The Standard Model of particle physics in other universes

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Abstract

The purpose of this paper is to demonstrate how the mathematical objects and structures associated with the particle physics in other universes, can be inferred from the mathematical objects and structures associated with the particle physics in our own universe. As such, this paper is a continuation of the research programme announced in McCabe (2004), which implemented this idea in the case of cosmology.

The paper begins with an introduction that outlines the structuralist doctrine which this research programme depends upon. Section 2 explains how free elementary particles in our universe correspond to irreducible representations of the double cover of the *local* space-time symmetry group $SO_0(3,1) \circledS \mathbb{R}^{3,1}$, and relates the configuration representation to the momentum representation. The difficulties of treating elementary particles in curved space-time, and the Fock space second-quantization are also explained. Section 2.1 explores the particle physics of universes in which the local symmetry group is the entire Poincare group $O(3,1) \circledS \mathbb{R}^{3,1}$ or the isochronous Poincare group $O^{\uparrow}(3,1) \circledS \mathbb{R}^{3,1}$. Section 2.2 considers free particles in universes with a different dimension or geometrical signature to our own. Section 3 introduces gauge fields, and, via Derdzinski's interaction bundle approach, explains how connections satisfying the Yang-Mills equations correspond to the irreducible representations for 'gauge bosons'. To explore the possible gauge fields, section 3.1 explains the classification of principal G-bundles over 4-manifolds, and section 3.2 expounds the structure theorem of compact Lie groups. Section 3.3 summarises the consequences for classifying gauge fields in other universes, and section 3.4 infers the structures used to represent interacting particles in other universes. The paper concludes in Section 3.5 by explaining the standard model gauge groups and irreducible representations which define interacting particle multiplets, and specifies the possibilities for such multiplets in other universes.

1 Introduction

The general interpretational doctrine adopted in this paper can be referred to as 'structuralism', in the sense advocated by Patrick Suppes (1969), Joseph Sneed

(1971), Frederick Suppe (1989), and others. This doctrine asserts that, in mathematical physics at least, the physical domain of a theory is conceived to be an instance of a mathematical structure or collection of mathematical structures. The natural extension of this principle proposes that an entire physical universe is an instance of a mathematical structure or collection of mathematical structures. In particular, each type of particle is considered to be an instance of some species of mathematical structure. One frequently finds in the literature the assertion that an elementary particle 'is' an irreducible, unitary representation of the local space-time symmetry group G, (e.g. Sternberg 1994, p149; Streater 1988, p144). As such, this is an expression of structuralism. Whilst the definition of structuralism is most often expressed in terms of the set-theoretical, Bourbaki notion of a species of mathematical structure, one could reformulate the definition in terms of other approaches to the foundations of mathematics, such as mathematical Category theory. One could assert that our physical universe is an object in a mathematical Category, or a collection of such objects. In particular, one could assert that each type of particle is an object in a mathematical Category.

Those expressions of structuralism which state that 'the' physical universe is an instance of a mathematical structure, tacitly assume that our physical universe is the only physical universe. If one removes this assumption, then structuralism can be taken as the two-fold claim that (i) our physical universe is an instance of a mathematical structure, and (ii), other physical universes, if they exist, are either different instances of the same mathematical structure, or instances of different mathematical structures. Given that mathematical structures are arranged in tree-like hierarchies, other physical universes may be instances of mathematical structures which are sibling to the structure possessed by our universe. In other words, the mathematical structures possessed by other physical universes may all share a common parent structure, from which they are derived by virtue of satisfying additional conditions. This would enable us to infer the mathematical structure of other physical universes by first generalizing from the mathematical structure of our own, and then classifying all the possible specializations of the common, generic structure.

Hence, it is the aim of this paper not only to explain how the particle world of our universe is an instance of certain mathematical structures, but to also emphasize how the mathematical structures of our particle world are special cases of more general mathematical structures. Throughout, the intention is to establish the mathematical structure possessed by the particle world in our own universe, and to use that to infer the nature of the particle world in other universes.

2 Free Particles

In relativistic quantum theory, the two basic types of thing which are represented to exist are matter fields and gauge force fields. A gauge force field mediates the interactions between the matter fields. Relativistic quantum theory is obtained by applying quantization procedures to classical relativistic particle mechanics and classical relativistic field theory. The typical quantization procedure can be broken down into first-quantization and second-quantization. In first-quantization, it is possible to represent interacting fields in a tractable mathematical manner. The first-quantized approach is empirically adequate to the extent that it enables one to accurately represent many of the structural features of the physical world. Second-quantization, quantum field theory proper, is required to generate quantitatively accurate predictions, but quantum field theory proper is incapable of directly representing interacting fields.

In the first-quantized theory, a matter field can be represented by a cross-section of a vector bundle, and a gauge force field can be represented by a connection upon a principal fibre bundle. A brief digression to explain these concepts may be helpful to the uninitiated: A fibre bundle is something which enables one to attach a structured set to each point of an underlying mathematical space, called the base space. In the case of interest to us, the base space will be a manifold. The structured set assigned to a point is called the fibre over that point, and the collection of all the fibres is the fibre bundle. Each fibre will be isomorphic to something called the typical fibre of the fibre bundle; one cannot have a fibre bundle which assigns non-isomorphic sets to different points of the base space. If each fibre is a vector space, then one has a vector bundle; if each fibre is a group, then one has a group bundle; and so on. A cross-section of a fibre bundle is something which allows one to pick out an element from each of the fibres. Thus, a cross-section of a vector bundle, for example, picks out an element from the vector space assigned to each point of the base. Now, a principal fibre bundle, without delving into its subtleties, is a fibre bundle in which there is a group, called the structure group, which acts as a group of transformations upon each of the fibres. A connection ∇ upon a fibre bundle enables one to define the notion of path-dependent parallelism between the different fibres. Thus, equipped with a connection, one can determine whether elements in fibres over two distinct points are parallel to each other along a particular path joining those points. In passing, note that a connection on a fibre bundle enables one to define a special operator called the covariant derivative.

The fact that, in the first-quantized theory, a matter field can be represented by a cross-section of a vector bundle, and a gauge force field can be represented by a connection upon a principal fibre bundle, is rather curious because the matter fields are obtained by quantizing the point-like objects of classical relativistic particle mechanics, whilst, at first sight, the gauge fields have undergone no quantization at all. If one treats the first-quantized matter fields as classical fields, and if one treats those matter fields as interacting with classical gauge fields, then there is no inconsistency. However, on both counts, such a treatment may be misleading.

Given that the matter fields in the first-quantized theory are the upshot of quantizing classical particles, they are provisionally interpretable as wave-functions, i.e they are provisionally interpretable as vectors in a quantum state space, coding probabilistic information. One of the outputs from the first quantized theory is a state space for each type of elementary particle, which becomes a so-called 'one-particle subspace' of the second-quantized theory. The vector bundle cross-sections which represent a matter field in the first-quantized theory, are vectors from the one-particle subspace of the second-quantized theory.

The connections which represent a gauge field can be shown, under a type of symmetry breaking called a 'choice of gauge', to correspond to cross-sections of a direct sum of vector bundles,[1] (Derdzinski 1992, p91). The cross-sections of the individual direct summands are vectors from the one-particle subspaces of particles called 'interaction carriers', or 'gauge bosons'. Hence, neither the matter fields nor the gauge fields of the first-quantized theory can be unambiguously treated as classical fields. Given these complexities, the terms 'particle' and 'field' will be used interchangeably throughout this paper, without the intention of conveying any interpretational connotations.

A particle is an elementary particle in a theory if it is not represented to be composed of other particles. All particles, including elementary particles, are divided into fermions and bosons according to the value they possess of a property called 'intrinsic spin'. If a particle possesses a non-integral value of intrinsic spin, it is referred to as a fermion, whilst if it possesses an integral value, it is referred to as a boson. The elementary matter fields are fermions and the interaction carriers of the gauge force fields are bosons. The elementary fermions in our universe number six leptons and six quarks. The six leptons consist of the electron and electron-neutrino (e, ν_e), the muon and muon-neutrino (μ, ν_μ), and the tauon and tauon-neutrino (τ, ν_τ). The six quarks consist of the up-quark and down-quark (u, d), the charm-quark and strange-quark (c, s), and the top-quark and bottom-quark (t, b). The six leptons have six anti-leptons, $(e^+, \overline{\nu}_e)$, $(\mu^+, \overline{\nu}_\mu)$, $(\tau^+, \overline{\nu}_\tau)$, and the six quarks have six anti-quarks $(\overline{u}, \overline{d})$, $(\overline{c}, \overline{s})$, $(\overline{t}, \overline{b})$. These fermions are partitioned into three generations. The first generation, (e, ν_e, u, d), and its anti-particles, is responsible for most of the macroscopic phenomena we observe. Triples of up and down quarks bind together with the strong force to form protons and neutrons. Residual strong forces between these hadrons bind them together to form atomic nuclei. The electromagnetic forces between nuclei and electrons leads to the formation of atoms and molecules. (Manin 1988, p3).

Free matter fields ('free particles') are matter fields which are idealized to be free from interaction with force fields. To specify the free elementary particles which can exist in a universe, (i.e. the free elementary 'particle ontology'), one specifies the *projective*, unitary, irreducible representations of the 'local' symmetry group of space-time.

Some explanation of the concepts here is again in order. A linear representation of a group assigns to each group element a one-to-one, structure-preserving

[1] The direct sum $\oplus_i V_i$ of a collection of vector spaces $\{V_i\}$ is the vector space formed by taking the sums of the vectors in the constituent spaces. The direct sum contains all the constituent vector spaces as linearly independent subspaces. A direct sum of a collection of vector bundles over the same underlying space possesses, over each point, a fibre consisting of the direct sum of the vector spaces over that point in each one of the constituent vector bundles.

transformation of a vector space. An irreducible representation is one for which there is no subspace, apart from the zero subspace and the entire vector space, which is closed under the action of the represented group elements. In other words, one cannot restrict the representation to a smaller subspace. An inner product on a vector space enables one to define the length of vectors and the angles between vectors; if a representation space is equipped with an inner product, then the linear transformations which leave this inner product between vectors unchanged are referred to as orthogonal transformations (in the case of a real vector space), or unitary transformations (in the case of a complex vector space). A unitary group representation is a representation which assigns a unitary transformation of a complex vector space to each group element. A projective representation of a group assigns to each group element a one-to-one, structure preserving transformation of the space of one-dimensional subspaces $\mathscr{P}(V)$ in a vector space V. In the case of interest in this paper, the vector space will be a Hilbert space $\mathscr{H}$, a special type of vector space equipped with an inner product, and the projective space in this case is considered to inherit its own product from the inner product on the Hilbert space. Defining a symmetry of $\mathscr{P}(\mathscr{H})$ to be a one-to-one mapping which preserves this product, each symmetry can be implemented by a unitary operator U on the Hilbert space,[2] but the unitary operator is only unique up to a complex number of unit modulus $z = e^{i\theta}$. A projective unitary representation is a projective representation implemented by unitary operators which are only determined up to a complex number of unit modulus, a so-called 'phase factor'. One can choose a unitary operator from each such equivalence class, but doing so does not generally define a unitary representation. Instead, such operators will satisfy the equation

$$U(g_1 \circ g_2) = \omega(g_1, g_2)U(g_1)U(g_2) \; ,$$

with $\omega(g_1, g_2)$ a complex number of unit modulus. If one can judiciously choose elements from each equivalence class so that $\omega = 1$ everywhere, then the projective unitary representation is said to be unitarizable as an ordinary representation. The pure states of a quantum system can be represented by the one-dimensional subspaces of a Hilbert space, hence the requirement here that an elementary system correspond to a projective, unitary, irreducible representation of the local symmetry group.[3]

To understand what the 'local' symmetry group of a space-time is, one needs to begin by understanding that the large-scale structure of a space-time is represented by a pseudo-Riemannian manifold $(\mathcal{M}, g)$. At each point x of a manifold $\mathcal{M}$ the set of all possible tangents to the curves which pass through that point form a special vector space, called the tangent vector space $T_x\mathcal{M}$. A metric tensor field g on the manifold assigns an inner product $\langle \; , \; \rangle$ to the tangent vector space at each point of the manifold, and this is considered to provide the manifold with a geometrical structure. The metric g has a signature (p, q) de-

[2] In general, the operator can be either unitary or anti-unitary.

[3] An *irreducible* projective representation is one for which there is no non-trivial subspace $W \subset \mathscr{H}$ such that $\mathscr{P}(W)$ is closed under the group representation.

termined by the number p of orthogonal unit vectors which have a positive inner product $\langle v, v \rangle > 0$, and the number q of orthogonal unit vectors which have a negative inner product $\langle w, w \rangle < 0$. The dimension n of the manifold $\mathcal{M}$, and the signature (p, q) of the metric g, determine the largest possible local symmetry group of the space-time. The automorphism group[4] of a tangent vector space $T_x\mathcal{M}$, equipped with the inner product $\langle \ , \ \rangle$, defines the largest possible local symmetry group of such a space-time, the semi-direct product $O(p, q) \circledS \mathbb{R}^{p,q}$. The semi-orthogonal group $O(p, q)$ is the group of linear transformations which preserve the inner product of a real vector space equipped with an inner product of signature (p, q). $\mathbb{R}^{p,q}$ is $(p + q)$-dimensional Euclidean space equipped with an inner product of signature (p, q), and treated as a group of translations in this context. A semi-direct product is a special type of product of two groups in which the first factor acts as a group of transformations upon the second factor in a particular way, (see Sternberg (1994), p135-136, for more details).

If there is no reason to restrict to a subgroup of $O(p, q) \circledS \mathbb{R}^{p,q}$, then one specifies the possible free elementary particles in such a universe by specifying the projective, unitary, irreducible representations of $O(p, q) \circledS \mathbb{R}^{p,q}$. As a consequence of $O(p, q) \circledS \mathbb{R}^{p,q}$ being a non-compact Lie group, these representations are infinite-dimensional.

In the case of our universe, the dimension is $n = 4$, and the signature is $(p, q) = (3, 1)$, indicating three spatial dimensions and one time dimension. An n-dimensional pseudo-Riemannian manifold such as this, with a signature of $(n-1, 1)$, is said to be a Lorentzian manifold. Each tangent vector space of a 4-dimensional Lorentzian manifold is isomorphic to Minkowski space-time, hence the automorphism group of such a tangent vector space is the Poincare group, $O(3, 1) \circledS \mathbb{R}^{3,1}$, the largest possible symmetry group of Minkowski space-time. In the case of our universe, the Lorentzian manifold appears to be equipped with a time orientation and a space orientation. This entails that time reversal operations, parity transformations (spatial reflections), and combinations thereof, are not considered to be local space-time symmetries. Hence, the local space-time symmetry group of our universe appears to be a subgroup of the Poincare group, called the *restricted* Poincare group, $SO_0(3, 1) \circledS \mathbb{R}^{3,1}$. The projective, unitary, irreducible representations of the restricted Poincare group are in bijective correspondence with the ordinary, unitary, irreducible representations of its universal covering group,[5] $SL(2, \mathbb{C}) \circledS \mathbb{R}^{3,1}$. In particular, every projective unitary representation of $SL(2, \mathbb{C}) \circledS \mathbb{R}^{3,1}$ is unitarizable, hence the projective unitary representations of $SO_0(3, 1) \circledS \mathbb{R}^{3,1}$ can be lifted to projective unitary representations of $SL(2, \mathbb{C}) \circledS \mathbb{R}^{3,1}$ with the aid of the covering map, and then unitarized. Thus, one specifies the free elementary particle ontology of our universe by specifying the ordinary, irreducible, unitary representations of $SL(2, \mathbb{C}) \circledS \mathbb{R}^{3,1}$.

