A relevantist's glance at Bell's theorem

Fernando Cano-Jorge and Luis Estrada-González

Abstract In this paper, we examine Routley/Sylvan's suggestion that the logic to be employed in quantum mechanics is a relevance logic, and how that would affect the plausibility of certain ideas in the scope of no-go theorems like Bell's.

1 Introduction

Richard Routley's (Sylvan since 1983) ultramodal project announced in "Ultralogic as universal?" [26] (also partly pursued and delineated through the whole volume *Exploring Meinong's Jungle and Beyond*) consists in rethinking and redoing the foundations of mathematics and science to get rid of paradoxes and other counterintuitive outcomes of the more traditional approaches. Such a reformation of foundations would require, according to Routley, rethinking and redoing not only the philosophies of mathematics and science, but most of philosophy up to now. The whole project would be based on a logic quite distinct from classical or standard logic. The target logic should be relevant —i.e. invalidates any argument where premises and conclusions have no content-connection—, paraconsistent —i.e. it invalidates

Fernando Cano-Jorge
Universidad Panamericana, Facultad de Filosofía, Augusto Rodin 498, Insurgentes Mixcoac, C.P. 03920, Benito Juárez, Ciudad de México, MÉXICO. e-mail: fcano@up.edu.mx

Luis Estrada-González
Institute for Philosophical Research-UNAM, Circuito Maestro Mario de la Cueva s/n, Ciudad Universitaria, C.P. 04510, Coyoacán, Ciudad de México, MÉXICO,
Faculty of Physico-Mathematical Sciences-BUAP, Av. San Claudio y 18 Sur, Col. San Manuel, Ciudad Universitaria, C.P. 72570, Puebla, MÉXICO. e-mail: loisayaxsegrob@comunidad.unam.mx

the argument from contradictories to arbitrary conclusions— and ultramodal, that is, not treating necessarily equivalent expressions as logically equivalent.

Since the announcement of the ultramodal project, Routley claimed that a logic in the relevance family could help in avoiding the anomalies of quantum theory. Following the analysis in [22], Routley argues that the alleged counterexamples to Distribution in quantum physics rely on paradoxes of implication, in particular $A \to (A \wedge (B \vee \sim B))$ and related *expansion principles*.

Many years later, Sylvan [31, p. 27] included explicitly, among the "future [i.e. 1995 onward] applications of relevance logics", an analysis of Bell's theorem. Were Sylvan right, this would open the door for re-evaluating certain results in the empirical sciences, just as having second thoughts on logic brought new life to certain ideas in mathematics, such as infinitesimals, naïve set theory or the provability of Church's thesis.[1]

However, no philosopher, logician or scientist tried to fulfill Sylvan's program, at least regarding the foundations of physics. In particular, no one tried to verify Sylvan's claim that the proof of Bell's theorem relies on expansion principles. In this paper, we will provide the basis for such a relevantist analysis of Bell's theorem and of its alleged implications. This would be the first step towards the development of the ultramodal program in the field of philosophy of physics.

The plan of the paper is as follows. In Section 2 we provide a brief reconstruction of Bell's theorem and the discussion surrounding it apt for a philosophical readership. In Section 3, we present a proof of Bell's inequality conveniently formulated in logical notation. We will see that Sylvan was right, and that the proof requires expansion (and suppression) principles. Finally, in Section 4 we discuss some prospects, both physical and philosophical, opened by an eventual rejection of certain steps in the proof of Bell's theorem. For readers not acquainted with relevance logics, we provide an appendix with Hilbert-style presentations of many basic relevance logics.

2 Bell's theorem

Bell's theorem is a response to the famous Einstein-Podolsky-Rosen paradox [10], which is the original realization of an incompatibility between the completeness of quantum theory and common sense assumptions about *locality*. The EPR argument concludes that the explanation of physical reality given by standard quantum theory is incomplete.

In a nutshell, the EPR argument shows that the following two assertions cannot both be true:

[1] [29, Ch. 3] provides a succinct recount of some intuitionistic mathematical theories incompatible with classical mathematics; [24] is still the *locus classicus* for inconsistent mathematics, much of it based on logics belonging to the relevance tradition.

(i) The description of physical phenomena by means of the wave function is a complete description.
(ii) The states of spatially separate objects are independent of each other.

Following Fine [11], the main assumptions involved in the argument are:

Definite Values: The eigenvalue corresponding to the eigenstate of a system is a value determined by the real physical state of that system.
Separability: Spatially separated systems have real physical states.
Locality (EPR): If systems are spatially separate, the measurement (or the absence of measurement) of one system does not directly affect the reality of the other.
Lemma: If quantities on separated systems have strictly correlated values, those quantities have definite values.
Completeness (Fine): If the description of systems by state vectors were complete, then definite values of quantities could be inferred from a state vector for the system itself or from a state vector for a composite of which the system is a part.

The argument's Lemma considers two entangled particles which have been separated by a space-like interval and are such that their spins are correlated; this, together with the assumptions, entails:

Conclusion: Separated systems as described have definite position and momentum values simultaneously (which goes against Heisenberg's Uncertainty Principle). Since this cannot be inferred from any state vector, the quantum mechanical description of systems by means of state vectors is incomplete.

According to Albert [2, p. 61], this is what Einstein, Podolsky and Rosen meant by "complete" when they asked if the quantum mechanical description of physical reality is complete:

Completeness (EPR) A description of the world is complete just in case nothing that's true about the world (nothing that's an "element of reality" of the world) gets left out of that description.

and it is in this sense that the EPR argument shows the incompatibility of propositions (i) and (ii); so one must be discarded.

