The Sure-Thing Principle

Jean Baccelli[*a] and Lorenz Hartmann[†b]

[a] *University of Oxford*
[b] *University of Basel*

September 28, 2023

Abstract

The Sure-Thing Principle famously appears in Savage's axiomatization of Subjective Expected Utility. Yet Savage introduces it only as an informal, overarching dominance condition motivating his separability postulate P2 and his state-independence postulate P3. Once these axioms are introduced, by and large, he does not discuss the principle any more. In this note, we pick up the analysis of the Sure-Thing Principle where Savage left it. In particular, we show that each of P2 and P3 is equivalent to a dominance condition; that they strengthen in different directions a common, basic dominance axiom; and that they can be explicitly combined in a unified dominance condition that is a candidate formal statement for the Sure-Thing Principle. Based on elementary proofs, our results shed light on some of the most fundamental properties of rational choice under uncertainty. In particular they imply, as corollaries, potential simplifications for Savage's and the Anscombe-Aumann axiomatizations of Subjective Expected Utility. Most surprisingly perhaps, they reveal that in Savage's axiomatization, P3 can be weakened to a natural strengthening of so-called Obvious Dominance.

[*]jean.baccelli@philosophy.ox.ac.uk
[†]lorenz.hartmann@unibas.ch

We are indebted to Ani Guerdjikova, two anonymous reviewers, Franz Dietrich, Yorgos Gerasimou, Edi Karni, Georg Nöldeke, Marcus Pivato, Teddy Seidenfeld, and Peter Wakker for helpful comments, criticisms, or suggestions. We also thank audiences at Johns Hopkins University, the University of Oxford, and the University of St Andrews. All errors and omissions are ours. Funding from the DFG project 426170771, the SNSF project 200915, the University of Oxford, and the University of Basel is gratefully acknowledged.

1 Introduction

1.1 Motivation

The Sure-Thing Principle (STP) famously appears in Savage's axiomatization of Subjective Expected Utility (SEU; Savage, 1954, 1972). In a section simply entitled "The Sure-Thing Principle", Savage introduces it as follows (1954, p. 21): "A businessman contemplates buying a certain piece of property. He considers the outcome of the next presidential election relevant to the attractiveness of the purchase. So, to clarify the matter to himself, he asks whether he would buy if he knew that the Democratic candidate were going to win, and decides that he would. Similarly, he considers whether he would buy if he knew that the Republican candidate were going to win, and again finds that he would do so. Seeing that he would buy in either event, he decides that he should buy, even though he does not know which event obtains . . . [E]xcept possibly for the assumption of simple ordering, I know of no other . . . principle governing decisions that finds such ready acceptance."

Savage does not directly employ the STP in his analysis, however. He leaves it as an informal, overarching dominance (also known as contingent—or case-by-case—reasoning) condition motivating several of his axioms. Most importantly, he uses it to motivate his separability postulate P2 and his state-independence postulate P3 (the formal statements of which we will recall shortly).[1] Savage comments (1954, p. 22): "The sure-thing principle cannot appropriately be accepted as a postulate in the sense that P1 [Savage's weak order, or "simple ordering", axiom] is because it would introduce new undefined technical terms referring to knowledge and possibility that would render it mathematically useless without still more postulates governing these terms. It will be preferable to regard the principle as a loose one that suggests certain formal postulates well articulated with P1." Once introduced, P2 and P3 supersede the STP in Savage's analysis. By and large (qualifications to follow), this is where Savage and the larger literature have left the principle, even more so, the investigation of the kinship of P2 and P3.

We pick up the analysis of the STP where Savage left it. We show that even without enriching his framework in any way (with primitive conditional preferences, knowledge operators, or anything else that would be additional to the traditional assumptions recapitulated in Sec. 2.1), more can be said than is currently known about the STP, and the kinship between P2 and P3 can be further confirmed. In particular, we show that each of P2 and P3 is equivalent to a dominance condition (Prop. 1); that they can be explicitly combined in a unified dominance condition that is a candidate formal

[1] Savage also uses the STP to motivate his postulate P7, that has a specific bite only when infinitely-valued options (known as "general acts") come into play. (On the relationships between P7 and the other Savage axioms in that case, see Hartmann, 2020; Frahm and Hartmann, 2023.) As detailed in Sec. 2.1, we here focus on finitely-valued options (the so-called "simple acts").

statement for the full STP (Prop. 2); and that they strengthen in different directions a common, basic dominance axiom (Prop. 3). These results imply, as corollaries, potential simplifications for Savage's and, incidentally, the Anscombe-Aumann (Anscombe and Aumann, 1963) axiomatizations of SEU (Cors. 2 and 3). In particular (Cor. 3), and this may be our most surprising result, we show that in Savage's axiomatization of SEU, P3 can be weakened to so-called Obvious Dominance (Li, 2017), suitably enriched with a strict clause. Obvious Dominance (further discussed on p. 12) is an extremely minimal rationality condition that has recently attracted considerable attention in various areas of economics, starting with mechanism design (via the notion of "obvious strategy-proofness"). It has been introduced and motivated as a dramatic weakening of stronger dominance conditions such as the one to which, taken alone, P3 proves equivalent—hence the surprising nature of the result referred to above. As we shall see, that result further underscores the importance of taking into account the overlap between P2 and P3, which is a key feature of our investigation of the STP. The first of the above results (Prop. 1) can essentially be found in Sec. 2.7 of Savage, 1954, and it is manifestly known in the literature. Still, even this result of ours improves on Savage's in minor ways (see p. 7 for details), and our other results (Props. 2 and 3; Cors. 2 and 3) are entirely new, to the best of our knowledge. Based on elementary proofs, together they shed light on the internal structure of and interplay between arguably the two most fundamental properties of rational choice under uncertainty. The modesty of the objective notwithstanding, no comparably comprehensive analysis of the STP (as originally envisaged by Savage) seems available in the current literature.

1.2 Literature Review

The function of this subsection is to acknowledge important preexisting work on the STP, broadly construed, while also emphasizing anew that this preexisting literature does not study the questions on which our paper focuses.

The STP may be the most scrutinized property in the literature if, as virtually all current decision theorists, one equates it with P2.[2] This is because P2 is the property generalized in most Non-Expected Utility models (the most famous generalization, the so-called "co-monotonic STP", is presented and discussed in Schmeidler, 1989; Gilboa, 1987; Chew and Wakker, 1996). But the STP is rarely discussed if, like Savage, one understands it as uniting P2 and P3.[3] Inspired especially by the second series of remarks by

[2]For philosophical discussions of the STP, thus understood, from the point of view of Causal Decision Theory and the like, see for instance Jeffrey, 1982; Pearl, 2016.

[3]It even seems that Savage discovered the principle starting from the P3, rather than the P2 side. The example in the 1950 letter to Samuelson which historians (see Moscati, 2016, p. 230) take to contain the first occurrence of the principle is, upon close inspection, under the scope of P3, not P2. (This follows from Prop. 1.b.) Contrast the more famous early occurrence in Friedman and Savage, 1952 (p. 468), that is arguably under the scope of P2.

Savage previously quoted, some have analyzed the STP using tools from epistemic logic (see, e.g., Samet, 2022 and the references therein; also Chew and Wang, 2022 for a recent proposal to combine the STP, thus approached, with Obvious Dominance). But here as well, the focus is on P2. Similarly with Ghirardato, 2002 (and the relevant parts of the dynamic consistency literature to which this reference belongs), where primitive notions of conditional preferences shed light on the nature of the STP—understood as P2. Valuable discussions of the STP that touch upon its potential difference from P2, yet not its relationship with P3, include Grant et al., 2000, Dietrich et al., 2021, Esponda and Vespa, 2021, and, in passing, Fleurbaey, 2010 (see its fn. 9). While evidently without the benefit of the more recent advances above, Sec. 2.7 of Savage, 1954 seems to remain the most complete analysis to date. We defer discussing exactly how we improve on Savage's own analysis until after we state our first main result (see p. 7). For now, we only wish to reiterate that the questions underlying our other main results and their corollaries—viz. how P2 and P3 can be combined with one another, on the one hand, and decomposed in terms of a common and a distinctive part, on the other hand—have not hitherto been considered in the literature, to the best of our knowledge.