Let us examine the selection of the restricted Poincare group in a little more detail. The largest possible local symmetry group of our space-time,

[4]The automorphisms of a structured set are the one-to-one maps of the set onto itself which preserve the structure possessed by that set.

[5]See Wald (1984), p345, for a definition of the universal covering group.

the Poincare group $O(3,1) \circledS \mathbb{R}^{3,1}$, is the group of diffeomorphic[6] isometries of Minkowski space-time $\mathcal{M}$, a semi-direct product of the Lorentz group $O(3,1)$ with the translation group $\mathbb{R}^{3,1}$. In contrast, the Lorentz group is the group of *linear* isometries of Minkowski space-time. The Poincare group is a disconnected group which possesses four components; one component contains the isometry which reverses the direction of time, another component contains the isometry which performs a spatial reflection (it reverses parity), another component contains the isometry which both reverses the direction of time and performs a spatial reflection, whilst the identity component $SO_0(3,1)\circledS\mathbb{R}^{3,1}$ preserves both the direction of time and spatial parity. The identity component $SO_0(3,1) \circledS \mathbb{R}^{3,1}$, the restricted Poincare group, is also referred to as the 'proper orthochronous' Poincare group, and is often denoted as $\mathscr{P}_+^\uparrow$ in the Physics literature. Similarly, the identity component of the Lorentz group, $SO_0(3,1)$, is variously referred to as the *restricted* Lorentz group, or the 'proper orthochronous' Lorentz group, and is often denoted as $\mathscr{L}_+^\uparrow$ in Physics literature.

In 1956 it was discovered that interactions involving the weak nuclear force violate parity symmetry, and, in 1964, a single weak interaction process, the decay of the K^0-meson, was discovered to violate time inversion symmetry. Thus, the physical evidence seems to indicate that our universe possesses a separate time orientation and space orientation, and that the local symmetry group of our space-time is therefore the restricted Poincare group $SO_0(3,1) \circledS \mathbb{R}^{3,1}$, the group of local symmetries which preserve time and space orientation. However, Geroch and Horowitz state that "the strongest conclusion to be drawn...using the presently observed symmetry violations in elementary particle physics, is that our spacetime must be total orientable. One cannot conclude from this, for example, that our spacetime must be separately time- and space-orientable," (1979, p229). If true, this would entail that the local space-time symmetry group is $SO(3,1)\circledS\mathbb{R}^{3,1}$. Whilst a Lorentzian manifold equipped with a separate time orientation and space orientation must also possess a space-time orientation, (a 'total' orientation), the converse is not true. The presence of a space-time orientation does not entail the presence of a separate time orientation and space orientation, and the absence of a space-time orientation does not entail the separate absence of a time orientation and space orientation. Letting T denote the operation of time inversion, and letting P denote the operation of parity reversal, one says that, whilst P and T entails PT, the converse is not true. Sternberg states that "the current belief is that the correct [local space-time symmetry] group has two components corresponding to simultaneous space and time inversion, but that this transformation must be accompanied by reversal of all electric charges," (1994, p150). The comments of Sternberg and Geroch-Horowitz seem to stem from the CPT theorem of second-quantized quantum field theory, which states that a combination of time reversal, parity transformation, and charge conjugation (C) provide a physical symmetry. If the local symmetry group of space-time were $SO(3,1) \circledS \mathbb{R}^{3,1}$, then PT would be a physical symmetry, but it is known that such is not the case in weak interactions. In

[6]Diffeomorphisms are the isomorphisms of manifolds.

fact, as Penrose points out: "Assuming CPT, we can regard C - the interchange of particles with their antiparticles - as equivalent to PT, so we can regard the antiparticle of some particle as being the 'space-time reflection' (PT) of that particle," (2004, p639). Given that $(PT)^2 = I$, and given that $CPT = I$, it follows that, in terms of local space-time symmetries, $C = PT$. Hence, Sternberg and Geroch-Horowitz have misinterpreted the CPT theorem, which is consistent with the notion that neither P, nor T, nor PT are physical symmetries, and is consistent with the belief that the restricted Poincare group $SO_0(3,1) \circledS \mathbb{R}^{3,1}$ is the local space-time symmetry group in our universe.

The physically relevant irreducible unitary representations of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$ are parameterized by one continuous parameter, the mass m, and one discrete parameter, the spin s. One can present these representations in either the momentum representation (the Wigner representation), or the configuration representation.

The Wigner approach uses the method of induced group representations, applied to semi-direct product groups. Given a semi-direct product $G = H \circledS N$, the method of induced representation can obtain, up to unitary equivalence, all the irreducible unitary representations of the group G. Given that the local symmetry group of space-time is a semi-direct product $G = SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$, the method of induced representation enables us to obtain, up to unitary equivalence, all the irreducible unitary representations of $G = SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$. In fact, the method of induced representation enables us to classify all the irreducible unitary representations of $G = SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$, and to provide an explicit construction of one case from each unitary equivalence class, (Emch 1984, p503). The first step of this construction is to find the orbits of the $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$-action on $\mathbb{R}^{3,1}$. These orbits coincide with the orbits of the $SO_0(3,1)$-action on $\mathbb{R}^{3,1}$. These orbits provide the base spaces of the bundles used to construct the representations, and they come in three types:

- For any $m \in \mathbb{R}_+$, there are the 'mass hyperboloids' $\mathscr{V}_m$, where $\mathscr{V}_m = \{p \in \mathbb{R}^{3,1} : \langle p, p \rangle = m^2 > 0\}$.
- There are the 'light cones' $\mathscr{V}_0 = \{p \in \mathbb{R}^{3,1} : \langle p, p \rangle = m^2 = 0\}$.
- Then, for any $m \in i\mathbb{R}_+$, there are the hypersurfaces $\mathscr{V}_m = \{p \in \mathbb{R}^{3,1} : \langle p, p \rangle = m^2 < 0\}$

The third type of orbit is not considered to be physically relevant. The parameterization of the physically relevant orbits by the continuous parameter m provides the first parameter used to classify the unitary irreducible representations of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$. The second step of the construction is to identify the isotropy groups of the $SL(2,\mathbb{C})$-action upon each orbit. The irreducible unitary representations of the isotropy groups provide the typical fibres of the bundles used to construct the representations. The $m^2 > 0$ orbits have isotropy group $SU(2)$, while the $m^2 = 0$ orbits have isotropy group $\widetilde{E}(2)$.[7] The discrete parameter s used to parameterize the irreducible unitary representations of these

[7] The double cover of $E(2) = SO(2) \circledS \mathbb{R}^2$, the group of 'motions' of the Euclidean plane.

isotropy groups provides the second parameter used to classify the physically relevant irreducible unitary representations of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$.

Whilst the tangent vector space $T_x\mathcal{M}$ at an arbitrary point x of a 4-dimensional Lorentzian manifold is isomorphic to Minkowski space-time, the cotangent vector space[8] $T^*_x\mathcal{M}$ can be treated as Minkowski energy-momentum space. In the Wigner approach, free particles of mass m and spin s correspond to vector bundles $E^{\pm}_{m,s}$ over mass hyperboloids and light cones $\mathscr{V}^{\pm}_m$ in Minkowski energy-momentum space $T^*_x\mathcal{M} \cong \mathbb{R}^{3,1}$. It is the Hilbert spaces $\mathscr{H}^{\pm}_{m,s}$ of square-integrable[9] cross-sections of these vector bundles $E^{\pm}_{m,s}$ which provide the physically relevant irreducible unitary representations of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$.

The irreducible unitary representation of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$ upon the space $\Gamma_{L^2}(E^+_{m,s})$ of square-integrable cross-sections of $E^+_{m,s}$, for a particle of mass m and spin s, is unique up to unitary equivalence. The anti-particle of mass m and spin s is represented by the conjugate representation on the space $\Gamma_{L^2}(E^-_{m,s})$ of square-integrable cross-sections of the vector bundle $E^-_{m,s}$. In other words, if the particle is represented by the Hilbert space $\mathscr{H}$, then the anti-particle is represented by the conjugate Hilbert space $\overline{\mathscr{H}}$. The two representations are related by an anti-unitary transformation.

Universes which differ in their local symmetry group, differ in the mathematical structure their particle world is an instance of. However, even with the local symmetry group fixed, the projective, unitary, irreducible representations of this group only determine the set of possible free particles. The actual free particles instantiated in a universe appears to be contingent. In our universe, only a finite number of elementary free particles, of specific mass and spin, have been selected from the infinite number of possible free elementary particles. Thus, universes with the same local symmetry group, and the same set of possible free particles, can be further sub-classified by the particular collection of actual free particles instantiated.

In the configuration representation, each irreducible unitary representation is constructed from a space of mass-m solutions, of either positive or negative energy, to a linear differential equation over Minkowski space-time $\mathcal{M}$. The Hilbert space of a unitary irreducible representation in the configuration representation is provided by the completion of a space of mass-m, positive or negative energy solutions, which can be Fourier-transformed into square-integrable objects in Minkowski energy-momentum space. Under Fourier transform, the positive-energy mass-m solutions become cross-sections on $T^*_x\mathcal{M}$ with support on the 'forward-mass' hyperboloid $\mathscr{V}^+_m$, and under Fourier transform, the negative-energy mass-m solutions become cross-sections on $T^*_x\mathcal{M}$ with support on the 'backward-mass' hyperboloid $\mathscr{V}^-_m$

Whilst the Wigner approach deals directly with the irreducible unitary rep-

[8] The cotangent vector space $T^*_x\mathcal{M}$ is said to be the *dual* vector space. i.e it is the vector space containing all the linear maps $f : T_x\mathcal{M} \to \mathbb{R}$.

[9] 'Square-integrable' simply means that the square of the modulus of the cross-section, when integrated over the base space, is finite. After taking a set of equivalence classes of square-integrable cross-sections, one obtains a Hilbert space, a special type of vector space which is equipped with an inner product, and which is complete as a metric space.

resentations of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$, the configuration space approach requires two steps to arrive at such a representation. In the configuration space approach, for each possible spin s, one initially deals with a non-irreducible, mass-independent representation of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$. For each spin s, there is a finite-dimensional vector space V_s, such that the mass-independent representation can be taken as the space of cross-sections $\Gamma(\eta)$ of a vector bundle η over $\mathcal{M}$ with typical fibre V_s.

The representations of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$ upon such $\Gamma(\eta)$ can be defined by a combination of the finite-dimensional irreducible representations of $SL(2,\mathbb{C})$, and the action of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$ upon the base space $\mathcal{M}$. The complex, finite-dimensional, irreducible representations of $SL(2,\mathbb{C})$ are indexed by the set of all ordered pairs (s_1, s_2), (Bleecker 1981, p77), with

$$(s_1, s_2) \in \frac{1}{2}\mathbb{Z}_+ \times \frac{1}{2}\mathbb{Z}_+ \ .$$

In other words, the finite-dimensional irreducible representations of $SL(2,\mathbb{C})$ form a family $\mathscr{D}^{s_1,s_2}$, where s_1 and s_2 run independently over the set $\{0, 1/2, 1, 3/2, 2, ...\}$. The number $s_1 + s_2$ is called the spin of the representation.

Now one can define, for each possible spin s, an infinite-dimensional, mass-independent representation of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$ upon $\Gamma(\eta)$. Letting $\psi(x)$ denote an element of $\Gamma(\eta)$, the representation is defined as

$$\psi(x) \rightarrow \psi'(x) = \mathscr{D}^{s_1,s_2}(A) \cdot \psi(\Lambda^{-1}(x-a)) \ ,$$

where it is understood that $A \in SL(2,\mathbb{C})$, $a \in \mathbb{R}^{3,1}$, Λ is shorthand for $\Lambda(A)$, and Λ is the covering homomorphism $\Lambda : SL(2,\mathbb{C}) \rightarrow SO_0(3,1)$.

These non-irreducible, mass-independent representations do not correspond to single particle species. Each space of vector bundle cross-sections represents many different particle species. To obtain the mass m, spin-s irreducible unitary representations of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$ in the configuration representation, one introduces linear differential equations, such as the Dirac equation or Klein-Gordon equation, which contain mass as a parameter. These differential equations are imposed upon the cross-sections in the non-irreducible, mass-independent, spin-s representation. Each individual particle species corresponds to cross-sections for a particular value of mass.

In terms of the Wigner representation, first quantization is the process of obtaining a Hilbert space of cross-sections of a vector bundle over $\mathscr{V}_m^{\pm}$. In terms of the configuration representation, first quantization is the two-step process of obtaining a vector bundle over $\mathcal{M}$, and then identifying a space of mass-m solutions.

There are two mathematical directions one can go after first quantization. Firstly, one can treat the Hilbert space obtained as the 'one-particle' state space, and one can use this Hilbert space to construct a Fock space. This is the process of second quantization. One defines creation and annihilation operators upon the Fock space, and thence one defines scattering operators. One can use the scattering operators to calculate the transition amplitudes between incoming and outgoing free states of a system involved in a collision process. Calculation of these transition amplitudes requires the so-called 'regularization' and 'renormalization' of perturbation series, but these calculations do enable one to obtain empirically adequate predictions. Nevertheless, a Fock space is a space of states for a free system. In the configuration representation, the space of 1-particle states is a linear vector space precisely because it is a space of solutions to the *linear* differential equation for a free system.

Although one could use either the Wigner representation or the configuration representation, second quantization conventionally uses a Wigner representation for the one-particle Hilbert spaces.

Given the single-particle Hilbert space $\mathscr{H}_{m,s}$ for a bosonic system, the Fock space is

$$\mathscr{F}_{m,s} = \bigoplus_{n=0}^{\infty} \mathscr{H}_{m,s}^{\odot n} \; ,$$

where $\mathscr{H}_{m,s}^{\odot n}$ is the n-fold symmetric tensor product of $\mathscr{H}_{m,s}$.

Given the single-particle Hilbert space $\mathscr{H}_{m,s}$ for a fermionic system, the Fock space is

$$\mathscr{F}_{m,s} = \bigoplus_{n=0}^{\infty} \mathscr{H}_{m,s}^{\wedge n} \; ,$$

where $\mathscr{H}_{m,s}^{\wedge n}$ is the n-fold anti-symmetric tensor product of $\mathscr{H}_{m,s}$.[10]

In both cases $\mathscr{H}^0 = \mathbb{C}^1$, the so-called vacuum sector, containing a distinguished non-zero vector $1 \in \mathbb{C}^1$, called the vacuum vector.

The irreducible, unitary representation of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$ on the single-particle space extends to a unitary representation on the Fock space, albeit a non-irreducible representation.

The other mathematical direction one can go, which conventionally uses the configuration representation, is to treat first-quantization as an end in itself. In the fibre bundle approach, a mass m, spin s particle can be represented by the mass-m cross-sections of a spin-s bundle η. This mass-independent bundle η can, following Derdzinski (1992), be referred to as a *free-particle bundle*. One can associate a vector bundle δ with a gauge field, which can, again following Derdzinski, be referred to as an *interaction bundle*. One can take the free-particle bundle η, and with the interaction bundle δ, one can construct an *interacting particle bundle* α. The mass-m cross-sections of this bundle represent the particle in the presence of the gauge field. This is the route of the first-quantized interacting theory. The first-quantized interacting theory is not empirically adequate, and it is not possible to subject the first-quantized interacting theory to second-quantization because the state space of an interacting

[10] See Prugovecki (1981), p305-306, for an explanation of symmetric and anti-symmetric tensor products, and Boothby (1986), p203-205, for an explanation of symmetric and anti-symmetric projection operations.

system is not a linear vector space; in the configuration representation, the space of states for an interacting 1-particle system consists of vector bundle cross-sections which satisfy a *non-linear* differential equation. Hence, there is no Fock space for an interacting system.