By itself, Locality (EPR) might appear to many as a quite reasonable assumption about the way things work in our world. After all, if one is convinced about Definite Values and Separability, then common sense dictates that one should also believe Locality (EPR) as a consequence, specially for systems largely distant from each other in space-time. However, the fact that it leads to violations of Heisenberg's Uncertainty Principle, a core tenet of quantum theory, is sufficient to show that a complete quantum theory cannot satisfy Locality (EPR), i.e. it is such that a measurement performed in particle 1 does affect the result of a measurement performed in particle 2 no matter how far apart are they from each other, as if the entangled particles could exchange super-luminal (faster than light) signals between them to "inform" their partner particle which value of the physical quantity for the state vector measured far apart from it must be. The local hidden-variables program constituted, thus, the only hope for a locally causal re-formulation of quantum theory.

Moreover, notice that Locality (EPR) might seem (quite reasonably) grounded on relativistic constraints: given two experiments E_1 and E_2 performed in two different locations separated by a space-like interval, no measurement performed in E_1 can affect the results of measurements performed in E_2, except by signals that travel faster than light (in line with what Einstein called "spooky action at a distance"), which of course seems to go against general relativity theory. We will return to the matter of compatibility between the non-locality of quantum physics and general relativity in Section 4.

In response to the paradox posed by Einstein, Podolsky, and Rosen, John S. Bell considered a quantum theory enriched with additional variables that completely specified the state of a given quantum system, and gave a mathematical definition of EPR's informal assumption of locality in order to prove that no physical theory satisfying his locality condition will be able to fully reproduce the statistical predictions of quantum theory.

Much has been said since Bell's original argument. Even Bell himself revisited the argument in [4]. And most importantly, the theorem has been generalized even further by Clauser, Horne, Shimony, and Holt (CHSH) in [9] so as to apply to realizable experiments. But since quantum theory has been regarded by many scientists and philosophers as a description of the physical nature of our world, Bell's theorem has led some to conclude, in a metaphysical fashion, that our world is non-local. This striking conclusion is commonly regarded as proof of the incompatibility between quantum physics and general relativity, raising suspicion against the soundness of Bell's theorem. Actually, the underlying assumptions of the proof have been called into philosophical and mathematical scrutiny throughout the literature ever since its publication, and most of the discussion of the proof can be carried out in terms of *factorizability* and *Bell-type inequalities*.

First, Bell [3] introduces the EPR argument as the result according to which quantum mechanics could not be a complete theory but should be supplemented by additional variables whose purpose is to restore locality into the theory. Let the parameter λ stand for a more complete specification of the physical state, playing the role of additional variables completing quantum theory; λ may be a variable, a set of variables or even a set of functions, and this makes no difference for Bell's proof. So given a pair of spin one-half particles moving freely in opposite directions, we have

$$A(\mathbf{a}, \lambda) = \pm 1, B(\mathbf{b}, \lambda) = \pm 1 \tag{1}$$

where A is the result of measuring the spin $\boldsymbol{\sigma_1}$ of particle 1 along a selected component in Stern-Gerlach magnets and $\mathbf{a}$ is some unit vector; and similarly for B, regarding particle 2.

The crucial assumption in Bell's proof is not about hidden-variables but about locality: the result B for particle 2 does not depend on the setting $\mathbf{a}$ of the magnet for particle 1, nor A on B. Accordingly, we have then

Locality (Bell) The probability of obtaining results A and B with settings $\mathbf{a}$ and $\mathbf{b}$ given λ can be written as the product of the probability of obtaining result A with setting $\mathbf{a}$ given λ and the probability of obtaining result B with setting $\mathbf{b}$ given λ:

$$P_{\mathbf{a},\mathbf{b}}(A, B|\lambda) = P_{\mathbf{a}}(A|\lambda)P_{\mathbf{b}}(B|\lambda)$$

This idea was already intended in Locality (EPR) but it was not formulated mathematically until Bell's paper as a factorizability condition on the expectation value of the product of the two components $\boldsymbol{\sigma_1} \cdot \mathbf{a}$ and $\boldsymbol{\sigma_2} \cdot \mathbf{b}$:

$$P(\mathbf{a}, \mathbf{b}) = \int d\lambda \rho(\lambda) A(\mathbf{a}, \lambda) B(\mathbf{b}, \lambda) \tag{2}$$

where $\rho(\lambda)$ is the probability distribution of λ.

The experimental settings in the form of (1) and the assumption of Locality (Bell) in the form of (2) entail a contradiction by means of the famous Bell inequality:

$$1 + P(\mathbf{b}, \mathbf{c}) \geq |P(\mathbf{a}, \mathbf{b}) - P(\mathbf{a}, \mathbf{c})| \tag{3}$$

where $\mathbf{c}$ is another unit vector.

Bell's inequality is key for the proof. It is the application of (2) in (3), for certain values, what helps Bell reach a contradiction, i.e. a violation of (3) in the form of $4(\epsilon + \delta) \geq \sqrt{2} - 1$, which requires to consider an array of three Stern-Gerlach magnets with different orientations where spin will be measured along three unit vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. Without Bell's inequality, equation (2) is not enough to derive the conclusion of the theorem.

To better grasp the workings of the inequality in Bell's theorem, we will use David M. Harrison's version of the proof, which in turn is a variant on d'Espagnat's version [12].[2] Consider a collection of macroscopic objects, each characterized by three independent two-valued parameters α, β and γ. For example, take a collection of animals and define the following parameters:

α	vertebrate	β	venomous	γ	warm-blooded
$\sim\alpha$	invertebrate	$\sim\beta$	non-venomous	$\sim\gamma$	cold-blooded

[2] There is an even simpler reconstruction in [6].