2 Analysis

2.1 Preliminaries

Let S be a *state space*, Σ a σ-algebra on S, and X a set of *consequences*. Elements of Σ are called *events*. For any event $E \in \Sigma$, $\overline{E}$ denotes its complement. Given $E \in \Sigma$, we refer to a finite Σ-measurable partition $\{E_1, \ldots, E_n\}$ of E simply as a *partition* of E. *Acts* are Σ-measurable mappings from S to X. More specifically, throughout this paper, by "acts" we mean *simple* (finite-valued) acts, the set of which we denote by F.[4,5] With the usual abuse of notation, X also denotes the set of *constant* acts.

Our primitive is a binary relation $\succcurlyeq$ over F, interpreted as the *preferences* of a decision-maker among acts. Its asymmetric and symmetric parts are denoted $\succ$ and $\sim$, respectively. In keeping with Savage's own nomenclature, the axioms—Savage preferred to say: the postulates, hence his nomenclature—which he imposes on $\succcurlyeq$ will be here referred to as the P-axioms. We always assume that $\succcurlyeq$ is complete and transitive—in other words, that Savage's

[4] Hence the fact that we do not discuss Savage's P7, despite its being also motivated by the STP. P7 becomes relevant only when *general* (infinitely-valued) acts are considered.

[5] Our main proofs are those of Props. 1 and 3, Lemmas 3, 4, and 6. Our proofs of Prop. 3.a and Lemma 6 hold for simple acts only. The other proofs hold for general acts.

axiom P1 holds.[6]

P1. *$\succcurlyeq$ is a weak order (i.e., complete and transitive).*

For $f, g \in F$ and $E \in \Sigma$, fEg denotes the act resulting in f on E and g on $\overline{E}$. An event $E \in \Sigma$ is *null* if $fEh \sim gEh$ for all $f, g, h \in F$. Otherwise E is *non-null*. An event $E \in \Sigma$ is *essential* if neither it nor its complement is null. While all of the above is standard, such is not the case of the following notation. First, for any $E \in \Sigma$, $f \in F$, we write $f(E)$ if and only if f is constant over E, and we also let $f(E)$ denote the constant consequence of f on E. Second, for any $f, g, h \in F$, ternary partition $\{E_1, E_2, E_3\}$ of S, fE_1gE_2h denotes the act equal to f on E_1, g on E_2, and h on E_3.

Next, we recall the statement of Savage's axioms P2 and P3.

P2. *For all $f, g, h, h' \in F$, $E \in \Sigma$, $fEh \succcurlyeq gEh \Leftrightarrow fEh' \succcurlyeq gEh'$.*

P3. *For all $x, y \in X$, non-null $E \in \Sigma$, $h \in F$, $x \succcurlyeq y \Leftrightarrow xEh \succcurlyeq yEh$.*

P2 and P3 are well appreciated to be logically independent.[7] Like P1, they are necessary conditions for SEU to hold, i.e., for the existence of a utility function $u : X \to \mathbb{R}$ and a probability measure P on (S, Σ) such that for all $f, g \in F$,

$$f \succcurlyeq g \Leftrightarrow \int_S u\left(f(s)\right) dP(s) \geq \int_S u\left(g(s)\right) dP(s).$$

Savage's Theorem refers to the axiomatization of SEU by P1, P2, P3 together with three axioms that can be left in the background of our analysis, viz. the comparative probability axiom P4, the non-triviality axiom P5, and the continuity axiom P6 (see Savage, 1954 and, for a modern exposition, Thm. 10.1 in Gilboa, 2009). *The Anscombe-Aumann Theorem* refers to a popular alternative axiomatization of the SEU representation, set in a different analytical framework that is a suitably construed mixture space. The axiomatization is in terms of a non-triviality assumption, the von Neumann - Morgenstern axioms of expected utility under risk, and the standard State-Wise Dominance condition (see Anscombe and Aumann, 1963; Thm. 14.1 in Gilboa, 2009).

[6] This is evidently a major restriction. It is in line with Savage's already quoted remark that the STP covers "certain formal postulates *well articulated with P1*" (1954, p. 22; emphasis added).

[7] For instance, an SEU model with ordinally state-dependent utilities satisfies P2 but not P3 (more details in, e.g., Baccelli, 2017, Sec. 2.3) and conversely, a Non-Expected Utility Probabilistically Sophisticated model (as defined in Machina and Schmeidler, 1992) satisfies P3 but not P2.

2.2 Results

2.2.1 From the STP to P2 and P3, and back

In this subsection we explain how, inspired by Savage's informal STP, one may see each of P2 and P3 as a dominance condition (Prop. 1). This clarifies how the STP supports both P2 and P3. Conversely, we also explain how these dominance conditions help formalizing the STP (Prop. 2). This clarifies how P2 and P3 can harness the full power of the STP.

Recall the reasoning of Savage's businessman quoted in the introduction. The core intuition is one of case-by-case reasoning. Now consider the following two axioms. They are dominance conditions—hence their being here labelled (like any other dominance condition to come) as D-axioms. They correspond to different ways of cashing out what it means to "kn[o]w" (Savage, 1954, p. 21) that an event occurs, and thus of sustaining case-by-case reasoning. In a nutshell, in the first case, knowing that an event obtains means suitably conditioning on it, while in the second case, it implies that uncertainty has been fully resolved.[8]

D2. *For all $f, g, h \in F$, partition $\{E_1, \ldots, E_n\}$ of S,*

i. *if $fE_ih \succcurlyeq gE_ih$ for all E_i, then $f \succcurlyeq g$;*

ii. *if in addition $fE_ih \succ gE_ih$ for some E_i, then $f \succ g$.*[9]

D3. *For all $f, g \in F$, partition $\{E_1, \ldots, E_n\}$ of S,*

i. *if $f(E_i) \succcurlyeq g(E_i)$ for all E_i, then $f \succcurlyeq g$;*

ii. *if in addition $f(E_i) \succ g(E_i)$ for some non-null E_i, then $f \succ g$.*

D3 is essentially Strong State-Wise Dominance, i.e., State-Wise Dominance (D3.i)—as in, say, the Anscombe-Aumann Theorem—enriched with a strict clause (D3.ii). Apart from the intended parallelism with D2, the main reason for the non-traditional phrasing adopted here is that a *state*-wise dominance property cannot be non-trivially enriched with a strict clause of the same kind when the state space is infinite and all states must be null—as is the case in, e.g., Savage's Theorem. By contrast, as D3 illustrates, a similarly inspired *event*-wise dominance property can be so enriched. D2 on the other hand, despite being a familiar condition, seems to have no established name in the literature.[10] The key point is that as the parallelism between the two axioms makes transparent, D2 is, like D3, a dominance or a monotonicity

[8] As regards the second axiom, recall that for any $E \in \Sigma$, $f \in F$, we write $f(E)$ if and only if f is constant over E, and that we also let $f(E)$ denote the constant consequence of f on E.

[9] In D2.ii, there is no need to assume that E_i be non-null; this follows from $fE_ih \succ gE_ih$.

[10] For some of the issues that arise in connection with infinitary variants of D2, see for instance Seidenfeld and Schervish, 1983.

property. While the latter expresses *ex post* dominance, the former arguably expresses *interim* dominance. This is in the sense that it effectively applies case-by-case reasoning to partial, rather than full, resolutions of uncertainty.

Now, recalling that P1 is assumed throughout our analysis, consider our first main results, viz. the equivalences in Prop. 1.[11,12]

Proposition 1.

a. P2 holds if and only if D2 holds;

b. P3 holds if and only if D3 holds.

Proof. See the Appendix. □

In light of Prop. 1, assuming P2 or D2—respectively, P3 or D3—in the statement of Savage's Theorem is merely a "matter of taste" (Savage, 1954, p. 26). There could be principled methodological reasons to choose one way rather than the other, however. For instance, the dominance format D2 and D3 clearly is the one under which the required properties look most attractive from a normative point of view. On the other hand, one may speculate that to that format, Savage preferred the one that most closely corresponds to the role the properties play in the proof of the existence of the SEU representation—the initial format P2 and P3. Incidentally, it is also the format under which the axiom are the easiest to test.

Be that all as it may, two further comments on Prop. 1 are in order. First, D2.i is demonstrably equivalent to D2.[13] Accordingly, in Prop. 1.a, D2.i could be stated equivalent to P2. By contrast, no similar analysis applies to D3, D3.i, and P3, since the second of these properties is strictly weaker than the first.[14] One of our additional results in the Appendix (Lemma 3) identifies the weakening of P3 to which D3.i is equivalent.

