Nonseparability and Quantum Chaos[*]

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Conventional wisdom has it that chaotic behavior is either strongly suppressed or absent in quantum models. Indeed, some researchers have concluded that these considerations serve to undermine the correspondence principle, thereby raising serious doubts about the adequacy of quantum mechanics. Thus, the quantum chaos question is a prime subject for philosophical analysis. The most significant reasons given for the absence or suppression of chaotic behavior in quantum models are the linearity of Schrödinger's equation and the unitarity of the time-evolution described by that equation. Both are shown in this essay to be irrelevant by demonstrating that the crucial feature for chaos is the nonseparability of the Hamiltonian. That demonstration indicates that quantum chaos is likely to be exhibited in models of open quantum systems. A measure for probing such models for chaotic behavior is developed, and then used to show that quantum mechanics has chaotic models for systems having a continuous energy spectrum. The prospects of this result for vindicating the correspondence principle (or the motivation behind it, at least) are then briefly examined.

1. Introduction. If quantum mechanics is a fundamental theory, then it should be able to characterize the behavior of systems at the macroscopic level. So, by hypothesis, quantum mechanics should be able to provide an adequate account of measurement processes, since such processes crucially involve macroscopic devices. Of course, the difficulties in obtaining such an account from quantum mechanics has been discussed extensively in the philosophical literature. By contrast, a closely related problem (being based on the same hypothesis), that of obtaining an adequate quantum mechanical account of the classically chaotic behavior of certain macroscopic systems, has received virtually no attention in this literature. The main purposes of this essay are to bring this problem to the attention of the philosophical community, and to propose a method for resolving it. The proposal is programmatic. A specific measure of chaos is characterized for open

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quantum systems. Attention is focused on one particular subclass of open systems, and a mathematical framework is proposed for characterizing their chaotic behavior. But, no specific quantum model of this subclass is shown to exhibit this type of chaos, nor is the connection between the proposed measure and classical measures of chaos demonstrated in this essay. The difficult nature of these problems is made manifest, however.

Some have used the quantum chaos problem to cast doubt on the correspondence principle and, as a result, on the empirical adequacy of quantum mechanics. Roughly, this principle asserts that quantum models of physical systems should become increasingly similar to the corresponding classical models as the size of the physical systems being modeled approaches macroscopic dimensions with empirical equivalence at the macroscopic level. The phrase "approaching macroscopic dimensions" is specified in several different (non-equivalent) ways. The most common are: as the quantum numbers get very large and as Planck's constant effectively goes to zero. The issue of how the principle is to be precisely formulated is an extremely delicate one.[1] It is not addressed in this essay since what matters here is the hypothesis that underlies the principle, which was characterized in the previous paragraph. Macroscopic systems and classical models of these systems exhibit chaotic behavior. Consequently, it should be possible to show that there are quantum models of these systems that do likewise, if quantum mechanics really $(q_1,...,q_N)$ is a fundamental theory.[2]

The usual approach to constructing chaotic quantum models is a "top-down" strategy. Researchers begin with simple classical models that are known to be chaotic, and then quantize the equations of motion by replacing classical functions with the corresponding quantum operators. Three standard paradigms of quantized classically-chaotic models are the periodically kicked rotor, the particle in a stadium, and the Rydberg atom in a strong magnetic field, as indicated in (Berry 1987), (Gutzwiller 1990), (Ford and Mantica 1992), and (Jensen 1992). These models manifest unusual behavior and give rise to some rather striking graphic images. But the authors of those essays cited above each indicate that the behavior that is exhibited by those models is not chaotic.

More generally, it seems considerable effort has failed to yield a single convincing case of a chaotic quantum model. Some, such as Ford and his associates, regard the failure to derive quantum chaos by means of the approach characterized above as grounds for modifying the correspondence

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[1] Two good discussions of the difficulties involved in giving a precise statement of this principle are (Berry 1994) and (Batterman 1992).

[2] The doubts that have been raised as to whether the correspondence principle requires this (Batterman 1991, 1992) are based on how Bohr understood and used it at a certain period, as opposed to how it is now understood and used. Standard accounts of the principle —such as in Section 12 of Chapter 1 of (Messiah 1961) or Chapter 2 of (d'Espagnat 1976)—entail that there must be quantum models that are empirically equivalent to classically chaotic models. That will suffice for the purpose at hand.

principle and even suggest that it serves to show the incompleteness or inadequacy of the quantum formalism (Ford et al. 1990), (Ford and Mantica 1992). To counter such pessimistic conclusions, it is useful to begin with two alleged explanations for the failure. Both seem to indicate that chaotic quantum models are, in principle, impossible. Showing the inadequacies of those explanations will serve to suggest a better explanation; namely, that for the most part researchers have been looking for quantum chaos in the wrong place.

One condition that is regarded by most as necessary for chaos in a classical system is that there be at least one nonlinear component in the equations of motion for the system that couples at least two variables together.[3] Some researchers have concluded from this that all quantum systems must be nonchaotic on the grounds that the equations of motion for quantum systems are assumed to be linear (Berry 1989). But, that conclusion is too hastily drawn because insufficient attention has been given to determining the crucial effect of the nonlinear terms in classical systems, and to determining whether that effect can arise in quantum mechanics without such terms.