Now let $N(\alpha_1, \ldots, \alpha_n)$ be the number of objects that are, or have, the properties α_i. That is, the α_i are sets of things and $N(\alpha_1, \ldots, \alpha_n)$ is the number of objects in the set $\alpha_1 \cap \ldots \cap \alpha_n$ (i.e. the set defined by the satisfaction of all properties α_i for each i). Then,

$$N(\alpha, \sim\beta) + N(\beta, \sim\gamma) \geq N(\alpha, \sim\gamma) \tag{4}$$

means, according to our example, that the number of non-venomous vertebrates added to the number of venomous cold-blooded animals must be larger or equal than the number of cold-blooded vertebrates. Notice that the first term of the left side, i.e. $N(\alpha, \sim\beta)$, includes both $N(\alpha, \sim\beta, \gamma)$ and $N(\alpha, \sim\beta, \sim\gamma)$, and that precisely $N(\alpha, \sim\beta, \sim\gamma)$ is included in the right side term $N(\alpha, \sim\gamma)$.

For suppose x is vertebrate and cold-blooded. Then, x is also either venomous or non-venomous.
Case 1: if x is venomous, then x is venomous and cold-blooded. Case 2: if x is non-venomous, then x is non-venomous and vertebrate.
Hence, x is either non-venomous and vertebrate, or x is venomous and cold-blooded. This argument is general, so for any α, β and γ as considered, and using ^{c} as set-theoretic complement to handle negation, $(\alpha \cap \gamma^c) \subseteq ((\alpha \cap \beta^c) \cup (\beta \cap \gamma^c))$, whence (4).

Since (4) is true for any collection of macroscopic objects, we may write

$$P(\alpha, \sim\beta) + P(\beta, \sim\gamma) \geq P(\alpha, \sim\gamma) \tag{5}$$

to represent the probability that in a random selection we find an object with the specified properties. Though (5) works just fine for macroscopic objects like the animals in our example, it should not be expected to apply to quantum-mechanical spin measurements. Take Harrison's settings:

α	$\lvert\uparrow\rangle$, $\theta = 0°$	β	$\lvert\uparrow\rangle$, $\theta = 45°$	γ	$\lvert\uparrow\rangle$, $\theta = 90°$
$\sim\alpha$	$\lvert\downarrow\rangle$, $\theta = 0°$	$\sim\beta$	$\lvert\downarrow\rangle$, $\theta = 45°$	$\sim\gamma$	$\lvert\downarrow\rangle$, $\theta = 90°$

where $|\uparrow\rangle$ and $|\downarrow\rangle$ represent spin up and spin down, respectively, and the different angles correspond to the rotation performed in the Stern-Gerlach apparatuses with respect to the z-axis in order to arrange them with three different orientations. Each apparatus stacked thus will only let spin up atoms pass through, so if we follow (5) then

$$P(\text{passes}\, \alpha, \text{not}\, \beta) + P(\text{passes}\, \beta, \text{not}\, \gamma) \geq P(\text{passes}\, \alpha, \text{not}\, \gamma) \tag{6}$$

But then (6) cannot be true for our quantum-mechanical setup because violations to Heisenberg's Uncertainty Principle occur: according to standard quantum theory, if we measure β first, half of the beam of atoms will pass through, but if we measure α first, the probability of passing through β is changed. This is a non-commutative relation between operators, precisely the kind of relation addressed in Heisenberg's Uncertainty Principle.

In order to sidestep this difficulty, following Harrison, consider an experiment in which object 1 is picked out randomly and then an object 2 is picked out *with each property opposite to* object 1. Then, $P(\alpha, \sim\beta)$ for object 1 is equal to the probability that object 1 has the property α and the corresponding object 2 has the property β; this may be written as $P(\alpha_1, \beta_2)$. Thus, (5) is modified accordingly as

$$P(\alpha_1, \beta_2) + P(\beta_1, \gamma_2) \geq P(\alpha_1, \gamma_2) \tag{7}$$

which is Bell's inequality.

The following remark in [12, p. 813] addresses the condition of picking object 2 with the opposite properties of object 1:

> The fact that for the two objects each has all the properties opposite to the other corresponds to the fact that in spin measurements if the two polarizers have the same orientation then the two atoms have opposite spins.

Finally, if we apply (7) to Harrison's setup, we have

$$P(|\uparrow\rangle_1^{0^\circ}, |\uparrow\rangle_2^{45^\circ}) + P(|\uparrow\rangle_1^{45^\circ}, |\uparrow\rangle_2^{90^\circ}) \geq P(|\uparrow\rangle_1^{0^\circ}, |\uparrow\rangle_2^{90^\circ})$$

which we can calculate using

$$P(|\uparrow\rangle_1^{\theta_1}, |\uparrow\rangle_2^{\theta_2}) = \frac{1}{2}\sin^2[(\theta_1 - \theta_2)/2]$$

which results in

$$0.146 \geq 0.250$$

establishing the desired violation of Bell's inequality.

In the next section we will see what it takes to prove this inequality and discuss whether any part of the proof can be called into question, as Sylvan thought.