Before proceeding further, some caveats can be added to the approach outlined above. Firstly, it is assumed that the free particle ontology of a universe equals the interacting particle ontology. In other words, although a realistic representation of particles involves representing their interaction with force fields, it is assumed that the set of particle types which could exist in a universe is determined by the free particle ontology.

It is also assumed that representations of the *local* symmetry group of space-time are an adequate means of determining the free elementary particle ontology. One could reason that elementary particles exist at small length scales, and the strong equivalence principle of general relativity holds that Minkowski space-time, and its symmetries, are valid on small length scales. i.e. the strong equivalence principle holds that the global symmetry group of Minkowski space-time is the local symmetry group of a general space-time. One can choose a neighbourhood U about any point in a general space-time, which is sufficiently small that the gravitational field within the neighbourhood is uniform to some agreed degree of approximation, (Torretti 1983, p136). Such neighbourhoods provide the domains of 'local Lorentz charts'. A chart in a 4-dimensional manifold provides a diffeomorphic[11] map $\phi : U \rightarrow \mathbb{R}^4$. If $\mathbb{R}^4$ is equipped with the Minkowski metric, a local Lorentz chart provides a map which is almost isometric, to some agreed degree of approximation, (ibid., p147). Unless the gravitational field is very strong, one can treat each elementary particle as 'living in' the domain of a local Lorentz chart within a general space-time $(\mathcal{M}, g)$. Unless the gravitational field is very strong, the fibre bundles employed in relativistic quantum theory are assumed to be fibre bundles over Minkowski space-time. This is done with the understanding that the base space of such bundles represents the domain of an arbitrary local Lorentz chart, rather than the whole of space-time. Hence, with the exception of the regions where the gravitational field is very strong, the elementary particles which exist in a general Lorentzian space-time still transform under the global symmetry group of Minkowski space-time, namely the Poincare group, or a subgroup thereof.

With the exception of regions where the gravitational field is very strong, a fully realistic representation of each individual elementary particle would begin with a Lorentzian manifold $(\mathcal{M}, g)$ which represents the entire universe, and would then identify a small local Lorentz chart which the particle 'lives in'. The particle would then be represented by the cross-sections and connections of bundles over this small local Lorentz chart. In terms of practical physics, this would be an act of representational *largesse*, but in terms of ontological

[11] A diffeomorphism is a one-to-one map which preserves continuity and differentiability. It is the isomorphism between differentiable manifolds.

considerations, it is important to bear in mind.

Where the gravitational field is very strong (i.e. where the space-time curvature is very large), it is no longer valid to assume that the gravitational field is uniform on the length scales at which elementary particles exist. Note that because gravity is geometrized in general relativity, it is consistent to speak of *free* elementary particles in a gravitational field. Where the gravitational field is very strong, it is not valid to assume that free elementary particles transform under the global symmetry group of Minkowski space-time. Where the gravitational field is very strong, elementary particles are represented, in the first-quantized theory, by fibre bundles over general, curved space-times. Again, this is done with the understanding that the base space of such bundles represents a small region of space-time, rather than the whole universe. These considerations weaken the assumption that the representations of the Poincare group are an adequate means of determining the free elementary particle ontology in a universe. Suppose, for the sake of argument, that the particle ontology does change in a region of curved space-time: in a general curved space-time, there might well be no isometry group at all, hence the possible elementary particles in such a space-time region could not be classified by the irreducible unitary representations of that region's space-time symmetry group. If the free elementary particle ontology is not determined in all regions of space-time by the irreducible unitary representations of the local symmetry group in the regions of weak gravitational field, then one would have to abandon a classification scheme based upon representations of space-time symmetry groups.

Given the absence, in general, of a symmetry group for a curved space-time, practitioners of quantum field theory in curved space-time take the linear field equations associated with particles of mass m and spin s in the Minkowski space-time 'configuration representation', and generalise them to curved space-times. The solutions of these equations can be considered to represent first-quantized free particles of mass m and spin s in curved space-time. Whilst the solutions of these linear equations correspond to unitary irreducible representations of the space-time symmetry group in the case of Minkowksi space-time, no such correspondence exists for the generalised equations. Moreover, for $s > 1$, there are reasons for thinking the solutions to these equations do not satisfy physical criteria. For example, such equations do not have a well-posed initial-value formulation (Wald 1984, p375). If particles of $s > 1$ can exist in regions of weak gravitational field, one presumes they can wander into regions of curved space-time, hence one should not conclude that $s > 1$ particles cannot exist in curved space-time. It is possible that the generalised equations provide the physically correct description of $s > 1$ particles in curved space-time, but simply do not provide the same degree of tractability as their Minkowski space-time counterparts. Alternatively, it is possible that the correct representation of $s > 1$ particles in curved space-time has not yet been found.

2.1 Parity

Whilst the largest possible local symmetry group of a universe with three spatial dimensions and one temporal dimensions is the full Poincare group $O(3,1) \circledS \mathbb{R}^{3,1}$, because our universe appears to possess a time orientation and a space orientation, the local symmetry group here is the restricted Poincare group $SO_0(3,1) \circledS \mathbb{R}^{3,1}$. This, however, appears to be a contingent fact about our universe. There are other possible universes, of the same dimension and geometrical signature to our own, which possess a larger local symmetry group. For example, in a universe in which space reflections are also local space-time symmetries, the local symmetry group would be the isochronous Poincare group $O^{\uparrow}(3,1) \circledS \mathbb{R}^{3,1}$. (Also called the orthochronous Poincare group). This group consists of both the identity component of the Poincare group, and the component which contains the operation of parity reversal, $\mathscr{P} : (x_0, x_1, x_2, x_3) \mapsto (x_0, -x_1, -x_2, -x_3)$. The possible free particles in such a universe are specified by the irreducible unitary representations of some double cover $H \circledS \mathbb{R}^{3,1}$ of the isochronous Poincare group, where H is a $\mathbb{Z}_2$-extension of $SL(2, \mathbb{C})$ to a group which double covers $O^{\uparrow}(3,1)$. (Sternberg 1994, p150-161).

In a universe in which space reflections $\mathscr{P}$, time reversals $\mathscr{T}$, and combinations thereof $\mathscr{P}\mathscr{T}$, are all local space-time symmetries, the local symmetry group is the entire Poincare group $O(3,1) \circledS \mathbb{R}^{3,1}$. The possible free particles in such a universe are specified by the irreducible unitary representations of some double cover $K \circledS \mathbb{R}^{3,1}$ of the entire Poincare group, where K is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-extension of $SL(2, \mathbb{C})$ to a group which double covers $O(3,1)$. There are eight such double covers of the Poincare group, (Sternberg 1994, p160-161). For each different choice of double cover K, one has a different family of unitary irreducible representations, hence for each different choice of double cover K, one has a different free-particle ontology.

One can treat $\{I, \mathscr{P}, \mathscr{T}, \mathscr{P}\mathscr{T}\}$ as the group $\mathbb{Z}_2 \times \mathbb{Z}_2$, (Sternberg 1994, p160). In any double cover of $O(3,1)$, let $\mathscr{L}_{\mathscr{P}}$ denote any one of the two elements which cover $\mathscr{P}$, let $\mathscr{L}_{\mathscr{T}}$ denote any one of the two elements which cover $\mathscr{T}$, and let $\mathscr{L}_{\mathscr{P}}\mathscr{L}_{\mathscr{T}}$ denote any one of the two elements which cover $\mathscr{P}\mathscr{T}$. Each one of the eight possible double covers of $O(3,1)$ is specified by a triple of values for the following three variables, (DeWitt-Morette *et al* 2000, p90):

$$\mathscr{L}_{\mathscr{P}}^2 = a, \quad \mathscr{L}_{\mathscr{T}}^2 = b, \quad (\mathscr{L}_{\mathscr{P}}\mathscr{L}_{\mathscr{T}})^2 = c$$

where $a, b, c \in \{1, -1\} = \mathbb{Z}_2$.

Two of the eight possible double covers are the well-known groups $Pin(3,1)$ and $Pin(1,3)$, closely related to Clifford algebras. In terms of the finite groups which cover $\{I, \mathscr{P}, \mathscr{T}, \mathscr{P}\mathscr{T}\}$ within a double cover of $O(3,1)$, one of the eight double covers uses $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, three use $\mathbb{Z}_2 \times \mathbb{Z}_4$, three use D_4, the dihedral group of order eight, and one uses the quaternionic group G_2.

Consider the case where space reflections are included in the local space-time symmetry group.[12] To find the irreducible unitary representations of $H \circledS \mathbb{R}^{3,1}$, one studies the orbits and isotropy groups of the $H \circledS \mathbb{R}^{3,1}$-action on $\mathbb{R}^{3,1}$. The physically relevant orbits of $H \circledS \mathbb{R}^{3,1}$ are the same as the physically relevant orbits of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$, namely the mass hyperboloids and light cones $\mathscr{V}_m$. However, the isotropy groups of these orbits change under the action of the enlarged group. For $m^2 > 0$ the isotropy group of a mass hyperboloid is an enlargement of $SU(2)$, and for $m = 0$ the isotropy group is an enlargement of $\widetilde{E}(2)$. However, there is more than one choice for an enlargement of $SL(2,\mathbb{C})$ that covers $O^\uparrow(3,1)$, and as a consequence, there is more than one choice for an enlargement of the isotropy group of a mass hyperboloid or light cone.

Consider first the case of $m^2 > 0$. Let $\overline{SL}(2,\mathbb{C})$ denote an enlargement of $SL(2,\mathbb{C})$ that covers $O^\uparrow(3,1)$, and let $\overline{SU}(2)$ denote the consequent enlargement of $SU(2)$. The parity transformation $\mathscr{P} \in O^\uparrow(3,1)$ is covered by a pair of elements $\pm\mathscr{L} \in \overline{SL}(2,\mathbb{C})$. In the case of $m^2 > 0$, the parity transformation $\mathscr{P}$ leaves the representative point $(m,0,0,0)$ in a mass hyperboloid fixed, hence $\pm\mathscr{L}$ belong to the enlarged isotropy group $\overline{SU}(2)$ of a mass hyperboloid.

Let I denote the identity element in $\overline{SU}(2)$. Bosonic representations of $\overline{SU}(2)$ (i.e. spin $s \in \mathbb{Z}$) map both $\pm I$ to Id, where Id denotes the identity transformation on the representation space. In contrast, fermionic representations of $\overline{SU}(2)$ map I to Id and $-I$ to $-Id$.

The different choices of enlargement $\overline{SU}(2)$ for the isotropy group correspond to whether (a) $\mathscr{L}^2 = I$ or (b) $\mathscr{L}^2 = -I$. In both cases, the finite dimensional, complex, irreducible representations of $\overline{SU}(2)$ are now parameterized not only by spin s, but also by parity $\epsilon = \pm 1$.

In the case of enlargement (a), an irreducible representation π must be such that

$$\pi(\mathscr{L}^2) = \pi(\mathscr{L})\pi(\mathscr{L}) = \pi(I) = Id$$

hence either $\pi(\mathscr{L}) = Id$ or $\pi(\mathscr{L}) = -Id$. This is true for both bosonic and fermionic representations. Depending upon whether $\pi(\mathscr{L}) = \pm Id$, the representation is said to have $\epsilon = \pm 1$ parity.

In the case of enlargement (b), one needs to distinguish between bosonic and fermionic representations. A bosonic representation of a double cover $\overline{SU}(2)$ is a representation which has been lifted from a representation of $O^\uparrow(3)$. In the case of enlargement (b), a bosonic representation sends $\mathscr{P} \in O^\uparrow(3)$ to $\epsilon \cdot Id$, and therefore sends $\pm\mathscr{L}$ to $\epsilon \cdot Id$, with $\epsilon = \pm 1$ being the parity of the bosonic representation. A fermionic representation π sends $-I$ to $-Id$, hence given that $\mathscr{L}^2 = -I$ in case (b),

$$\pi(\mathscr{L}^2) = \pi(\mathscr{L})\pi(\mathscr{L}) = \pi(-I) = -Id$$

It follows that $\pi(\mathscr{L}) = \epsilon i \cdot Id$, where $\epsilon = \pm 1$ is the parity of the representation.

Thus, in the case of $m^2 > 0$, if the local symmetry group of space-time includes parity transformation, then the irreducible representations of $H \circledS \mathbb{R}^{3,1}$

[12] The treatment here closely follows Sternberg 1994, p153-154.

are not parameterized by mass and spin alone. Rather, they are parameterized by mass, spin and parity. Hence parity ('handedness') is an invariant property of a massive particle precisely when the local symmetry group of space-time includes parity transformations.

In the case of $m^2 > 0$, the method of induced representation, applied to the isochronous Poincare group, yields vector bundles $E^+_{m,s,\epsilon}$ with the same base space and the same typical fibre as those produced for the restricted Poincare group, but with different representations of an enlarged isotropy group upon the typical fibre.

Now consider the case of $m = 0$. The parity transformation $\mathscr{P} \in O^\uparrow(3,1)$, and therefore $\mathscr{L} \in \overline{SL}(2,\mathbb{C})$, do not leave an arbitrary point in a light cone, such as $(1,0,0,1)$, fixed. Hence $\mathscr{L}$ does not belong to the isotropy group of a light cone. To label the different choices of isotropy group one must instead introduce an element $\mathscr{R} = \mathscr{U}\mathscr{L}$ with

$$\mathscr{U} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

$\mathscr{R}$ does belong to the isotropy group of a light cone. Either $\mathscr{R}^2 = \mathscr{U}^2\mathscr{L}^2 = I$ or $\mathscr{R}^2 = \mathscr{U}^2\mathscr{L}^2 = -I$, depending upon the choice of isotropy group.

Whilst the irreducible representations of $\widetilde{E}(2)$ are parameterized by $s \in \frac{1}{2}\mathbb{Z}$, the irreducible representations of an enlarged light cone isotropy group are parameterized by $t \in \frac{1}{2}\mathbb{Z}_+$. For $t \neq 0$, the t-representation can be decomposed into a direct sum of the $s = t$ and $s = -t$ representations of $\widetilde{E}(2)$. The t-representation maps the group element $\mathscr{R} = \mathscr{U}\mathscr{L}$ to a linear transformation which sends the $s = t$ representation of $\widetilde{E}(2)$ into the $-s$ representation. Hence, whilst the irreducible representations of $\widetilde{E}(2)$ are 1-dimensional, the irreducible representations of an enlarged light cone isotropy group are 2-dimensional for $t \neq 0$. In the case of $m = 0$, the method of induced representation, applied to the isochronous Poincare group, yields vector bundles $E^+_{0,t}$ with the same base space, but different typical fibres, from those obtained for the restricted Poincare group. In the case of the photon, a particle of mass $m = 0$ and spin/helicity $t = 1$, the bundle $E^+_{0,1}$ possesses $s = 1$ and $s = -1$ sub-bundles which correspond to the right-handed and left-handed polarizations of a photon. These sub-bundles correspond to the $E^+_{m,s} = E^+_{0,1}$ and $E^+_{m,s} = E^+_{0,-1}$ bundles used in the representations of the restricted Poincare group.

The case of $t = 0$ depends upon the choice of the enlarged isotropy group. For one choice there will be two 1-dimensional representations, whilst for the other choice, there will be four 1-dimensional representations.

When $H = SL(2,\mathbb{C})$, i.e. when parity transformation is not a local space-time symmetry, the spin s representation and spin $-s$ representation, (assuming $s > 0$), for a zero mass particle, are interpreted to represent right-handed and left-handed versions of the same particle. The particle associated with the spin s representation is said, for example, to have right-handed parity, whilst the particle associated with the spin $-s$ representation is said to have left-handed parity. The choice of handedness is purely conventional; the important point is that the two representations correspond to opposite handedness. The interpretation that the s representation and the $-s$ representation correspond to opposite parities is drawn from the fact that these representations are interchanged when the symmetry group is enlarged to include parity transformations.