[11]Prop. 1, hence also Prop. 2, holds for general acts.

[12]Occasionally, one can find in the literature the claim that the STP, construed exclusively as P2, is "stronger" than State-Wise Dominance (see, e.g., Mongin and Pivato, 2016, p. 727). In light of Prop. 1 and the already noted logical independence of P2 and P3 (see fn. 7), that extends to P2 and D3.i alone, these claims call for interpretation. They should not be understood as logical statements, but as informal comparisons of normative strength. A widely received view is indeed that while P2 is—among the properties specific to uncertainty—the most contentious requirement of rationality, State-Wise Dominance is essentially uncontroversial. In light of Prop. 1, the latter claim is debatable inasmuch as one can debate that ordinal state-independence is a requirement of rationality (more on this in, e.g., Karni, 2008).

[13]This follows from the fact that each of D2.i and D2 proves equivalent to P2. Further, D2 proves equivalent to the special case of D2.i referring to a binary partition.

[14]For instance, Maxmin Expected Utility (Gilboa and Schmeidler, 1989) and Choquet Expected Utility (Schmeidler, 1989; Gilboa, 1987) satisfy D3.i but in general not D3.

Second, it is instructive to compare the results in Prop. 1 to those in Sec. 2.7 of Savage, 1954.[15] Savage proves that P2 $\Rightarrow$ D2 (1954, Thm. 2) but, whatever the reason, he does not discuss the converse. We have not found the full equivalence elementarily proved in the literature; but it is undoubtedly a folk theorem. We merely extend Savage's proof in the obvious way.[16] On the other hand, Savage does prove the full equivalence P3 $\Leftrightarrow$ D3 (1954, Thm. 3)—an equivalence of which the larger literature is clearly aware of.[17] Yet Savage suggests, and we have not found it denied in the literature, that it holds only when P2 is assumed, while our result shows that P2 is not needed—an important aspect on which our next subsection will elaborate.

Meanwhile, Prop. 1 helps one discern how P2 and P3 can be explicitly combined in a unified condition expressing, as much as can be within Savage's framework, the full STP—a simple question which we have not seen raised in the literature. While based on the original conditions P2 and P3, it is unclear how such a unification could be achieved, it is entirely transparent based on their equivalent dominance format D2 and D3. As registered in Prop. 2 below, the answer—simplified as much as possible—is given by D4, the hybrid dominance condition stated next. The condition is hybrid in the sense that, much like Savage with his initial businessman example, it leaves open the exact kind of dominance reasoning employed.

D4. *For all $f, g, h \in F$, partition $\{E_1, \dots, E_n\}$ of S,*

i. *if for all E_i either $fE_ih \succcurlyeq gE_ih$ or $f(E_i) \succcurlyeq g(E_i)$, then $f \succcurlyeq g$;*

ii. *if in addition $f(E_i) \succ g(E_i)$ for some non-null E_i, then $f \succ g$.*

Proposition 2. *P2 and P3 hold if and only if D4 holds.*

Proof. Immediate from Prop. 1. □

Since it combines P2 and P3, D4 covers patterns of dominance reasoning justified by neither axiom alone. For instance, for any $f, g, h \in F$, $E \in \Sigma$, D4 justifies concluding $f \succcurlyeq g$ from $fEh \succcurlyeq gEh$ and $f(\overline{E}_i) \succcurlyeq g(\overline{E}_i)$ for each

[15] Savage focuses on the strengthenings of D2 and D3 featuring conditional antecedents and consequents. These strengthenings are only apparent, however, for the conditional—as in Savage—and unconditional—as in our paper—conditions are demonstrably equivalent. (This follows from the fact that they each prove equivalent to P2 and P3, respectively.) Accordingly, we ignore this difference in the clarifications to follow.

[16] Contrast our simple proof and the more involved treatments in, e.g., Marschak, 1986 (in a generalization of Savage's framework), LaValle, 1992 (over decision trees), or Zimper, 2008 (in the context of decision-making under risk). As these papers illustrate, and as recently emphasized among others by Li et al. (2023), the fact that separability is equivalent to a form of dominance or monotonicity has been known for a long time.

[17] For instance, witness the different names (viz. "State-Wise Dominance", "Monotonicity," or "State-Independence") which the literature has given to the last axiom in the Anscombe-Aumann characterization of SEU.

cell $\overline{E}_i$ of some partition $\{\overline{E}_1, \ldots, \overline{E}_n)\}$ of $\overline{E}$—a particularly useful schema. Taken in isolation from one another, neither P2 nor P3 could support that conclusion. In terms of Savage's initial businessman example, the above pattern of dominance reasoning could correspond to the following scenario. Assume that there is not only a Democrat and a Republican, but also an Independent candidate. For simplicity, further assume that the election of either candidate would determine exactly one consequence for the act of buying or not buying the property. Let E_1 (respectively: E_2; E_3) denote the event that the Republican (respectively: the Democrat; the Independent) candidate wins, and let f (respectively: g) denote the act of buying (respectively: not buying) the property. For instance because of the tax policy to which the Republican candidate would have committed, it might be that the businessman has the preference $f(E_1) \succcurlyeq g(E_1)$. While for similar reasons he may well strictly prefer $g(E_2)$ to $f(E_2)$ but $f(E_3)$ to $g(E_3)$, it might also be that for some other business venture h, he has the preference $fE_2 \cup E_3h \succcurlyeq gE_2 \cup E_3h$. From these two heterogeneous pieces of data, D4 justifies concluding $f \succcurlyeq g$, which taken alone neither P2 nor P3 could.[18] This also manifests, more generally, that D4 seems flexible enough to capture many possible explications of the businessman example motivating Savage's introduction of the STP.

Finally, since it exactly corresponds to the conjunction of P2 and P3, D4 can replace these axioms in the statement of Savage's Theorem. This is recorded in Cor. 1 below.

Corollary 1. *In the statement of Savage's Theorem, keeping all the other axioms, P2 and P3 can be conjointly replaced by D4.*

Proof. Immediate from Prop. 2. □

2.2.2 The STP: P2, P3, and their intersection

In this subsection we further investigate the internal structure of the STP, starting from the observation that P2 and P3 have an intersection, and that in that sense the STP has a basic core. Given our previous results, our main result in this section (Prop. 3) suggests that three distinct properties underpin the STP. To further highlight the difference with Savage's own P-axioms or the equivalent dominance conditions stated as the D-axioms, we will label these more basic properties as the A-axioms.

Previously (on p. 7), we noted that to show the equivalence of P3 and D3, it is unnecessary to assume P2. To best appreciate why such is the case,

[18] The models mentioned in fn. 7 could be invoked here once again to illustrate this point.

one may observe that the weak implication of P2 stated next, A1, is also implied by P3 alone.

A1. *For all* $x, y \in X$, $E \in \Sigma$, $h, h' \in F$, $xEh \succcurlyeq yEh \Leftrightarrow xEh' \succcurlyeq yEh'$.

A1 weakens P2 by ranging only over acts with constant (instead of general, i.e. variable) non-common parts while it weakens P3 by focusing on only one event at a time (instead of reasoning across non-null events). In neither case is this supposed to isolate an uncontroversial implication of the stronger axiom; this is only to highlight that these stronger axioms do intersect, and that to this extent the STP has a basic core. Instructively, as further noted in the Appendix (Lemma 4), A1 proves equivalent to an especially minimal dominance property (D1), that is transparently a special case of D2 as well as, less transparently, D3.

The fact that A1 is an implication of both P2 and P3[19] naturally raises the following question. How exactly is the former property to be strengthened to obtain either of the latter properties—or both, to reach the full STP? As an inspection of the three axioms makes clear, P2 strengthens the invariance in A1 by requiring that it also hold over acts with variable (non-constant) non-common parts, while P3 strengthens it by requiring, instead, that it also hold across different (non-null) events.[20] The properties characterizing these orthogonal strengthenings are introduced next.[21]

A2. *For any* $f, g, h, h' \in F$, $x, y \in X$, *partition* $\{E_1, E_2, E_3\}$ *of* S, *if either* $fE_1h \succ gE_1h$ *and* $xE_2h \prec yE_2h$, *or* $fE_1h \prec gE_1h$ *and* $xE_2h \succ yE_2h$, *then* $fE_1xE_2h \succcurlyeq gE_1yE_2h \Leftrightarrow fE_1xE_2h' \succcurlyeq gE_1yE_2h'$.