An alternative approach, one that is not considered in detail here, is to look for quantum chaos by introducing a nonlinear term in Schrödinger's equation. The feasibility of that approach is questionable due to conflicts with the theory of relativity that seem to arise when such terms are introduced into that equation. A rather general framework for introducing nonlinear terms into quantum mechanics was developed in (Weinberg 1989). It was later discovered independently in (Gisin 1990) and (Polchinski 1991) that the framework allows for superluminal signaling in the EPR situation. Gisin suggests that the assumption of spontaneous collapse might provide a way out, noting that this approach is outside of Weinberg's framework. It is also outside of the framework of this essay—see the concluding section of this essay for an explation of this claim.

In classical mechanics, a linear system always has a separable Hamiltonian (meaning that it can be separated into a sum of Hamiltonians, one element in the sum for each subsystem) and, as a consequence, is nonchaotic. So, a classical Hamiltonian is nonseparable only if there is a nonlinear coupling term. Some coupling terms can be eliminated in classical mechanics by a canonical transformation. Those that cannot be eliminated in that way render the Hamiltonian nonseparable. Thus, the key to chaos in classical systems is better characterized as the nonseparability of the Hamiltonian rather than the nonlinearity of the equations of motion.[4]

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[3] In (Chirikov 1992) it is argued that this is not so for equations that have explicit time-dependence. But, the explicit occurrence of time can be eliminated in such cases, see pp. 33-36 of (Tabor 1989) or pp. 73-76 of (Hilborn 1994). Doing so shows there is a nonlinear term of the appropriate type.

[4] Some may be inclined to think that the key condition is integrability rather than separability. The issue is confusing because "integrability" is used in at least two distinct ways. One is equivalent to separability, the other is more general. The more general use is better for some contexts. For the purposes of this essay, however, the narrower one is better because it is less complicated and the complications introduced in considering the more general notion would serve no purpose here. The three concepts mentioned in this note are discussed more fully in Section 2.

Focusing on the nonseparability of the Hamiltonian rather than the nonlinearity of the equations of motion in connection with chaos in classical systems is an important step in locating quantum systems that are capable of exhibiting chaotic behavior. It should be rather obvious to researchers in the foundations of quantum physics that there are nonseparable quantum Hamiltonians, since that is a necessary condition for deriving the measurement problem. Such Hamiltonians makes it possible for an object-apparatus compound to evolve from a tensor product state to a nonseparable (or equivalently entangled) state.[5]

Another feature of the quantum formalism that might be used to explain why quantum models are in general nonchaotic is that the time evolution described by Schrödinger's equation corresponds to a unitary transformation,[6] meaning that they do not change the angle of separation (the inner product) or the distance (the square modulus of the difference) between vectors that correspond to two distinct quantum states. This invariance is taken to mean that all closed quantum systems are completely insensitive to initial conditions. By contrast, chaotic classical models exhibit extreme sensitivity to initial conditions, meaning that the separation between two classical states diverges exponentially on average. That extreme sensitivity is regarded by most as necessary for classical chaos. Again, it appears that quantum systems cannot exhibit chaotic behavior.

But, there are at least three reasons for not drawing that conclusion based on unitarity. First, only closed quantum systems, meaning those that are for all practical purposes in isolation from others, undergo unitary time-evolution. It is well known that open systems, those that are in significant interaction with other systems, undergo nonunitary time evolution. That point is crucial for the purposes of this essay since it may not be possible to treat quantum systems that are sufficiently close to macroscopic dimensions as closed systems; that is to say, the negligibility of a system's interaction with its environment may vary inversely with the size of the system and become non-negligible as the system approaches macroscopic dimensions (Joos and Zeh 1985). Moreover, that point also dovetails rather nicely with the point made above concerning linearity and nonseparability in quantum mechanics since the Hamiltonian of a compound quantum system is nonseparable if and only if the time evolution of its components is nonunitary. It may also serve to

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[5] Some quantum systems are in a nonseparable state due to permutation symmetries rather than an interaction; so, the two notions of nonseparability (one for Hamiltonians, the other for states) are quite distinct, though somewhat related. To avoid (rather than introduce, it is hoped) further confusion, it is worth mentioning here another notion of separability that is associated with the formalism of quantum mechanics. A Hilbert space can be either separable or nonseparable. In that context the notion of separability is topological. It is not related to the other two. These three notions of separability in quantum mechanics are discussed in more detail in Sections 3-4.

[6] The criticism is developed in Section 12.3 of (Hilborn 1994). Hilborn considers the possibility of avoiding the limitation imposed by unitary time evolution by looking for chaotic behavior in open quantum systems, then gives it a hasty dismissal based on the linearity of quantum mechanics.

explain why researchers have had little success in locating chaos in quantum systems: the time evolution of open quantum systems has received little attention in the quantum chaos literature.

Second, the notion of "state" in quantum mechanics is radically different from the corresponding notion in classical mechanics. In classical mechanics, the state of a closed system is represented by a point in phase space. Phase space has two degrees of freedom for each generalized coordinate. The phase space of a compound system is merely the Cartesian product of the phase spaces of its components. The state of an open system is also represented as a point in phase spaces. In quantum mechanics, the state of a closed system is represented by a unit vector in Hilbert space. Hilbert space only has one degree of freedom for each generalized coordinate. The Hilbert space of a compound system is the tensor product of the Hilbert spaces of its components. The state of an open system cannot be represented by a vector in Hilbert space but it can be represented by a mixed density operator.[7]

It is worth elaborating here on some of the differences in representation between closed and open quantum systems. A unit vector in Hilbert space may be thought of as a point on the surface of a unit hypersphere. Unitary time evolution then corresponds to continuous motion on the surface of the hypersphere. A mixed density operator in Hilbert space may be thought of as corresponding to a point that is inside the hypersphere rather than on its surface. Nonunitary time evolution may be thought of as a continuous variation of the point's distance from the origin as well as its angular speed of rotation. Not all points inside the unit hypersphere correspond to a density matrix, since no point inside the hypersphere corresponding to the completely mixed states (mixed states in which N orthogonal components of the density matrix each have the weight 1/N, where N is the dimension of the Hilbert space) can represent a mixed density matrix.