When parity transformation is not a local space-time symmetry, the parity $\epsilon = \pm 1$ of a zero mass particle is an invariant property of the particle, determined by the irreducible unitary representation of $SL(2, \mathbb{C}) \circledS \mathbb{R}^{3,1}$ with which it is associated. By contrast, when parity *is* a local space-time symmetry, parity is not an invariant property of a zero mass particle. The representation space can be decomposed into a left-handed subspace and a right-handed subspace, so such a particle can possess parity as a property, but the parity of such a particle can change as the state of the particle changes, and it can change under a change of reference frame that involves the parity transformation $\mathscr{P}$. *Contra* Sternberg (1994, p155) it *does* make sense to speak of the handedness of a zero-mass particle when parity is a local space-time symmetry, but the point is that the handedness is not an *invariant* property under such a local space-time symmetry group.

Let us agree to define an intrinsic property of an object to be a property which the object possesses independently of its relationships to other objects. Let us also agree to define an extrinsic property of an object to be a property which the object possesses depending upon its relationships with other objects. These definitions of intrinsicness and extrinsicness may not be adequate to cover intrinsic and extrinsic properties in general, (Weatherson 2002, Section 2.1), but they are adequate for dealing with physical quantities and the values of physical quantities, the types of properties which philosophers refer to as 'determinables' and 'determinates' respectively, (Swoyer 1999). If the value of a quantity possessed by an object can change under a change of reference frame, then the value of that quantity must be an extrinsic property of the object, not an intrinsic property. The value of such a quantity must be a relationship between the object and a reference frame, and under a change of reference frame, that relationship can change.

To reiterate, when parity-reversal is a space-time symmetry, the parity of a zero-mass particle can change under a change of reference frame. Hence, when parity-reversal is a space-time symmetry, parity is not possessed intrinsically by a zero-mass particle. When parity-reversal is a space-time symmetry, the parity possessed by a zero-mass particle depends upon the reference frame, hence the parity of a zero-mass particle is a relationship between the particle and a reference frame.

This is distinct from the claim that parity itself, as an abstract property, is an extrinsic property when parity-reversal is a space-time symmetry, (Weatherson 2002, Section 1.2). Recall that when parity-reversal is a space-time symmetry, parity is an invariant property of massive particles. An invariant property cannot change under a change of reference frame, hence parity is possessed intrinsically by massive particles when parity-reversal is a space-time symmetry.

When parity-reversal does not belong to the local space-time symmetry group, a zero-mass particle possesses parity intrinsically, and a massive particle does not possess parity at all.

2.2 Free Particles in other universes

Universes of a different dimension and/or geometrical signature, will possess a different local symmetry group, and will therefore possess different sets of possible free particles. Moreover, even universes of the same dimension and geometrical signature will not necessarily possess the same sets of possible free particles. The preceding section demonstrated that the dimension and geometrical signature of a universe do not alone determine the possible free particles which can exist in a universe. The dimension and geometrical signature merely determines the largest possible local symmetry group, and universes with different orientation properties will possess different local symmetry groups, and, perforce, will possess different sets of possible free particles.

Recall that for an arbitrary pseudo-Riemannian manifold $(\mathcal{M}, g)$, of dimension n and signature (p, q), the largest possible local symmetry group is the semi-direct product $O(p, q) \circledS \mathbb{R}^{p,q}$. If there is no restriction to a subgroup of this, then one specifies the possible free elementary particles in such a universe by specifying the *projective*, unitary, irreducible representations of $O(p, q) \circledS \mathbb{R}^{p,q}$. For a universe which has p spatial dimensions and q temporal dimensions, and which has, in addition, a space orientation and a time orientation, the local space-time symmetry group will be the semi-direct product $SO_0(p, q) \circledS \mathbb{R}^{p,q}$. Given an arbitrary semi-direct product $G \circledS V$, with V a finite-dimensional real vector space, every projective, unitary, irreducible representation of $G \circledS V$ lifts to a unitarizable irreducible representation of the universal cover $\widetilde{G} \circledS V$ if G is a semisimple group. For $p + q > 2$, $SO_0(p, q)$ is a connected simple group, hence every projective, unitary, irreducible representation of $SO_0(p, q) \circledS \mathbb{R}^{p,q}$, for $p + q > 2$, corresponds to an ordinary, unitary, irreducible representation of the universal cover $\widetilde{SO_0}(p, q) \circledS \mathbb{R}^{p,q}$.[13] Thus, the elementary particles in a universe which has p spatial dimensions and q temporal dimensions, and which has, in addition, a space orientation and a time orientation, will correspond to a subset of the irreducible, unitary representations of $\widetilde{SO_0}(p, q) \circledS \mathbb{R}^{p,q}$. As before, the method of induced representation, applied to semi-direct products, can be used to classify these representations, and to explicitly generate the energy-momentum space representations. The first step is to find the orbits of the $\widetilde{SO_0}(p, q) \circledS \mathbb{R}^{p,q}$-action on $\mathbb{R}^{p,q}$. Given that $\mathbb{R}^{p,q}$ is an abelian group, acting upon itself by conjugation, these orbits coincide with the orbits of the $\widetilde{SO_0}(p, q)$-action on $\mathbb{R}^{p,q}$. The orbits of the $\widetilde{SO_0}(p, q)$-action on $\mathbb{R}^{p,q}$ correspond to the orbits of $SO_0(p, q)$ of $\mathbb{R}^{p,q}$. As before, these orbits are hypersurfaces in $\mathbb{R}^{p,q}$, and, as before, they come in three types:

- For any $m \in \mathbb{R}_+$, there are the hypersurfaces $\mathscr{V}_m$, where $\mathscr{V}_m = \{p \in \mathbb{R}^{p,q} :$

[13] Note that for $p, q > 2$, whilst $Spin(p, q)$ is a double cover of $SO_0(p, q)$, it is not the universal cover.

$\langle p,p\rangle = m^2 > 0\}$.[14]

- There are the null hypersurfaces, $\mathscr{V}_0 = \{p \in \mathbb{R}^{p,q} : \langle p,p\rangle = m^2 = 0\}$.
- Then, for any $m \in i\mathbb{R}_+$, there are the hypersurfaces $\mathscr{V}_m = \{p \in \mathbb{R}^{p,q} : \langle p,p\rangle = m^2 < 0\}$

Unless $p = 1$ or $q = 1$, each such hypersurface consists of a single connected component. Hence, the special case of $Spin(3,1) \circledS \mathbb{R}^{3,1}$, where the $m^2 > 0$ mass hyperboloids consist of two components, (a forward mass hyperboloid and a backward mass hyperboloid), each of which is a separate orbit of the group action, is not typical. In addition, it is only in the case that $q = 1$ that the $m^2 > 0$ orbits are spacelike hypersurfaces. In general, the $m^2 > 0$ orbits and $m^2 < 0$ orbits will be timelike hypersurfaces. i.e. the tangent vector space at each point will be spanned by timelike and spacelike vectors. The spacelike surfaces in $\mathbb{R}^{p,q}$ will be p-dimensional, and unless $q = 1$, they will not be hypersurfaces.

The continuous parameter m, which parameterizes the orbits of the $\widetilde{SO}_0(p,q)$-action on $\mathbb{R}^{p,q}$, provides the first parameter to classify the irreducible unitary representations of $\widetilde{SO}_0(p,q) \circledS \mathbb{R}^{p,q}$. The next step is to identify the isotropy groups of the $\widetilde{SO}_0(p,q)$-action on $\mathbb{R}^{p,q}$. The isotropy group of the $\widetilde{SO}_0(p,q)$-action on the $m^2 > 0$ orbits is $\widetilde{SO}_0(p,q-1)$, and the isotropy group of the $\widetilde{SO}_0(p,q)$-action on the $m^2 < 0$ orbits is $\widetilde{SO}_0(p-1,q)$. This means that the isotropy group of the $\widetilde{SO}_0(p,q) \circledS \mathbb{R}^{p,q}$-action on the $m^2 > 0$ orbits is $\widetilde{SO}_0(p,q-1) \circledS \mathbb{R}^{p,q}$, and the isotropy group of the $\widetilde{SO}_0(p,q) \circledS \mathbb{R}^{p,q}$-action on the $m^2 < 0$ orbits is $\widetilde{SO}_0(p-1,q) \circledS \mathbb{R}^{p,q}$. In the special case of $Spin(3,1) \circledS \mathbb{R}^{3,1}$, the isotropy group of the action of $Spin(3,1)$ on the $m^2 > 0$ orbits is $Spin(3) = SU(2)$. By virtue of being a compact group, $SU(2)$ has finite-dimensional irreducible unitary representations, which can be parameterized by a discrete parameter. However, in the general case, unless $p = 1$ or $q = 1$, the isotropy groups of the $\widetilde{SO}_0(p,q)$-action will be non-compact, and all the non-trivial irreducible unitary representations of such groups are infinite-dimensional. Hence, in addition to the first continuous parameter, one must deal with the additional continuous parameters, (and the possible additional discrete parameters), required to classify the infinite-dimensional irreducible unitary representations of the isotropy groups, $\widetilde{SO}_0(p,q-1) \circledS \mathbb{R}^{p,q}$ and $\widetilde{SO}_0(p-1,q) \circledS \mathbb{R}^{p,q}$.[15]

In conclusion, the free elementary particles which exist in universes with a different number of spatial and/or temporal dimensions to our own, are not parameterized by mass and spin. That particular parameterization is a unique consequence of the dimension and geometrical signature of our own universe.

This conclusion can be reinforced by considering the configuration space approach. Recall that when there are three spatial dimensions and one temporal

[14] There is no significance here in the choice of 'p' to denote both an element of energy-momentum space, and the number of spatial dimensions in the geometrical signature (p,q).

[15] Private communications with Veeravalli Varadarajan and Shlomo Sternberg.

dimension, one begins by defining a bundle which possesses a spin-s representation of $Spin(3,1) \cong SL(2,\mathbb{C})$ upon its typical fibre. Whilst the complex finite-dimensional irreducible representations of $SL(2,\mathbb{C})$ are indexed by pairs of spins (s_1, s_2), with $s = s_1 + s_2$, this does not generalize to the case of $\widetilde{SO_0}(p,q)$ relevant to universes with an arbitrary number of space and time dimensions.

Let us briefly digress to explain this. The representations of $\widetilde{SO_0}(p,q)$ correspond to the representations of the lie algebra $\mathfrak{so}(p,q)$, and the complex representations of $\mathfrak{so}(p,q)$ correspond to the representations of its complexification $\mathfrak{so}(p,q) \otimes \mathbb{C}$. Moreover, $\mathfrak{so}(p,q) \otimes \mathbb{C} \cong \mathfrak{so}(n) \otimes \mathbb{C}$, for $p+q=n$. In the case of $\mathfrak{so}(3,1)$, this means that its complex, finite dimensional, irreducible representations correspond to the complex, finite-dimensional, irreducible representations of $\mathfrak{so}(4)$. Now, it happens that $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, hence the complex, finite-dimensional, irreducible representations of $\mathfrak{so}(3,1)$ correspond to the finite-dimensional irreducible representations of $\mathfrak{so}(4) \otimes \mathbb{C} \cong (\mathfrak{su}(2) \otimes \mathbb{C}) \oplus (\mathfrak{su}(2) \otimes \mathbb{C})$, (Derdzinski 1992, Section 3.4). The finite-dimensional irreducible representations of $\mathfrak{su}(2) \otimes \mathbb{C}$ are indexed by their spin, and the irreducible representations of a lie algebra direct sum correspond to tensor products of the irreducible representations of the summands, hence the finite-dimensional irreducible representations of $(\mathfrak{su}(2) \otimes \mathbb{C}) \oplus (\mathfrak{su}(2) \otimes \mathbb{C})$ correspond to pairs of spins (s_1, s_2). Sadly, $\mathfrak{so}(n)$ is not in general a direct sum of copies of $\mathfrak{su}(2)$, hence the complex, irreducible, finite-dimensional representations of $\mathfrak{so}(p,q)$ cannot, in general, be indexed by a tuple of spins. As a consequence, the complex, irreducible, finite-dimensional representations of $\widetilde{SO_0}(p,q)$ also cannot be indexed by pairs of spins. Thus, the vector bundles possessing representations of $\widetilde{SO_0}(p,q)$ upon their typical fibres cannot be labelled as spin-s free particle bundles.

Whilst the parameterization of elementary particles varies from one universe to another, it remains true that the dimension, signature, and orientation of a space-time determine the spectrum of possible free elementary particles which can exist in that space-time. Moreover, it is also true that the dimension, signature, and orientation of a space-time determine the fundamental differential equations which govern the behaviour of free elementary particles. One cannot vary the dimension-signature-orientation of a space-time whilst holding fixed the fundamental laws of physics for free systems in that space-time. The laws of physics for free systems, such as the Klein-Gordon equation, the Dirac equation and the Weyl equation, are the configuration space expressions of the irreducible unitary representations of the local space-time symmetry group. For a fixed particle type in a fixed space-time, the differential equations governing that particle cannot be varied. In particular, the signature of the space-time metric is reflected in the signature of the partial differential equations governing the free elementary particles in that space-time. The Klein-Gordon equation, Dirac equation and Weyl equation in our universe are hyperbolic partial differential equations precisely because the signature of our space-time is Lorentzian. Universes with, for example, metric tensors of Riemannian signature, will possess elementary particles that satisfy elliptic partial differential equations in the configuration representation.

There is a history of argument which attempts to explain why we observe three spatial dimensions, by pointing out that if the laws of physics are held fixed, and the number of spatial dimensions is changed, then it is not possible for stable planetary orbits, or stable atoms and molecules to exist. Callender (2005) correctly emphasizes that if the laws of physics are allowed to vary, then stable systems can exist in universes with a different number of spatial dimensions, and he asserts that "physical law is such a weak kind of necessity compared with conceptual or metaphysical necessity." However, Callender fails to recognize that the particle types in a universe are determined by the irreducible unitary representations of the local space-time symmetry group, and the fundamental laws of elementary particles are simply the configuration space expressions of those irreducible unitary representations. One is not making the glib assumption that the laws of physics in our universe must be the same as those in any other universe; rather, there is a conceptual link between the signature, dimension and orientation of a space-time and the fundamental laws of physics that govern particles within that space-time.

Let us briefly explain how the signature of space-time is reflected, via the inverse Fourier transform from the unitary representations on energy-momentum space, in the differential equations on configuration space. A function $\phi(p)$ with, for example, support on a forward mass hyperboloid in Minkowski energy-momentum space, will satisfy the equation $(||p||^2 - m^2)\phi(p) = 0$. Such a function is the Fourier transform of a function $\psi(x)$ on Minkowski configuration space which satisfies the equation $(-\partial_0^2 + \partial_1^2 + \partial_2^2 + \partial_3^2 - m^2)\psi(x) = 0$. To see this, note first that for any function $\phi(p)$ on a mass hyperboloid in Minkowski energy-momentum space, there exists a function $\psi(x)$ on Minkowski configuration space which is the inverse Fourier transform of $\phi(p)$:[16]

$$\psi(x) = \frac{1}{(2\pi)^2} \int e^{-i\langle p,x \rangle} \phi(p) d^4p$$

It follows that for $j = 1, 2, 3$, the Fourier transform of $\partial_j \psi$ is $ip_j\phi$ because

$$\begin{aligned}\partial_j \psi(x) &= \frac{1}{(2\pi)^2} \int \partial_j [e^{-i\langle p,x \rangle}] \phi(p) d^4p \\ &= \frac{1}{(2\pi)^2} \int e^{-i\langle p,x \rangle} ip_j \phi(p) d^4p\end{aligned}$$

Similarly, for $j = 0$, the Fourier transform of $\partial_j \psi$ is $-ip_j\phi$. For $j = 0, 1, 2, 3$, the Fourier transform of $\partial_j^2 \psi$ is $-p_j^2 \phi$ because $(-ip_j)^2 = (ip_j)^2 = -p_j^2$.