A3. *For all* $x, y \in X$, *essential* $E \in \Sigma$, $xEy \succcurlyeq y \Leftrightarrow yEx \succcurlyeq y$.

As will be seen shortly, A2 captures the part of P2 that P3 cannot deliver. The axiom considers a basic pattern of preference conflict over some event. Indeed, calling E the union of E_1 and E_2, the axiom ranges over acts a, b such that $aE_1h \succ bE_1h$ but $aE_2h \prec bE_2h$ or the other way around—hence our referring to a "preference conflict", viz. over E—with a and b constant on at least one of E_1 and E_2—hence our calling the conflict pattern "basic".[22] What A2 states is that whatever the settlement of that preference conflict (i.e., whether it is a or b that over E the decision-maker prefers), it cannot depend

[19] One may further conjecture that A1 is the strongest property common to, and in that sense the common core of, P2 and P3.

[20] The latter fact may be seen most clearly by considering, instead of P3, the following logically equivalent property: For all $x, y \in X$, non-null $E, E' \in \Sigma$, $h, h' \in F$, $xEh \succcurlyeq yEh \Leftrightarrow xE'h' \succcurlyeq yE'h'$.

[21] As regards the first axiom, recall that for any $f, g, h \in F$, ternary partition $\{E_1, E_2, E_3\}$ of S, we let fE_1gE_2h denote the act equal to f on E_1, g on E_2, and h on E_3.

[22] The example developed after D4 is readily adapted to illustrate exactly such a pattern.

on the common values assigned to a and b outside the event of interest (i.e., h, h', etc. on $\overline{E} = E_3$). In that sense, A2 is a kind of trade-off consistency axiom.[23] P3 being effectively, as D3 illustrates, a unanimity axiom, it has no bearing on such patterns of conflicting preferences. Conversely, A3 captures the part of P3 that P2 cannot deliver. It is an especially minimal ordinal state-independence axiom. The common part of the acts it considers is restricted to one of the two consequences involved in their non-common part (y, in the above notation). Against that background, A3 states only that consequences must be ordered in the same way across any essential event and its complement (viz., for any such event, $xEy \succcurlyeq yEy$ if and only if $x\overline{E}y \succcurlyeq y\overline{E}y$).[24] P2 has no bearing on such issues of cross-event consistency.

These axioms lead us to our next main results, that are the equivalences in Prop. 3.[25]

Proposition 3.

a. P2 holds if and only if A1 and A2 hold;

b. P3 holds if and only if A1 and A3 hold.

Proof. See the Appendix. $\square$

Prop. 3 thus suggests that three distinct ideas underpin the STP as captured by D4, namely, to recap: a property that can be seen as a particularly basic dominance condition (A1); a kind of trade-off consistency requirement (A2); and an especially minimal ordinal state-independence axiom (A3).

Prop. 3 also helps identifying potential simplifications for Savage's and, incidentally, the Anscombe-Aumann axiomatizations of SEU. To prepare for including the latter axiomatization to our analysis, we note that the Appendix (Lemma 5) provides a decomposition of D3.i comparable to the one which, given Prop. 1, Prop. 3 provides for D3. In particular, this decomposition establishes that the part of D3.i which P2 cannot deliver is WA3, the weakening of A3 introduced next. Unlike the stronger A3, WA3 excludes only strict preference reversals (as in $xEy \succ yEy$ but $x\overline{E}y \prec y\overline{E}y$) in the ordering of consequences across one essential event and its complement.

WA3. *For all $x, y \in X$, essential $E \in \Sigma$, $xEy \succ y \Rightarrow yEx \succcurlyeq y$.*

We record the links between D3.i and WA3 as stated in Lemma 1.

Lemma 1. *Assume P2. Then, D3.i holds if and only if WA3 holds.*

[23] Because we find it illuminating here, we thus use freely a phrase established for other purposes elsewhere in the literature; see in particular Köbberling and Wakker, 2003.

[24] The axiom ranges only over essential (rather than over all non-null) events. As the details of the proof of Prop. 3.b indicate, P3 could be immediately thus generalized as well.

[25] Prop. 3.b holds for general acts. Hence so does Cor. 2.b.

Proof. See the Appendix. □

The following corollaries of Prop. 3 and Lemma 1 can now be stated.[26]

Corollary 2.

a. *In the statement of Savage's Theorem, keeping all the other axioms, P2 can be weakened to A2.*

b. *In the statement of Savage's Theorem, keeping all the other axioms, P3 can be weakened to A3.*

c. *In the statement of the Anscombe-Aumann Theorem, keeping all the other axioms, D3.i can be weakened to WA3.*

Proof. Immediate from Prop. 3, Lemma 1, and (for Cor. 2.c) the well known fact that in the Anscombe-Aumann framework, P2 is implied by the Weak Order and the von Neumann - Morgenstern Independence properties alone. □

In light of Cor. 2, and to echo the comments we made after Prop. 1, assuming P2 or A2—respectively, P3 or A3—in the statement of Savage's Theorem is another "matter of taste" (Savage, 1954, p. 26), and a similar remark applies to D3.i, WA3, and the Anscombe-Aumann Theorem.[27]

2.2.3 The STP, P3, and Obvious Dominance

In this subsection we zoom in on one particularly interesting implication of our analysis pertaining to P3, specifically. Given our previous results, our main result in this section (Lemma 2) indicates that starting from P2, there is a surprising way of bridging the full STP, alternative to assuming either the maximal axiom P3 or the minimal axiom A3.

Our starting point is the observation that in principle, variants of Cor. 2 can be obtained with P2 (respectively, P3; D3.i) being replaced by any of its weakening that would still be strong enough to imply A2 (respectively,

[26] As regards Cor. 2.b (and similarly for our later Cor. 3.a.), recall that we here consider Savage's Theorem for simple acts. In this result, unlike in Savage's Theorem for general acts (see Hartmann, 2020), P3 is not redundant. Accordingly, weakening it is not otiose.

[27] Admittedly, it might not always be wise to assume only the leanest axioms possible. For instance, P2 and P3 are arguably more conceptually transparent than A2 and A3. Such is especially true of P2, compared to the less demanding but more cumbersome—our proposed interpretation in terms of trade-off consistency notwithstanding—A2.

A3; WA3). Indeed, an interesting illustration pertaining to P3 involves the dominance axiom stated next.[28]

D5. *For all $f, g \in F$, partition $\{E_1, \ldots, E_n\}$ of S,*

i. *if $f(E_i) \succcurlyeq g(E_j)$ for all E_i, E_j, then $f \succcurlyeq g$;*
ii. *if in addition $f(E_i) \succ g(E_i)$ for some non-null E_i, then $f \succ g$.*

D5 is essentially Obvious Dominance (Li, 2017) suitably enriched with a strict clause (D5.ii). Its weak clause (D5.i) simply states that if the worst case of f is better than the best case of g, then f must be better than g overall.[29] Considered normatively, this makes for an especially compelling "sure-thing principle", if any. In the context of decision theory, the axiom is of interest also because, under appropriate background conditions (see, e.g., Chambers and Echenique, 2016, Sec. 8.4; Zhang and Levin, 2017, Thm. 1), it proves to underpin the existence of an *Act-Dependent* SEU representation. Virtually all non-expected utility models fall within that category (on which see especially Cerreia-Vioglio et al., 2011, Cor. 3).

In the Appendix (Lemma 6), we provide decompositions of D5 and D5.i comparable to the ones previously provided for D3 and D3.i. Based on these decompositions, one may claim what follows.

Lemma 2. *Assume P2. Then,*

a. P3 holds if and only if D5 holds;

b. D3.i holds if and only if D5.i holds.

Proof. See the Appendix. □

The main interest of Lemma 2 may reside in Cor. 3 below, which is the previously announced illustration. Bearing in mind what we previously wrote about Act-*Dependent* SEU, note that the results to follow pertain, by contrast, to the existence of the classical Act-*Independent* SEU representation.

