Writing the equation in this way makes it natural to think of Λ as a constant of nature. We might, of course, alternatively re-write the same equation as:

$$G_{\mu\nu} = -g_{\mu\nu}\Lambda \tag{24}$$

In this context, it is natural to think of the term Λ as a dynamical source for the Ricci degrees of freedom. But these are, of course, completely trivial re-writings of the same equation and no one would take there to be a genuine interpretational problem in which of the two equations we focus our attention on in the context of general relativity taken in isolation. Some physicists, however, see there to be good reasons *coming from outside the theory* to treat the cosmological constant vs. dark energy interpretational question as a substantive physical issue.[13]

This requirement for reference to external physical or physics principles could also be taken to be required for the case of our two interpretations of Ricci-Weyl relationship. Within general relativity the equations themselves do not give priority to the differing dynamical and non-dynamical interpretations. However, in a wider theoretical context, and considering in particular the relevance to quantization, the distinction between the interpretation may in fact ground genuine physical differences.[14] It is in this sense Spacetime Conventionalism 4 becomes an profitable interpretational viewpoint that offers insight into alternative heustitic strategies for theory extension, rather than a problematic underdetermination of spacetime structure.

6. THE BACH CONJECTURE

Spacetime Conventionalism 5

(1) The *geometric structure* of spacetime is not an empirically-determined truth.
(2) Physical differences with regard which tensor fields play the role of geometric structure, arise through different conventions the decomposition of the Einstein tensor into conformally invariant and non-conformally invariant parts.

[13]For the historical details see Huterer (2011) and Peebles and Ratra (2003).
[14]For example, whereas it is generally the case that kinematical restrictions are converted to super-selection rules in quantization, dynamical restrictions, such as conservation laws, are applied as quantum nomological restrictions, which allows for their violation subject to the uncertainty principle. See Gryb and Thébault (2016) for more discussion in the context of the problem of time and the cosmological constant.

The final form of conventionalism we will consider is, in a sense, a marriage of the formulation of Reichenbachian conventionalism in the spirit of Weatherall and Manchak (2014) with the idea discussed above of understanding conformal invariant tensors as representing the geometric structure of spacetime. The view is, however, to our knowledge entirely novel. Furthermore, it leads to a precise mathematical conjecture the truth of which is as yet unproven.

The basic starting point for this particular strategy is to look to re-express the Einstein equation in a conformally invariant form. That is, seek to show that Einstein's equation is true if and only if,

$$B_{\mu\nu} = 8\pi T_{\mu\nu} + A_{\mu\nu}, \tag{25}$$

where $T_{\mu\nu}$ is the matter-energy tensor appearing in Einstein's equation, $B_{\mu\nu}$ is a conformally invariant tensor, and $A_{\mu\nu}$ is a further tensor that is not conformally invariant. So far this is a purely a formal exercises in analysis. The interpretation move is then to take $B_{\mu\nu}$ to characterise facts about pure spacetime structure, which is, by assumption, understood to be conformally invariant. By contrast, the non-conformally invariant objects $T_{\mu\nu}$ and $A_{\mu\nu}$ are taken to characterise a matter-energy field and a further 'universal effect', restrictively. Clearly, by the theorem of Weatherall and Manchak (2014), $A_{\mu\nu}$ will not be expressible as a Newtonian force. We are thus explicitly making use of our more liberal Carnapian perspective on conventionalism.

The re-formulation (25) we are looking for to fulfil this interpretative requirement is indeed possible, by defining $B_{\mu\nu}$ to be the *Bach tensor*[15] which can be expressed in terms of the Weyl and Ricci tensors as:

$$B_{\mu\nu} := \nabla^{\sigma}\nabla^{\rho}C_{\rho\mu\nu\sigma} + \tfrac{1}{2}C_{\rho\mu\nu\sigma}R^{\rho\sigma}. \tag{26}$$

In four dimensions, the Bach tensor is well-known to be conformally invariant (Bach 1921). Thus, defining $A_{\mu\nu} := B_{\mu\nu} - G_{\mu\nu}$, we find that Equation (25) holds if and only if the Einstein Equation does, as desired.

The pivotal issue in the context of Spacetime Conventionalism 5, is then whether the Bach tensor $B_{\mu\nu}$ is the *unique* conformally invariant two-place tensor, at least up to a multiplicative constant. Adopting the interpretation of spacetime structure and matter-energy above: uniqueness would imply that this decomposition allows one to distinguish uniquely between spacetime structure ($B_{\mu\nu}$) and

[15]Named for Rudolf Bach, not the Baroque German composer. Although, the former was in fact a pseudonym for Rudolf Förster. See Appendix A for discussion of the relation between the Bianchi identities, Bach tensor and the Cotton and Schouten tensors.

matter-energy ($T_{\mu\nu}$) and the 'universal effect' ($A_{\mu\nu}$); and, *non*-uniqueness would amount to a sense in which the distinction between spacetime structure and matter-energy is conventional, in a sense closely related to that suggested by Reichenbach (1928) and Carnap (1966).