Third, in light of the other two reasons, it may be necessary to consider alternative formulations of classical mechanics or quantum mechanics, such as a Hilbert space formulation of classical mechanics (Koopman 1931) or a phase space formulation of quantum mechanics (Wigner 1932), in order to determine whether quantum mechanics has an appropriate counterpart to classical chaos, such as sensitive dependency on initial conditions (in the classical sense of exponentially diverging trajectories).[8]

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[7] Some may be inclined not to regard open systems as having a state, but that position is too extreme. The state of a system is the condition of the system with respect to each of its observable physical quantities, and a mixed density operator that is obtained for an open system specifies just that. If, for example, the state of a compound closed system is known, then it is always possible to derive from it a mixed density operator for each of its components. Such operators are sometimes referred to as "improper mixtures," as in Section 7.2 of (D'Espagnat 1976).

[8] It has been suggested that quantum mechanics suppresses chaotic behavior because it makes no sense in quantum mechanics to talk about exponentially diverging trajectories on smaller and smaller scales due to the uncertainty relations (Batterman 1991). But, it is not clear what is to be said about limitations imposed by the uncertainty relations. Bohm's ontological interpretation

Although alternative formulations may facilitate the discovery of those counterparts by placing each theory in the same formal setting (in an attempt to minimize the differences between them), those methods are more difficult to use and are much less familiar than the standard one. It is more reasonable to search for chaotic behavior in open quantum systems using the standard formalism together with a reasonably close analogue to one of the standard indicators of classical chaos such as sensitive dependence on initial conditions, algorithmic complexity, continuous power spectrum, or mixing (Eckmann and Ruelle 1985). After presenting a more technical exposition of the considerations above, I introduce a method for probing open quantum systems for a reasonable analogue to classical chaotic behavior, and then use it to locate quantum chaos.

One other potential source of confusion is worth discussing now. One may be inclined to regard the considerations above as just an elaboration of a point made by Ford and his associates (Ford 1989), (Ford et al 1990). Ford believes that the absence of quantum chaos has nothing to do with linearity or unitarity on the grounds that Liouville's equation in classical statistical mechanics can give rise to chaotic behavior despite its being linear and the corresponding time evolution being unitary. But, the considerations here go considerably beyond Ford's observation. What Ford misses is that Liouville's equation cannot characterize chaotic behavior unless the total energy is the only integral of the motion. If it there are others besides the total energy, then the system is not ergodic; and ergodicity is a necessary condition for chaos in classical statistical mechanics.[9] If the total energy is the only constant of the motion, then the Hamiltonian is nonseparable.

2. Nonseparability and Chaos in Classical Mechanics. Hamilton's formulation of the equations of motion for classical systems is the relevant one for discussing quantum mechanics since Schrödinger's famous equation of motion is modeled on that formulation.[10] The state of a system of particles with 2N degrees of freedom is specified using a set of generalized coordinates $(q_1,\ldots,q_N)$ and a corresponding set of generalized momenta $(p_1,\ldots,p_N)$, where $p_i = \partial L/\partial \dot{q}_i$ and $\sum_{i=1}^{N}(1/2)m_i\dot{q}_i^2 - V(q_1,\ldots,q_N)$. Its time evolution is determined by solving Hamilton's equations which consists of 2N first-order differential equations: $\dot{q}_i = \partial H/\partial p_i$ and $\dot{p}_i = -\partial H/\partial q_i$ for $i = 1,\ldots,N$ and the Hamiltonian H given by $H = \sum_{i=1}^{N}(p_i^2/2m_i) + V(q_1,\ldots,q_N)$.

In most cases, it is impossible to solve Hamilton's equations analytically (Helleman 1980). But, it is possible to do so for a very special class of Hamiltonian systems, the integrable systems.

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makes particle trajectories intelligible within the quantum framework by introducing the idea of a quantum potential (Bohm and Hiley 1993).

[9] See pp. 171-172 of (Helleman 1980), pp. 173-174 of (Tabor 1989), and pp. 215-224 of (Reichl 1980).

[10] See Section 5.3 of (Jammer 1966) or see Section 15.2 of (Merzbacher 1970) for details.

A system having 2N degrees of freedom is integrable if and only if there are *N* constant functions $(f_1,\ldots,f_N)$ of the generalized coordinates and momenta (i.e., constants of the motion) that are differentiable, independent, globally defined, and in involution (i.e., the Poisson bracket of any pair of f_i's is zero). Each constant of the motion may be used to decrease the number of degrees of freedom by one via a canonical transformation, which is characterized below. There are other notions of integrability aside from the one characterized above for Hamiltonian systems that are discussed in the literature, as in (Flaschka et al. 1991). The condition for Hamiltonian systems is referred to in the broader context as "Arnol'd-Liouville integrability." The narrower context will suffice for the purposes here; i.e., "integrability" is short for Arnol'd-Liouville integrability.