[28] By contrast, for a weakening of one of the two classical axioms P2, P3 that is *not* strong enough to imply the relevant minimal property referred to in Cor. 2, one may mention the co-monotonic restriction of P2, that underlies Choquet Expected Utility. It is too weak to imply WA1 (featured in Lemma 5). Consequently, replacing D3.i by WA3 in an axiomatization of Choquet Expected Utility would lead to a more general model.

[29] D5.i is equivalent to the following unconditional variant of Savage's P7. (In appreciating the axiom, recall once again that we write $f(E)$ if and only if f is constant over E.) For any $f, g \in F$, if $f \succcurlyeq g(E)$ for all $E \in \Sigma$, then $f \succcurlyeq g$, and if $f(E) \succcurlyeq g$ for all $E \in \Sigma$, then $f \succcurlyeq g$.

Corollary 3.

a. *In the statement of Savage's Theorem, keeping all the other axioms, P3 can be weakened to D5.*

b. *In the statement of the Anscombe-Aumann Theorem, keeping all the other axioms, D3.i can be weakened to D5.i.*

Proof. Immediate from Lemma 2 (and the remark about the Anscombe-Aumann framework already made in the proof of Cor. 2). □

Lemma 2 thus also implies that starting from P2, to reach the full force of the STP as captured by D4, it suffices to assume the weak and therefore especially compelling dominance property D5.

3 Conclusion

In this note, we have investigated the diversity and vindicated the unity of the Sure-Thing Principle as originally envisaged by Savage—viz., over simple (finitely-valued) acts, as the combination of his separability postulate P2 and his state-independence postulate P3. We have done so without enriching his framework in any way, be it with primitive conditional preferences, with knowledge operators, or with anything additional to the structural assumptions made in the traditional Savage literature. Among other results, we have shown that each of P2 and P3 is equivalent to a dominance condition (Prop. 1); that they can be explicitly combined in a unified dominance condition that is a candidate formal statement for the full STP (Prop. 2); and that they strengthen in different directions a common, basic dominance axiom (Prop. 3). These results, that hold under Savage's P1 weak order assumption, are visually summarized in Diagram 1, displayed next.

$$
\begin{tikzcd}
\text{A1 and A2} \arrow[d, Leftrightarrow, "\text{Prop. } 3.a"] & & \text{A1 and A3} \arrow[d, Leftrightarrow, "\text{Prop. } 3.b"] & & \\
\text{P2} \arrow[d, Leftrightarrow, "\text{Prop. } 1.a"] & \text{and} & \text{P3} \arrow[d, Leftrightarrow, "\text{Prop. } 1.b"] \arrow[rr, Leftrightarrow, "\text{Prop. } 2"'] & & \text{D4} \\
\text{D2} & & \text{D3} & &
\end{tikzcd}
$$

Diagram 1: Main Results

From these and our other results (see especially our surprising Cor. 3, that involves Obvious Dominance) emerges a clearer conceptual picture of the variety of dominance conditions imposed by Subjective Expected Utility, as well as a better understanding of its axiomatic underpinnings. Directions for future research include, of course, investigating the Sure-Thing Principle over general (possibly infinitely-valued) acts. This is natural since many but not all our proofs hold unchanged for general acts, and because Savage also uses the Sure-Thing Principle to motivate his axiom P7, on which his theorem relies when such acts are at stake. The main result in Hartmann, 2020 may be taken to suggest that in that case, one could focus on Savage's P2 and P7, i.e. ignore P3, on the account that the latter axiom demonstrably follows from the former two. But this is true if one is ready to assume also Savage's P4, which is arguably (also in Savage's own view) unrelated to the Sure-Thing Principle. Another more methodological direction for future research is to further investigate the intersections of the logically independent axioms underpinning Subjective Expected Utility (or its generalizations), as we did here focusing on P2 and P3. For instance, a close inspection of the proof of Hartmann, 2020 reveals that it hinges on the identification of an intersection between Savage's P3 and P4. The rest of Savage's axiomatic system could be systematically revisited, looking for such intersections and the further light they may shed on the foundations of Subjective Expected Utility (or its generalizations).

References

Anscombe, F., and R. Aumann (1963): "A Definition of Subjective Probability," *The Annals of Mathematical Statistics*, 34(1), 199–205.

Baccelli, J. (2017): "Do Bets Reveal Beliefs?," *Synthese*, 194(9), 3393–3419.

Cerreia-Vioglio, S., P. Ghirardato, F. Maccheroni, M. Marinacci, and M. Siniscalchi (2011): "Rational Preferences under Ambiguity," *Economic Theory*, 48(2-3), 341–375.

Chambers, C., and F. Echenique (2016): *Revealed Preference Theory.* New York: Cambridge University Press.

Chew, S. H., and P. Wakker (1996): "The Comonotonic Sure-Thing Principle," *Journal of Risk and Uncertainty*, 12, 5–27.

Chew, S. H., and W. Wang (2022): "Generalizing Obvious Dominance Using the Sure-thing Principle," working paper.

Dietrich, F., A. Staras, and R. Sugden (2021): "Savage's Response to Allais as Broomean Reasoning," *Journal of Economic Methodology*, 28(2), 143–164.

Esponda, I., and E. Vespa (2021): "Contingent Preferences and the Sure-Thing Principle: Revisiting Classic Anomalies in the Laboratory," working paper.

Fleurbaey, M. (2010): "Assessing Risky Social Situations," *Journal of Political Economy*, 118(4), 649–680.

Frahm, G., and L. Hartmann (2023): "Erratum to 'Savage's P3 is Redundant'," *Econometrica*, 91(3), 33–33.

Friedman, M., and L. Savage (1952): "The Expected Utility Hypothesis and the Measurability of Utility," *The Journal of Political Economy*, 60(6), 463–474.

Ghirardato, P. (2002): "Revisiting Savage in a Conditional World," *Economic Theory*, 20(1), 83–92.

Gilboa, I. (1987): "Expected Utility with Purely Subjective Non-Additive Probabilities," *Journal of Mathematical Economics*, 16(1), 65–88.

——— (2009): *Theory of Decision under Uncertainty*. New York: Cambridge University Press.

Gilboa, I., and D. Schmeidler (1989): "Maxmin Expected Utility with Non-Unique Prior," *Journal of Mathematical Economics*, 18(2), 141–153.

Grant, S., A. Kajii, and B. Polak (2000): "Decomposable Choice under Uncertainty," *Journal of Economic Theory*, 92(2), 169–197.

Hartmann, L. (2020): "Savage's P3 Is Redundant," *Econometrica*, 88(1), 203–205.

Jeffrey, R. (1982): "The Sure-Thing Principle," in *PSA: Proceedings of the Biennial Meeting of the Philosophy of Science Association*, vol. 2, pp. 718–730. Cambridge University Press.

Karni, E. (2008): "State-Dependent Utility," in *The Handbook of Rational and Social Choice*, ed. by P. Anand, P. Pattanaik, and C. Puppe, pp. 223–238. Oxford: Oxford University Press.

Köbberling, V., and P. Wakker (2003): "Preference Foundations for Nonexpected Utility: A Generalized and Simplified Technique," *Mathematics of Operations Research*, 28(3), 395–423.

LaValle, I. (1992): "Small Worlds and Sure Things: Consequentialism by the Back Door," in *Utility Theories: Measurements and Applications*, ed. by W. Edward, pp. 109–136. New York: Springer.

Li, C., K. Rohde, and P. Wakker (2023): "The Deceptive Beauty of Monotonicity, and the Million-Dollar Question: Row-First or Column-First Aggregation?," working paper.

Li, S. (2017): "Obviously Strategy-Proof Mechanisms," *American Economic Review*, 107(11), 3257–87.

Machina, M., and D. Schmeidler (1992): "A More Robust Definition of Subjective Probability," *Econometrica*, 60(4), 745–780.

Marschak, T. (1986): "Independence versus Dominance in Personal Probability Axioms," in *Uncertainty, Information, and Communication: Essays in Honor of Kenneth J. Arrow, Volume III*, ed. by W. Heller, R. Starr, and D. Starrett, pp. 129–71. Cambridge: Cambridge University Press.

Mongin, P., and M. Pivato (2016): "Social Evaluation under Risk and Uncertainty," in *The Oxford Handbook of Well-Being and Public Policy*, ed. by M. Adler, and M. Fleurbaey, pp. 711–745. Oxford: Oxford University Press.