Hence, the Fourier transform of $(-\partial_0^2 + \partial_1^2 + \partial_2^2 + \partial_3^2 - m^2)\psi(x)$ is $(p_0^2 - p_1^2 - p_2^2 - p_3^2 - m^2)\phi(p)$. Using the notation $(-\partial_0^2 + \partial_1^2 + \partial_2^2 + \partial_3^2 - m^2)\psi(x) \equiv (\Box - m^2)\psi(x)$, one has:

[16] Note that the indefinite Minkowski space inner product $\langle p, x \rangle = p_0 t - p_1 x_2 - p_2 x_2 - p_3 x_3$ is being used to define the Fourier transform here.

$$
\begin{aligned}
(\Box - m^2)\psi(x) &= \frac{1}{(2\pi)^2}\int (-\partial_0^2 + \partial_1^2 + \partial_2^2 + \partial_3^2 - m^2)[e^{-i\langle p,x\rangle}]\phi(p)d^4p \\
&= \frac{1}{(2\pi)^2}\int e^{-i\langle p,x\rangle}(-(-ip_0)^2 + (ip_1)^2 + (ip_2)^2 + (ip_3)^2 - m^2)\phi(p)d^4p \\
&= \frac{1}{(2\pi)^2}\int e^{-i\langle p,x\rangle}(p_0^2 - p_1^2 - p_2^2 - p_3^2 - m^2)\phi(p)d^4p
\end{aligned}
$$

Thus, the inverse Fourier transform of $(||p||^2 - m^2)\phi(p)$ is $(-\partial_0^2 + \partial_1^2 + \partial_2^2 + \partial_3^2 - m^2)\psi(x)$. Flip one of the metric signs from positive to negative, or vice versa, and the sign of the corresponding differential operator on configuration space must clearly also change.

Partial differential equations (PDEs) can be classified by the eigenvalues of their component matrices, (Tegmark 1998, p19). For example, consider the case of a second-order linear partial differential equation in an n-dimensional space-time:

$$
\left[\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}(x)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + c(x)\right] u(x) = f(x) \ ,
$$

Assuming that $A(x)$ is a symmetric real $n \times n$ matrix, it must have n real eigenvalues, and one can classify a second-order linear partial differential equation in terms of the signs of the eigenvalues of $A(x)$. The PDE at x is defined to be

- elliptic if either all the eigenvalues of $A(x)$ are positive or all the eigenvalues of $A(x)$ are negative,
- hyperbolic if one eigenvalue is negative and all the other positive, or vice versa,
- ultrahyperbolic if at least two eigenvalues are positive and the others negative, or vice versa, and
- parabolic if at least one eigenvalue is zero, entailing that $A(x)$ is singular, $\det A(x) = 0$.

If the space-time signature is Riemannian, such as $(++\cdots+)$ or $(--\cdots-)$, then the elementary particles in such a space-time will satisfy elliptic partial differential equations in the configuration representation. If the space-time signature is Lorentzian, such as $(++\cdots-)$ or $(--\cdots+)$, then the elementary particles in such a space-time will satisfy hyperbolic partial differential equations in the configuration representation. If the space-time signature is like $(++\cdots--)$, then the elementary particles in such a space-time will satisfy ultrahyperbolic partial differential equations in the configuration representation. Finally, if the space-time signature is degenerate, with at least one zero in the signature, then the elementary particles in such a space-time will satisfy parabolic partial differential equations in the configuration representation. The Schrodinger equation of non-relativistic quantum mechanics is a parabolic partial differential equation, which follows from the fact that space-time is represented by the degenerate metric tensor of Newton-Cartan space-time in non-relativistic quantum mechanics.

The problem of finding a solution u to a partial differential equation on a particular domain, with a particular set of boundary conditions or initial conditions, is said to be 'well-posed' if:

- At least one solution exists.
- At most one solution exists. i.e. the solution is unique.
- The solution depends continuously upon the boundary conditions and initial conditions.

If there are too many boundary or initial conditions, a solution will not exist, and the problem is said to be 'overdetermined'. If there are too few boundary or initial conditions, the solution will not be unique, and the problem is said to be 'underdetermined'. If the solution does not depend continuously upon the boundary and initial conditions, then the problem is said to be unstable. If the problem of solving a partial differential equation fails any of these conditions, it is said to be an 'ill-posed' problem.

Domains tend to be classified in this context as open or closed, meaning non-compact or compact topology. One can treat initial conditions as special types of boundary conditions; initial conditions are the boundary conditions on a spacelike part of the boundary to a region of space-time. A region of space-time can be compact or non-compact, and the initial boundary can be compact or non-compact too. Boundary conditions tend to be classified as:

- Dirichlet - The solution u is specified upon the boundary.
- Neumann - The derivative of the solution with respect to the normal vector field on the boundary is specified.
- Robin/Mixed/Cauchy - A conjunction or linear combination of the solution and its normal derivative is specified on the boundary.

The problem of solving the different types of partial differential equation is well-posed or ill-posed depending upon both the type of boundary conditions specified, and the type of domain. Whilst the situation is largely unresolved for ultrahyperbolic equations,[17] the following results are known for the other equation types (see, for example, Morse and Feshbach (1953), Sec.6.1):

- Dirichlet.
  - Open domain
    1. Hyperbolic: underdetermined

[17] Private communication with V.G.Romanov.

2. Elliptic: underdetermined
3. Parabolic: unique and stable in one direction

- Closed domain
  1. Hyperbolic: overdetermined
  2. Elliptic: unique and stable
  3. Parabolic: overdetermined

- Neumann.
  - Open domain
    1. Hyperbolic: underdetermined
    2. Elliptic: underdetermined
    3. Parabolic: unique and stable in one direction
  - Closed domain
    1. Hyperbolic: overdetermined
    2. Elliptic: unique and stable with additional constraints
    3. Parabolic: overdetermined

- Robin/Mixed/Cauchy.
  - Open domain
    1. Hyperbolic: unique and stable
    2. Elliptic: ill-posed
    3. Parabolic: overdetermined
  - Closed domain
    1. Hyperbolic: overdetermined
    2. Elliptic: overdetermined
    3. Parabolic: overdetermined

Hence, the well-posedness of the partial differential equations which govern free elementary particles in universes with a different number of spatial or temporal dimensions to our own, clearly depends upon the topology of the domains in question, and the type of the boundary conditions specified.

One qualification should be added to the results above: Cauchy data on a non-compact spacelike hypersurface provides a well-posed problem for a hyperbolic equation, but Asgeirsson's theorem implies that Cauchy data on a timelike hypersurface is ill-posed for such an equation. This theorem also has ramifications for ultrahyperbolic equations; because there are at least two spatial dimensions and at least two temporal dimensions for such equations, spacelike hypersurfaces do not exist, and Cauchy data on a timelike hypersurface does not provide a well-posed problem for an ultrahyperbolic equation either. However, Cauchy data on a combination of null hypersurfaces and other types of hypersurface can provide a well-posed problem for a hyperbolic equation, so the same may prove to be the case for ultrahyperbolic equations.

Tegmark claims that in the universes where all n dimensions are spatial, or all n dimensions are temporal, the elliptic differential equations would not enable the physics in such universes to have predictive power. He acknowledges (1998, p19, paragraph d) that an elliptic equation on a closed domain leads to a well-posed problem, but seems to regard such a situation as a boundary value problem, distinct from an initial data problem. Tegmark assumes that an initial data problem must involve an open domain, and points out that elliptic PDEs are ill-posed on such domains: "specifying only u on a non-closed surface gives an underdetermined problem, and specifying additional data, e.g., the normal derivative of u, generally makes the problem overdetermined," (ibid., p19, footnote). However, there is no reason why initial data problems must have non-compact initial boundaries rather than compact initial boundaries, and in a universe in which all dimensions are spatial, or in which all dimensions are temporal, it seems artificial to make a distinction between boundary value problems and initial data problems. Elliptic partial differential equations do not provide well-posed problems given Cauchy data on open or closed domains, and they do not provide well-posed problems on open domains given any type of boundary data, but they do provide well-posed problems on closed domains with either Dirichlet or Neumann boundary conditions. Thus, it is still possible to make inferences about the interior of bounded regions from information on the boundary. Tegmark treats prediction as simply the use of local observations to make inferences about other parts of the pseudo-Riemannian manifold, whatever its dimension and signature. If one accepts this definition of prediction, then the differential equations which govern free elementary particles in universes where all n dimensions are spatial, or all n dimensions are temporal, would still have predictive power under certain circumstances.

There is also an implicit assumption in Tegmark's reasoning that the physics in a universe only has predictive power if the equations which govern the elementary particles have predictive power. The signature of the space-time geometry may determine the type of the equations which govern the elementary particles, but it doesn't determine the type of all the PDEs governing all the physical processes in a space-time. For example, in our own space-time, where the fundamental equations for free particles are hyperbolic, the diffusion equation, which governs certain higher-level statistical and thermodynamical processes, is parabolic.

3 Gauge fields

In the Standard Model, each gauge force field corresponds to a compact connected Lie Group G, called the gauge group. A gauge field with gauge group G can either be represented by a connection on a principal fibre bundle P with structure group G, or by a connection on a vector bundle δ equipped with a so-called 'G-structure'.

Given a complex vector bundle δ of fibre dimension n, any matrix sub-group $G \subset GL(n, \mathbb{C})$ acts freely, from the right, upon the set of bases in each fibre.

Treating a basis as a row vector, $(e_1, ..., e_n)$, it is mapped by $g \in G$ to another basis $(e'_1, ..., e'_n)$ by matrix multiplication:

$$(e'_1, ..., e'_n) = (e_1, ..., e_n)g = (g_{j1}e_j, ..., g_{jn}e_j)$$

Needless to say, one sums over repeated indices in this expression.

In general, the G-action upon the set of bases in each fibre will possess multiple orbits. The selection of one particular orbit of this G-action, in each fibre of δ, is called a G-structure in δ, (Derdzinski 1992, p81-82). A vector bundle δ equipped with a G-structure is sometimes referred to as a 'G-bundle'.

Geometrical objects in each fibre of a vector bundle, such as inner products and volume forms, can be used to select a G-structure. For example, suppose δ is a complex vector bundle of fibre dimension n: the unitary group $U(n)$ acts freely upon the set of bases in each fibre, and there are multiple orbits of the $U(n)$-action in each fibre, but if each fibre is equipped with a positive-definite Hermitian inner product, then the inner product singles out the orbit consisting of orthonormal bases. By stipulating that an inner product selects the orbit of orthonormal bases, one defines a bijection between inner products and $U(n)$-structures. For any orbit of the $U(n)$-action, there is an inner product with respect to which that orbit consists of orthonormal bases. Given any basis $(e_1, ..., e_n)$, one can define an inner product which renders that basis an orthonormal basis by stipulating that the matrix of inner products between the vectors in the basis has the form $diag \ \{1, 1, ..., 1\}$. Given any pair of vectors v, w, they can be expressed as $v = c_1e_1 + \cdots + c_ne_n$ and $w = a_1e_1 + \cdots + a_ne_n$ in this basis, and their inner product is now defined to be

$$\langle v, w\rangle = \langle c_1e_1 + \cdots + c_ne_n, a_1e_1 + \cdots + a_ne_n\rangle = c_1a_1 + \cdots + c_na_n$$

Once an inner product has been defined which renders $(e_1, ..., e_n)$ orthonormal, all the other bases which can be obtained from $(e_1, ..., e_n)$ under the action of $U(n)$ must themselves be orthonormal.

This bijection between inner products and $U(n)$-structures is, however, merely conventional.[18] Given the specification of an inner product, the convention is that a basis belongs to the $U(n)$-structure if the matrix of inner products between its constituent vectors has the form $diag \ \{1, 1, ..., 1\}$. Given the specification of an inner product, one could alternatively fix an arbitrary positive-definite Hermitian matrix, and stipulate that a basis belongs to the $U(n)$-structure if the matrix of inner products between its constituent vectors equals the chosen positive-definite Hermitian matrix. This would provide an alternative bijection between inner products and $U(n)$-structures.

Following Derdzinski, we shall refer to a complex vector bundle δ equipped with a G-structure as an *interaction bundle*. The electromagnetic force corresponds to a complex line bundle λ equipped with a $U(1)$-structure; the electroweak force corresponds to a complex vector bundle ι, of fibre dimension 2,

[18] Private communication with Andrzej Derdzinski

equipped with a $U(2)$-structure; and the strong force corresponds to a complex vector bundle ρ, of fibre dimension 3, equipped with an $SU(3)$-structure.

In the interaction bundle picture, (Derdzinski 1992, p81-83), there is no need to introduce a principal fibre bundle P to define a gauge connection, a choice of gauge, or a gauge transformation. One introduces a bundle $G(\delta)$ of automorphisms of each fibre of δ, and a bundle $\mathfrak{g}(\delta)$ of endomorphisms[19] of each fibre of δ. A cross-section of $G(\delta)$ specifies an automorphism of each fibre of δ, and a cross-section of $\mathfrak{g}(\delta)$ specifies an endomorphism of each fibre of δ. Given the G-structure in each fibre of δ, typically a Hermitian inner product, perhaps in tandem with a volume form, an automorphism or endomorphism of each fibre δ_x is a mapping which preserves this structure.

Each fibre of $G(\delta)$ is a Lie group, and each fibre of $\mathfrak{g}(\delta)$ is a Lie algebra. $G(\delta)$ is said to be a Lie group bundle, and $\mathfrak{g}(\delta)$ is said to be a Lie algebra bundle. Each fibre of $G(\delta)$ is isomorphic to the matrix Lie group $G \subset GL(n, \mathbb{C})$, and each fibre of $\mathfrak{g}(\delta)$ is isomorphic to the matrix Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(n, C)$, but the isomorphisms are not canonical. It is necessary to fix a basis in a fibre δ_x to establish an isomorphism between $G(\delta)_x$ and $G \subset GL(n, \mathbb{C})$. Similarly, it is necessary to fix a basis in a fibre δ_x to establish an isomorphism between $\mathfrak{g}(\delta)_x$ and $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{C})$.

In the interaction bundle picture, a gauge transformation is a cross-section of $G(\delta)$. Hence, a gauge transformation selects, at each point x, an automorphism α_x of the fibre δ_x; a gauge transformation is a bundle automorphism which respects the G-structure in each fibre. The (infinite-dimensional) group of all such automorphisms $\mathcal{G} = \Gamma(G(\delta))$ is the group of gauge transformations. A cross-section of $G(\delta)$ also acts upon the Lie algebra bundle of endomorphisms $\mathfrak{g}(\delta)$. At each point x, the automorphism α_x acts adjointly, as an inner automorphism upon $\mathfrak{g}(\delta)_x$, mapping an endomorphism T into $\alpha_x T \alpha_x^{-1}$.

Free gauge fields, represented by G-connections[20] ∇^δ on an interaction bundle δ, must satisfy the free-field Yang-Mills equations, (Derdzinski 1992, p84),

$$div\ R^{\nabla^\delta} = 0$$

R^{∇^δ} is the curvature two-form of the connection ∇^δ.

The space of G-connections on δ is an affine space.[21] In fact, it is the space of smooth cross-sections of an affine bundle $\mathscr{C}(\delta)$, whose translation space is the set of smooth cross-sections of $T^*\mathcal{M} \otimes \mathfrak{g}(\delta)$. The space of G-connections on δ which satisfy the free-field Yang-Mills equations correspond to a subspace of the space of cross-sections of $\mathscr{C}(\delta)$.

