Are Maxwell gravitation and Newton-Cartan theory theoretically equivalent?

Eleanor March*

12/08/2023

Abstract

A recent flurry of work has addressed the question whether Maxwell gravitation and Newton-Cartan theory are theoretically equivalent. This paper defends the view that there are plausible interpretations of Newton-Cartan theory on which the answer to the above question is "yes". Along the way, I seek to clarify what is at issue in this debate. In particular, I argue that whether Maxwell gravitation and Newton-Cartan theory are equivalent has nothing to do with counterfactuals about unactualised matter, *contra* the appearance of previous discussions in the literature. Nor does it have anything to do with spacetime and dynamical symmetries, *pace* recent claims by Jacobs (2023). Instead, it depends on some rather subtle questions concerning how facts about the geodesics of a connection acquire physical significance, and the distinction between dynamical and kinematic possibility.

Acknowledgements: I would like to thank Adam Caulton for many helpful comments and discussions on the material presented here.

*Balliol College, University of Oxford. eleanor.march@balliol.ox.ac.uk

1 Introduction

It is well known that Newtonian gravitation admits, in addition to its usual static and kinematic shift symmetries, a symmetry known as Trautman gauge symmetry, in which the connection and gravitational potential are altered. Moreover, it is often claimed that just as kinematic shift symmetry motivates the transition from Newtonian to Galilean spacetime, so does Trautman gauge symmetry motivate the transition to a geometrised formulation of Newtonian gravitation, known as Newton-Cartan theory.[1]

Recently however, Saunders (2013) and Dewar (2018) have challenged this orthodoxy – arguing that Maxwellian spacetime is the appropriate setting which encapsulates the lessons of Trautman gauge symmetry. But whilst the relationship between Newton-Cartan theory and Galilean gravitation has been widely discussed, aspects of the relationship between Newton-Cartan theory and Maxwell gravitation remain unclear. In particular, there is little consensus on the extent to which Maxwell gravitation has less structure than Newton-Cartan theory, or whether the two should be viewed as competitors at all.[2] Moreover, such questions have important implications for wider debates about theoretical equivalence, theoretical underdetermination, and how symmetries bear on the interpretation of theories.

Here, I aim to address some of these issues. First, I review some details of Maxwell gravitation and Newton-Cartan theory, as well as some preliminary results concerning the relationship between them. I then, in section 3, turn to the interpretation of these results. I discuss the fact that the models of these two theories are not in one-to-one correspondence, and clarify how this relates to the issue of test particles and counterfactuals about unactualised matter. Section 4 aims to diffuse Jacobs's (2023) recent argument that Maxwell gravitation and Newton-Cartan theory have different spacetime and dynamical symmetry groups. Finally, in sections 5 and 6, I use the resources of category theory to discuss how this relates to the question of theoretical equivalent. Section 7 concludes.

1. See for example Stachel (2007), Knox (2014), Read & Møller-Nielsen (2020).
2. A recent selection of competing views on the subject: Saunders (2013), Knox (2014), Weatherall (2016), Dewar (2018), Wallace (2020), Jacobs (2023).

2 Maxwell gravitation and Newton-Cartan theory

Let M be a smooth four-manifold (assumed connected, Hausdorff, and paracompact). A temporal metric t_a on M is a smooth, closed, non-vanishing 1-form;[3] a spatial metric h^{ab} on M is a smooth, symmetric, rank-$(2,0)$ tensor field which admits, at each point in M, a set of four non-vanishing covectors $\overset{i}{\sigma}_a$, $i = 0, 1, 2, 3$, which form a basis for the cotangent space and satisfy $h^{ab}\overset{i}{\sigma}_a\overset{j}{\sigma}_b = 1$ for $i = j = 1, 2, 3$ and 0 otherwise. A spatial and temporal metric are compatible iff $h^{an}t_n = 0$. We say that a vector field σ^a is spacelike iff $t_n\sigma^n = 0$, and timelike otherwise. Given the structure defined here, t_a induces a foliation of M into spacelike hypersurfaces, and relative to any such hypersurface, h^{ab} induces a unique spatial derivative operator D such that $D_a h^{bc} = 0$.[4] We say that h^{ab} is flat just in case for any such spacelike hypersurface, D commutes on spacelike vector fields, so that $D_{[a}D_{b]}\sigma^c = 0$ for all spacelike vector fields σ. Finally, let ∇ be a connection on M. We say that ∇ is compatible with the metrics just in case $\nabla_a t_b = 0$ and $\nabla_a h^{bc} = 0$.

The first spacetime setting we will consider for Newtonian gravitation theory is Galilean spacetime. This is a structure $\langle M, t_a, h^{ab}, \nabla\rangle$, where ∇ is a flat, compatible connection. Let $\langle M, t_a, h^{ab}, \nabla\rangle$ be a Galilean spacetime, T^{ab} the Newtonian mass-momentum tensor for whichever matter fields are present, and ϕ a scalar field (which represents the gravitational potential). Then $\langle M, t_a, h^{ab}, \nabla, T^{ab}, \phi\rangle$ is a model of *Galilean gravitation* just in case

$$\nabla_n T^{na} = -\rho\nabla^a\phi \tag{1a}$$

$$\nabla_n\nabla^n\phi = 4\pi\rho \tag{1b}$$

where $\rho := T^{nm}t_n t_m$ is the scalar mass density field.

In what follows, we will be interested in the following transformation one can

3. Here and throughout, abstract indices are written in Latin script; component indices are written in Greek script, with the exception of i, j, k, which are reserved for the spatial components of tensor fields in some coordinate basis; and the Einstein summation convention is used. Round brackets denote symmetrisation, square brackets antisymmetrisation.

4. See Weatherall (2018, 37-38) and Malament (2012, §4.1) for further details.

make on models of Galilean gravitation, known as *Trautman gauge symmetry*:

$$\nabla \to (\nabla, t_b t_c \nabla^a \psi) \tag{2a}$$

$$\phi \to \phi + \psi \tag{2b}$$

where $\nabla^a \nabla^b \psi = 0$.[5] This is a symmetry of Galilean gravitation, in the sense that $\mathfrak{M}$ is a model of Galilean gravitation just in case all its Trautman gauge symmetry-related cousins are. Trautman gauge symmetry-related models agree on T^{ab}, so at least appear to be empirically indistinguishable.[6] One might therefore wonder if there are theories of Newtonian gravitation which collapse the distinction between Trautman gauge symmetry-related models of Galilean gravitation. As is well known, the answer to this question is "yes", and there are in fact two such theories – Maxwell gravitation, and Newton-Cartan theory.

I will begin by introducing Newton-Cartan theory. Let $\langle M, t_a, h^{ab} \rangle$ be a non-relativistic spacetime, ∇ a compatible derivative operator on M, and T^{ab} the mass-momentum tensor for whichever matter fields are present. Then $\langle M, t_a, h^{ab}, \nabla, T^{ab} \rangle$ is a model of *Newton-Cartan theory* just in case

$$\nabla_n T^{na} = 0 \tag{NCT1}$$

$$R_{ab} = 4\pi\rho t_a t_b \tag{NCT2}$$

$$R^{a\ c}_{\ b\ d} = R^{c\ a}_{\ d\ b} \tag{NCT3}$$

$$R^{ab}_{\ \ cd} = 0. \tag{NCT4}$$

Maxwell gravitation requires some further groundwork. This theory is set on Maxwellian spacetime, which is supposed to be equipped with a standard of rotation, but *not* a standard of absolute acceleration. But whilst the metrics and connection are by now standard notions, the rotation standard is not, and

5. For details, see Malament (2012, §4). The notation here follows Malament (2012, proposition 1.7.3): $\nabla' = (\nabla, C^a_{\ bc})$ iff for all smooth tensor fields $\alpha^{a_1 \dots a_r}_{\quad\ \ b_1 \dots b_s}$ on M,

$$\begin{aligned}(\nabla'_n - \nabla_n)\alpha^{a_1 \dots a_r}_{\quad\ \ b_1 \dots b_s} = {} & \alpha^{a_1 \dots a_r}_{\quad\ \ m b_2 \dots b_s} C^m_{\ n b_1} + \dots + \alpha^{a_1 \dots a_r}_{\quad\ \ b_1 \dots b_{s-1} m} C^m_{\ n b_s} \\ & - \alpha^{m a_2 \dots a_r}_{\quad\ \ b_1 \dots b_s} C^{a_1}_{\ \ nm} - \dots - \alpha^{a_1 \dots a_{r-1} m}_{\quad\ \ b_1 \dots b_s} C^{a_r}_{\ \ nm}.\end{aligned}$$

6. As such, Trautman gauge symmetry is at least an epistemic symmetry in Dasgupta's (2016) sense.

stands in need of further comment. This was originally introduced by Weatherall (2018): if t_a, h^{ab} are compatible temporal and spatial metrics on M, a standard of rotation $\circlearrowleft$ compatible with t_a and h^{ab} is a map from smooth vector fields ξ^a on M to smooth, antisymmetric rank-$(2,0)$ tensor fields $\circlearrowleft^b \xi^a$ on M, such that

1. $\circlearrowleft$ commutes with addition of smooth vector fields;
2. Given any smooth vector field ξ^a and smooth scalar field α, $\circlearrowleft^a (\alpha\xi^b) = \alpha \circlearrowleft^a \xi^b + \xi^{[b}d^{a]}\alpha$;
3. $\circlearrowleft$ commutes with index substitution;
4. Given any smooth vector field ξ^a, if $d_a(\xi^n t_n) = 0$ then $\circlearrowleft^a \xi^b$ is spacelike in both indices; and
5. Given any smooth spacelike vector field σ^a, $\circlearrowleft^a \sigma^b = D^{[a}\sigma^{b]}$.