Moscati, I. (2016): "Retrospectives: How Economists Came to Accept Expected Utility Theory: The Case of Samuelson and Savage," *Journal of Economic Perspectives*, 30(2), 219–36.

Pearl, J. (2016): "The Sure-Thing Principle," *Journal of Causal Inference*, 4(1), 81–86.

Samet, D. (2022): "The Impossibility of Agreeing to Disagree: An Extension of the Sure-Thing Principle," *Games and Economic Behavior*, 132, 390–399.

Savage, L. (1954): *The Foundations of Statistics*. New York: Wiley (First Edition).

——— (1972): *The Foundations of Statistics*. New York: Dover (Second Edition).

Schmeidler, D. (1989): "Subjective Probability and Expected Utility without Additivity," *Econometrica*, 57(3), 571–587.

Seidenfeld, T., and M. Schervish (1983): "A Conflict Between Finite Additivity and Avoiding Dutch Book," *Philosophy of Science*, 50(3), 398–412.

Zhang, L., and D. Levin (2017): "Bounded Rationality and Robust Mechanism Design: An Axiomatic Approach," *American Economic Review*, 107(5), 235–239.

Zimper, A. (2008): "Revisiting Independence and Stochastic Dominance for Compound Lotteries," *The BE Journal of Theoretical Economics*, 8(1), 1–11.

4 Appendix

Proof of Proposition 1

Proof.

a. ($\Rightarrow$) Assume $fE_ih \succcurlyeq gE_ih$ for all E_i. By P2, take $h_1 = f$ and, for all $i \in \{2, \dots, n\}$, $h_i = gE_1 \cup \dots \cup E_{i-1}f$ to conclude by P1 that $f \succcurlyeq g$, with a strict consequent if any of the antecedents is strict. In more detail, the first preference reads as $f \succcurlyeq gE_1f$, the second as $gE_1f \succcurlyeq gE_1 \cup E_2f$, and so on, with the final one reading as $gE_1 \cup \dots E_{i-1}f \succcurlyeq g$, so that $f \succcurlyeq g$ (or, if any of the preceding preferences is strict, $f \succ g$) follows by P1.

($\Leftarrow$) Assume $fEh \succcurlyeq gEh$. By P1, $hEh' \succcurlyeq hEh'$. Hence, by D2, $fEh' \succcurlyeq gEh'$.

b. ($\Rightarrow$) Assume $f(E_i) \succcurlyeq g(E_i)$ for all E_i. By P3 or the definition of null events, $f(E_i)E_ih_i \succcurlyeq g(E_i)E_ih_i$ for all E_i, with a strict preference if $f(E_i) \succ g(E_i)$ and E_i is non-null, and $\{h_1, \dots, h_n\}$ any collection of acts. So take $h_1 = f$ and, for all $i \in \{2, \dots, n\}$, $h_i = gE_1 \cup \dots \cup E_{i-1}f$ to conclude by P1 that $f \succcurlyeq g$, with a strict preference $f \succ g$ if $f(E_j)E_jh_j \succ g(E_j)E_jh_j$ for some E_j.

($\Leftarrow$) [$\Rightarrow$] Assume $x \succcurlyeq y$. By D3, $xEf \succcurlyeq yEf$ for any E, therefore for any non-null E in particular. [$\Leftarrow$] Assume $xEf \succcurlyeq yEf$ for some non-null E. If $y \succ x$, by D3, $yEf \succ xEf$, a contradiction. Hence, it must be the case that $x \succcurlyeq y$.

$\square$

Proof of Lemma 3

LRP3. *For all $x, y \in X$, non-null $E \in \Sigma$, $h \in F$, $x \succcurlyeq y \Rightarrow xEh \succcurlyeq yEh$.*

Lemma 3. *LRP3 holds if and only if D3.i holds.*

Proof.

(⇒) Similar to the proof of (⇒) in Prop. 1.b.

(⇐) We prove the contrapositive. Assume $yEh \succ xEh$. If $x \succcurlyeq y$, then by D3.i, $xEh \succcurlyeq yEh$, a contradiction. Hence, $y \succ x$.

□

Proof of Lemma 4

D1. *For all $f, g, h \in F$, partition $\{E_1, \ldots, E_n\}$ of S,*

i. *if $f(E_i)E_ih \succcurlyeq g(E_i)E_ih$ for all E_i, then $f \succcurlyeq g$;*
ii. *if in addition $f(E_i)E_ih \succ g(E_i)E_ih$ for some E_i, then $f \succ g$.*

Lemma 4. *A1 holds if and only if D1 holds.*

Proof.

(⇒) Similar to the proof of (⇒) in Prop. 1.a.

(⇐) Assume $xEf \succcurlyeq yEf$. From this assumption and the fact that $fEx \succcurlyeq fEx$ by P1, it follows from D1 that $x \succcurlyeq yEx$. Now assume $yEg \succ xEg$ for a contradiction. From this assumption and the fact that $gEx \succcurlyeq gEx$ by P1, it follows from D1 that $yEx \succ x$, a contradiction. Thus, it must be the case that $xEg \succcurlyeq yEg$.

□

Proof of Proposition 3

Proof.

a. (⇒) Trivial.

(⇐) The proof is by induction on the number of consequences that are possibly non-common between fEh and gEh.

Induction basis. The base case is $fEh \succcurlyeq gEh$ with f and g having one possibly non-common consequence. It follows from A1 that $fEh' \succcurlyeq gEh'$.

Induction step. Assume P2 holds for all acts having n possibly non-common consequences. Consider the case where $n + 1$ consequences are possibly non-common. So assume $fEh = aE_1xE_2h \succcurlyeq bE_1yE_2h = gEh$, with $\{E_1, E_2\}$ a partition of E and a and b having n possibly non-common consequences on E_1. Either $aE_1h \succcurlyeq bE_1h$ or $xE_2h \succcurlyeq yE_2h$ holds, for otherwise by P1 and the induction hypothesis $bE_1yE_2h \succ aE_1xE_2h$ would follow, a contradiction. If both $aE_1h \succcurlyeq bE_1h$ and $xE_2h \succcurlyeq yE_2h$ hold, then $aE_1xE_2h' \succcurlyeq bE_1xE_2h'$ by P1 and the induction hypothesis. If $aE_1h \prec bE_1h$ and $xE_2h \sim yE_2h$ or if $aE_1h \sim bE_1h$ and $xE_2h \prec yE_2h$, then by P1 and the induction hypothesis $bE_1yE_2h \succ aE_1xE_2h$, a contradiction as above. So the only remaining cases to consider are $aE_1h \prec bE_1h$ and $xE_2h \succ yE_2h$, and $aE_1h \succ bE_1h$ and $xE_2h \prec yE_2h$. In both cases A2 implies from $aE_1xE_2h \succcurlyeq bE_1yE_2h$ that $aE_1xE_2h' \succcurlyeq bE_1yE_2h'$. Thus $fEh' \succcurlyeq gEh'$ holds in all cases.

b. ($\Rightarrow$) Trivial.

($\Leftarrow$)

[$\Rightarrow$] Assume $yEf \succ xEf$. Consider first the case where E is essential. A1 then implies both $y \succ xEy$ and $yEx \succ x$. From the latter, A3 implies $xEy \succ x$. The former and P1 then imply $y \succ x$. Consider next the case where $\overline{E}$ is null. Then by definition for all f, g, h, $f\overline{E}h \sim g\overline{E}h$. Thus it holds in particular that $y \sim yEf$ and $x \sim xEf$, so that given $yEf \succ xEf$, by P1, $y \succ x$ also follows.

[$\Leftarrow$] Assume $xEf \succcurlyeq yEf$ with E non-null. If E is essential, A1 implies both $x \succcurlyeq yEx$ and $xEy \succcurlyeq y$. From the latter, A3 implies $yEx \succcurlyeq y$. The former and P1 then imply $x \succcurlyeq y$. Consider next the case where $\overline{E}$ is null. Then by definition for all f, g, h, $f\overline{E}h \sim g\overline{E}h$. Thus it holds in particular that $x \sim xEf$ and $y \sim yEf$, so that given $xEf \succcurlyeq yEf$, by P1, $x \succcurlyeq y$ also follows.

□

Proof of Lemma 1

Proof. Immediate from Lemma 5 below.