A closely related class of systems that also may be solved analytically are the separable systems. A Hamiltonian system of equations that is expressed in terms of the set of generalized coordinates and momenta $(q_1,\ldots,q_N,p_1,\ldots,p_N)$ is separable if and only if there is a set of generalized coordinates and momenta $(Q_1,\ldots,Q_N,P_1,\ldots,P_N)$ such that: (1) each new coordinate Q_i and momentum P_i is an analytic function of the old ones, $Q_i = Q_i(q_1,...,q_N,p_1,...,p_N)$ and $P_i = P_i(q_1,...,q_N,p_1,...,p_N)$, (2) Hamilton's equations has the same form in the new set of coordinates and momenta as it did in the old (which means, by definition, that the transformation is canonical), and (3) the Hamiltonian is an analytic function of the P_i only (effectively eliminating the interactions). Thus, a system having N degrees of freedom is separable (meaning completely separable) if and only if there is a canonical transformation of $H(p,q)$ to $H(P,Q)$ that reduces the system to a set of N one-body problems or, equivalently, $H(P,Q)=\sum_{i=1}^{N} H_i(P_i,Q_i)$.

The relation between integrability and separability is logical equivalence. It is obvious that separability is a sufficient condition for integrability: the partial Hamiltonians of a separated system, that is to say the $H_i(P_i,Q_i)$ obtained by reducing the system to N one-body problems, are the constants of the motion which are in involution with each other. The converse is not so obvious. But, it turns out that the constants of the motion of an integrated system may be used to convert it into a separated system via a canonical transformation: see pp. 169-172 of (Helleman 1980) or pp. 322-325 of (Whittaker 1944). Texts indicating that separability is a more restrictive condition than integrability, as in (Tabor 1989), pp. 70-79, and (Boccaletti and Pucacco 1996), pp. 117-124, are using the broader notion of "integrability" mentioned above.

An example of a separable system is a pair of harmonic oscillators coupled by a spring. The coordinates are $q=(q_1,q_2)$ and the momenta are $p=(p_1,p_2)$. The potential energy for the system is $V(q)=\frac{1}{2}\left(q_1^2+q_2^2\right)-\frac{1}{2}q_1q_2$, and the kinetic energy is $T(p)=\frac{1}{2}\left(p_1^2+p_2^2\right)$. The Hamiltonian $H(q,p)=\frac{1}{2}\left(p_1^2+p_2^2+q_1^2+q_2^2\right)-\frac{1}{2}q_1q_2$ is separable; that is to say, the coupling term $\frac{1}{2}q_1q_2$ can be eliminated by a canonical transformation. That is accomplished by using these sets of generalized coordinates $Q = (Q_1, Q_2)$ and momenta $P = (P_1, P_2)$, where $Q_1 = \frac{1}{\sqrt{2}}(q_1 + q_2)$ and $Q_2 = \frac{1}{\sqrt{2}}(q_1 - q_2)$. In the new coordinates, $V(q)$ is transformed into $V(Q) = \frac{1}{4}Q_1^2 + \frac{3}{4}Q_2^2$ and $T(p)$ is transformed into $T(P) = \frac{1}{2}P_1^2 + \frac{1}{2}P_2^2$. The change of coordinates converts the coupled pair into two independent oscillators, and that shows the separability of the original Hamiltonian; i.e., $H(Q,P) = H(Q_1,P_1) + H(Q_2,P_2)$, with $H(Q_1,P_1) = \frac{1}{2}P_1^2 + \frac{1}{4}Q_1^2$ and $H(Q_2,P_2) = \frac{1}{2}P_2^2 + \frac{3}{4}Q_2^2$.

Separable Hamiltonians give rise to time evolution that is either periodic or quasi-periodic. That is to say, they cannot give rise to chaotic behavior. A canonical example of a nonseparable Hamiltonian that is thought, on the basis of numerical solutions, to be capable of giving rise to chaotic behavior is the Henon-Heiles system,[11] whose time evolution is governed by the Hamiltonian $H(q,p) = \frac{1}{2}\left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) + \frac{1}{2}q_1 q_2^2 - \frac{1}{3}q_1^3$. The system has been studied by fixing the energy, thereby reducing the system to three independent degrees of freedom, and then by examining Poincaré maps for the motion of the system.

A Poincaré map consists of intersection points of a trajectory of the system on the constant energy surface with a suitably oriented plane; one side of the plane is arbitrarily chosen, and only points from which the trajectory emerges with respect to that side of the plane are recorded. Maps of many trajectories for the same energy are often superposed to show concisely the character of the motion. But, there are only three types of maps that arise for Hamiltonian systems.[12] The map may consist of a finite number of points in which case the motion is periodic, or it may consist of an indefinite number of points forming a closed curve in which case the motion is quasiperiodic; and in both cases, there is another constant of the motion besides the energy. The map may also have an indefinite number of points that appear to fill some regions of the plane. Extreme cases of this type are those values of the energy in which each trajectory appears to wander over the entire energy surface; that is, the system appears to be ergodic. But, ergodicity is not the crucial factor. What is crucial is that these wandering trajectories diverge exponentially (Lunsford and Ford 1972). That means there is an extreme sensitivity to initial conditions, and that is one of the key indicators of chaotic behavior in classical systems (other indicators are briefly characterized in Section 4).

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[11] This system was first discussed in (Henon and Heiles 1964). For additional discussion, see pp. 173-175 of (Helleman 1980), pp. 224-227 of (Reichl 1980), pp. 337-342 of (Tabor 1989), or pp. 343-350 of (Hilborn 1994).

3. Nonseparability and Chaos in Quantum Mechanics. The crucial points made in Sections 1 and 2 concerning classical systems are that such systems can exhibit chaotic behavior, that the nonseparability of the Hamiltonian is a necessary condition for that behavior, and that the Hamiltonian must have nonlinear terms to be nonseparable. Since most classical systems have nonseparable Hamiltonians, (parameter dependent) chaotic behavior is a rather ubiquitous feature of those systems. Consequently, there should be many quantum systems that exhibit chaotic behavior, given the correspondence principle, which was briefly characterized in Section 1.