One can then define a Maxwellian spacetime as a structure $\langle M, t_a, h^{ab}, \circlearrowleft\rangle$, where $\circlearrowleft$ is compatible with t_a and h^{ab}.

Now fix a spacetime $\langle M, t_a, h^{ab}\rangle$, and let ∇ and $\circlearrowleft$ be a connection and standard of rotation on M, both compatible with the metrics. Following March (2023), I will say that a standard of rotation and connection are *compatible* just in case they agree with one another in the following sense: for any vector field η^a on M, $\nabla^{[a}\eta^{b]} = \circlearrowleft^a \eta^b$.[7] Likewise, a connection ∇ is compatible with a spacetime $\langle M, t_a, h^{ab}, \circlearrowleft\rangle$ just in case it is compatible with the metrics and $\circlearrowleft$. Finally, a spacetime $\langle M, t_a, h^{ab}, \circlearrowleft\rangle$ is *flat derivative operator compatible* just in case some flat derivative operator is compatible with $\langle M, t_a, h^{ab}, \circlearrowleft\rangle$. As Weatherall (2018, proposition 1) proves, a spacetime is flat derivative operator compatible just in case h^{ab} is flat and there exists a unit timelike vector field ξ^a on M such that $\circlearrowleft^a \xi^b = 0$ and $\pounds_\xi h^{ab} = 0$.[8]

Finally, we need to say something about the Newtonian mass-momentum tensor T^{ab}. We have already seen that we can extract the scalar mass density field ρ from T^{ab} using the temporal metric. But in both Galilean gravitation and Newton-Cartan theory, we also used derivative operators to extract vector

7. See Weatherall (2018) for details; the basic fact is that any connection determines a unique compatible standard of rotation, but a standard of rotation does not similarly determine a unique compatible connection.

8. Here and throughout, $\pounds$ denotes the Lie derivative.

fields from T^{ab}. In Maxwell gravitation, we will likewise want to extract vector fields from T^{ab}, but without the use of derivative operators. To do this, we first impose the "Newtonian mass condition": whenever $T^{ab} \neq 0$, $T^{nm}t_n t_m > 0$. This captures the idea that the matter fields we are interested in are massive, in the sense that there can only be non-zero mass-momentum in spacetime regions where the mass density is strictly positive. Since T^{ab} is symmetric, the Newtonian mass condition guarantees that whenever $T^{ab} \neq 0$, we can uniquely decompose T^{ab} as

$$T^{ab} = \rho \xi^a \xi^b + \sigma^{ab} \tag{4}$$

where $\xi^a = \rho^{-1} t_n T^{na}$ is a smooth unit timelike future-directed vector field (interpretable as the net four-velocity of the matter fields F), and σ^{ab} is a smooth symmetric rank-$(2,0)$ tensor field which is spacelike in both indices (interpretable as the stress tensor for F).

We can now introduce Maxwell gravitation. Let $\langle M, t_a, h^{ab}, \circlearrowleft \rangle$ be a Maxwellian spacetime, and T^{ab} the Newtonian mass-momentum tensor for whichever matter fields are present. Then $\langle M, t_a, h^{ab}, \circlearrowleft, T^{ab} \rangle$ is a model of *Maxwell gravitation* just in case

(i) $\langle M, t_a, h^{ab}, \circlearrowleft \rangle$ is flat derivative operator compatible; and

(ii) For all points $p \in M$ such that $\rho \neq 0$, the following equations hold at p:

$$\pounds_\xi \rho - \frac{1}{2}\rho \hat{h}_{mn} \pounds_\xi h^{mn} = 0 \tag{MG1}$$

$$\frac{1}{3}\sum_{i=1}^{3} \overset{i}{\lambda}_r \xi^n \Delta_n (\xi^m \Delta_m \overset{i}{\lambda}{}^r) = -\frac{4}{3}\pi\rho - \frac{1}{3} D_m(\rho^{-1} D_n \sigma^{nm}) \tag{MG2}$$

$$\pounds_\xi (\circlearrowleft^c \xi^a) + 2(\circlearrowleft^n \xi^{[c}) \hat{h}_{nm} \pounds_\xi h^{a]m} + \circlearrowleft^c (\rho^{-1} D_n \sigma^{na}) = 0, \tag{MG3}$$

where $\hat{h}_{ab}$ is the spatial metric relative to ξ^a,[9] the $\overset{i}{\lambda}{}^a$ are three orthonormal connecting fields for ξ^a, and Δ is the "restricted derivative operator" defined in Weatherall (2018). This acts on arbitrary spacelike vector fields σ^a at a point

9. That is, the unique symmetric tensor field on M such that $\hat{h}_{an}\xi^n = 0$ and $h^{an}\hat{h}_{nb} = \delta^a{}_b - t_b \xi^a$.

p according to

$$\eta^n \Delta_n \sigma^a := \pounds_\eta \sigma^a + \sigma_n \circlearrowleft^n \eta^a - \frac{1}{2}\sigma_n \pounds_\eta h^{an} \tag{5}$$

where η^a is a unit timelike vector at p (the Lie derivative is taken with respect to any extension of η^a off of p). It also has the property that $\eta^n \Delta_n \sigma^a = \eta^n \nabla_n \sigma^a$ for any derivative operator ∇ compatible with $\circlearrowleft$ (Weatherall 2018, 37).[10]

The relation between Maxwell gravitation and Newton-Cartan theory is characterised by the following two results (March 2023):

Proposition 1. *Let $\langle M, t_a, h^{ab}, \nabla, T^{ab}\rangle$ be a model of Newton-Cartan theory. Then there exists a unique standard of rotation $\circlearrowleft$ such that ∇ is compatible with $\circlearrowleft$ and $\langle M, t_a, h^{ab}, \circlearrowleft, T^{ab}\rangle$ is a model of Maxwell gravitation.*

Proposition 2. *Let $\langle M, t_a, h^{ab}, \circlearrowleft, T^{ab}\rangle$ be a model of Maxwell gravitation. Then there exists a unique equivalence class of derivative operators $[\nabla]$ such that:*

- *All the $\nabla \in [\nabla]$ are compatible with $\circlearrowleft$;*
- *For any two ∇, $\nabla' \in [\nabla]$, $\nabla' = (\nabla, t_b t_c \sigma^a)$, where σ^a is a spacelike and twist-free vector field which satisfies $\nabla_n \sigma^n = 0$ and $\rho\sigma^a = 0$;*
- *For any $\nabla \in [\nabla]$, $\langle M, t_a, h^{ab}, \nabla, T^{ab}\rangle$ is a model of Newton-Cartan theory.*

Corollary 2.1. *Let $\langle M, t_a, h^{ab}, \circlearrowleft, T^{ab}\rangle$ be a model of Maxwell gravitation such that at all points $p \in M$, $\rho \neq 0$. Then there exists a unique derivative operator ∇ such that $\langle M, t_a, h^{ab}, \nabla, T^{ab}\rangle$ is a model of Newton-Cartan theory.*

3 On geodesics, counterfactuals, and surplus structure

As such, the relationship between Maxwell gravitation and Newton-Cartan theory is not altogether straightforward. Whenever ρ is nowhere vanishing, each

10. For further details on this way of presenting Maxwell gravitation, including the interpretation of the equations (MG) and its relation to Dewar's (2018) theory by the same name, see March (2023).

model of Maxwell gravitation is uniquely associated with a model of Newton-Cartan theory, and *vice versa*. But typically, a model of Maxwell gravitation does not carry enough information to fix a unique Newton-Cartan connection in regions where $\rho = 0$. And given the geodesic principle, these models of Newton-Cartan theory *prima facie* disagree as to the allowed trajectories for test particles in regions where $\rho = 0$. As a result, one might think that Newton-Cartan theory draws its distinctions finer than Maxwell gravitation; such is precisely Saunders's concern when he writes

> Take possible worlds each with only a single, structureless particle. Depending on the connection, there will be infinitely-many distinct trajectories, infinitely-many distinct worlds of this kind. But in [Maxwellian terms], as in Barbour-Bertotti theory, there is only one such world – a trivial one, in which there are no meaningful predications of the motion of the particle at all. Only for worlds with two or more particles can distinctions among motions be drawn. From the point of view of the latter theories, the fault lies with introducing a non-trivial connection – curvature – without any source, unrelated to the matter distribution. At a deeper level, it is with introducing machinery – a standard of parallelism for time-like vectors, defined even for a single particle – that from the point of view of a relationalist conception of particle motions is unintelligible. (Saunders 2013, 46-47)

This has lead a number of authors to suggest that the difference between Maxwell gravitation and Newton-Cartan theory has to do with counterfactuals about the behaviour of unactualised matter. For example, Dewar (2018, 266) claims that "the distinction at issue [in whether Newton-Cartan theory draws its distinctions finer than Maxwell gravitation] is whether unactualised dispositions may properly be considered as empirically respectable properties." Similarly, Wallace (2020, 29) attempts to diffuse Saunders's concern by noting that the models of Newton-Cartan theory which correspond to a single model of Maxwell gravitation disagree only as to the unactualised trajectories of test particles in regions where $\rho = 0$, so that "insofar as these counterfactuals [about the behaviour of unactualised test matter] are indeterminate (perhaps because a Humean view of laws [...] is assumed) so is the Newton-Cartan connection."