In the interaction bundle picture, a choice of gauge corresponds to the selection, in each fibre of δ, of a basis which respects the G-structure. e.g. if

[19]The endomorphisms of a structured set are the maps of the set into itself which are not necessarily one-to-one, but which do preserve the structure of the set.

[20]A G-connection is a connection on a bundle equipped with a G-structure, which is 'compatible' with the G-structure. i.e the geometrical objects that define the G-structure are rendered parallel with respect to the connection.

[21]An affine space is a set which is acted upon transitively and effectively by the additive group structure of a vector space.

the G-structure consists of an inner product in each fibre, then the selected basis in each fibre should be orthonormal. A choice of gauge renders the space of connections canonically isomorphic with the space of smooth cross-sections of $T^*\mathcal{M} \otimes (\mathcal{M} \times \mathfrak{g})$. In other words, a choice of gauge enables one to treat a G-connection on δ, as a Lie-algebra valued one-form A on the base space $\mathcal{M}$, a so-called gauge field connection 'pull-down'.

A gauge field connection pull-down A transforms under a configuration space representation of the local space-time symmetry group, which in our universe is $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$, and it also transforms under a representation of the group of gauge transformations $\mathcal{G} = \Gamma(G(\delta))$. Whilst there is a finite-dimensional representation of $SL(2,\mathbb{C}) \times G$ upon $\mathbb{R}^{3,1} \otimes \mathfrak{g}$, the typical fibre of the translation space bundle $T^*\mathcal{M} \otimes \mathfrak{g}(\delta)$, and whilst this representation uses the adjoint representation of G on $\mathfrak{g}$, the representation of the infinite-dimensional group $\mathcal{G}$ is upon $\Gamma(\mathfrak{g}(\delta))$.

Given the representation of $SL(2,\mathbb{C}) \times G$ upon the typical fibre $\mathbb{R}^{3,1} \otimes \mathfrak{g}$ of the gauge field translation space bundle $T^*\mathcal{M} \otimes \mathfrak{g}(\delta)$, the selection of a basis in $\mathfrak{g}$, or the restriction of the representation to $SL(2,\mathbb{C}) \times Id$, enables one to decompose this representation as a direct sum

$$\bigoplus^{dim\ \mathfrak{g}} \mathbb{R}^{3,1}$$

i.e. one decomposes the representation into a direct sum of $dim\ \mathfrak{g}$ copies of the representation of $SL(2,\mathbb{C})$ on $\mathbb{R}^{3,1}$.

So, a choice of gauge renders $\mathscr{C}(\delta)$, the affine bundle housing the G-connections on δ, canonically isomorphic with $T^*\mathcal{M} \otimes (\mathcal{M} \times \mathfrak{g})$, and the selection of a basis in $\mathfrak{g}$ then enables one to decompose $\mathcal{M} \times \mathfrak{g}$ as the direct sum

$$\bigoplus^{dim\ \mathfrak{g}} (\mathcal{M} \times \mathbb{R}^1),$$

which, in turn, enables one to decompose $T^*\mathcal{M} \otimes (\mathcal{M} \times \mathfrak{g})$ as the direct sum, (Derdzinski 1992, p91):

$$\bigoplus^{dim\ \mathfrak{g}} T^*\mathcal{M}$$

$T^*\mathcal{M}$ is the configuration space bundle for 'real vector bosons', neutral particles of spin 1. A spin-s configuration space bundle possesses, upon its typical fibre, either a complex, finite-dimensional, irreducible representation of $SL(2,\mathbb{C})$ from the $\mathscr{D}^{s_1,s_2}$ family, for $s = s_1 + s_2$, or a direct sum of such representations. Given that $T^*\mathcal{M}$ is a real vector bundle, it cannot possess upon its typical fibre a member of the $\mathscr{D}^{s_1,s_2}$ family of complex representations, but it does possess the real representation of $SL(2,\mathbb{C})$ which complexifies to the $\mathscr{D}^{1/2,1/2}$ representation. In this sense, $T^*\mathcal{M}$ is a spin-1 configuration space bundle.

The differential equations for a spin 1 bundle, (Derdzinski 1992, p19), consist of the Klein-Gordon equation,

$$\Box\psi = m^2\psi$$

and the divergence condition

$$div\ \psi = 0$$

Under a choice of gauge, the cross-sections of the affine bundle $\mathscr{C}(\delta) \cong \bigoplus^{dim\ \mathfrak{g}} T^*\mathcal{M}$ which satisfy the free-field Yang-Mills equations, correspond to the mass 0 solutions to these equations. This is easiest to see in the case of electromagnetism, where a choice of gauge selects an isomorphism $\mathscr{C}(\lambda) \cong T^*\mathcal{M}$ which maps a connection ∇ to a real vector potential A. The Maxwell equations are the Yang-Mills equations in the special case of electromagnetism, and with the Lorentz choice of gauge, the Maxwell equations upon a real vector potential,

$$\Box A = 0, \quad div\ A = 0,$$

clearly correspond to the differential equations for a spin 1 particle of mass 0.

Hence, under a choice of gauge, from the space of $U(1)$ connections satisfying the free-field Maxwell equations, one can construct a space which is the inverse Fourier transform of the space of single photon spates $\Gamma_{L^2}(E^+_{0,1})$ in the Wigner representation.

In our universe the 'gauge bosons', or 'interaction carriers' of a gauge field are the spin 1, mass 0, Wigner-representations of $SL(2,\mathbb{C})$ Ⓢ $\mathbb{R}^{3,1}$, which inverse Fourier transform into spaces constructed from mass 0 cross-sections of spin 1 bundles such as $T^*\mathcal{M}$. These spin 1 bundles belong to a decomposition such as $\bigoplus^{dim\ \mathfrak{g}} T^*\mathcal{M}$ of the affine bundle $\mathscr{C}(\delta)$ housing the G-connections on δ. For gauge fields which undergo spontaneous symmetry breaking, the decomposition changes slightly from $\bigoplus^{dim\ \mathfrak{g}} T^*\mathcal{M}$, (See McCabe 2005, Section 4.6).

Given that a choice of gauge renders the affine bundle $\mathscr{C}(\delta)$ isomorphic to the translation space bundle $T^*\mathcal{M} \otimes \mathfrak{g}(\delta)$, and given that a choice of Lie algebra basis then enables one to decompose the translation space bundle into separate interaction carrier bundles, one might refer to the translation space bundle as the interaction carrier bundle. In the case of the strong force, with $G = SU(3)$, one has $dim\ SU(3) = 8$, therefore one has 8 strong force interaction carriers; namely, the gluons. In the case of the electroweak force, with $G = U(2)$, one has $dim\ U(2) = 4$, therefore one has 4 interaction carriers: the photon γ, the $W^\pm$ particles, and the Z^0 particle.

Note that whilst the interaction carriers can be defined by irreducible representations of $SL(2,\mathbb{C})$ Ⓢ $\mathbb{R}^{3,1}$ alone in the Wigner representation, cross-sections of the interaction carrier bundle $T^*\mathcal{M} \otimes \mathfrak{g}(\delta)$ transform under both $SL(2,\mathbb{C})$ Ⓢ $\mathbb{R}^{3,1}$ and $\mathcal{G}$. For example, the space of single-photon states in the Wigner representation is the Fourier transform of a space of $U(1)$-connections in the configuration representation *modulo gauge transformations*. Gauge bosons in the Wigner representation do not transform under the group of gauge transformations. Note also that it is only under symmetry breaking that the interaction carrier bundle breaks into a direct sum of bundles housing the inverse Fourier transforms of the Wigner representations.

Mark that there is some distortion of meaning when people say that the interaction carriers of a gauge field 'belong to' the adjoint representation of the gauge group G. The interaction carriers of a gauge field belong to an infinite-dimensional representation of $(SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}) \times \mathcal{G}$, which is certainly not the same thing as the finite-dimensional adjoint representation of G. To reiterate, it is the representation of $SL(2,\mathbb{C}) \times G$ upon the typical fibre of $T^*\mathcal{M} \otimes \mathfrak{g}(\delta)$ which uses the finite-dimensional adjoint representation of G, tensored with a finite-dimensional representation of $SL(2,\mathbb{C})$ on $\mathbb{R}^{3,1}$.

Thus, in the case of the strong force, the gluons belong to an infinite-dimensional representation of $(SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}) \times \mathcal{G}$, with $\mathcal{G} = \Gamma(SU(\rho))$. However, the representation of $SL(2,\mathbb{C}) \times SU(3)$ upon the typical fibre of the translation bundle $T^*\mathcal{M} \otimes \mathfrak{su}(\rho)$ *does* use the eight-dimensional adjoint representation of $SU(3)$, tensored with a finite-dimensional representation of $SL(2,\mathbb{C})$ on $\mathbb{R}^{3,1}$. In the case of the electroweak force, the interaction carriers of the unified electroweak force belong to an infinite-dimensional representation of $(SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}) \times \mathcal{G}$, with $\mathcal{G} = \Gamma(U(\iota))$. One has a representation of $SL(2,\mathbb{C}) \times U(2)$ upon the typical fibre of the translation bundle $T^*\mathcal{M} \otimes \mathfrak{u}(\iota)$, and this representation *does* use the four-dimensional adjoint representation of $U(2)$.

3.1 Classification of Principal G-Bundles

Whilst each gauge field corresponds to a particular compact and connected Lie group G, the choice of a particular G does not, in general, determine a unique principal fibre bundle P with structure group G, or a unique vector bundle δ with a G-structure. In other words, the choice of a gauge group does not uniquely determine the mathematical object upon which the representation of a gauge field is dependent.

In the case of a 4-dimensional manifold $\mathcal{M}$, it is possible, for any Lie group G, to classify all the principal G-bundles over $\mathcal{M}$, (DeWitt, Hart and Isham, 1979, pp199-201). Although the purview of this paper extends to space-times of arbitrary dimension, let us consider the classification over 4-manifolds for illustrative purposes.

Suppose that G is a simply connected Lie group. In this case, the principal G-bundles over a four-dimensional manifold $\mathcal{M}$ are classified by the elements of the fourth cohomology group over the integers $H^4(\mathcal{M};\mathbb{Z})$ of the manifold $\mathcal{M}$. In the event that $\mathcal{M}$ is compact and orientable, $H^4(\mathcal{M};\mathbb{Z}) = \mathbb{Z}$, hence the principal G-bundles, for a simply connected Lie group G over a compact and orientable 4-manifold, are in one-to-one correspondence with the integers. In the event that $\mathcal{M}$ is either non-compact or non-orientable, $H^4(\mathcal{M};\mathbb{Z}) = \{Id\}$. This means that for a simply connected Lie group G, all the principal G-bundles over a non-compact or non-orientable 4-manifold are trivial bundles, isomorphic to $\mathcal{M} \times G$.

In the special case where the simply connected Lie group G is a special unitary group $SU(n)$, the element of $H^4(\mathcal{M};\mathbb{Z})$ which corresponds to a particular principal $SU(n)$-bundle, is the second Chern class of that bundle. For different principal $SU(n)$-bundles, the second Chern class of the bundle corresponds to different cohomology equivalence classes of the base manifold $\mathcal{M}$. The case of a special unitary group is of relevance to the Standard Model, where $SU(2)$ is involved with the electroweak force, and $SU(3)$ is the gauge group of the strong force.

Turning to non-simply connected Lie groups, take the case where G is a unitary group $U(n)$. In the case that $G = U(1)$, the set of inequivalent principal $U(1)$-bundles over any 4-manifold $\mathcal{M}$ is in one-to-one correspondence with the elements of the second cohomology group over the integers $H^2(\mathcal{M};\mathbb{Z})$. The element of $H^2(\mathcal{M};\mathbb{Z})$ which corresponds to a particular principal $U(1)$-bundle is the first Chern class of that bundle. This case is relevant to the electromagnetic force, which has gauge group $U(1)$.

In the case of $U(n)$, for $n > 1$, the set of inequivalent principal $U(n)$-bundles over any 4-manifold $\mathcal{M}$ is in one-to-one correspondence with the elements of $H^2(\mathcal{M};\mathbb{Z}) \oplus H^4(\mathcal{M};\mathbb{Z})$. The case of relevance to the Standard Model is $G = U(2)$, the gauge group of the electroweak force.

To reiterate, these results demonstrate that, in general, the choice of principal fibre bundle or interaction bundle is not determined by the gauge group. In the case of the electromagetic force, there are many principal $U(1)$-bundles $\{P_i : i \in H^2(\mathcal{M};\mathbb{Z})\}$ over a 4-dimensional space-time $\mathcal{M}$, and for each different bundle P_i, the standard representation of $U(1)$ on $\mathbb{C}^1$ defines a different interaction bundle $\lambda_i = P_i \times_{U(1)} \mathbb{C}^1$ equipped with a $U(1)$-structure. Similarly, in the case of the electroweak force, there are many principal $U(2)$-bundles $\{Q_i : i \in H^2(\mathcal{M};\mathbb{Z}) \oplus H^4(\mathcal{M};\mathbb{Z})\}$ over a 4-dimensional space-time $\mathcal{M}$, and for each different bundle Q_i, the standard representation of $U(2)$ on $\mathbb{C}^2$ defines a different interaction bundle $\iota_i = Q_i \times_{U(2)} \mathbb{C}^2$ equipped with a $U(2)$-structure.

In the case of the strong force, with simply connected gauge group $SU(3)$, there are, in general, many principal $SU(3)$-bundles $\{S_i : i \in H^4(\mathcal{M};\mathbb{Z})\}$ over a 4-dimensional space-time $\mathcal{M}$, and for each different bundle S_i, the standard representation of $SU(3)$ on $\mathbb{C}^3$ defines a different interaction bundle $\rho_i = S_i \times_{SU(3)} \mathbb{C}^3$ equipped with a $SU(3)$-structure. However, in the case of a non-compact or non-orientable 4-dimensional manifold, $H^4(\mathcal{M};\mathbb{Z}) = \{Id\}$, and the only principal $SU(3)$-bundle is therefore $S = \mathcal{M} \times SU(3)$.

Because Minkowski space-time is contractible, (meaning it can be continuously deformed to an individual point), all its cohomology groups are trivial. This entails that in the special case of the Standard Model over Minkowski space-time the choice of a gauge group G determines a unique principal G-bundle, and all the interaction bundles are trivial.

3.2 The Structure of Compact Groups

The Structure Theorem for connected compact Lie groups, (Simon 1996, p155; Hofmann and Morris 1998, p204-207), entails that any compact, connected Lie group G is isomorphic to a quotient of a finite direct product,

$$G \cong L_1 \times L_2 \times \cdots \times L_r \times \mathbb{T}^p / D$$

where each L_i is a compact, simple, and simply connected Lie group, $\mathbb{T}^p$ is a p-dimensional torus, and D is a finite central subgroup of $L_1 \times L_2 \times \cdots \times L_r \times \mathbb{T}^p$.

Before deriving this structural decomposition theorem, let us define some of the terms in our discourse. Recall that each Lie Group G possesses a Lie algebra $\mathfrak{g}$ isomorphic to the tangent vector space at the identity element of the Lie group. A Lie algebra is a vector space equipped with an antisymmetric, bilinear product operation [,] called the Lie bracket. An abelian Lie algebra is such that $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$. An ideal in a Lie algebra is a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ which is such that $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{h}$, $Y \in \mathfrak{g}$. An ideal is also said to be an invariant subalgebra.