Initially, it seems that quantum mechanics could not characterize systems that exhibit chaotic behavior since it is generally assumed that quantum Hamiltonians must be linear operators.[13] But, that impression is misguided. Quantum Hamiltonians can be nonseparable without having nonlinear terms, and the notion of separability is exactly similar to the one given in Section 2 for classical physics; i.e., the Hamiltonian for N interacting systems is separable if and only if there is a canonical transformation that serves effectively to eliminate the interaction terms, thereby reducing the Hamiltonian to a sum of N Hamiltonians, one for each subsystem.[14] The corresponding time evolution operator derived from the Hamiltonian is a tensor product of N time-evolution operators, in which case tensor product states evolve to other tensor product states.

Examples of nonseparable Hamiltonians for pairs of interacting N-state quantum systems are introduced below. Before doing so, it is worth discussing a related but distinct notion of nonseparability that has received much attention in the literature. The state function of a set of quantum systems can evolve from a tensor product of state functions (one element in the product for each member of the set) to a superposition of tensor product states that cannot be transformed into a tensor product state. The resulting state for the set is said to be nonseparable. It should be clear, given the previous discussion, that such an evolution can take place only if the Hamiltonian for the system is nonseparable. Some quantum systems (any collection of fermions) need not interact to exist in a nonseparable state, the nonseparability being due to permutation symmetries.[15] Such systems cannot evolve from a nonseparable state to a tensor product state due to related

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[12] For dissipative systems, a fourth possibility arises: the map may, for certain values of the set of parameters involved, have a fractal-like structure. Such a map is indicative of the existence of a strange attractor.

[13] An example of a nonlinear operator is L, where $L|\phi_t\rangle$ is expressed in the position representation as $\left(Log|\phi(r,t)|^2\right)\phi(r,t)$.

[14] Canonical transformations in quantum mechanics are unitary transformations. Moreover, a unitary transformation is referred to as a "canonical transformation" whether or not it has a classical canonical analog—see pp. 343-345 of (Merzbacher 1970).

[15] See any textbook discussion of identical particles and permutation symmetries, such as Chapter 9 of (Dirac 1958), pp. 207-224.

symmetry requirements imposed on the Hamiltonian. So, the two notions of nonseparability, one for Hamiltonians the other for state functions, are related but are not equivalent.

When the state of a compound system is nonseparable, it cannot be described by a state function.[16] By contrast, the state of a compound classical system is always separable (Park 1968). That difference is widely regarded as being profound. Nonseparable states are crucially involved in many of the problems that are at the core of foundational research in quantum mechanics: the measurement problem, the EPR paradox, violations of Bell's inequality, and conundrums connected with "identical" particles.

Another reason for thinking that there are no chaotic quantum systems was mentioned in Section 1. The time evolution operator U that is derived from the quantum Hamiltonian H is unitary, and unitary time evolution of a quantum state is completely insensitive to initial conditions; but, extreme sensitivity to initial conditions is generally regarded as necessary for chaotic behavior. The unitarity condition is most obvious when H is time-independent (i.e., for conservative systems). It is then possible to give a simple formal solution to Schrödinger's equation,[17] $i\hbar(d/dt)|\phi_t\rangle = H|\phi_t\rangle$, the formal solution being $|\phi_t\rangle = e^{-iH(t-t')/\hbar}|\phi_{t'}\rangle$. The operator $e^{-iH(t-t')/\hbar}$ is clearly unitary, and is denoted as $U(t,t')$, or more simply as *U*. The insensitivity can be seen by noting that the inner product is invariant under a unitary transformation; i.e., $\langle\psi_t|\phi_t\rangle = \langle\psi_{t'}|\phi_{t'}\rangle$, since $|\phi_t\rangle = U(t,t')|\phi_{t'}\rangle$ and $\langle\psi_t| = \langle\psi_{t'}|U^{-1}(t,t')$. That means the separation between the two points on the unit hypersphere corresponding to those vectors are fixed with respect to one another. Alternatively, suppose that $|\psi_{t'}\rangle = (1+\varepsilon)|\phi_{t'}\rangle$ for some arbitrarily small real ε. The norm of the difference between $|\phi_{t'}\rangle$ and $|\psi_{t'}\rangle$ equals ε^2, and unitary transformations are norm preserving (being a special case of the previous argument).

The considerations above offer important clues as to where to look for chaotic behavior in quantum systems. If a nonseparable Hamiltonian is crucial for chaos and unitary time evolution is completely insensitive to initial conditions whether or not the quantum Hamiltonian is separable, then there must be some other way of exploiting nonseparability. The key is to examine features of the time evolution of the state of a component of an interacting pair of quantum systems whose time

[16] Of course, the component systems may each be described by density operators that are derived from the state vector for the compound system. It is the time evolution of density operators corresponding to the component of a pair of interacting systems that will be of substantial interest below in connection with the quantum chaos question.

[17] For the time-dependent case, the operator may be formally represented as a product of infinitesimal unitary operators, which is a unitary operator. See pp. 310-312 of (Messiah 1961), or other graduate-level texts.

evolution is described by a nonseparable Hamiltonian. In such cases, the time evolution of the component is not unitary (nor is it a similarity transformation, more generally). A system that interacts with another in this way is referred to as an "open system." A measure for characterizing the behavior of such systems is proposed below. It is first used to characterize the behavior of a component of a pair of interacting systems having discrete spectra. The proposed measure indicates that the behavior of these systems is no worse than quasiperiodicity. It is then applied to a component of a pair of interacting systems having continuous spectra, and indicates that most quantum models of this sort are chaotic. That serves as a partial vindication of the correspondence principle. What remains is a challenge: to show that quantum chaos is responsible for classical chaos.