However, this cannot be the whole story. After all, if we wish to evaluate counterfactuals about what would happen were matter introduced to empty spacetime regions, there is another obvious strategy – which is to note that, in realistic cases, introducing matter into empty regions will perturb the mass-momentum tensor slightly, so that ρ is no longer vanishing. If we wish to account for this perturbation to the mass-momentum tensor, then this requires that we move to a different model of Newton-Cartan theory from whichever one we are taking to represent the actual world – which will give determinate predictions for the behaviour of this unactualised matter. Indeed, this is precisely the strategy one must take when evaluating counterfactuals about unactualised matter within Maxwell gravitation, since the theory does not equip empty spacetime regions with test particle trajectories.

Note that whilst this strategy for evaluating counterfactuals about unactualised matter is at odds with physics practice, it is strikingly similar to possible worlds analyses of counterfactuals familiar from metaphysics. As an example, consider Lewis's (1973, 1973, 1979) account. According to Lewis, the counterfactual 'If it were the case that A, then it would be the case that C.' is true just in case some world where both A and C are true is more similar to the actual world than any world where A is true but C is false. Similarity amongst worlds, for Lewis, is to be evaluated using the following criteria, in order of most to least importance (Lewis 1979):

- Avoid widespread, diverse violations of actual law.
- Maximise the region of perfect match of particular fact.
- Minimise small, simple violations of actual law.

In practice, then, Lewis's prescription for evaluating counterfactuals about the behaviour of unactualised matter is as follows: take a world which is a perfect duplicate of the actual world before some time t,[11] insert a small violation of

11. Lewis (1979) claims that his similarity ordering ensures that worlds which are perfect duplicates before time t but diverge thereafter will be more similar than worlds which differ before t but are perfect duplicates after t. Whilst this is controversial (see Elga 2001), I am assuming that this does in fact work as intended.

actual law at t to introduce unactualised matter into the region of interest, and then evolve the laws forward. But now compare this to how one must evaluate counterfactuals about unactualised matter in Maxwell gravitation. We take the model that we are using to represent the actual world. We discontinuously modify T^{ab} at some time t to insert unactualised matter into the region of interest – thereby violating at least (MG1). And then we evolve the laws forward to examine how it behaves. If one thinks that Lewis's account is adequate, then it is this method – and not the use of test particles – which is the correct way to evaluate counterfactuals about unactualised matter.

All this suggests that what is at issue, in discussions of whether Newton-Cartan theory draws its distinctions finer than Maxwell gravitation, is nothing to do with counterfactuals about the behaviour of unactualised matter *per se*. Given a suitably realistic treatment, both Newton-Cartan theory and Maxwell gravitation are able to make perfectly good sense of such counterfactuals, and to do so without invoking test particles. As such, the relevant question is not whether counterfactuals about unactualised matter are indeterminate in either of the two theories, *pace* Wallace. Nor is it whether unactualised dispositions constitute empirical content, *pace* Dewar.

Rather, the salient difference is that in Newton-Cartan theory, it is not only facts about the mass-momentum tensor and standard of rotation which are represented explicitly in the theory's formalism, but also facts about the Newton-Cartan connection. Moreover, these facts about the Newton-Cartan connection are equipped, via the geodesic principle, with a physical interpretation in terms of test particle trajectories. At issue is precisely the legitimacy of including these facts as a fundamental part of our physical theories. Providing that ρ is nowhere vanishing, this is harmless – but as we have seen, there is no unique way to draw in these trajectories in regions where $\rho = 0$. And then the introduction of a Newton-Cartan connection to regions where $\rho = 0$ begins to look like surplus structure, and disagreements about the geodesics of this connection like distinctions without differences.

As I see it, this way of stating the concern seems much closer to Saunders's comments than the readings offered by Dewar (2018) and Wallace (2020) in terms of counterfactuals about unactualised matter. After all, as the foregoing analysis shows, proponents of Maxwell gravitation as much as anyone are able to make sense of counterfactuals about unactualised matter. They may not think that this constitutes empirical content, but then again they may – nothing in the theory mandates either view. However, what *does not* make sense, from the perspective of Maxwell gravitation, is talk of test particle trajectories which cannot be defined from those of material bodies.[12] And this is precisely where Saunders locates the source of the problem – "with introducing a non-trivial connection curvature without any source, unrelated to the matter distribution."

Nevertheless, one might wonder: what is the closest we can get to test particle trajectories in empty spacetime regions within Maxwell gravitation? For this, it is instructive to recall how it is that test particle trajectories are derived in Newton-Cartan theory. The central result here is Weatherall's (2011) Newtonian geodesic theorem (where I have modified his statement of the theorem slightly to match the terminology used here):

Proposition 3 (Weatherall, 2011)**.** *Let $\langle M, t_a, h^{ab}\rangle$ be a non-relativistic spacetime, ∇ a compatible derivative operator on M and suppose that M is oriented and simply connected. Suppose also that $R^{ab}{}_{cd} = 0$. Let $\gamma : I \to M$ be a smooth curve. Suppose that given any open subset O of M containing $\gamma[I]$, there exists a smooth symmetric field T^{ab} on M such that:*

- *T^{ab} satisfies the Newtonian mass condition;*
- *$\nabla_n T^{na} = 0$;*
- *supp(T^{ab}) $\subset O$; and*
- *There is at least one point in O at which $T^{ab} \neq 0$.*

Then γ is a timelike curve that can be reparametrised as a geodesic.

Corollary 3.1 (Weatherall, 2011)**.** *Let $\langle M, t_a, h^{ab}\rangle$ be a non-relativistic spacetime, ∇ a compatible derivative operator on M and suppose that M is oriented.*

12. Note that this is not in conflict with Wallace's (2020, §4) argument that vector relationism – of which Maxwell gravitation is the continuum limit (see March 2023) – contains emergent inertial structure. The inertial structure which Wallace considers is defined by considering the behaviour of a dynamically isolated system of particles embedded in a larger universe; clearly this construction fails in regions where there is in fact no matter present.

Suppose also that M is spatially flat and $R^{ab}{}_{cd} = 0$. For any $p \in M$, there exists a neighbourhood of p, Q, such that if $\gamma : I \to Q$ is a smooth curve, and for any open subset O of Q containing $\gamma[I]$ there exists a smooth symmetric field T^{ab} on M satisfying the above conditions, then γ is a timelike curve that can be reparametrized as a geodesic (segment).

The interpretation of proposition 3 is as follows. Fix a Newton-Cartan spacetime which satisfies (NCT4). Then the only curves in that spacetime which are apposite to represent the worldlines of test particles, in the sense that they may be traversed by an arbitrarily small, non-interacting mass distribution, are timelike geodesics. Corollary 3.1 states that, without imposing global topological constraints on M, the same result holds locally.

We can then use a similar construction to Weatherall's geodesic theorem to say something about how we might recover an analogue of test particles within Maxwell gravitation. However, the result is limited in a significant way. In Weatherall's geodesic theorem, we consider the behaviour of test matter in a fixed Newton-Cartan spacetime $\langle M, t_a, h^{ab}, \nabla \rangle$. If we are imagining that this spacetime is a model of Newton-Cartan theory for some T^{ab}, then this amounts to choosing a fixed background matter distribution. But we cannot straightforwardly do the same in Maxwell gravitation. This is for two reasons. First, we cannot use the spacetime structure of Maxwell gravitation to encode facts about the background matter distribution as we can in Newton-Cartan theory. Secondly, even if we fix a background matter distribution, we cannot express the condition that the test matter T^{ab} satisfies $\nabla_n T^{na} = 0$ with respect to (one of) the Newton-Cartan connections induced by this background matter distribution using only the structure of Maxwellian spacetime. We can, however, do so indirectly – by modelling both the test and background matter separately and demanding that the resulting structure be a model of Maxwell gravitation (and invoking proposition 2). Thus we have the following result:

Proposition 4. *Let $\langle M, t_a, h^{ab}, \circlearrowleft, T^{ab} \rangle$ be a model of Maxwell gravitation, and suppose that M is oriented and simply connected. Let $\gamma : I \to M$ be a smooth curve, and let $T^{ab} = \tilde{T}^{ab} + \tau^{ab}$ with $supp(\tilde{T}^{ab}) \cap supp(\tau^{ab}) = \varnothing$. Moreover, suppose that given any open subset O of M containing $\gamma[I]$, we have:*

- *supp*$(\tau^{ab}) \subset O$*; and*

- *There is at least one point in* O *at which* $\tau^{ab} \neq 0$*.*

Then there exists a unique equivalence class of derivative operators $[\nabla]$ *such that*