3 The logical basis of Bell's theorem and some irrelevancies in the proof

Following Harrison's analysis, the logical treatment of Bell's inequality is more straightforward. Recall (4):

$$N(\alpha, \sim\beta) + N(\beta, \sim\gamma) \geq N(\alpha, \sim\gamma)$$

Let us rewrite it as follows, which will improve readability when we translate it below into set-theoretic terms:

$$N(\alpha, \sim\gamma) \leq N(\alpha, \sim\beta) + N(\beta, \sim\gamma)$$

As we have seen, $N(\alpha_1, \ldots, \alpha_n)$ is the set $\alpha_1 \cap \ldots \cap \alpha_n$. This means that (4) can be interpreted set-theoretically as follows, given the usual interpretations of addition as symmetric difference, negation as set complement and the partial ordering as set inclusion:

$$(\alpha \cap \gamma^c) \subseteq ((\alpha \cap \beta^c) \cup (\beta \cap \gamma^c)) \cap (((\alpha \cap \beta^c) \cap (\beta \cap \gamma^c))^c) \tag{8}$$

Let Px, Qx and Rx be the defining predicates of the sets α, β and γ, respectively, and then treat intersection as conjunction, union as disjunction, set complement as negation and inclusion as deducibility, as customary:

$$Px \wedge \sim Rx \vdash ((Px \wedge \sim Qx) \vee (Qx \wedge \sim Rx)) \wedge \sim ((Px \wedge \sim Qx) \wedge (Qx \wedge \sim Rx)) \tag{9}$$

Moreover, since we are working simply in the monadic fragment of first-order (classical) logic and quantifiers are playing no substantial role, we can treat any of the predicates as atomic and simplify to

$$P \wedge \sim R \vdash ((P \wedge \sim Q) \vee (Q \wedge \sim R)) \wedge \sim ((P \wedge \sim Q) \wedge (Q \wedge \sim R)) \tag{10}$$

although from to time to time we will use the more expressive notation according to our needs.

(10) strikingly resembles an expansion principle, as Sylvan noticed. Let us make terminology clear with the following definitions:

0-expansion principle: $A \rightarrow B$ is a *0-expansion principle* if and only if (i) A is a proper subformula of B and (ii) $A \leftrightarrows B$ is valid in a logic at least as strong as (classical) **S4**.[3]

Dually,

0-suppression principle: $A \rightarrow B$ is an *0-suppression principle* if and only if (i) B is a proper subformula of A and (ii) $A \leftrightarrows B$ is valid in a logic at least as strong as **S4**.

In rigor, (10) cannot be counted as an expansion principle because it is written not in arrow but in rule form. Consider the following two generalizations of the notion of expansion principle:

1-expansion principle: $A_1, \ldots, A_n \vdash B$ is a *1-expansion principle* if and only if (i) all the A_i's are proper subformulas of B and (ii) $A_1, \ldots, A_n \dashv\vdash B$ is valid in a logic at least as strong as (classical) **S4**.

[3] As in classical modal logic, strict implication $A \prec B$ is defined as $\Box(A \supset B)$, where $\supset$ is material implication from classical logic and $\Box$ is the necessity connective of normal modal logic, and strict co-implication $A \leftrightarrows B$ is defined as $(A \prec B) \wedge (B \prec A)$. Actually, Sylvan counted as "a logic of strict implication" any extension of Feys' **S1**^{o}, but **S4** is the working logic for him most of the times.

2-expansion principle: 'If $A_1, \ldots, A_n \vdash B$ then $C_1, \ldots, C_m \vdash D$' is a *2-expansion principle* if and only if (i) all the A_i's and C_j's are proper subformulas of D and (ii) '$A_1, \ldots, A_n \vdash B$ if and only if $C_1, \ldots, C_m \vdash D$' is valid in a logic at least as strong as (classical) **S4**.[4]

Consider now the following proof of Bell's inequality (4):
Suppose

$$0 \leq N(\alpha, \sim\beta, \gamma) + N(\sim\alpha, \beta, \sim\gamma)$$

which is arithmetically true. Now, add $N(\alpha, \sim\beta, \sim\gamma) + N(\alpha, \beta, \sim\gamma)$ to both sides of the inequality:
$0+(N(\alpha, \sim\beta, \sim\gamma)+N(\alpha, \beta, \sim\gamma)) \leq (N(\alpha, \sim\beta, \gamma)+N(\sim\alpha, \beta, \sim\gamma))+(N(\alpha, \sim\beta, \sim\gamma)+N(\alpha, \beta, \sim\gamma))$

Given that $0 + N(x_1, \ldots, x_n) = N(x_1, \ldots, x_n)$ for any $N(x_1, \ldots, x_n)$, the inequality now is
$N(\alpha, \sim\beta, \sim\gamma)+N(\alpha, \beta, \sim\gamma) \leq (N(\alpha, \sim\beta, \gamma)+N(\sim\alpha, \beta, \sim\gamma))+(N(\alpha, \sim\beta, \sim\gamma)+N(\alpha, \beta, \sim\gamma))$

Now, given that $N(\alpha, \sim\beta, \sim\gamma) + N(\alpha, \beta, \sim\gamma) = (N(\alpha, \sim\gamma) \times (N(\sim\beta) + N(\beta))$ and, furthermore, that $N(\sim\beta) + N(\beta) = 1$ and $N(x_1, \ldots, x_n) \times 1 = N(x_1, \ldots, x_n)$ for any $N(x_1, \ldots, x_n)$, one obtains
$N(\alpha, \sim\gamma) \leq (N(\alpha, \sim\beta, \gamma) + N(\sim\alpha, \beta, \sim\gamma)) + (N(\alpha, \sim\beta, \sim\gamma) + N(\alpha, \beta, \sim\gamma))$

Then, assuming commutativity and associativity for addition, one gets
$N(\alpha, \sim\gamma) \leq (N(\alpha, \sim\beta, \gamma) + N(\alpha, \sim\beta, \sim\gamma)) + (N(\sim\alpha, \beta, \sim\gamma) + N(\alpha, \beta, \sim\gamma))$

Finally, repeating some of the assumptions above,

$$N(\alpha, \sim\gamma) \leq N(\alpha, \sim\beta) + N(\beta, \sim\gamma)$$

which is Bell's inequality.