The simplest compound quantum system is an interacting pair s_1+s_2 of nonidentical two-state systems, s_1 and s_2. If the Hamiltonian is nonseparable, the time-evolved state of s_1+s_2 will be nonseparable most of the time. That means the states of s_1 and s_2 are mixed states most of the time. So, it is appropriate to use density operator notation. Let the initial state of s_1+s_2 be $\rho = |\phi\rangle\langle\phi|$, where $|\phi\rangle = (r_1 \quad r_2 \quad r_3 \quad r_4)^{\mathrm{T}}$ and the Hamiltonian H be the diagonal matrix whose set of elements on the diagonal is the set of eigenvalues $\{a_1, a_2, a_3, a_4\}$ of H. The Hamiltonian is separable iff there is a set of four numbers $\{\alpha, \beta, \gamma, \delta\}$ that simultaneously satisfies this set of equations $\{a_1 = \alpha+\gamma,\ a_2 = \alpha+\delta,\ a_3 = \beta+\gamma,\ a_4 = \beta+\delta\}$. For most sets of eigenvalues, there does not exist such a set; i.e., from a formal standpoint, most Hamiltonians are nonseparable. The Hamiltonian is time independent; so, the unitary operator U that characterizes the time-evolution of s_1+s_2 is the diagonal matrix whose set of elements on the diagonal is $\{e^{-ia_1t}, e^{-ia_2t}, e^{-ia_3t}, e^{-ia_4t}\}$.[18] The time evolved state of s_1+s_2 is $\rho(t) = U\rho U^{-1}$. Thus, the time evolved state of s_1 is $\rho_1(t) = \mathrm{Tr}^{s_2}(\rho(t))$; that is, it is obtained from $\rho(t)$ by tracing out the degrees of freedom associated with s_2.

The question to address now is whether there is a suitable measure for the complexity of the time evolution of s_1. In his review of density operator techniques (Fano 1957), Fano briefly characterizes three relevant information measures for that purpose: (1) $\mathrm{Tr}(\rho\cdot\ln(\rho))$, (2) $\ln(\mathrm{Tr}(\rho^2))$, and (3) $\mathrm{Tr}(\rho^2)$. He notes that (1) is closely related to an entropy measure: the von Neumann entropy $-k\mathrm{Tr}(\rho\cdot\ln(\rho))$, which is clearly the case if units are chosen so that Boltzmann's constant k=1.[19] The

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[18] In order to simplify the form of U here and other expressions that follow, t′ is set to 0 and a conversion to natural units is made so that $\hbar = 1$.

[19] Two properties, additivity and subadditivity, may be used to characterize key differences between the three concepts: (1) is both additive and subadditive, (2) is additive but not subadditive, (3) is

remaining difference of a minus sign serves to relate an increase in entropy with a decrease in information and conversely. More recent developments provide important connections between (2) and another type of entropy measure, and similarly for (3). (2) is closely related to one of Renyi's α-entropies $(1-\alpha)^{-1}\ln(\mathrm{Tr}(\rho^{\alpha}))$, the one obtained by setting $\alpha=2$, and (3) is closely related to one of Daróczy's β-entropies $(2^{1-\beta}-1)^{-1}(\mathrm{Tr}(\rho^{\beta})-1)$, the one corresponding to $\beta=2$.[20] Clearly, an entropy measure is an appropriate measure of the complexity of the time evolution of a system. But, which one is best to use? The matter is a context dependent one. From a formal point of view, the von Neumann entropy has the most desirable features (Wehrl 1978). Perhaps that is why it is used most often in the literature. But, there are other considerations aside from formal ones, such as pragmatic considerations, and in some situations it is desirable to give a higher priority to pragmatic considerations over formal ones.

In what follows $D_2(\rho)$ denotes the Daróczy entropy for $\beta=2$; i.e., $D_2(\rho) \equiv 2(1-\mathrm{Tr}(\rho^2))$. $D_2(\rho)$ is used rather than von Neumann's entropy concept to assess the character of the time evolution of open quantum systems because expressions obtained using the former can be reduced analytically to a simpler form via trigonometric linearization unlike the latter, and the resulting expression can be related to classically chaotic features. I take it as a given that it is most desirable to determine whether quantum chaos exists using analytic methods rather than approximation methods or, if that is not possible, to use the former rather than the latter for as long as possible.

Consider again the open system s_1 that is discussed above, and assume for the sake of simplicity that the components of $|\phi\rangle$ are real numbers.[21] It then follows that

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neither additive nor subadditive. For a thorough discussion of those properties and others that are used to characterize entropy concepts, see (Wehrl 1978). Nonadditive entropies are discussed at length in (Behara 1990)—see the next footnote.

[20] Von Neumann's entropy was first introduced in (von Neumann 1927), Renyi's α-entropies in (Renyi 1960), and Daróczy's β-entropies in (Daróczy 1970). Axiomatic presentations of each of these notions is given in (Behara 1990)—see pp. 39-77 for the first (referred to there as "Shannon entropy"), pp. 78-115 for the second, and pp. 122-170 for the third (referred to as "polynomial entropy") . Behara also discusses nonpolynomial algebraic entropies (pp. 172-190). $D_2(\rho)$ is closely related to another entropy concept, one of the Tsallis q-entropies $(1-q)^{-1}(\mathrm{Tr}(\rho^{q})-1)$ for q=2 (Tsallis 1988). The Tsallis and Renyi entropy concepts have recently been discussed in connection with classical statistical mechanics—see (Mariz 1992) and (Ramshaw 1992, 1993).