- *All the* $\nabla \in [\nabla]$ *are compatible with* $\circlearrowleft$*;*

- *For any two* ∇*,* $\nabla' \in [\nabla]$*,* $\nabla' = (\nabla, t_a t_b \sigma^c)$*, where* σ^a *is a spacelike and twist-free vector field such that* $\nabla_n \sigma^n = 0$ *and* $\rho\sigma^a = 0$*;*

- *For any* $\nabla \in [\nabla]$*,* $\langle M, t_a, h^{ab}, \nabla, T^{ab}\rangle$ *is a model of Newton-Cartan theory;*

- $\gamma[I]$ *is a timelike curve which can be reparametrised as a geodesic with respect to any* $\nabla \in [\nabla]$*; and*

- $\sigma^a = 0$ *on* $\gamma[I]$*.*

Proof. That there exists a unique equivalence class of derivative operators satisfying the first three conditions follows immediately from proposition 2. So let ∇ be an arbitrary member of $[\nabla]$, and consider the tuple $\langle M, t_a, h^{ab}\rangle$. Clearly, this is a non-relativistic spacetime, with M oriented and simply connected; moreover, ∇ is a compatible derivative operator on M which, since $\langle M, t_a, h^{ab}, \nabla, T^{ab}\rangle$ is a model of Newton-Cartan theory, satisfies $R^{ab}{}_{cd} = 0$. And since $\mathrm{supp}(\tilde{T}^{ab}) \cap \mathrm{supp}(\tau^{ab}) = \varnothing$, τ^{ab} is a symmetric tensor field which satisfies the Newtonian mass condition and $\nabla_n \tau^{na} = 0$. It follows that, since all the conditions of Weatherall's geodesic theorem are met, γ is a timelike curve that can be reparametrised as a geodesic with respect to ∇. Moreover, since ∇ is arbitrary, we must have that for any two ∇, $\nabla' \in [\nabla]$, if ξ^a is the tangent vector field to γ, then

$$\begin{aligned}\xi^n \nabla'_n \xi^a &= \xi^n \nabla_n \xi^a - \xi^n \xi^m t_n t_m \sigma^a \\ &= -\sigma^a \\ &= 0,\end{aligned}$$

so that $\sigma^a = 0$ along $\gamma[I]$. $\square$

On the face of it, proposition 4 appears very similar to Weatherall's geodesic theorem. However, there is a significant disanalogy. In proposition 3, the derivative operator ∇ with respect to which γ is a geodesic is relatively unconstrained. In particular, it can be a Newton-Cartan connection for any given matter distribution. But this is not the case in proposition 4. Here, ∇ can only be a Newton-Cartan connection for some matter distribution if that distribution also includes the test matter represented by τ^{ab}. Whereas Weatherall's geodesic theorem applies to test matter which has not already been explicitly represented in the formalism, proposition 4 does not.

As such, it is important to be clear as to what proposition 4 allows us to say. As τ^{ab} becomes arbitrarily close to zero, the connection will become arbitrarily close to a Newton-Cartan connection for the background matter distribution, which does not include test particles. And in each case, the curve γ traversed by τ^{ab} will be a geodesic of that connection. So γ will become arbitrarily close to a geodesic of some Newton-Cartan connection for the background matter distribution.

But proposition 4 does not tell us *which* Newton-Cartan connection this will be. To see this, note that although all the connections in proposition 4 agree on γ (and anywhere else where $\rho \neq 0$), they need not agree off of γ. So the connection that ∇ most closely approximates, as τ^{ab} becomes arbitrarily small, will depend on how one initially chooses ∇ in regions where $\rho = 0$. And here, we have exactly the same freedom as in proposition 2. As a result, proposition 4 does not provide a means of recovering a unique congruence of geodesic curves, for some given matter distribution.

Now consider what we should say about models of Newton-Cartan theory which differ only as to the connection in regions where $\rho = 0$. On the face of it, these represent physically distinct possibilities. After all, given the geodesic principle, these models literally interpreted make different predictions for the behaviour of test particles in regions where $\rho = 0$.

But in virtue of what do facts about the behaviour of test particles in empty spacetime regions count as physical facts? A natural answer would be: in virtue of the fact that test particles represent the limiting case of the kind of construction used in proposition 4, as the influence of test matter on non-test matter becomes negligible. That is, we model the behaviour of some particle plus background matter distribution, and then consider what happens as the particle mass becomes much smaller than that of the rest of the matter. But we have just seen that this procedure fails to fix a unique Newton-Cartan connection in regions where $\rho = 0$. Moreover, pairs of models which disagree only as to the connection in these regions are not guaranteed to be isomorphic (since σ^a need not be rigid). This suggests that the physical significance of the Newton-Cartan connection in regions where $\rho = 0$ is more subtle than it first appears.

We can also say something more in support of this worry. As March (2023, 24) argues, the equations (NCT) are equivalent to the conjunction of the equations (MG), the flat derivative operator compatibility condition, and (the geometrised version of) Newton's second law

$$\rho \xi^n \nabla_n \xi^a = -\nabla_n \sigma^{na}, \tag{NII}$$

with $\circlearrowleft$ now interpreted as the unique standard of rotation compatible with ∇. From this perspective, the only difference between Maxwell gravitation and Newton-Cartan theory is the presence of (NII), whose role is essentially to provide a partial gauge-fixing of the connection. In regions where $\rho \neq 0$, this furnishes the connection with a physical interpretation – namely, as the unique connection relative to which fluid elements obey (the geometrised version of) Newton's second law. But in regions where $\rho = 0$, (NII) fails to provide any additional constraint on the connection at all.[13] So we cannot give an analogous physical interpretation to the connection in regions where $\rho = 0$. And we have just seen that models which differ as to the connection in regions where $\rho = 0$ can both lay equal claim to represent the limiting case of the construction used in proposition 4.

How else might the Newton-Cartan connection in regions where $\rho = 0$ acquire physical significance? An obvious response would be to say that the Newton-Cartan connection in regions where $\rho = 0$ represents counterfactuals about unactualised matter. But as discussed earlier, this would be a mistake:

13. Given the Newtonian mass condition, that is.

counterfactuals about unactualised matter can be represented perfectly well elsewhere in the theory, without invoking test particles. Moreover, if we do insist on evaluating counterfactuals about unactualised matter using test particles, then we seem to face a troubling indeterminism as to which Newton-Cartan connection is the physically salient one.

This suggests a view on which the Newton-Cartan connection only has physical significance in regions where $\rho \neq 0$. In these regions, it can be given an interpretation through the equation (NII). As a result, models which differ only as to the Newton-Cartan connection in regions where $\rho = 0$ represent the same physical state of affairs. Under this interpretation, Newton-Cartan theory might exhibit representational redundancy, but would draw its distinctions no finer than Maxwell gravitation.

Of course, there are other interpretations available. We could say that the geodesics of the Newton-Cartan connection in empty spacetime regions represent physical structure, but that models which differ only as to the connection in regions where $\rho = 0$ represent the same physical state of affairs. Or, we could say that these models represent distinct physical states of affairs, and look for some other interpretative principle to justify this. However, both these approaches face the problem of articulating just what physical structure the Newton-Cartan connection in empty spacetime regions is supposed to represent. Clearly, it is nothing to do with the matter distribution in question. Nor is it anything to do with counterfactuals, as emphasised above.

There are two plausible options here. We have seen that the $\rho \to 0$ limit of the kind of construction used in proposition 4 to model test particles fails to fix a unique connection in regions where $\rho = 0$. But we could still say that this limit is represented equally, but redundantly, by each of these models. The immediate difficulty here would be to explain what the physical significance of this limit is, if it is not unique. This is particularly severe once we have recognised that test particles need not be used to evaluate counterfactuals about unactualised matter.

The other option would be to appeal to the fact that one way of fixing a unique Newton-Cartan connection in regions where $\rho = 0$ is via a choice of boundary conditions. If we expect these to come endowed with a physical interpretation (perhaps because we are modelling a subsystem of a larger universe), then at least in practice, this would explain why it is sometimes appropriate to interpret models which differ only as to the Newton-Cartan connection in regions where $\rho = 0$ as physically distinct. However, it is not then clear what we are supposed to say about the fact that models of Newton-Cartan theory can also be used to represent complete physical histories. Given these problems, I take it that the first view – on which the geodesics of the Newton-Cartan connection in regions where $\rho = 0$ do not represent anything physical – is the most attractive.

As such, the position defended here parts company from those of Wallace, Dewar, and Saunders in an important respect. For these authors, it is assumed from the outset that facts about the geodesics of the Newton-Cartan connection *automatically* have physical significance. Hence, Saunders claims that Newton-Cartan theory draws distinction without differences, Dewar suggests that at issue is whether or not such differences are *empirical* (rather than physical) differences, and Wallace claims that we may avoid Saunders's conclusion by declaring the Newton-Cartan connection indeterminate in regions where $\rho = 0$. By contrast, I have argued that the problem arises at an earlier stage. Proper attention needs to be paid to *how* it is that the geodesics of the Newton-Cartan connection come to have physical significance, and how differences in the geodesics of the Newton-Cartan connection come to represent physical differences, in deciding whether or not Newton-Cartan theory draws its distinctions finer than Maxwell gravitation. One could simply declare at the outset that facts about the geodesics of the Newton-Cartan connection are physical facts. But once we pay proper attention to the details of Newton-Cartan theory, it is less clear that this interpretation is tenable.[14]

14. Though it is worth noting that there are other considerations one might bring to bear on this. For example, considered as the non-relativistic limit of GR, one might think that the Newton-Cartan connection inherits physical significance from the fact that the connection in GR arguably *does* represent physical structure (perhaps because it can sustain gravitational wave solutions in vacuum regions). For further discussion of the use of intertheoretic relations to constrain interpretative judgements, see Linnemann & Read (2021).