Let us isolate the assumptions behind the proof. Let E be any arithmetical term. For any w, x, y and z:

$0 \leq N(x, \sim y, z) + N(x, y, \sim z)$
If $N(x_1, \ldots, x_n) \leq N(y_1, \ldots, y_m)+N(z_1, \ldots, z_i)$ then $N(x_1, \ldots, x_n)+E \leq N(y_1, \ldots, y_m) + N(z_1, \ldots, z_i) + E$
$0 + N(x_1, \ldots, x_n) = N(x_1, \ldots, x_n)$
$N(x_1, \ldots, x_n)+(N(y_1, \ldots, y_m)+N(z_1, \ldots, z_i)) = N(x_1, \ldots, x_n)+(N(z_1, \ldots, z_i)+N(y_1, \ldots, y_m)) = (N(x_1, \ldots, x_n) + N(y_1, \ldots, y_m)) + N(z_1, \ldots, z_i)$
$N(x, \sim y, z) + N(x, y, z) = N(x, z) \times (N(\sim y) + N(y))$

This last assumption in turn depends on

$N(x, x) = N(x)$
$N(\sim x) + N(x) = 1$

and

[4] The corresponding generalizations of the notion of suppression principles are left to the reader.

$x \times 1 = x$

In logical terms, these assumptions amount, respectively, to the following ones[5]:
(R1) $\bot \vdash (A \wedge \sim B \wedge C) \veebar (A \wedge B \wedge \sim C)$
(R2) If $A \vdash B$ then $(A \vee C) \wedge \sim (A \wedge C) \vdash (B \vee C) \wedge \sim (A \wedge C)$
(R3) $(\bot \vee A) \wedge \sim (\bot \wedge A) \dashv\vdash A$
(R4) $A \veebar (B \veebar C) \dashv\vdash A \veebar (C \veebar B) \dashv\vdash (A \veebar B) \veebar C$
(R5) $A \wedge A \dashv\vdash A$
(R6) $A \vee \sim A \dashv\vdash \top$
(R7) $A \wedge \top \dashv\vdash A$

What is the connection of all this with the expansion principles? Suppose further that the following hold, for any A, B and C:
(R8) $\bot \vee A \dashv\vdash A$
(R9) $\bot \wedge A \dashv\vdash \bot$
(R10) $\sim \bot \dashv\vdash \top$
(R11) $A \wedge (B \vee C) \dashv\vdash (A \wedge B) \vee (A \wedge C)$
Then, making the appropriate substitutions in (R3)'s right-to-left direction and using (R8), one gets

$$A \vdash A \wedge \sim (\bot \wedge A)$$

and then, using (R9) and (R10),

$$A \vdash A \wedge \top$$

Finally, using (R6) we get

$$A \vdash A \wedge (B \vee \sim B)$$

and using (R11) we get

$$A \vdash ((A \wedge B) \vee (A \wedge \sim B))$$

At this point one can blame (R11), Distribution, which is the usual suspect in the logical discussions on quantum mechanics. Nonetheless, a relevantist has a different diagnosis. Routley explained informally the invalidity of $B \rightarrow ((B \wedge A) \vee (B \wedge \sim A))$ thus:

> For suppose B holds in an incomplete situation where neither A nor $\sim A$ holds. Then $A \vee \sim A$ fails to hold, and likewise $(B \wedge A) \vee (B \wedge \sim A)$ fails to hold; so the antecedents of the implications are not sufficient for their respective consequents; and the implications are falsified. [27, p. 28]

[5] Where '$A \veebar B$' stands for exclusive disjunction, that is, $(A \vee B) \wedge \sim (A \wedge B)$. Actually, much weaker assumptions would suffice, as in some cases only one direction of the inter-deducibility —either left to right or vice versa— would be enough and there are some redundancies in the assumptions below. Nonetheless, we give these slightly stronger and redundant assumptions to facilitate the exposition. No further important assumptions are being hidden by this simplification of exposition, though.

Hence, Routley identifies the problem in the steps prior to the use of Distribution, namely, in the validity of $A \vdash A \wedge \top$, and of $A \vdash A \wedge (B \vee \sim B)$.

Mittelstaedt gave a similar reason to reject $B \to (B \wedge A) \vee (B \wedge \sim A))$ in the quantum realm, namely:

> If a property $P([B])$ is known to pertain to the object system, then one must not assume that, in addition, an arbitrary property $P([A])$ pertains to the system or not. If $P([A])$ is incommensurable with $P([B])$, then $P([A])$ cannot be tested by experiment. This is, however, less important. The essential point is that under the conditions described here the property $P([A])$ is not only subjectively unknown to the observer but it is objectively undecided whether the system possesses $P([A])$ or $P(\sim[A])$. ([23, p. 237]; italics in the original.)[6]

Thus,

> In the language of classical physics the strong pragmatic preconditions of value definiteness and unrestricted availability of propositions lead to classical logic, in particular to the equivalence $B = (B \wedge A) \vee (B \wedge \sim A)$. However, if this law is applied to quantum-mechanical propositions which are not commensurable, then one comes into conflict with the nonobjectifiability of propositions in quantum mechanics.