[21] In the more general case where the components of $|\phi\rangle$ are complex, there is an additional variable term involving $\mathrm{Sin}[(a_1 - a_2 - a_3 + a_4)t]$, which means that the number of distinct frequencies involved in the motion does not change in the more general case.

$$D_2(\rho_1(t)) = 2 - 2\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\sum_{l=1}^{2}\langle\alpha_i\beta_j|\phi(t)\rangle\langle\phi(t)|\alpha_k\beta_j\rangle\langle\alpha_k\beta_l|\phi(t)\rangle\langle\phi(t)|\alpha_i\beta_l\rangle$$
$$= 2 - 2\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\sum_{l=1}^{2} r_{2(i-1)+j}e^{-ia_{2(i-1)+j}t} r_{2(k-1)+j}e^{ia_{2(k-1)+j}t} r_{2(k-1)+l}e^{-ia_{2(k-1)+l}t} r_{2(i-1)+l}e^{ia_{2(i-1)+l}t}$$
$$= 4(r_1^2 r_4^2 + r_2^2 r_3^2) - 8r_1 r_2 r_3 r_4 \mathrm{Cos}[(a_1 - a_2 - a_3 + a_4)t].$$

On the first line $|\phi(t)\rangle = U|\phi\rangle$, $\{\langle\alpha_1|,\langle\alpha_2|\} \equiv \{(1 \quad 0),(0 \quad 1)\}$ and similarly for $\{\langle\beta_1|,\langle\beta_2|\}$. The third line clearly indicates that the time evolution of s_1 is periodic (i.e., stable or not chaotic), when the Hamiltonian for s_1+ s_2 is nonseparable.

To generalize the above for n-state systems let $m=n^2$, $|\phi\rangle = (r_1 \quad \cdots \quad r_m)^T$, H and U be the diagonal matrices with nonzero elements $\{a_1,\cdots,a_m\}$ and $\{e^{-ia_1 t},\cdots,e^{-ia_m t}\}$, respectively. Then,

$$D_2(\rho_1(t)) = 2 - 2\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\langle\alpha_i\beta_j|\phi(t)\rangle\langle\phi(t)|\alpha_k\beta_j\rangle\langle\alpha_k\beta_l|\phi(t)\rangle\langle\phi(t)|\alpha_i\beta_l\rangle$$
$$= 2 - 2\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} r_{n(i-1)+j}\, r_{n(i-1)+l}\, r_{n(k-1)+j}\, r_{n(k-1)+l}\, e^{-i(a_{n(i-1)+j} - a_{n(i-1)+l} - a_{n(k-1)+j} + a_{n(k-1)+l})t}$$
$$= \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\sum_{k=j+1}^{n}\sum_{l=i+1}^{n}\Big[(2r_{n(i-1)+j}r_{n(l-1)+k})^2 + (2r_{n(i-1)+k}r_{n(l-1)+j})^2$$
$$- 8r_{n(i-1)+j}\, r_{n(i-1)+k}\, r_{n(l-1)+j}\, r_{n(l-1)+k} Cos\big(a_{n(i-1)+j} - a_{n(i-1)+k} - a_{n(l-1)+j} + a_{n(l-1)+k}\big)t\Big]$$

The number of distinct frequencies involved in the time evolution of $D_2(\rho_1(t))$ for the component s_1 of a pair of n-state systems s_1+s_2 is $\frac{1}{4}n^2(n-1)^2$, where n is any positive integer.[22] The behavior of such systems is either periodic or quasi-periodic, but not chaotic. The same goes for the limiting case in which n goes to infinity. That means it is necessary to look for chaotic behavior in pairs of systems having continuous spectra.

In the next section, $D_2(\rho_1(t))$ is derived for the continuous case. The derivation is formal rather than mathematical, meaning that Dirac's delta function is implicitly used in the derivation. A rigorous mathematical derivation would require formulating the problem in a rigged Hilbert space. That formulation is beyond the scope of this essay. A brief characterization of the notion of a rigged Hilbert space formalism will be given after a preliminary association is suggested between the measure used above and one of the standard indicators of chaotic behavior in classical physics.

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[22] That may be seen by noting that $\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\sum_{k=j+1}^{n}\sum_{l=i+1}^{n} 1 = \frac{1}{4}n^2(n-1)^2$.

4. Continuous Spectra and Quantum Chaos. To obtain the Daróczy entropy measure for the continuous case, it is necessary to replace discrete indices with continuous ones. The resulting expression is

$$D_2(\rho_1(t)) = 2 - 2\int\int\int\int \langle\alpha(\xi)\beta(\zeta)|\psi(t)\rangle\langle\psi(t)|\alpha(\xi')\beta(\zeta)\rangle\langle\alpha(\xi')\beta(\zeta')|\psi(t)\rangle\langle\psi(t)|\alpha(\xi)\beta(\zeta')\rangle \mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}\xi'\,\mathrm{d}\zeta'$$

$$= 2 - 2\int\int\int\int r(\xi,\zeta) r(\xi',\zeta) r(\xi,\zeta') r(\xi',\zeta') e^{-i(a(\xi,\zeta) - a(\xi',\zeta) - a(\xi,\zeta') + a(\xi',\zeta'))t} \mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}\xi'\,\mathrm{d}\zeta'$$

where the integrals are from λ to υ, $r(\xi,\zeta) = r((\upsilon-\lambda)\xi+\zeta)$ and similarly for $a(\xi,\zeta)$. The index term $(\upsilon-\lambda)$ corresponds to the difference between the lower and upper limits on the continuum of energy values. There is a lower limit $a(\lambda)$ since energies are always positive-valued. The upper limit $a(\upsilon)$ may be justified by supposing that the initial state of the compound system is a superposition of a continuum of energy eigenstates with some finite upper limit. It follows that there can be a continuum of distinct frequencies involved in the time evolution of the Daróczy entropy measure, and this is regarded as being indicative of quantum chaos. If that identification is correct, then the time evolution of the component of a pair of interacting quantum systems in the continuous domain is in most cases chaotic (for at least some ranges of parameter values).