4 On spacetime and dynamical symmetries

The foregoing discussion suggests that there are plausible interpretations of Newton-Cartan theory on which models which disagree only as to the connection in regions where $\rho = 0$ represent the same physical state of affairs. This would avoid the worry that Maxwell gravitation and Newton-Cartan theory are inequivalent because Newton-Cartan theory draws its distinctions finer than Maxwell gravitation.

However, Jacobs (2023) has recently presented another argument that Newton-Cartan theory and Maxwell gravitation are inequivalent, on the basis that the two theories have different spacetime and dynamical symmetry groups. Jacobs begins his analysis by defining an "active" version of the dynamic shift – analogous to the standard kinematic and static shifts – which produces a linear time-dependent acceleration of the matter content of the original solution. Since these active dynamic shifts are a dynamical symmetry but not a spacetime symmetry of Galilean gravitation, the theory violates Earman's (1989, 46) "adequacy conditions" on the construction of spacetime theories. These demand that there be a match between the spacetime and dynamical symmetries of a theory, in the following sense:

SP1: Any dynamical symmetry of T is a spacetime symmetry of T.

SP2: Any spacetime symmetry of T is a dynamical symmetry of T.

Jacobs then goes on to argue that, although both Newton-Cartan theory and Maxwell gravitation restore SP1, they do so in different ways. In moving to Maxwell gravitation, we enlarge the spacetime symmetries from the Galilei to the Maxwell group. Meanwhile, in moving to Newton-Cartan theory, we employ the opposite strategy – restricting the dynamical symmetries to the Galilei group. For Jacobs, this means that Maxwell gravitation and Newton-Cartan theory are inequivalent: the two theories disagree as to what the dynamical symmetries are, and even if it is *sometimes* possible to define a unique Newton-Cartan connection from a model of Maxwell gravitation, this is not true of all models of the theory, even less so the kinematically possible models (KPMs).

Never mind the question whether theories with different spacetime and dynamical symmetry groups can be equivalent (I for one would not question this). Instead, I wish to focus on Jacobs's technical claim here, viz. the symmetry groups of Maxwell gravitation and Newton-Cartan theory. I claim that this rests on a mistake. The spacetime symmetries of a theory are standardly defined as the automorphism group of its absolute objects, where the absolute objects "are supposed to be the same in each dynamically possible model" (Earman 1989, 45). In arguing that the spacetime symmetries of Newton-Cartan theory are the Galilei group, Jacobs assumes that the Newton-Cartan connection is an absolute object (Jacobs 2023, proposition 3). But in Newton-Cartan theory, much like in general relativity, the connection is not an absolute object; its value depends on the matter distribution we are considering.[15] Wallace (2020, 28) makes a similar observation, noting that "in Newton-Cartan theory, the connection does double duty, imposing both the rotation standard (a piece of absolute structure) and the inertial structure (something dynamical and contingent)".[16]

This presents a serious problem for Jacobs's argument that Maxwell gravitation and Newton-Cartan theory are inequivalent, and likewise for his claim that the two theories represent different ways of restoring Earman's SP1. If only the standard of rotation associated with ∇ is invariant across the DPMs of Newton-Cartan theory, then its spacetime symmetry group is in fact the same as that of Maxwell gravitation. Not only that, but the two theories also share the same dynamical symmetry group. In particular, if $h : M \to M$ is a diffeomorphism generated by an arbitrary Maxwell transformation,[17] then the induced map $\langle M, t_a, h^{ab}, \nabla, T^{ab}\rangle \to \langle M, t_a, h^{ab}, h^*\nabla, h^*T^{ab}\rangle$ preserves both solutionhood of the equations (NCT), and all the absolute objects.

Of course, this requires that we allow dynamical symmetry transformations to act on the connection – a piece of spacetime structure – as well as the mat-

15. As is obvious from (NCT2). To put the point pithily, taking the Newton-Cartan connection to be an absolute object would mean taking there to be only one nomically possible mass density field according to the theory.

16. To see this, note that any pair of compatible connections which determine the same standard of rotation are related by a transformation of the form $\nabla \to (\nabla, t_b t_c \sigma^a)$, for some spacelike σ^a (Weatherall 2018, proposition 1). Meanwhile, any pair of compatible connections which satisfy (NCT3) and (NCT4) are related by a transformation of the form $\nabla \to (\nabla, t_b t_c \sigma^a)$, where σ^a is spacelike and twist-free (Dewar 2018, proposition 4).

17. That is, transformations of the form $t \to t + \tau$, $x^i(t) \to R^i{}_j x^j(t) + a^i(t)$, where x^μ is an arbitrary Maxwellian coordinate system on M.

ter distribution. I will merely point out that this is completely standard; it is precisely the notion of dynamical symmetry implicit in the claim that the dynamical symmetries of general relativity are the full diffeomorphism group.

Still, there is one part of Jacobs's analysis which does carry over intact. Arbitrary Maxwell transformations of the mass-momentum tensor preserve solutionhood of the equations (MG). But they do not preserve solutionhood of the equations (NCT). In general, if $\langle M, t_a, h^{ab}, \nabla, T^{ab} \rangle$ is a solution of the equations (NCT), then $\langle M, t_a, h^{ab}, \nabla, h^* T^{ab} \rangle$ will violate at least (NII), where $h : M \to M$ is a diffeomorphism generated by an arbitrary Maxwell transformation. *Prima facie*, this reveals an important difference between the two theories: once we move to consider the entire space of KPMs, there will be non-solutions of Newton-Cartan theory which correspond to solutions of Maxwell gravitation. I will return to this argument in section 6.

5 Maxwell gravitation, Newton-Cartan theory, and categorical equivalence

The foregoing discussion suggests that there are plausible interpretations of Newton-Cartan theory on which models which disagree only as to the connection in regions where $\rho = 0$ are physically equivalent. It also suggests that the spacetime and dynamical symmetry groups of both Maxwell gravitation and Newton-Cartan theory are the Maxwell group. What, in this case, should we say about whether Maxwell gravitation and Newton-Cartan theory are equivalent?

For this, it will be useful to have a formal standard of theoretical equivalence to hand. The standard of theoretical equivalence which I will employ here is one which has been brought to bear on a number of debates in philosophy of physics in recent years,[18] and it is called *categorical equivalence*. This requires some groundwork. The use of category theory as a tool for investigating the relationships between physical theories is motivated by the fact that the class of a theory's models often has – or can be given – the structure of a category.

18. See Rosenstock, Barrett, & Weatherall (2015), Weatherall (2016), Barrett (2019), Nguyen, Teh, & Wells (2020).

Whilst there is some variance in how exactly this category is defined,[19] one straightforward way of doing this is to take the objects of this category to be the theory's models, and its arrows to be inter-model relationships which preserve physical content.

According to the criterion of theoretical equivalence we will consider, two theories are equivalent just in case their associated categories of models are "isomorphic" in the following precise sense:

Categorical equivalence: theories T_1 and T_2 are equivalent just in case there exists an equivalence of categories between the categories of models of T_1 and T_2 which preserves empirical content.

Two categories $\mathbf{T}_1$ and $\mathbf{T}_2$ are equivalent just in case there exist functors $F : \mathbf{T}_1 \to \mathbf{T}_2$, $G : \mathbf{T}_2 \to \mathbf{T}_1$ such that FG is isomorphic to $\mathrm{id}_{\mathbf{T}_2}$, and likewise GF is isomorphic to $\mathrm{id}_{\mathbf{T}_1}$.[20] As such, one might take categorical equivalence to capture the idea that we can translate between T_1 and T_2, in a way that preserves empirical content, and that these translations are – up to isomorphism – inverses of each other. But categorical equivalence does not require that the translation between the models of T_1 and T_2 be unique. For example, consider the relationship between Galilean gravitation and Newton-Cartan theory. We know (from the Trautman geometrisation and recovery theorems) that each model of Galilean gravitation is uniquely associated with a model of Newton-Cartan theory, but not *vice versa*. Typically, we can only recover a model of Galilean gravitation from a model of Newton-Cartan theory up to Trautman gauge symmetry – transformations of the form (2). But if we interpret Trautman gauge symmetry-related models of Galilean gravitation as physically equivalent – which amounts to taking the arrows in our category-theoretic representation of Galilean gravitation to include not only diffeomorphisms, but also transformations of the form (2) – then as Weatherall (2016) shows, the two theories will still be categorically equivalent. Whether this is sufficient for theoretical

19. For example, when Barrett and Halvorson (2022) talk about theories as categories, they have in mind specifically first-order theories, and categories whose objects are the theory's models, and whose arrows are elementary embeddings. My approach here is more in the spirit of Weatherall (2016), Barrett (2019), Nguyen, Teh, & Wells (2020).

20. Or equivalently, just in case there exists a functor $F : \mathbf{T}_1 \to \mathbf{T}_2$ which is full, faithful, and essentially surjective. For details, see for example Weatherall (2017) and the references therein.

equivalence is an issue which we will return to in section 6.