An ideal is the Lie algebra equivalent of a closed, normal subgroup of a connected Lie group. A closed subgroup $H \subset G$ of a Lie group G is a Lie subgroup under the inclusion mapping $i : H \to G$. Hence, one can equivalently refer to either a closed subgroup or a Lie subgroup of a Lie group. If $H \subset G$ is a Lie subgroup of a connected Lie group G, with $\mathfrak{g}$ denoting the Lie algebra of G, and $\mathfrak{h} \subset \mathfrak{g}$ denoting the Lie algebra of H, then H is a normal subgroup of G if and only if $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, (Fulton and Harris 1991, p122).

A Lie algebra $\mathfrak{g}$ is defined to be simple if $dim\ \mathfrak{g} > 1$ and $\mathfrak{g}$ contains no nontrivial ideals, (Fulton and Harris 1991, p122). A connected Lie group can be defined to be simple if its Lie algebra is simple, or equivalently, if it contains no non-trivial, closed, *connected* normal subgroups. Under this definition, a simple connected Lie group *can* possess non-trivial, closed, normal subgroups, but if they exist they must be discrete. If a simple connected Lie group G possesses a non-trivial, discrete, closed normal subgroup $H \subset G$, then the Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is an ideal, but $\mathfrak{h} = \{0\}$, consistent with the fact that there is no *non-trivial* ideal in $\mathfrak{g}$. Although H here is a non-trivial subgroup, $H \neq \{1\}$, the identity component of H is trivial, $H_0 = exp(\mathfrak{h}) = \{1\}$, (Hofmann and Morris 1998, p193-194). Given a Lie group G and a closed, normal subgroup H, the quotient G/H is a Lie group. A Lie group which has no non-trivial, closed, normal subgroups, has no quotient Lie groups. As defined here, a simple connected Lie group *can* have quotient Lie groups, but they can only be quotient Lie groups with respect to a discrete subgroup.

An abelian Lie algebra cannot be simple because any Lie subalgebra of an abelian Lie algebra must be an ideal. For any subalgebra $\mathfrak{h}$ of an abelian Lie algebra $\mathfrak{g}$, $[X, Y] = 0 \in \mathfrak{h}$, for all $X \in \mathfrak{h}$, $Y \in \mathfrak{g}$. Correspondingly, every connected subgroup of a connected, abelian Lie group must be a normal subgroup. Hence, an abelian Lie group cannot be simple.

A semisimple Lie algebra can be defined as a Lie algebra which has no non-trivial abelian ideals, but it will be more useful to characterise it as a Lie algebra which is the direct sum of simple Lie algebras. The only non-trivial ideals of a semisimple Lie algebra are the non-abelian direct summands. Semisimple Lie groups are the direct products of simple Lie groups. The only non-trivial normal subgroups of a semisimple Lie group are the factors of the product. Needless to say, every simple Lie algebra is semisimple, and every simple Lie group is a semisimple Lie group.

Now, the Lie algebra of a compact, connected Lie group can be decomposed as the direct sum of a semisimple Lie algebra and an abelian Lie algebra. In other words, the Lie algebra of a compact, connected Lie group can be decomposed as a direct sum of simple Lie algebras and abelian Lie algebras. Hence a compact, connected Lie group G must be locally isomorphic to a direct product

$$G_1 \times G_2 \times \cdots \times G_r \times \mathbb{T}^p$$

where each G_i is a simple, compact Lie group, and $\mathbb{T}^p$ is the p-fold direct product of $U(1) = \mathbb{T}$, the unique compact connected 1-dimensional abelian Lie group.

In more economical notation, a compact connected Lie group must be locally isomorphic to a direct product

$$K \times N$$

where K is a compact, semisimple Lie group, and N is a compact abelian Lie group.

The only connected 1-dimensional Lie groups are $\mathbb{R}^1$ and $U(1) = \mathbb{T}$, both of which are abelian. Every connected abelian Lie group, of arbitrary dimension, is isomorphic to a direct product of these two 1-dimensional Lie groups. Any compact connected abelian Lie group, of dimension p, is isomorphic to the direct product of p copies of $U(1) = \mathbb{T}$. Hence the abelian factor N in the above decomposition must be $\mathbb{T}^p$ for some integer p.

Locally isomorphic groups share the same universal cover, hence the universal cover of G must equal the universal cover of $G_1 \times G_2 \times \cdots \times G_r \times \mathbb{T}^p$. The universal cover of a direct product is given by the product of the individual universal covers, hence the universal cover of G must be

$$\widetilde{G} \cong \widetilde{G}_1 \times \widetilde{G}_2 \times \cdots \times \widetilde{G}_r \times \mathbb{R}^p$$

where each $\widetilde{G}_i$ is a simple and simply connected Lie group, and $\mathbb{R}^1$ is the universal cover of $U(1) = \mathbb{T}$.

It can be proven that a compact connected Lie group which has the property of being semisimple, must have a compact universal covering group.[22] As a trivial consequence of being simple, each G_i must also be semisimple, hence each of the $\widetilde{G}_i$ must be compact as well as simple and simply connected. $U(1) = \mathbb{T}$ is not semisimple, by virtue of being abelian, hence there is no inconsistency with the fact that $\mathbb{R}^1$ is non-compact.

In more economical notation, one can express the universal cover of a compact connected Lie group G as

$$\widetilde{G} \cong \widetilde{K} \times \mathbb{R}^p$$

[22]Private communication with Karl H.Hofmann

where $\widetilde{K}$ is a compact, semisimple, simply connected Lie group.

Any connected Lie group G must be isomorphic to a quotient $\widetilde{G}/J$ of the universal cover, where J is a discrete central subgroup of $\widetilde{G}$, hence

$$G \cong \widetilde{G}/J = \widetilde{G}_1 \times \cdots \times \widetilde{G}_r \times \mathbb{R}^p/J$$

The centre $Z(\widetilde{G})$ of the universal cover is given by

$$Z(\widetilde{G}) = Z(\widetilde{G}_1) \times \cdots \times Z(\widetilde{G}_r) \times \mathbb{R}^p$$

From the 'Finite Discrete Centre Theorem', (Hofmann and Morris 1998, p180), it can be proven that the universal covering group of a compact connected semisimple Lie group must have a finite centre, hence each $Z(\widetilde{G}_i)$ is finite.[23] If at least one of the $Z(\widetilde{G}_i)$ is non-trivial, this entails that the centre $Z(\widetilde{G})$ of the universal covering group has multiple components. Although $\widetilde{G}$ is connected, it is perfectly possible for its centre $Z(\widetilde{G})$ to have multiple components. The identity component of the centre $Z_0(\widetilde{G})$ is

$$Z_0(\widetilde{G}) = Id \times \cdots \times Id \times \mathbb{R}^p$$

If $G \cong \widetilde{G}/J$, then the centre $Z(G)$ must be isomorphic to $Z(\widetilde{G})/J$, hence

$$Z(G) \cong Z(\widetilde{G}_1) \times \cdots \times Z(\widetilde{G}_r) \times \mathbb{R}^p/J$$

Once again, although G is connected, it is perfectly possible for its centre $Z(G)$ to have multiple components. Because G is compact, its centre $Z(G) \cong Z(\widetilde{G})/J$ must also be compact. Every compact *connected* abelian Lie group must be a product of p copies of $\mathbb{T}$, hence the *identity component* of the centre, $Z_0(G)$, must be $\mathbb{T}^p$.[24]

Whilst J belongs to the centre $Z(\widetilde{G})$ of the universal cover, it is not necessarily contained within the identity component of the centre. Hence, we can introduce a further subgroup $F \subset J$ which is the subgroup of J that belongs to $Z_0(\widetilde{G})$.[25] Defining

$$F = Z_0(\widetilde{G}) \cap J = (Id \times \cdots \times Id \times \mathbb{R}^p) \cap J$$

we obtain

$$\widetilde{G}_1 \times \cdots \times \widetilde{G}_r \times \mathbb{T}^p \cong \widetilde{G}_1 \times \cdots \times \widetilde{G}_r \times \mathbb{R}^p/F$$

and then

$$G \cong \widetilde{G}/J \cong (\widetilde{G}/F)/(J/F) \cong \widetilde{G}_1 \times \cdots \times \widetilde{G}_r \times \mathbb{T}^p/D$$

where $D = J/F$ is a finite central subgroup of $\widetilde{G}_1 \times \cdots \times \widetilde{G}_r \times \mathbb{T}^p$. This is the structure theorem, with $\widetilde{G}_i = L_i$. The quotient group $D = J/F$ is finite

[23] Private communication with Karl H.Hofmann

[24] Private communication with Karl H.Hofmann

[25] Private communication with Karl H.Hofmann

because it is a discrete, and therefore closed subgroup of the compact group $\widetilde{G}_1 \times \cdots \times \widetilde{G}_r \times \mathbb{T}^p$.

The only compact, simple, and simply connected Lie groups are the special unitary groups $SU(n),\ n \geq 2$, the symplectic groups $Sp(n),\ n \geq 2$, the spin groups $Spin(2n+1),\ n \geq 3$, the spin groups $Spin(2n),\ n \geq 4$, and the five exceptional Lie groups $E_6,\ E_7,\ E_8,\ F_4$, and G_2, (Simon 1996, p151). The reason that the list of Spin groups begins at $Spin(7)$ is that $Spin(3) \cong SU(2)$, $Spin(4) \cong SU(2) \times SU(2)$, (a non-simple group anyway), $Spin(5) \cong Sp(2)$, and $Spin(6) \cong SU(4)$, (ibid., p152). The list of symplectic groups begins at $Sp(2)$ because $Sp(1) \cong SU(2)$, (ibid., p144).

Given this exhaustive, non-repetitious list of compact, simple, and simply connected Lie groups, it follows that each one of the L_i in the structural decomposition of a compact connected Lie group,

$$G \cong L_1 \times L_2 \times \cdots \times L_r \times \mathbb{T}^p/D,$$

is a copy of one of the groups in this list. Multiple copies of the same group are, of course, permitted.

Whilst individual gauge force fields are said to have a gauge group, the Standard Model itself is said to have a gauge group by virtue of the fact that it collects together, and partially unifies, all the non-gravitational interactions. Of the infinite number of possible compact connected Lie groups available, the Standard Model of the gauge force fields in our universe corresponds to the case where

$$\widetilde{K} = L_1 \times L_2 \times \cdots \times L_r = SU(3) \times SU(2),$$

and where $p = 1$. Hence, the universal cover of the gauge group G of the Standard Model is

$$\tilde{G} = SU(3) \times SU(2) \times \mathbb{R}^1,$$

and the gauge group of the Standard Model is some quotient group

$$G \cong (SU(3) \times SU(2) \times \mathbb{R}^1)/J$$

where J is a discrete central subgroup of $SU(3) \times SU(2) \times \mathbb{R}^1$.

Hence, the gauge group G of the Standard Model is such that

$$G \cong (SU(3) \times SU(2) \times U(1))/D$$

where D is a finite central subgroup of $SU(3) \times SU(2) \times U(1)$.

In the Standard Model for the particle world in our universe, a collection of finite-dimensional, irreducible representations of $SU(3) \times SU(2) \times U(1)$ define the interacting elementary particle 'multiplets'. In the case of the structural decomposition of the Standard Model gauge group, the finite central subgroup

D is the subgroup which acts trivially in all of these representations. It happens that $D \cong \mathbb{Z}_6$, hence when G denotes the Standard Model gauge group for our universe,

$$G \cong (SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6$$

The gauge group of the Standard Model admits a non-trivial structural decomposition precisely because the strong and electroweak gauge fields are not unified in the Standard Model. The defining characteristic of a gauge field Grand Unified Theory (GUT), is that it postulates a *simple*, connected and compact Lie group as the gauge group of our universe. Such a theory unifies the strong and electroweak forces by postulating that the simple GUT gauge group contains $(SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6 \cong S(U(3) \times U(2))$ as a subgroup. The most basic example which satisfies this criterion is $SU(5)$. In effect, GUTs suggest that the non-trivial structural decomposition of the Standard Model gauge group is the result of spontaneous symmetry breaking.

A universe with non-unified gauge fields different to our own, would have a Standard Model in which $SU(3) \times SU(2)$ is replaced by a different finite product $L_1 \times L_2 \times \cdots \times L_r$, in which each L_i is a copy of one of the infinite number of compact, simple, simply connected Lie groups available. One could use special unitary groups of higher dimension, spin groups, symplectic groups, or even copies of the exceptional groups. One could also replace $U(1) = \mathbb{T}$ with an alternative compact abelian Lie group $\mathbb{T}^p$.

A possible universe with extra, non-unified forces in addition to the non-unified strong and electroweak forces present in our own universe, would have a gauge group G with the following structural decomposition:

$$G \cong L_1 \times \cdots \times L_j \times SU(3) \times SU(2) \times U(1)/D$$

where j is the number of extra, non-unified forces.

3.3 Gauge fields in other universes

At first sight, the gauge force fields which can exist in a universe appear to be independent of the space-time dimension, signature and orientation. Given a space-time $\mathcal{M}$ of arbitrary dimension, signature, and orientation, a gauge field in such a universe is specified by the selection of a compact connected Lie group G. With the exception of regions in which the gravitational field is very strong, a gauge field in such a universe can be represented by a connection upon a principal G-bundle or interaction bundle over the Minkowski space-time $\mathbb{R}^{p,q}$ of the relevant dimension and signature.

The structure theorem of compact connected Lie groups enables one to decompose any such group into a quotient of a direct product of simple, simply connected compact groups, and a compact abelian group. These groups can be given an exhaustive, non-repetitious listing, hence one apparently obtains a classification of gauge fields in other universes.

This approach is, however, deceptive. Recall that the interaction carriers ('gauge bosons') of a gauge force field correspond to (integer spin) unitary irreducible representations of the local space-time symmetry group. A physically legitimate gauge field must be such that, under a choice of gauge, the space of its G-connections which satisfy the free Yang-Mills equations, must decompose into a direct sum of unitary irreducible representations of the local space-time symmetry group. Given a choice of gauge, and given a choice of basis in the lie algebra of G, the space of G-connections will always decompose as

$$\bigoplus^{dim\ G} T^*\mathcal{M}$$

In the special case of a universe with three spatial dimensions and one time dimension, the cotangent bundle $T^*\mathcal{M}$ possesses upon its typical fibre, isomorphic to $\mathbb{R}^{3,1}$, a spin-1 irreducible representation of $Spin(3,1) \cong SL(2,\mathbb{C})$. In the case of universes with an arbitrary number of space and time dimensions, a space of G-connections will still decompose as $\bigoplus^{dim\ G} T^*\mathcal{M}$, but the typical fibre of the cotangent bundle $T^*\mathcal{M}$ will be isomorphic to $\mathbb{R}^{p,q}$, and will possess an irreducible representation of $\widetilde{SO}_0(p,q)$. As already pointed out in section 2.2, bundles equipped with representations of $\widetilde{SO}_0(p,q)$ upon their typical fibres cannot, in general, be interpreted as spin-s free particle bundles. In particular, $T^*\mathcal{M}$ cannot be interpreted as a spin-1 free particle bundle in the case of an arbitrary space-time $\mathcal{M}$.

3.4 Interactions

In the first-quantized interacting theory, the interaction bundles and free-particle bundles are conventionally bundles over Minkowski configuration space. A free particle of mass m and spin s is represented by a positive-energy, mass-m solution ϕ of a linear differential equation, (Derdzinski 1992, p84),

$$\mathscr{P}(x, \phi(x), (\nabla^\eta \phi)(x), (\nabla^{\eta^2} \phi)(x), ...) = 0 \ ,$$

imposed upon the cross-sections of a spin-s free particle bundle η. ∇^η here is the Levi-Civita connection on η.