□

WA1. For all $x, y \in X$ s.t. $x \succcurlyeq y$, $E \in \Sigma$, $h, h' \in F$, $xEh \succcurlyeq yEh \Rightarrow xEh' \succcurlyeq yEh'$.

Lemma 5. *D3.i holds if and only if WA1 and WA3 hold.*

Proof.

($\Rightarrow$) Trivial.

(⇐) Consider the basic case where where f and g differ on at most one event of the partition. So consider $x, y \in X$, $f \in F$ and $E \in \Sigma$, and assume $x \succcurlyeq y$. P1 implies that either $x \succcurlyeq xEy$ or $xEy \succcurlyeq y$ must hold, for otherwise $y \succ x$ would hold. Assume that $xEy \succcurlyeq y$ holds. Then WA1 implies $xEf \succcurlyeq yEf$. Assume that $x \succcurlyeq xEy$ holds. If $x \sim xEy$, then by P1 $xEy \succcurlyeq y$ follows, so that again $xEf \succcurlyeq yEf$ follows from WA1. If $x \succ xEy$, then $x \succcurlyeq yEx$ follows either from the definition of null events, if E is null, or from WA3 (contraposed, with substitution of variables), if E is non-null. WA1 then implies $xEf \succcurlyeq yEf$. So $x \succcurlyeq y$ implies $xEf \succcurlyeq yEf$ in the above basic case. From that, the general case follows by iteratively using P1 finitely may times. Specifically, the baseline argument delivers $f \succcurlyeq gE_1f$, then $gE_1f \succcurlyeq gE_1 \cup E_2f$, and so on, finally $gE_1 \cup \dots E_{i-1}f \succcurlyeq g$, so that $f \succcurlyeq g$ follows from P1.

□

Proof of Lemma 2

Proof. Immediate from Lemma 6 below.

□

A4. *For any $E \in \Sigma$, for any $f, g \in F$ and $x, y \in X$ such that for all $A, B \in \Sigma$, $f(A) \succcurlyeq x \succcurlyeq y \succcurlyeq g(B)$, $fEx \succcurlyeq gEx \Leftrightarrow fEy \succcurlyeq gEy$ and $gEx \succcurlyeq fEx \Leftrightarrow gEy \succcurlyeq fEy$.*

WA4. *For any $E \in \Sigma$, for any $f, g \in F$ and $x, y \in X$ such that for all $A, B \in \Sigma$, $f(A) \succcurlyeq x \succcurlyeq y \succcurlyeq g(B)$, $fEx \succcurlyeq gEx \Leftrightarrow fEy \succcurlyeq gEy$.*

Lemma 6.

a. *D5 holds if and only if A4 and A3 hold;*

b. *D5.i holds if and only if WA4 and WA3 hold.*

Proof.

b. (⇒) Consider first WA3. Assume $xEy \succ y$ with E essential. D5.i (contraposed) and P1 imply $x \succ y$, so that D5.i implies $yEx \succcurlyeq y$. Consider next WA4. On the assumptions of the axiom, D5.i implies the conjunction, hence a fortiori the equivalence, of $fEx \succcurlyeq gEx$ and $fEy \succcurlyeq gEy$.

(⇐) For an act $h = (x_1, E_1; x_2, E_2; \dots; x_n, E_n)$ with (without loss of generality) $x_1 \succcurlyeq x_2 \succcurlyeq \cdots \succcurlyeq x_n$, we say that the *weak internality condition* holds if $x_1 \succcurlyeq h \succcurlyeq x_n$. To show D5.i, it suffices to show that every act satisfies the weak internality condition. We show that the weak internality condition holds via induction on the size of the partition of the state space with respect to which the act is defined.

Induction base: We need to show that for all $x, y \in X$, if $x \succcurlyeq y$ then for all $E \in \Sigma$, we have $x \succcurlyeq xEy \succcurlyeq y$.

Case 1: $x \sim y$. Assume, for a contradiction, $xEy \succ y$. WA4 then implies $x \succ yEx$ hence $x \succcurlyeq yEx$. Also from $xEy \succ y$, WA3 (or the definition of null events if $\overline{E}$ is null) implies $yEx \succcurlyeq y$. As $x \sim y$, $x \succcurlyeq yEx$ together with $yEx \succcurlyeq y$ implies by P1 that $xEy \sim y$, a contradiction to the assumption we started with. Thus, $y \succcurlyeq xEy$ must hold. The claim $xEy \succcurlyeq y$ is proved similarly. Thus, $xEy \sim y$, so that by P1 and $x \sim y$, $x \succcurlyeq xEy \succcurlyeq y$.

Case 2: $x \succ y$. By P1 either $x \succ yEx$ or $yEx \succ y$ must hold, for otherwise $y \succcurlyeq x$ would follow. If $x \succ yEx$ holds, then $x \succcurlyeq xEy$ follows either from the definition of null events, if $\overline{E}$ is null, or from WA3 (contraposed, with substitution of variables), if $\overline{E}$ is non-null. Also from $x \succ yEx$, WA4 implies $xEy \succcurlyeq y$ (for if not, WA4 contraposed would imply $yEx \succ x$, a contradiction). If $yEx \succ y$ holds, then WA4 implies $x \succ xEy$ hence $x \succcurlyeq xEy$ more generally. Also from $yEx \succ y$, WA3 (or the definition of null events) implies $xEy \succcurlyeq y$. Thus, the weak internality condition always holds.

Induction step: We need to show that if for all acts defined with respect to an i-ary partition of the state space, the weak internality condition holds, then it also holds for all acts defined with respect to an $i+1$-ary partition of the state space. So consider any act $h = (x_1, E_1; x_2, E_2; \ldots; x_i, E_i; x_{i+1}, E_{i+1})$ such that $x_1 \succcurlyeq \cdots \succcurlyeq x_{i+1}$. We need to show $h \succcurlyeq x_{i+1}$. ($x_1 \succcurlyeq h$ is similar.) The induction step implies that $x_i E_{i+1} h \succcurlyeq x_i$. As $x_i \succcurlyeq x_{i+1}$, the induction base implies that $x_i \succcurlyeq x_i E_{i+1} x_{i+1}$. Thus, P1 implies that $x_i E_{i+1} h \succcurlyeq x_i E_{i+1} x_{i+1}$. As by the induction step and P1 $x_i E_{i+1} h \succcurlyeq x_i \succcurlyeq x_{i+1}$, by WA4, $h \succcurlyeq x_{i+1}$ follows (for if not, WA4 contraposed would imply $x_i E_{i+1} x_{i+1} \succ x_i E_{i+1} h$, a contradiction).

a. (⇒) Consider first A3. Assume $xEy \succcurlyeq y$ with E essential. If $y \succ x$, then by D5.ii $y \succ xEy$ would follow. So $x \succcurlyeq y$ must hold, so that by D5.i $yEx \succcurlyeq y$ follows. The converse is proved similarly.