In order to say whether Maxwell gravitation and Newton-Cartan theory are categorically equivalent, we first need to say something about the categories of models associated to these theories. I will take these to include two kinds of arrows:

- Isomorphisms induced by automorphisms of the theory's absolute objects
- Gauge transformations which do not fall under the above

For Maxwell gravitation, this gives us the following category:

MG: Objects are models of Maxwell gravitation, arrows are diffeomorphisms which preserve the metrics and standard of rotation.

However, for Newton-Cartan theory, the views considered here suggest four possible categories. On the one hand, there is the (more straightforward) question about the absolute objects of Newton-Cartan theory – namely, the metrics, and the standard of rotation associated to ∇. However, we have seen that Jacobs incorrectly takes the Newton-Cartan connection itself to be an absolute object in his analysis, and it is of some interest to see what happens if we do so here. On the other hand, there is the question about whether models of Newton-Cartan theory which differ only as to the connection in regions where $\rho = 0$ represent the same physical state of affairs. I have argued that these models *are* physically equivalent, but one might also take them to be inequivalent (for example, as Saunders does). Together, this gives us the following categories:

NCT$_1$: Objects are models of Newton-Cartan theory, arrows are diffeomorphisms that preserve the metrics and Newton-Cartan connection.

NCT$_2$: Objects are models of Newton-Cartan theory, arrows are pairs (χ, σ^a), where σ^a is a spacelike and twist-free vector field which satisfies $\nabla_n \sigma^n = 0$ and $\rho\sigma^a = 0$, and χ is a diffeomorphism which preserves the metrics and (gauge-transformed) Newton-Cartan connection $(\nabla, t_b t_c \sigma^a)$.

NCT$_3$: Objects are models of Newton-Cartan theory, arrows are diffeomorphisms which preserves the metrics and standard of rotation associated with ∇.

$\mathbf{NCT}_4$: Objects are models of Newton-Cartan theory, arrows are pairs (χ, σ^a), where σ^a is a spacelike and twist-free vector field which satisfies $\nabla_n \sigma^n = 0$ and $\rho\sigma^a = 0$, and χ is a diffeomorphism which preserves the metrics and standard of rotation associated with the (gauge-transformed) Newton-Cartan connection $(\nabla, t_b t_c \sigma^a)$.

Categories $\mathbf{NCT}_1$ and $\mathbf{NCT}_2$ result from (incorrectly) taking the Newton-Cartan connection to be an absolute object; in $\mathbf{NCT}_3$ and $\mathbf{NCT}_4$ only the metrics and standard of rotation associated with ∇ are absolute objects. In $\mathbf{NCT}_1$ and $\mathbf{NCT}_3$, models which differ only as to the connection in regions where $\rho = 0$ are interpreted as physically inequivalent; in $\mathbf{NCT}_2$ and $\mathbf{NCT}_4$ they are equivalent.

I will begin by considering $\mathbf{NCT}_1$ and $\mathbf{NCT}_2$. It is straightforward to show that neither of these categories are equivalent to $\mathbf{MG}$:

Proposition 5. *Let $F : \mathbf{NCT}_1 \to \mathbf{MG}$ be the functor which takes each model of Newton-Cartan theory to its corresponding Maxwell model, as given in proposition 1, and takes each arrow to an arrow generated by the same diffeomorphism. Then F is not full.*

Proof. Let $\mathfrak{M} = \langle M, t_a, h^{ab}, \nabla, T^{ab}\rangle$ be an object in $\mathbf{NCT}_1$, and let $\chi : M \to M$ be any diffeomorphism which preserves the metrics and satisfies $\chi^*\nabla = (\nabla, t_b t_c \sigma^a)$, where σ^a is a (non-zero) spacelike vector field.[21] By construction, $\chi : F(\mathfrak{M}) \to \chi^* F(\mathfrak{M})$ is an arrow in $\mathbf{MG}$ which is not the image of any arrow in $\mathbf{NCT}_1$ under F. □

Proposition 6. *Let $F : \mathbf{NCT}_2 \to \mathbf{MG}$ be the functor which takes each model of Newton-Cartan theory to its corresponding Maxwell model, as given in proposition 1, and each arrow $(\chi, \sigma^a) \to \chi$. Then F is not full.*

Proof. This is almost identical to the proof of proposition 5. Let $\mathfrak{M} = \langle M, t_a, h^{ab}, \nabla, T^{ab}\rangle$ be an object in $\mathbf{NCT}_2$, and suppose that $T^{ab} \neq 0$. Let $\chi : M \to M$ be any diffeomorphism which preserves the metrics and satisfies $\chi^*\nabla = (\nabla, t_b t_c \sigma^a)$, where σ^a is a spacelike vector field such that $\sigma^a \neq 0$ for at least one point p

21. Such exist. For example, if x^μ is a Maxwellian coordinate system on M, then in general, diffeomorphisms induced by arbitrary Maxwell transformations have this property.

where $\rho \neq 0$ (again, such exist). Since arrows in $\mathbf{NCT}_4$ at least preserve the Newton-Cartan connection in regions where $\rho \neq 0$, $\chi : F(\mathfrak{M}) \to \chi^* F(\mathfrak{M})$ is an arrow in **MG** which is not the image of any arrow in $\mathbf{NCT}_2$ under F. $\square$

Propositions 5 and 6 capture Jacobs's argument that Maxwell gravitation and Newton-Cartan theory are inequivalent. In taking the Newton-Cartan connection to be an absolute object in $\mathbf{NCT}_1$ and $\mathbf{NCT}_2$, we have taken the spacetime and dynamical symmetries of the theory to be the Galilei group. But precisely what goes wrong in propositions 5 and 6 is that there are non-trivial automorphisms of Maxwellian spacetime which correspond neither to Galilean transformations, nor gauge transformations of the Newton-Cartan connection, nor compositions of the two. In both cases, this means that F fails to be full, and so in the terminology of Baez et al. (2006) forgets structure.[22] This might be taken to vindicate Jacobs's claim that theories with different symmetry groups are inequivalent because they have "different structures" (Jacobs 2023, 13).

However, as argued in section 4, there are convincing reasons to think that the Newton-Cartan connection is *not* an absolute object, and hence that Maxwell gravitation and Newton-Cartan theory have the same spacetime and dynamical symmetry groups. This means that it is not $\mathbf{NCT}_1$ and $\mathbf{NCT}_2$, but $\mathbf{NCT}_3$ and $\mathbf{NCT}_4$ which are the appropriate category-theoretic representations of Newton-Cartan theory:

Proposition 7. *Let $F : \boldsymbol{NCT}_3 \to \boldsymbol{MG}$ be the functor which takes each model of Newton-Cartan theory to its corresponding Maxwell model, as given in proposition 1, and each arrow to an arrow generated by the same diffeomorphism. Then F is not full.*

Proof. Let $T^{ab} = 0$ and consider the objects $\mathfrak{M} = \langle M, t_a, h^{ab}, \nabla, T^{ab} \rangle$ and $\mathfrak{M}' = \langle M, t_a, h^{ab}, (\nabla, t_b t_c \nabla^a \phi), T^{ab} \rangle$ in $\mathbf{NCT}_3$, where $\phi = e^x e^y \sin(\sqrt{2} z)$ in some Maxwellian coordinate system x^μ on M and ∇ is flat.[23] Now consider the arrow $\mathrm{id} : F(\mathfrak{M}) \to F(\mathfrak{M}')$ in **MG**. I claim that this is not the image of any arrow $\chi : \mathfrak{M} \to \mathfrak{M}'$ in $\mathbf{NCT}_3$. For this, note that ∇ transforms as $\nabla \to (\nabla, t_b t_c \sigma^a)$

22. For more on the connection between the failure of F to be full and (amount of) structure, see Barrett (2022).
23. I take this example from Dewar (2018, 265).

under the action of any Maxwell transformation on ∇, where σ^a is a spacelike vector field which is twist-free and rigid ($\nabla^a \sigma^b = 0$). $\nabla^a \phi$ is not rigid. $\square$

$\mathbf{NCT}_3$ and $\mathbf{MG}$ are not equivalent as categories. This is a result of the failure of unique recovery we see in proposition 2. Since F is not full, one might take this to capture Saunders's idea that the fact that we cannot in general define a unique Newton-Cartan connection from a model of Maxwell gravitation means that Newton-Cartan theory has surplus structure over Maxwell gravitation. However, $\mathbf{NCT}_4$ and $\mathbf{MG}$ *are* equivalent as categories:

Proposition 8. *There exists an equivalence of categories between* $\boldsymbol{NCT}_4$ *and* $\boldsymbol{MG}$ *which preserves empirical content.*