But the similarities between Routley's and Mittelstaedt's accounts end here. While Mittelstaedt blames Distribution, our (R11), in the argument from $B \to (B \wedge (A \vee \sim A))$ to $B \to (B \wedge A) \vee (B \wedge \sim A))$, Routley blames the premise itself, which is an expansion principle. And seemingly that should have been Mittelstaedt's attitude as well. Recall what he said: "If a property $P([B])$ is known to pertain to the object system, then one must not assume that, in addition, an arbitrary property $P([A])$ pertains to the system or not. If $P([A])$ is incommensurable with $P([B])$, then $P([A])$ cannot be tested by experiment".

A referee correctly observes that Mittelstaedt has Birkhoff and von Neumann's quantum logic in mind in this argument. In Birkhoff and von Neumann's quantum logic, disjunction is quite different than in classical logic and relevance logic: for instance, $A \vee_q \sim A$ may hold even if neither A nor $\sim A$ hold, where $\vee_q$ is the disjunction connective from quantum logic. Thus, blaming Distribution, a principle involving disjunctions, may seem like the correct choice to Mittelstaedt. But note that Routley's general project of applying relevance logics in diverse areas like mathematics, physics and philosophy in order to avoid paradoxical arguments provides a different diagnosis and perhaps a more general solution to the problem at hand by blaming irrelevant implications used in any kind of argument instead of relying just in some particularities of quantum mechanics. (Or any other area of inquiry for that matter.) Most likely Mittelstaedt would not be convinced by Routley's argument, favoring a more "specific" approach to the difficulties in quantum mechanics, but assessing the possibility of agreement between them is beyond the scope of our discussion.

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[6] We have introduced some notational changes, indicated in brackets, for uniformity.

The logical inspection of Bell's argument suggests that the condition of Locality (Bell) not only is incompatible with Bell-type inequalities as the original results show, but it turns out that Bell-type inequalities depend on logical irrelevancies. Though expansion principles are valid according to classical (Boolean) logic, they lead to paradoxical conclusions —like the very inequality— and hence should be avoided in science. This is precisely the motivation behind the development and application of different relevance logics.

4 A different physics?

The fact that Bell's inequality requires the acceptance of principles criticized by the relevance logics tradition is yet another instance of classical (Boolean) logic leading to paradoxical arguments. If we can reasonably stop Bell's argument by these means, then there is hope that we could find a local hidden-variables theory that can restore local causality into quantum theory.[7] The interest in finding such a theory rests on different motivations.

First, it is a fundamental quantum doctrine that a measurement does not, in general, reveal a preexisting value of the measured property; rather, the outcome of a measurement is brought into being by the act of measurement itself. This is the *measurement problem* and the hidden-variables program is an effort to construct a deeper level of description of the world, in which properties such as position and velocity do have simultaneous values, even though nature has conspired to prevent us from ascertaining them both at the same time [21, p. 803].

Moreover, John S. Bell re-discovered the fact that von Neumann's 1932 no-hidden-variables proof was based on wrong assumptions, something Grete Hermann discovered first in 1935 [21, p. 805], meaning there was no reason, after all, to discard hidden-variables theories as impossible. Bell even noted in his famous 1964 paper that Bohm's quantum theory was an example of a valid hidden variables theory —and an example of a non-local theory as well. So properties of individual systems may posses values prior to the measurement that reveals them regardless of there being any law enabling us to predict at an earlier time what those values will be.[8]

[7] This approach resembles Putnam's move: keep realist, but change logic. Putnam's view received much criticism (see [30]), but it need not apply here, since it is not strictly necessary to use the proposed change of logic to search for local realistic explanations. However, our point is merely that the relevance approach is yet another viable research program.

[8] A referee points out that von Neumann's proof gives a clear operational procedure to justify his derivation, and there are recent defenses of von Neumann's theorem. (See [1].) Mermin's observation is that von Neumann imposed the assumption that the value of an observable C is equal to the value of an observable A plus the value of an observable B, even when A and B do not commute, which is a wrong move since when A and B do not commute they do not have simultaneous eigenvalues and hence cannot be simultaneously

On the other hand, Bell's theorem is taken to establish that no local-realistic theory —of the kind considered by Bell, following, e.g., EPR's concepts of Definite Values and Separability plus his own mathematical formulation of Locality— can reproduce the empirical predictions of quantum physics for a certain class of experiments, so one must accept the real existence of faster-than-light causation, in apparent conflict with general relativity. However, and akin to the views presented here, in recent works like [13] one can find arguments for the idea that any theory aimed to explain the violation of Bell's inequalities must give up classical logic and the use of classical probabilities, and that there is a simple hidden-variables model holding to non-Boolean versions of Locality, which violates Bell's inequalities even for an ideally perfect Bell experiment. We will return briefly to the matter of probability at the end of this section.[9] But notice that if the relevantist analysis is correct, then we cannot rule out the possibility of reproducing the empirical predictions of quantum mechanics in a local-realistic theory.

On the matter of faster-than-light causation which violates general relativity, a few words of caution are in order, though. According to Norsen [25],

1. Local causality is often confused with local *signaling*, i.e., exclusively slower-than-light signaling, a human activity that requires a causal connection between the sending event and the receiving event, some measure of control over appropriate beables on the part of the sender, and some measure of access to appropriate beables on the part of the recipient.
2. The requirement that theories prohibit the possibility of super-luminal signaling is all that is imposed in relativistic quantum field theory; this requirement is that field operators at space-like separation commute.
3. The above requirement of quantum field theory is much weaker than the prohibition of faster-than-light causal influences.
4. Conflating local causality and local signaling is a prevalent mistake of Bell's commentators and it has lead to a double-standard in which alternatives to ordinary quantum mechanics are dismissed as non-local on the grounds that they include violations of relativistic causality.