Consider next A4. Given we already know that D5.i implies WA4, only the second clause of A4 needs arguing for here. We show $fAx \succ gAx \Rightarrow fAy \succ gAy$. (The converse is similar.) To show this, the following claim is key. Under the assumptions of the axiom, if $fAx \succ gAx$, then with $\{E_1, \ldots, E_n\}$ the coarsest partition of S common to f and g, there exists some E_i, with $E_i \cap A \neq \varnothing$ and $E_i \cap A$ non-null, such that $f(E_i) \succ g(E_i)$. Assume otherwise. Then, denoting $\{E_k, \ldots, E_m\}$ the elements of the partition having non-empty intersection with A, for all $i = k, \ldots, m$, either $g(E_i) \succcurlyeq f(E_i)$, or $f(E_i) \succ g(E_i)$ but E_i is null. We start by showing that the case where $g(E_i) \succcurlyeq f(E_i)$ for all $i = k, \ldots, m$ is impossible. As we know that $f(E_i) \succcurlyeq g(E_j)$ for all $i, j = 1, \ldots, n$, this case would imply that $g(E_i) \succcurlyeq f(E_i) \succcurlyeq g(E_j) \succcurlyeq f(E_j)$ hence $g(E_i) \succcurlyeq f(E_j)$ for all $i, j = k, \ldots, m$. Besides, as $f(E_i) \succcurlyeq x$ for all $i = 1, \ldots, n$ hence $g(E_i) \succcurlyeq x$ for all $i = k, \ldots, m$, it would thus hold that for all $E, E' \in \{E_k, \ldots, E_m, A^c\}$, $(gAx)(E) \succcurlyeq (fAx)(E')$. Thus by D5.i $gAx \succcurlyeq fAx$ would follow, contradicting that $fAx \succ gAx$. So for our assumption to hold, it must be that for some $i = k, \ldots, m$, $f(E_i) \succ g(E_i)$, but any such E_i is null. We show that this case also leads to a contradiction. Let A_1 stand for the union of all the indices satisfying the preceding condition, with A_2 its complement with respect to A. (If A_2 is empty, then by definition of null events the contradiction with $fAx \succ gAx$ is immediate.) Then by definition of null events, with z the worst consequence obtained under g over S, $zA_1fA_2x \sim fAx$. But following the same reasoning as in the first case, for all $E, E' \in \{E_k, \ldots, E_m, A^c\}$, $(gAx)(E) \succcurlyeq (zA_1fA_2x)(E')$. Thus by D5.i $gAx \succcurlyeq zA_1fA_2x$ would follow, again contradicting the assumption that $fAx \succ gAx$ (since $zA_1fA_2x \sim fAx$ has already been established). So there must be some E_i having non-empty and non-null intersection with A such that $f(E_i) \succ g(E_i)$. Thus, together with the assumption that $f(E_i) \succcurlyeq y \succcurlyeq g(E_j)$ for all $i, j = 1, \ldots, n$, by D5.ii, $fAy \succ gAy$ follows, as was to be shown.

($\Leftarrow$) Given that A3 $\Rightarrow$ WA3 and A4 $\Rightarrow$ WA4, in light of Lemma 6.b, only the strict clause of D5 needs arguing for.

Consider two acts $f, g \in F$, with their coarsest common partition $\{E_1, \ldots, E_n\}$ of S, such that (first half of the assumption) $f(E_i) \succcurlyeq g(E_j)\ \forall i, j \in \{1, \ldots, n\}$ and assume that (second half of the assumption) there exists a non-null $k \in \{1, \ldots, n\}$ such that $f(E_k) \succ g(E_k)$. Assume first that $f(E_i) \succ g(E_j)$ for all $i, j \in \{1, \ldots, n\}$. Then from P1, A3 $\Rightarrow$ WA3 and A4 $\Rightarrow$ WA4, and Lemma 6.b, one can conclude $f \succ g$ without even considering the second half of the assumption. So assume $f(E_i) \sim g(E_j)$ for some $i, j \in \{1, \ldots, n\}$ and consider now the second half of the assumption. From $f(E_i) \sim g(E_j)$, it must be that for all $l, m \in \{1, \ldots, n\}$, $f(E_l) \succcurlyeq f(E_i)$ and $g(E_i) \succcurlyeq g(E_m)$, for otherwise $g(E_m) \succ f(E_l)$ would follow, in contradiction with the assumption that $f(E_i) \succcurlyeq g(E_j)\ \forall i, j \in \{1, \ldots, n\}$. Furthermore, from $f(E_k) \succ g(E_k)$, it must be that $f(E_k) \succ f(E_i)$ or $g(E_j) \succ g(E_k)$, for otherwise $g(E_k) \succcurlyeq f(E_k)$ would follow. Consequently, to establish the strict conclusion of D5, it suffices to establish that the following *strict internality* condition holds: For any act $h = (x_1, E_1; x_2, E_2; \ldots; x_n, E_n)$, with $\{E_1, \ldots, E_n\}$ all non-null and $x_1 \succcurlyeq x_2 \succcurlyeq \cdots \succcurlyeq x_n$, if $x_1 \succ x_n$, then $x_1 \succ h \succ x_n$. In proving the condition, all the E_i can indeed be taken non-null without loss of generality. For instance, if E_1 is null, by definition of null events, the act $h = (x_1, E_1; x_2, E_2; x_3, E_3; \ldots; x_n, E_n)$ is indifferent to the act $h' = (x_2, E_1 \cup E_2; x_3, E_3; \ldots; x_n, E_n)$. Accordingly, instead of h, defined with respect to a partition of the state space not all the cells of which are non-null, one may always indifferently consider h', defined with respect to a partition of the state space all the cells of which are non-null. Now, under the strict internality condition, if $f(E_k) \succ f(E_i)$, then $f \succ f(E_i)$ so that given that $f(E_i) \succcurlyeq g(E_i)$ and $g(E_i) \succcurlyeq g$ by the weak internality condition, $f \succ g$ follows. Similarly, if $g(E_j) \succ g(E_k)$, then $g(E_j) \succ g$ by the strict internality condition so that given that $f \succcurlyeq f(E_i)$ by the weak internality condition and $f(E_i) \succcurlyeq g(E_j)$, $f \succ g$ follows.

One may establish that the strict internality condition holds by induction on the size of the partition of the state space with respect to which the act is defined.

Induction base: Assume $x \succ y$. We need to show that for any essential $E \in \Sigma$, $x \succ xEy \succ y$. By P1, for any E, either $x \succ xEy$ or $xEy \succ y$ must hold, for otherwise $y \succcurlyeq x$ would follow. Assume $x \succ xEy$. Then (the second clause of) A4 implies $yEx \succ y$. A3 then implies $xEy \succ y$. Assume $xEy \succ y$. Then $x \succ xEy$ similarly follows from A4 and A3. So $x \succ xEy \succ y$ holds in all cases.

Induction step: We need to show that if for all acts defined with respect to an i-ary partition of the state space, the strict internality condition holds, then it also holds for all acts defined with respect to an $i+1$-ary partition of the state space. So consider an act $h = (x_1, E_1; \ldots; x_i, E_i; x_{i+1}, E_{i+1})$, with $\{E_1, \ldots, E_i, E_{i+1}\}$ all non-null, $x_1 \succcurlyeq x_2 \succcurlyeq \cdots \succcurlyeq x_i \succcurlyeq x_{i+1}$, and $x_1 \succ x_{i+1}$. We need to show that $x_1 \succ h \succ x_{i+1}$ holds.

Consider first the claim that $h \succ x_{i+1}$ must hold. By P1, one of $x_1 \succ x_i$ or $x_i \succ x_{i+1}$ must hold, for otherwise $x_{i+1} \succcurlyeq x_1$ would follow. Assume first $x_1 \succ x_i$. By the induction hypothesis, $x_iE_{i+1}h \succ x_i$. By Lemma 6.b, $x_i \succcurlyeq x_iE_{i+1}x_{i+1}$. Hence by P1, $x_iE_{i+1}h \succ x_iE_{i+1}x_{i+1}$. A4 then implies $h \succ x_{i+1}$ (for otherwise, given that $x_i E_{i+1} h \not\succcurlyeq x_i \not\succcurlyeq x_{i+1}$, the second clause of A4 would imply $x_i E_{i+1} x_{i+1} \not\succcurlyeq x_i E_{i+1} h$, a contradiction). Now assume $x_i \succ x_{i+1}$. By the induction hypothesis, $x_i \succ x_i E_{i+1} x_{i+1}$. By Lemma 6.b, $x_i E_{i+1} h \not\succcurlyeq x_i$. Hence by P1, $x_i E_{i+1} h \succ x_i E_{i+1} x_{i+1}$. A4 again implies $h \succ x_{i+1}$.

Consider next the claim that $x_1 \succ h$ must hold. By P1, one of $x_1 \succ x_2$ or $x_2 \succ x_{i+1}$ must hold, for otherwise $x_{i+1} \not\succcurlyeq x_1$ would follow. Assume first $x_1 \succ x_2$. By the the induction hypothesis, $x_2 E_1 x_1 \succ x_2$. By Lemma 6.b, $x_2 \not\succcurlyeq x_2 E_1 h$. Hence by P1, $x_2 E_1 x_1 \succ x_2 E_1 h$. A4 implies $x_1 \succ h$. Now assume $x_2 \succ x_{i+1}$. By the induction hypothesis, $x_2 \succ x_2 E_1 h$. By Lemma 6.b, $x_2 E_1 x_1 \not\succcurlyeq x_2$. Hence by P1, $x_2 E_1 x_1 \succ x_2 E_1 h$. A4 again implies $x_1 \succ h$.

$\square$