Proof. Let $F : \mathbf{NCT}_4 \to \mathbf{MG}$ be the functor which takes each model of Newton-Cartan theory to its corresponding Maxwell model, as given in proposition 1, and each arrow $(\chi, \sigma^a) \to \chi$. F preserves empirical content since it preserves T^{ab}, and by proposition 4 is essentially surjective. It remains to show that F is full and faithful. First, let $\mathfrak{M} = \langle M, t_a, h^{ab}, \nabla, T^{ab} \rangle$, $\mathfrak{M}' = \langle M', t'_a, h'^{ab}, \nabla', T'^{ab} \rangle$ be two objects in $\mathbf{NCT}_4$. Suppose that there exist distinct arrows (χ, σ^a), $(\chi', \sigma'^a) : \mathfrak{M} \to \mathfrak{M}'$, and suppose for contradiction that $\chi = \chi'$. Then $\sigma^a \neq \sigma'^a$, since the arrows were assumed distinct. But then $(\nabla, t_b t_c \sigma^a) \neq (\nabla, t_b t_c \sigma'^a)$, so that $\chi_*(\nabla, t_b t_c \sigma^a) \neq \chi'_*(\nabla, t_b t_c \sigma'^a)$, so by contradiction $\chi \neq \chi'$ and F is faithful. Finally, let $\chi : F(\mathfrak{M}) \to F(\mathfrak{M}')$ be an arrow in $\mathbf{MG}$. Since $\chi_* \circlearrowright = \circlearrowright'$, we know that $\chi_* \nabla$ and ∇' are rotationally equivalent, so that $\chi_* \nabla = (\nabla', t'_b t'_c \sigma'^a)$, where σ'^a is a spacelike vector field on M' (Weatherall 2018, proposition 1). It follows that $\nabla' = \chi_*(\nabla, -t_b t_c \chi^* \sigma'^a)$, where we have used the fact that χ preserves the metrics. Now consider the tuple $\langle M, t_a, h^{ab}, (\nabla, -t_b t_c \chi^* \sigma'^a), T^{ab} \rangle$. This is an object in $\mathbf{NCT}_4$, since it maps to $\mathfrak{M}'$ under χ. Moreover, it agrees with $\mathfrak{M}$ on the metrics and mass-momentum tensor. It follows that $\chi^* \sigma'^a$ is spacelike, twist-free, and satisfies $\rho \chi^* \sigma^a = 0$ and $\nabla_n \chi^* \sigma'^n) = 0$ (see the proof of proposition 2). So $(\chi, -\chi^* \sigma'^a) : \mathfrak{M} \to \mathfrak{M}'$ is an arrow in $\mathbf{NCT}_4$ which maps to χ under F. Hence F is full. $\square$

6 Categorical equivalence and theoretical equivalence

We have seen how propositions 5, 6, and 7 can be used to give category-theoretic realisations of Jacobs's and Saunders's arguments that Newton-Cartan theory – interpreted after $\mathbf{NCT}_1$, $\mathbf{NCT}_2$, or $\mathbf{NCT}_3$ – is inequivalent to Maxwell gravitation. But the interpretation of Newton-Cartan theory which I have advocated for here is $\mathbf{NCT}_4$. Proposition 8 tells us that we can, in a precise sense, translate between this theory and Maxwell gravitation up to physical content preserving (i.e. gauge) transformations. Is this sufficient for theoretical equivalence?

The issue is fraught. For example, Glymour (1977) suggests a stronger *definitional equivalence* criterion, according to which two theories are equivalent just in case we can define from each model of the first theory a unique model of the second, and *vice versa*.[24] In particular, Glymour wants to claim that a theory with gauge freedoms cannot be equivalent to any rival theory in which these gauge freedoms are eliminated. This is because the former theory will contain additional untested hypotheses regarding the existence (and determinate magnitudes) of the gauge quantities. As an example, he takes Newton-Cartan theory and Galilean gravitation, noting the non-uniqueness of Trautman recoveries.

However, in my view, there remains more to be said. If we think that the connection and gravitational field in Galilean gravitation represent physical fields – which take physically distinct configurations in Trautman gauge symmetry-related models of the theory – then Glymour's claim might well be compelling. But this interpretation of the theory is not mandatory. For example, Knox (2011, 2014) argues that even within Galilean gravitation, the Newton-Cartan connection encodes the (local) structure of inertial frames. As a result, for Knox, the best interpretation of Galilean gravitation is one in which only the Newton-Cartan connection has physical significance; the Galilean connection and gravitational field are merely a less-than-perspicuous way of stating facts about the Newton-Cartan connection. And under this kind of interpretation,

24. Strictly speaking, this is not definitional equivalence proper, but rather the analogue of definitional equivalence which Glymour suggests for theories "formulated as sets of covariant equations" (Glymour 1977, 230).

it does seem appropriate to regard Newton-Cartan theory and Galilean gravitation as equivalent – a result which definitional equivalence seems unable to capture.

Now, there is room to argue that I have not been entirely fair to Glymour here. What motivates Glymour's definitional equivalence criterion is the thought that equivalent theories ought to be intertranslatable. But if one thinks that only the Newton-Cartan connection has physical significance in Galilean gravitation, then presumably one only ought to demand translatability up to the Newton-Cartan connection. As such, one can implement Glymour's criterion as follows. First, we reformulate Galilean gravitation in terms of the Newton-Cartan connection (the result is just Newton-Cartan theory under a different name), and *only then* appeal to Glymour's criterion – which of course now judges the two theories equivalent.

But once the problem has been stated in these terms, it should be clear that it is really an instance of a more general worry. Definitional equivalence leaves no conceptual room for us to interpret theories as containing structure which does not represent anything physical, nor for us to interpret gauge symmetry-related models of a theory as physically (rather than merely empirically) equivalent. Hence why, if we interpret Galilean gravitation *a la* Knox, definitional equivalence demands that we first reformulate the theory to remove the gauge quantities. Meanwhile, our interpretative practices *do* seem to make room for theories which do not wear their interpretation on their sleeve in this way. As such, the concern here is not merely that Glymour's criterion presents a problem for Knox's interpretation of Galilean gravitation; rather, it is that if Glymour is correct, then Knox's view *does not even make sense*. More generally, it is that to endorse definitional equivalence as a criterion of theoretical equivalence is to place unreasonable restrictions on interpreting a theory.

Of course, the interpretation of a theory is not unconstrained by its formalism. This is precisely why formal criteria of theoretical equivalence have been so fruitful. But despite this, there is a good deal of flexibility in how we interpret theories – which elements of a theory's formalism we take to represent elements of reality, and which differences between a theory's models we take to represent physical differences. It is this flexibility which definitional equivalence fails to capture.

Which takes us back to category theory, and categorical equivalence. Category theory has the resources to distinguish between theories which share the same formalism but have different interpretations, insofar as these interpretations can be realised through different choices of arrows. And if categorical equivalence is our standard of theoretical equivalence, then this choice of arrows does sometimes make a difference to whether or not two theories are equivalent.

As such, my final aim here is to say something in support of the verdict which categorical equivalence gives us in proposition 8, in much the same way as we did for Newton-Cartan theory and Galilean gravitation. There, we made use of Knox's interpretation of Galilean gravitation, on which only the Newton-Cartan connection has physical significance, to motivate the idea that the two theories are equivalent. But this has an obvious analogy for $\mathbf{NCT}_4$. For this, it is instructive to recall some of the discussion in section 3. There, we noted that we are always free to rewrite the equations of Newton-Cartan theory as follows: we replace the equations (NCT) with the equations (MG), the flat derivative operator compatibility condition, and (NII) (with $\circlearrowright$ now interpreted as the unique standard of rotation compatible with ∇) (March 2023). This makes it apparent that only the standard of rotation, rather than the connection, is needed for the internal dynamics of the matter distribution. Moreover, the degrees of freedom of ∇ not fixed by $\circlearrowright$ now figure only in the equation (NII). As such, we are always free to interpret (NII) as providing a (partial) fixing of these remaining degrees of freedom, rather than as a constraint on T^{ab} itself.

Should we say that Newton-Cartan theory is equivalent to Maxwell gravitation, in this case? I will approach this question roundaboutly, beginning with a remark made by Dewar (2018). In his discussion of Maxwell gravitation and Newton-Cartan theory, Dewar notes that a model of Newton-Cartan theory where $\rho \neq 0$ "carries a [...] form of redundancy: provided we know the standard of rotation associated to ∇, and provided we know the character of T^{ab}, we can "fill in the blanks" to reconstruct ∇ itself" (Dewar 2018, 264). He likens this feature of Newton-Cartan theory to comments made by Pooley (2013, §4.5) about the redundancy of standard presentations of Newtonian spacetime: given a Newtonian spacetime $\langle M, t_a, h^{ab}, \nabla, \xi^a \rangle$, we are always free to define ∇ from the remaining structure in the theory.

However, I would like to suggest that the kind of redundancy we see in Newton-Cartan theory is much more akin to the fact that Newtonian gravitation – restricted to the island universe sector, and coupled with the additional assumption that the centre of mass of the universe is at absolute rest – also has a certain redundancy to it. Given a Galilean spacetime and the mass-momentum tensor, we can always define ξ^a as the unique vector field which results from parallel transporting the centre of mass velocity field throughout all spacetime. ξ^a is irrelevant to the internal dynamics of the matter distribution, just as the irrotational degrees of freedom of ∇ are in Newton-Cartan theory. Notice also that in both cases, the choice of gauge sometimes results in a failure of unique recovery. Just as (NII) does not fix a unique connection when $\rho = 0$, so does the demand that ξ^a is the centre of mass velocity field fail to fix a unique vector field outside of the island universe sector, where the centre of mass is not well-defined. And there is also an obvious parallel to Jacobs's discussion of Maxwell gravitation and Newton-Cartan theory. Kinematic shift symmetry in Newtonian gravitation is – via Earman's SP1 – standardly taken as motivation for the move from Newtonian to Galilean spacetime. But we can also restore SP1 by restricting the dynamical symmetries to the Newtonian group. Now, it might *appear* that we can accomplish this by demanding that the centre of mass of the universe be at absolute rest. But by tying the standard of rest to facts about the matter distribution in this way, it is no longer an absolute object. As a result, the spacetime (and dynamical) symmetries of the theory remain the Galilei group.