So the apparently essential conflict between quantum theory and general relativity, the two fundamental pillars of physics, may turn out to be a misconception. See [20] for a detailed discussion of this issue.

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measured in order to calculate C. Our observation of von Neumann's mistake does not bear on wrong logical or even ontological assumptions, but on wrong physical assumptions, and that, as Mermin argues [21, p. 804], many generations of students who might have been tempted to try to construct hidden-variables theories were beaten into submission by von Neumann's 1932 claim that it could not be done, and that Hermann was completely ignored in her time.

[9] Related to this, see [15], where they show that the correlations of the observables involved in the Bohm–Bell type experiments can be expressed as correlations of classical random variables. We thank an anonymous referee for pointing out these references to us.

Finally, what about a logical inspection of generalizations of Bell's theorem, in order to sustain our general claim about Bell-type inequalities? *Perfect correlation* of spin measurements of the two particles is yet another assumption in Bell's proof. The CHSH argument [9, p. 881] proceeds without it:

> Defining the correlation function $P(\mathbf{a},\mathbf{b}) = \int_\Gamma A(\mathbf{a},\lambda)B(\mathbf{b},\lambda)\rho(\lambda)d\lambda$, where Γ is the total λ space, we have
>
> $$|P(\mathbf{a},\mathbf{b}) - P(\mathbf{a},\mathbf{c})| \leq 1 - \int_\Gamma B(\mathbf{b},\lambda)B(\mathbf{c},\lambda)\rho(\lambda)d\lambda$$
>
> Suppose that for some $\mathbf{b}'$ and $\mathbf{b}$ we have $P(\mathbf{b}',\mathbf{b}) = 1-\delta$, where $0 \leq \delta \leq 1$. Experimentally interesting cases will have δ close to but not equal to zero. Here we avoid Bell's experimentally unrealistic restriction that for some pair of parameters $\mathbf{b}'$ and $\mathbf{b}$ there is perfect correlation (i.e., $\delta = 0$). [...] And therefore
>
> $$|P(\mathbf{a},\mathbf{b}) - P(\mathbf{a},\mathbf{c})| \leq 2 - P(\mathbf{b}',\mathbf{b}) - P(\mathbf{b}',\mathbf{c})$$
>
> In the experiment proposed below $P(\mathbf{a},\mathbf{b})$ depends only on the parameter difference $\mathbf{b}-\mathbf{a}$. Defining $\alpha \equiv \mathbf{b}-\mathbf{a}$, $\beta \equiv \mathbf{c}-\mathbf{b}$, and $\gamma \equiv \mathbf{b}-\mathbf{b}'$ we have
>
> $$|P(\alpha) - P(\alpha+\beta)| \leq 2 - P(\gamma) - P(\beta+\gamma)$$

This is the CHSH inequality. Again, Locality (Bell) as their "correlation function" is applied to a Bell-type inequality to reach a contradiction but this time the argument considers testable physical scenarios that involve no perfect correlation between measurements.

In [4, p. 179], Bell proves the CHSH inequality and shows that his own original inequality (3) follows from it for certain values. He writes the general form of the CHSH inequality as

$$|P(\mathbf{a},\mathbf{b}) - P(\mathbf{a},\mathbf{b}')| + |P(\mathbf{a}',\mathbf{b}') + P(\mathbf{a}',\mathbf{b})| \leq 2 \tag{11}$$

where $\mathbf{a}'$ and $\mathbf{b}'$ are alternative settings of the instruments.

It is easy to see that (11) depends on expansion principles as well. Let us rewrite (11) as

$$|P(\mathbf{a_1},\mathbf{b_1}) - P(\mathbf{a_1},\mathbf{b_2})| + |P(\mathbf{a_2},\mathbf{b_2}) + P(\mathbf{a_2},\mathbf{b_1})| \leq 2 \tag{12}$$

which we may translate as

$$|N(\alpha_1,\beta_1) - N(\alpha_1,\beta_2)| + |N(\alpha_2,\beta_2) + N(\alpha_2,\beta_1)| \leq 2 \tag{13}$$

At first glance, expansion principles would also be involved in the proof of this inequality, but this is left for further work.

Though we ask about "a different physics" in this section, note that the relevance logics approach suggested by Routley is perfectly compatible with the slogan "unperformed experiments have no values" (which is widely spread among many working quantum physicists), whence the physics may not be quite different after all. What should really be different, following Routley, is the probability theory behind quantum theory, and in particular behind Bell's theorem. If the relevantist criticism to Bell's theorem is right, then the relevantist approach to quantum theory should make a corresponding change in probability theory; after all, the Expansion Principles noted in Bell's proof are expressed in probabilistic terms.[10]

Routley was aware of this. In [26, §12], he gave a theory of logical probability based on relevance logics by applying measure theory to De Morgan lattices, the algebraic structure underlying relevance logics, instead of Boolean lattices, the algebraic structure underlying classical logic and classical probability theory. In Routley's theory, the probability measure μ of a proposition A is defined as the sum of the measure of the worlds/situations where A is true:

$$\mu(A) = \sum_{a \in \{k \in K \mid I(A,k)=1\}} \mu(a)$$

where K is a non-empty set of worlds and I is an interpretation function relativized to worlds. This theory is meant to be a theory of partial entailment restricted to relevantist criteria, whence paradoxes of implication may not be carried over to conditional probabilities, and can accommodate inconsistent information without trivializing the theory since relevance logics are paraconsistent. The relevance logic-based theory of probability differs notably from the classical theory in the failure of $\mu(\sim A) = 1 - \mu(A)$ and other classical theorems involving negation. In particular, the equation $\mu(A) = \mu(A \wedge B) + \mu(A \wedge \sim B)$, which is the Expansion Principle involved in Bell's proof, fails to hold in Routley's probability theory.