When an interaction is 'switched on', one must deal with pairs (ψ, ∇^δ), where ψ is a cross-section of an interacting-particle bundle α, and ∇^δ is a connection on an interaction bundle δ, (ibid., p84). Such pairs must satisfy coupled field equations, consisting of (i) the interacting field equation upon the cross-sections ψ of α, and (ii) the coupled Yang-Mills equation upon the curvature R^{∇^δ} of the connection ∇^δ on δ:

$$\mathscr{P}(x, \psi(x), ((\nabla^\eta \otimes \nabla^\delta)\psi)(x), ((\nabla^\eta \otimes \nabla^\delta)^2\psi)(x), ...) = 0$$

$$div\ R^{\nabla^\delta} = C_0 J(\psi)$$

The move from η to α, and the move from the use of ∇^η in the free field equation, to the use of $(\nabla^\eta \otimes \nabla^\delta)$ in the interacting field equation, is often referred to as the 'minimal coupling substitution'. These coupled equations are non-linear, entailing that the set of all pairs (ψ, ∇^δ) which solve the coupled equations does not possess a linear vector space structure.

Thus, an interacting particle of mass m and spin s is represented by a positive energy, mass-m solution of a ∇^δ-dependent differential equation imposed upon the cross-sections of a spin-s interacting-particle bundle α. A spin-s interacting particle bundle α is a construction from a spin-s free-particle bundle η, and an interaction bundle δ. In the simplest case, if the free-particle bundle is η, and the interaction bundle is δ, then the interacting-particle bundle will be the tensor product $\alpha = \eta \otimes \delta$.

Recall that for the Standard Model over curved space-time, a gauge group G does not, in general, determine a unique interaction bundle δ, hence, in general, a spin-s particle interacting with a group-G gauge field does not have a unique interacting-particle bundle, even if one assumes the simplest type of interacting particle bundle $\eta \otimes \delta$. Instead, one has a family of interaction bundles δ_i, and a consequent family of interacting-particle bundles $\eta \otimes \delta_i$.

The infinite-dimensional group of gauge transformations $\mathcal{G} = \Gamma(G(\delta))$ acts upon the space of sections $\Gamma(\delta)$, thence it acts upon the space of sections $\Gamma(\alpha)$ of an interacting particle bundle. Hence, a spin-s interacting particle transforms under the action of the infinite-dimensional group of gauge transformations. Whilst a free-particle corresponds to an irreducible representation of the local space-time symmetry group, a particle with a gauge force field switched on transforms under both the local space-time symmetry group, and the infinite-dimensional group of gauge transformations. However, the precise sense in which an interacting particle transforms under these groups needs to be clarified.

Whilst a free particle in our universe corresponds to a unitary, irreducible representation of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$, a particle interacting with a gauge field of gauge group G does *not* correspond to a unitary, irreducible representation of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1} \times G$. One could find, and classify, all the unitary, irreducible representations of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1} \times G$, as an extension of the Wigner classification: All the irreducible unitary representations of compact groups are finite-dimensional, so one could set about taking all the tensor products of the unitary, irreducible, infinite-dimensional representations of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$ with the unitary, irreducible, *finite*-dimensional representations of G, to obtain all the unitary, irreducible representations of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1} \times G$.[26] However, these vector space representations do not correspond with the state spaces of interacting particles, which are non-linear. Interacting particles are not the unitary irreducible representations of any group. An interacting particle ψ in our universe does not transform under a representation of $SL(2,\mathbb{C}) \times G$ or a representation of $(SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}) \times G$. Rather, it transforms under a group action of $SL(2,\mathbb{C}) \circledS \mathbb{R}^{3,1}$, and a group action of $\mathcal{G} = \Gamma(G(\delta))$.

An interacting particle in a universe with an arbitrary number of space and

[26] Private communication with Heinrich Saller

time dimensions would transform under a group action of $\widetilde{SO}_0(p,q) \circledS \mathbb{R}^{p,q}$, and a group action of $\mathcal{G} = \Gamma(G(\delta))$, where the space of G-connections which satisfy the free Yang-Mills equations must decompose as a direct sum of unitary irreducible representations of $\widetilde{SO}_0(p,q) \circledS \mathbb{R}^{p,q}$.

3.5 Standard Model Gauge Groups and Representations

In the Standard Model of the particle world in our universe, a select collection of finite-dimensional irreducible representations of $SL(2,\mathbb{C})\times SU(3)\times SU(2)\times U(1)$ are said to define the elementary particle multiplets. A particle multiplet in our universe can be represented by an interacting particle bundle α or interaction carrier bundle $T^*\mathcal{M}\otimes\mathfrak{g}(\delta)$ which possesses a finite-dimensional irreducible representation of $SL(2,\mathbb{C})\times SU(3)\times SU(2)\times U(1)$ upon its typical fibre. The gauge bosons and each generation of interacting elementary fermions are partitioned into multiplets by a collection of finite-dimensional irreducible representations of $SU(3) \times SU(2) \times U(1)$, each of which is tensored with a finite-dimensional irreducible representation of $SL(2,\mathbb{C})$.

In general, given a product group $G\times H$, a finite-dimensional, irreducible representation $r : G \to Aut\ V_1$, and a finite-dimensional, irreducible representation $s : H \to Aut\ V_2$, the tensor product representation $r\otimes s : G\times H \to Aut\ (V_1\otimes V_2)$ is also a finite-dimensional, irreducible representation. Furthermore, every finite-dimensional, irreducible representation of $G \times H$ is equivalent to such a tensor product representation, (Sternberg 1994, p371). Hence, extending this to a three-fold group product, every finite-dimensional, irreducible representation of $SU(3) \times SU(2) \times U(1)$ is a tensor product of finite-dimensional, irreducible representations of the component groups. We therefore need to study the finite-dimensional, irreducible representations of each component group, $SU(3)$, $SU(2)$, and $U(1)$.

In general, the finite-dimensional, irreducible representations of $SU(n)$, for $n \geq 2$, are parameterized by the elements of the cartesian product

$$(\frac{1}{n}\mathbb{Z}_+)^{n-1}$$

In other words, each finite-dimensional, irreducible representation of $SU(n)$ is parameterized by a sequence of rational numbers $(s_1, ..., s_{n-1})$ called the 'spins' of the representation, (Derdzinski 1992, p132-134).

Hence, in the special case of $SU(3)$, the finite-dimensional, irreducible representations are parameterized by the elements of the cartesian product

$$(\frac{1}{3}\mathbb{Z}_+) \times (\frac{1}{3}\mathbb{Z}_+)$$

and in the special case of $SU(2)$, the finite-dimensional, irreducible representations are parameterized by the elements of

$$\frac{1}{2}\mathbb{Z}_+$$

In other words, the finite-dimensional, irreducible representations of $SU(3)$ are parameterized by pairs (s_1, s_2), each of which is a non-negative integral multiple of 1/3, and the finite-dimensional, irreducible representations of $SU(2)$ are parameterized by single numbers s, each of which is a non-negative integral multiple of 1/2.

The finite-dimensional, irreducible representations of $U(1)$ can be parameterized by the integers $\mathbb{Z}$, or by $\frac{1}{n}\mathbb{Z}$, according to convenience. For the representation of weak hypercharge and electric charge, indexing the representations of $U(1)$ by $\frac{1}{3}\mathbb{Z}$ is particularly convenient.

Given that every finite-dimensional, irreducible representation of $SU(3) \times SU(2) \times U(1)$ is a tensor product of finite-dimensional, irreducible representations of the component groups, it follows that the equivalence classes of finite-dimensional, irreducible representations of $SU(3)\times SU(2)\times U(1)$ will be indexed by some subset of the cartesian product

$$(\frac{1}{3}\mathbb{Z}_+) \times (\frac{1}{3}\mathbb{Z}_+) \times (\frac{1}{2}\mathbb{Z}_+) \times \frac{1}{n}\mathbb{Z}$$

Excluding the gauge boson representations, which utilise the adjoint representations, the Standard Model in our universe only uses finite-dimensional, irreducible representations of $SU(3)\times SU(2)\times U(1)$ which are tensor products of either the standard representation or trivial representation of $SU(3)$ with finite-dimensional, irreducible representations of $SU(2) \times U(1)$. Furthermore, those finite-dimensional, irreducible representations of $SU(2) \times U(1)$ are themselves tensor products of either the standard representation or trivial representation of $SU(2)$ with finite-dimensional, irreducible representations of $U(1)$.

The standard representation of $SU(3)$ is merely one of a countably infinite family of finite-dimensional, irreducible representations parameterized by $(\frac{1}{3}\mathbb{Z}_+)\times(\frac{1}{3}\mathbb{Z}_+)$. The standard representation of $SU(3)$ is indexed as the $(1/3, 0)$ representation. Similarly, the standard representation of $SU(2)$ is merely one of a countably infinite family of finite-dimensional, irreducible representations parameterized by $(\frac{1}{2}\mathbb{Z}_+)$. The standard representation of $SU(2)$ is indexed as the $s = 1/2$ representation.

Because only the trivial and standard representations of $SU(3)$ and $SU(2)$ are used to specify the elementary fermion multiplets in the Standard Model, it is practical, and notationally much simpler, to denote the representations of $SU(3)\times SU(2)\times U(1)$ with the dimension, rather than the 'spins', of the $SU(3)$ and $SU(2)$ representations. Thus, for the trivial representation, a simple '1' can be used, and for the standard representation of $SU(n)$, a simple 'n' can be used.

Using this notation, the particles, or, more accurately, the parts of the state spaces of the particles in, for example, the first fermion generation, are partitioned into multiplets by the finite-dimensional, irreducible representations of $SU(3)\times SU(2)\times U(1)$ in the following way, (Baez 1998 and Baez 1999; Schücker 1997, p30-31):

- The neutrino and the 'left-handed' part of the state-space of the electron (ν_L, e_L), transform according to the (1,2,-1) irreducible representation of $SU(3) \times SU(2) \times U(1)$. i.e. The tensor product of the trivial representation of $SU(3)$ with the 2-dimensional standard representation of $SU(2)$ with the 1-dimensional representation of $U(1)$ with hypercharge -1.

- The left-handed part of the state-spaces of the up quark and down quark (u_L, d_L) transform according to the (3,2,1/3) representation. i.e. The tensor product of the standard representation of $SU(3)$ with the 2-dimensional standard representation of $SU(2)$ with the 1-dimensional representation of $U(1)$ with hypercharge 1/3.

- The right-handed part of the state-space of the electron e_R transforms according to the (1,1,-2) representation.

- The right-handed part of the state-space of the up quark u_R transforms according to the (3,1,4/3) representation.

- The right-handed part of the state-space of the down quark d_R transforms according to the (3,1,-2/3) representation.

The list of particle multiplets here is based upon the assumption that the neutrino is massless. If, as current evidence indicates, the neutrino does possess mass, the right-handed neutrino forms an additional singlet corresponding to the trivial $(1, 1, 0)$ representation of $SU(3) \times SU(2) \times U(1)$.

There is a tacit understanding in the specification of the mulitplets above that each representation of $SU(3) \times SU(2) \times U(1)$ is tensored with an irreducible, finite-dimensional representation of $SL(2, \mathbb{C})$. An elementary fermion multiplet in the electroweak-unified Standard Model, typically contains parts of the state spaces of one or more of the elementary fermions which exist after electroweak symmetry breaking. Conversely, different parts of the state space of an elementary fermion after electroweak symmetry breaking can correspond to different irreducible representations of $SL(2, \mathbb{C}) \times SU(3) \times SU(2) \times U(1)$.

As ever, it must be emphasized that the state spaces of interacting fermions and gauge bosons are not finite-dimensional, nor are they the vector space representations, reducible or irreducible, of any group. The finite-dimensional irreducible representations of the Standard Model gauge group, $SU(3) \times SU(2) \times U(1)$, either correspond to representations upon the typical fibres of interacting-particle bundles, or to adjoint representations upon the typical fibres of interaction carrier bundles, whilst the state spaces of interacting fermions and gauge bosons are constructed from cross-section spaces of these bundles.

If an interaction bundle δ possesses a finite-dimensional representation of $SU(3) \times SU(2) \times U(1)$ upon its typical fibre, then given a free-particle bundle η equipped with a finite-dimensional representation of $SL(2, \mathbb{C})$ upon its typical fibre, the interacting particle bundle α constructed from δ and η will possess a finite-dimensional representation of $SL(2, \mathbb{C}) \times SU(3) \times SU(2) \times U(1)$ upon its typical fibre. If the representation of $SU(3) \times SU(2) \times U(1)$ is irreducible, if the representation of $SL(2,\mathbb{C})$ is irreducible, and if the interacting particle bundle is the tensor product $\alpha = \eta \otimes \delta$, then the representation of $SL(2,\mathbb{C}) \times SU(3) \times SU(2) \times U(1)$ upon the typical fibre of α will also be irreducible.

The only two free-particle bundles used in the Standard Model multiplets in our universe are σ_L and σ_R, the left-handed and right-handed Weyl spinor bundles, respectively. These bundles possess upon their typical fibres the $(1/2,0)$ and $(0,1/2)$ complex, finite-dimensional, irreducible representations of $SL(2,\mathbb{C})$. The interacting-particle bundles which correspond to elementary fermion multiplets in the Standard Model, are obtained by tensoring a Weyl spinor bundle with an interaction bundle that possesses an irreducible finite-dimensional representation of $SU(3) \times SU(2) \times U(1)$.

Given that the interacting elementary fermions with which we are most familiar are the interacting elementary fermions which exist after electroweak symmetry breaking, when the gauge group has changed from $SU(3)\times SU(2)\times U(1)$ to $SU(3)\times U(1)_Q$,[27] the interacting elementary fermions with which we are most familiar correspond to interacting particle bundles which possess a representation of $SL(2,\mathbb{C}) \times SU(3) \times U(1)_Q$ upon their typical fibre. The representation of $SL(2,\mathbb{C})$ upon the typical fibre of such a bundle is often a reducible direct sum representation, corresponding to the Dirac spinor bundle $\sigma = \sigma_L + \sigma_R$.

Universes with a gauge group other than $SU(3) \times SU(2) \times U(1)$, or a quotient thereof, will possess different force fields to those that exist in our own universe, and different sets of possible interacting elementary particles, interaction carriers, and elementary particle multiplets. Given a universe with a group of non-translational local space-time symmetries $SO_0(p,q)$, and a gauge group $L_1 \times \cdots \times L_r \times \mathbb{T}^p$, where each L_i is a simple, simply connected, compact Lie group, the set of finite-dimensional irreducible representations of $\widetilde{SO_0}(p,q) \times L_1 \times \cdots \times L_r \times \mathbb{T}^p$ would define the set of elementary particle multiplets.

Each irreducible representation of $L_1 \times \cdots \times L_r \times \mathbb{T}^p$ is a tensor product of irreducible representations of the individual factors. The individual L_i-representations will be representations of special unitary groups, spin groups, symplectic groups, or one of the exceptional groups. The irreducible representations of each family of simple, simply connected, compact Lie groups, can be classified in much the same way that the irreducible representations of $SU(n)$ can be classified. Hence, with the combination of the structural decomposition theorem for a compact, connected Lie group, and the classification of the irreducible representations of any simple, simply connected, compact Lie group, one can classify the particle multiplets of any universe in which the gauge group is assumed to be a compact, connected Lie group.

However, even with the gauge group G fixed, the finite-dimensional, irreducible representations of this group only determine the set of possible particle multiplets in a universe. The set of actual particle multiplets instantiated in

[27] $U(1)_Q$ being the gauge group of the electromagnetic force.

a universe appears to be contingent. In our universe, only a finite number of particle multiplets have been selected from the countably infinite number of possible finite-dimensional, irreducible representations of $SU(3) \times SU(2) \times U(1)$. Thus, universes with the same gauge group, and the same set of *possible* particle multiplets, can be further sub-classified by the particular collection of *actual* particle multiplets instantiated.

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