Now, compare this version of Newtonian gravity theory, in which we demand that the centre of mass of the universe is at absolute rest, to Galilean gravitation. The only difference between the two is that in the former theory, we have promoted a particularly convenient choice of gauge – the practice of taking the centre of mass of the universe as a reference frame – to a dynamical law. Clearly this is harmless, providing that we do not then interpret the centre of mass velocity field as ontologically subsistent spacetime structure. Moreover, the fact that the "standard of rest" so defined is not an absolute object guards against precisely this mistake. Rather, it suggests an interpretation on which the vector field ξ^a is simply an additional piece of structure introduced to represent (somewhat redundantly) the centre of mass velocity of the universe.

The analogy to Newton-Cartan theory and Maxwell gravitation is immediate. From the perspective of Maxwell gravitation, the decision to work with a connection with respect to which (NII) holds amounts simply to a choice of gauge. But in moving to Newton-Cartan theory, we promote this gauge-fixing to a dynamical law. My claim is just that to the extent that one thinks that this modified version of Newtonian gravitation is equivalent to Galilean gravitation, one should also think that Newton-Cartan theory, interpreted after $\mathbf{NCT}_4$, is equivalent to Maxwell gravitation.

One final point. At the end of section 6, we noted that there appear to be non-solutions of Newton-Cartan theory which correspond to solutions of Maxwell gravitation, so that we cannot translate between Maxwell gravitation and Newton-Cartan theory in a way that preserves solutionhood. The view developed here points to one possible response to this concern. Thus far, I have described the move from Maxwell gravitation to Newton-Cartan theory as a matter of gauge-fixing the Newton-Cartan connection by imposing (NII) as a dynamical constraint. But we could go further, and interpret (NII) as a *kinematic* constraint. This would avoid the problem of non-solutions of Newton-Cartan theory in which the centre of mass of the universe is accelerated mapping to solutions of Maxwell gravitation. It would be consistent with the idea that the move from Maxwell gravitation to Newton-Cartan theory simply involves a choice of gauge, this time imposed equally across the KPMs. And it fits naturally with the suggestion that the Newton-Cartan connection should not be interpreted as ontologically subsistent spacetime structure, but rather has its physical significance in virtue of the equation (NII). If (NII) is a dynamical constraint, then it is not clear how we should interpret ∇ outside of the DPMs. But if (NII) is a kinematic constraint, then ∇ can be given a consistent physical interpretation throughout the entire space of KPMs.

If this is right, then the suggestion that (NII) should be interpreted as a choice of gauge is more radical than it first appears. It also requires a discussion of the distinction between kinematic and dynamical possibility, which I do not have space to attempt here. A proper treatment of these issues will have to wait for another time.

7 Conclusions

The appropriate spacetime setting for Newtonian gravitation theory has long been a topic of foundational interest in philosophy of physics. Moreover, the task of finding this spacetime is often seen as a straightforward matter of following Earman's principles, alongside standard "symmetry-to-(un)reality" inferences, wherein the symmetry-variant structure of a theory is interpreted as physically unreal, and excised from the theory's formalism. On the orthodox view, different theories of Newtonian gravitation are seen as successive improvements upon one another, as more and more spacetime structure is eliminated.

However, Maxwell gravitation and Newton-Cartan theory suggest a more nuanced picture. Both can motivated by the symmetries of Galilean gravitation. But they differ as to how the symmetry in question should best be formalised (as Trautman gauge symmetry, or dynamic shift symmetry), and they differ as to what the moral of this symmetry should be. Whereas Newton-Cartan theory reconceptualises the Galilean connection and gravitational potential as redundantly describing a single entity, Maxwell gravitation eliminates both from the formalism altogether.

I have argued that there are plausible interpretations of Maxwell gravitation and Newton-Cartan theory on which they are equivalent. But the question is subtle. Above all, there remains further work to be done. How does the distinction between kinematic and dynamical possibility relate to questions of interpretation and theoretical equivalence? What is the relationship between the KPMs of Maxwell gravitation and Newton-Cartan theory, if (NII) is a kinematic constraint? And what, if anything, is the connection between the suggestion that (NII) should be taken as a choice of gauge, and the well-worn debate over the status of Newton's second law? The work raised by Newtonian gravitation theory, it appears, is far from over.

References

Baez, John, Toby Bartels, James Dolan, and David Corfield. 2006. "Property, structure, and stuff." http://math.ucr.edu/home/baez/qg-spring2004/discussion.html.

Barrett, Thomas William. 2019. "Equivalent and inequivalent formulations of classical mechanics." *British Journal for the Philosophy of Science* 70 (4): 1167–1199.

Barrett, Thomas William. 2022. "How to count structure." *Noûs* 56 (2): 295–322.

Barrett, Thomas William, and Hans Halvorson. 2022. "Mutual translatability, equivalence, and the structure of theories." *Synthese* 200 (3). https://doi.org/10.1007/s11229-022-03733-8.

Dasgupta, Shamik. 2016. "Symmetry as an epistemic notion (twice over)." *British Journal for the Philosophy of Science* 67 (3): 837–878.

Dewar, Neil. 2018. "Maxwell gravitation." *Philosophy of Science* 85 (2): 249–270.

Earman, John. 1989. *World Enough and Space-Time.* MIT Press.

Elga, Adam. 2001. "Statistical mechanics and the asymmetry of counterfactual dependence." *Philosophy of Science* 68 (3): S313–S324.

Glymour, Clark. 1977. "The epistemology of geometry." *Noûs* 11 (3): 227–251.

Jacobs, Caspar. 2023. "Are dynamical shifts dynamical symmetries?" *Philosophy of Science,* https://doi.org/10.1017/psa.2023.13.

Knox, Eleanor. 2011. "Newton-Cartan theory and teleparallel gravity: the force of a formulation." *Studies in the History and Philosophy of Modern Physics* 42:264–275.

Knox, Eleanor. 2014. "Newtonian spacetime structure in light of the equivalence principle." *British Journal for the Philosophy of Science* 65 (4): 863–880.

Lewis, David. 1973a. *Counterfactuals.* Basil Blackwell.

Lewis, David. 1973b. "Counterfactuals and comparative possibility." *Journal of Philosophical Logic* 2 (4): 418–446.

Lewis, David. 1979. "Counterfactual dependence and time's arrow." *Noûs* 13 (4): 455–476.

Linnemann, Neil, and James Read. 2021. "On the status of Newtonian gravitational radiation." *Foundations of Physics* 51 (53). https://doi.org/10.1007/s10701-021-00453-w.

Malament, David. 2012. *Topics in the Foundations of General Relativity and Newtonian Gravitation Theory.* University of Chicago Press.

March, Eleanor. 2023. "Maxwell gravitation without reference to equivalence classes of derivative operators." [Preprint]. http://philsci-archive.pitt.edu/id/eprint/22404.

Nguyen, James, Nicholas Teh, and Laura Wells. 2020. "Why surplus structure is not superfluous." *British Journal for the Philosophy of Science* 71 (2): 665–695.

Pooley, Oliver. 2013. "The reality of spacetime." Unpublished manuscript.

Read, James, and Thomas Møller-Nielsen. 2020. "Motivating dualities." *Synthese* 197 (1): 263–291.

Rosenstock, Sarita, Thomas William Barrett, and James Owen Weatherall. 2015. "On Einstein algebras and relativistic spacetimes." *Studies in the History and Philosophy of Modern Physics* 52:309–316.

Saunders, Simon. 2013. "Rethinking Newton's *Principia*." *Philosophy of Science* 80 (1): 22–48.

Stachel, John. 2007. "The story of Newstein, or: is gravity just another pretty force?" In *The Genesis of General Relativity Volume 4,* edited by Jürgen Renn and Matthias Schemmel, 1041–1078. Springer.

Wallace, David. 2020. "Fundamental and emergent geometry in Newtonian physics." *British Journal for the Philosophy of Science* 71 (1): 1–32.

Weatherall, James Owen. 2011. "The motion of a body in Newtonian theories." *Journal of Mathematical Physics* 52 (3). https://doi.org/10.1063/1.3556608.

Weatherall, James Owen. 2016a. "Are Newtonian gravity and geometrised Newtonian gravity theoretically equivalent?" *Erkenntnis* 81 (5): 1073–1091.

Weatherall, James Owen. 2016b. "Maxwell-Huygens, Newton-Cartan, and Saunders-Knox spacetimes." *Philosophy of Science* 83 (1): 82–92.

Weatherall, James Owen. 2017. "Categories and the foundations of classical space-time theories." In *Categories for the Working Philosopher,* edited by Elaine Laundry, 329–348. OUP.

Weatherall, James Owen. 2018. "A brief comment on Maxwell(/Newton)[-Huygens] spacetime." *Studies in the History and Philosophy of Modern Physics* 63:34–38.