Fascinatingly, although this question is both mathematically precise and foundationally important, its status appears, as yet, unsettled. Essentially, the issue is that more generally associated with finding the conformal invariants in the theory of Lorentzian manifolds:

> "Classically known conformally invariant tensors include the Weyl conformal curvature tensor, which plays the role of the Riemann curvature tensor, its three-dimensional analogue the Cotton tensor, and the Bach tensor in dimension four. Further examples are not so easy to come by." (Fefferman 2012, p.1)

Thus, although review of the literature does not reveal any other known conformal invariants in four-dimensions, there is no proof of the absence of such alternatives either. One might thus propose that the debate over conventionality and non-conventionality, as described above, is not just a matter of philosophical debate, but of settling a precise mathematical conjecture, whose truth is not yet known:

Bach Conjecture. *The Bach tensor the unique (up to a multiplicative constant) conformally invariant rank 2 tensor field on a four-dimensional Lorentzian manifold.*

If the Bach Conjecture is false, then Spacetime Conventionalism 5 is true; if the Bach Conjecture is true, then Spacetime Conventionalism 5 is false. We thus take the proof or refutation of the Bach conjecture to be an important problem in the foundations of general relativity that warrants considered attention.

7. SUMMARY

One might reasonably judge proposals for the conventionality of spacetime structure in terms of their respective novelty, interestingness, and plausibility. Our discussion has characterised and critically evaluated five formulations of the conventionality thesis and we hope the reader will agree that all five have the qualities of being both novel and interesting. The first three options considered have been argued to be implausible on grounds that vary from direct refutation (Spacetime Conventionalism 3), to provable falsity under reasonable assumptions (Spacetime Conventionalism 1), to rejection on the grounds of the lack of physical salience of the central undetermined object (Spacetime Conventionalism 2). Nevertheless, each of the three views are worth of consideration, not least since they illustrate important morals regarding the interplay between empirical and mathematical structures within the foundations of spacetime theory. The final two forms of conventionalism we considered to be novel, interesting, and plausible. Moreover, in terms of analysis of the dynamical role of the Bianchi identities (Spacetime Conventionalism 4) and proof or refutation of the Bach conjecture (Spacetime Conventionalism 5), they each open up an new direction of enquiry that we may hope to be profitably pursued in future researches.

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A

APPENDIX A. CONFORMAL TENSORS AND THE BIANCHI IDENTITIES

The Bianchi identities can also be related to the Cotton Tensor $\mathcal{C}_{\rho\sigma\nu}$ (Cotton 1899) and Bach Tensor B_{ab} as follows. Following García et al. (2004), we can decompose curvature into irreducible representations with respect to pseudo-orthogonal group for certain dimensions, n, as follows:

$$n = 1 \quad \rightarrow R_{\alpha\beta} = 0 \tag{27}$$

$$n = 2 \quad \rightarrow R_{\alpha\beta} = Scalar_{\alpha\beta} \tag{28}$$

$$n = 3 \quad \rightarrow R_{\alpha\beta} = Scalar_{\alpha\beta} + \widetilde{R}_{\alpha\beta} \tag{29}$$

$$n \geq 4 \quad \rightarrow R_{\alpha\beta} = Scalar_{\alpha\beta} + \widetilde{R}_{\alpha\beta} + C_{\alpha\beta} \tag{30}$$

where $\widetilde{R}$ denotes traceless Ricci part of the decomposition and every tensor is in their Weyl 2-form. For $n = 3$, the Cotton tensor paper which is given by $DR_{\alpha\beta} = DC_{\alpha\beta} + \frac{2}{n-2}\upsilon_{[\alpha} \wedge \mathcal{C}_{\beta]} = 0$ where D denotes exterior covariant derivative and υ denotes coframe of a Riemannian space of n dimensions.

$$\mathcal{C}_{\rho\sigma\nu} = 2\left(\nabla_{[\alpha}R_{\beta]\gamma} - \frac{1}{2\,(n-1)}\nabla_{[\alpha}Rg_{\beta]\gamma}\right) \tag{31}$$

and for $n = 4$, the Bach tensor can be defined as

$$B_{\alpha\beta} := \nabla^{\mu}\mathcal{C}_{\alpha\mu\beta} + L^{\mu\nu}C_{\alpha\mu\beta\nu} \tag{32}$$

where $L^{\mu\nu}$ is called Schouten tensor and given by

$$L_{\mu\nu} = R_{\mu\nu} - \frac{1}{2\,(n-1)}Rg_{\mu\nu}. \tag{33}$$