That theory remains underdeveloped and would require some adjustments to be really applicable to issues in quantum mechanics. The topic of probability theory formulated relevantly is beyond the scope of this paper, although see [8], but it should not escape from attention that it is a necessary development to be worked out in order to follow Routley's views on quantum mechanics.[11]

Finally, if Bell's theorem fails in the relevantist approach, is there any hope for a *complete* quantum theory in Einstein's sense? Said otherwise, is there any hope for a quantum theory in which the wave function of a given system is indeed a complete description of the system at hand? We believe it is too soon to attempt an answer to this question. Other limitative theorems from

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[10] In the literature, it has been explicitly stated that quantum probabilities are a very specific kind of non-Kolmogorovian probabilistic calculus, so our proposal of changing probability theory is by far not new in the quantum foundations community; see, for instance, [28], [16] or [14]. In such frameworks, the rules of classical probability theory only hold inside a measurement context, but will fail when propositions taken from different measurement contexts are considered.

[11] Non-classical theories of probability based on relevance logics and also on logics of formal inconsistency and other paraconsistent logics have been offered before. See [18], [19], [7] and [17].

quantum theory, like Kochen-Specker, require a relevantist analysis which we also leave for future research. However, since the relevantist approach does not discard the possibility of a hidden-variables local quantum theory by blocking Bell's argument, the possibility of finding a quantum theory that offers a complete description of the wave function of a system (without violating locality) is in principle restored. (Though, of course, this is not the only strategy available to incorporate locality into quantum theory.) The price to pay, though, is a logical revision and recasting of some mathematical theories used in physics.

Many more issues regarding the implications, both philosophical and physical, of the failure of Bell's theorem could be brought into discussion but we will limit ourselves to the logical observations made here since more work is needed to follow the consequences of a relevantist approach to quantum physics.

5 Conclusions

After giving a brief, philosopher-friendly reconstruction of Bell's theorem and the discussion surrounding it, we presented a proof of Bell's inequality conveniently formulated in logical notation. We found that the proof does require expansion (and suppression) principles, as Richard Sylvan guessed. An eventual rejection of some of the steps in the proof opens some prospects for rethinking certain parts of quantum physics.

Greatly abstract and theoretical areas of inquiry, such as philosophy and mathematics, seem to be fields very prone to revenants. Nowadays you can have actual infinities, infinitesimals or Meinongian ontologies in good health, to name a few, several decades or even centuries after being buried. Many of these cases have been accompanied with an appropriate logic. Is there any chance of an afterlife for local hidden-variables theories from relevance's hand?

Our highly speculative answer at this point is that there might be hope for such theories if we carry out a relevantist revision of standard probability theory and other forms of logical and mathematical reasoning spread throughout science, as Sylvan pointed out originally. The option of revising probability theory has been noticed —and rejected– for example in [32], but his arguments might be a bit quick. A more careful analysis of this option is left for further work.

Acknowledgements Work supported by the PAPIIT project IG400422. The second author also acknowledges the support of a DGAPA sabbatical grant. We thank the editors and the anonymous referees for their useful comments and suggestions which greatly helped to improve our typescript. We also benefited from discussing a previous version of this paper at the e-Workshop "Philosophical Issues of Quantum Mechanics", especially from Carl Hoefer's questions; Elías Okon's philosophy of physics seminar at UNAM; Virginia

Aspe-Armella's philosophy teachers seminar at Universidad Panamericana, specially from José Luis Rivera's comments; and Julio César Rubén Romo-Cruz's physico-mathematical research seminar at Universidad Panamericana.

Appendix

Here we present Hilbert-style versions of many (quantifier-free) basic relevant logics, following [5], where the axioms and rules for quantificational extensions can be found.

Axiom schemas

1. $A \to A$
2. $(A \wedge B) \to A$
3. $(A \wedge B) \to B$
4. $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$
5. $A \to (A \vee B)$
6. $B \to (A \vee B)$
7. $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$
8. $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$
9. $\sim\sim A \to A$
10. $(A \to\sim B) \to (B \to\sim A)$
11. $((A \to B) \wedge (B \to C)) \to (A \to C)$
12. $A \vee \sim A$
13. $(A \to\sim A) \to\sim A$
14. $(A \to B) \to ((B \to C) \to (A \to C))$
15. $(A \to B) \to ((C \to A) \to (C \to B))$
16. $(A \to (A \to B)) \to (A \to B)$
17. $A \to ((A \to B) \to B)$
18. $(A \to B) \to ((A \wedge C) \to (B \wedge C))$
19. $(A \to B) \to ((A \vee C) \to (B \vee C))$

Rules

1. $A, A \to B \vdash B$
2. $A, B \vdash A \wedge B$
3. $A \to B, C \to D \vdash ((B \to C) \to (A \to D))$
4. $A \to\sim B \vdash B \to\sim A$

Logics

B = A1–A9, R1–R4.
DW = **B** plus A10 minus R4.
DJ = **DW** plus A11 .
DK = **DJ** plus A12.

DL = **DJ** plus A13 minus A12.
TW = **DW** plus A14 and A15 minus R3.
TJ = **TW** plus A11.
TK = **TJ** plus A12.
T = **TK** plus A13 and A16 minus A11 and A12.
RW = **TW** plus A17 minus A15.
R = **T** plus A17 minus A13 and A15. = **RW** plus A16.

Model-theoretic presentations for these logics can be found in [27, Ch. 2, Ch. 4].

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