The result above leads to some important questions such as whether it can be demonstrated that the notion of quantum chaos characterized above is responsible for or at least serves to explain the occurrence of chaos at the macroscopic level. In other words, is the time evolution of the Daróczy entropy measure closely related to one of the standard indicators that are used to identify chaos in classical models? A preliminary answer to this question will be given along with some suggestions for future research along these lines.

Among the classical measures are sensitive dependency on initial conditions, continuous power spectrum, positive Kolmogorov entropy, and algorithmic complexity. Providing a sketch of each of these notions will suffice for giving a preliminary answer to the questions raised above. A model exhibits sensitive dependency on initial conditions if small differences in the initial state leads to exponentially diverging trajectories on average. A model has a continuous power spectrum if the square modulus of the Fourier transform of the description of its time evolution involves a continuum of frequencies. The Kolmogorov entropy is the average rate of change of the entropy of the system, which is zero for stable systems (periodic or quasiperiodic) and positive for chaotic systems (Atmanspacher and Scheingraber 1987).[23] Finally, a model is algorithmically complex if

[23] A quantum counterpart to Kolmogorov entropy has recently been formulated (Connes et al. 1987). Nonmathematicians may find the level of discussion in that essay too technical; in that case, see (Wehrl 1991). If this proves to be the definitive formal counterpart and to be operative as well,

its time-evolution recipe requires at least N bits of input information about the initial state of the system to obtain N bits of output information about its future state. These features of classical models and the relations between them are discussed in (Eckmann and Ruelle 1985).

There is more than just a superficial connection between the continuous power spectrum indicator in classical mechanics and the continuum of frequencies involved in the time evolution of Daróczy entropy measure since energy and entropy are closely related notions. The significance of this rather deep analogy for the correspondence principle will require further analysis in order to establish a rigorous connection between these two notions. It may be necessary to translate the quantum results into another formalism and to consider connections with other indicators of chaos in classical systems. Of the measures of chaos in classical systems mentioned above, it seems that the Kolmogorov entropy may also be relevant since the measure used above is an entropy measure. Relevant alternative formalisms are the Hilbert space formulation of classical physics (Koopman 1931) and phase space formulations of quantum physics, as in (Wigner 1932) or in (Bohm and Hiley 1993).[24] It is difficult to say at this point which of the three approaches holds the most promise, though there are some presumptions in favor of the third approach. Establishing a connection with exponential divergence of trajectories seems most promising in that approach, and has shown some promise in its use to address the quantum chaos question.[25]

In addition to establishing connections such as those characterized above, it seems that a more rigorous mathematical derivation of the formal result obtained above is needed since Dirac's delta-function and the associated formalism can only be regarded as heuristic devices. A simple illustration of this point follows. Suppose that a system has an observable with a continuous spectrum $A = \int \alpha(\xi)|\alpha(\xi)\rangle\langle\alpha(\xi)|d\xi$ and that the system is in the state $\rho = |\alpha(\xi')\rangle\langle\alpha(\xi')|$, which corresponds to an eigenfunction of A. The expression Tr(ρA) ought to be the expectation value for A, meaning that it should be $\alpha(\xi')$; but, the expression Tr(ρA) is meaningless in this instance. Consider the following formal derivation:

then research along the lines developed here may still serve as an heuristic due to the pragmatic advantage of the Daróczy entropy which was indicated above.

[24] See pp. 357-359 of (Holland 1993) for an informative comparison and contrast of the two phase space formulations of quantum mechanics.

[25] In (Parmenter and Valentine 1995) and (Dewdney and Malik 1996), it is shown that a chaotic causal trajectory may be associated with a wave function corresponding to a quantum state that is not evolving in a chaotic manner. The approach here is quite different in that it attempts to show that quantum states can undergo chaotic behavior.

$$
\begin{aligned}
\mathrm{Tr}(\rho A) &= \int \langle \gamma(\zeta)|\rho A|\gamma(\zeta)\rangle\, d\zeta \\
&= \iint \langle \gamma(\zeta)|\alpha(\xi')\rangle\langle \alpha(\xi')|\alpha(\xi)\rangle\langle \alpha(\xi)|\gamma(\zeta)\rangle\, d\xi\, d\zeta \\
&= \int \alpha(\xi)\langle \alpha(\xi)|\alpha(\xi')\rangle\langle \alpha(\xi')|\alpha(\xi)\rangle\, d\xi \\
&= \int \alpha(\xi)\delta(\xi-\xi')\delta(\xi-\xi')\, d\xi
\end{aligned}
$$

The problem shows itself in the last line. The product of a delta function with itself is meaningless. This and related absurdities are avoided in the RHS formulation of QM, which is regarded as a mathematical realization of Dirac's formalism. Note that the absurdity does not arise if the state of the system is a superposition $\rho = \int c(\xi)c(\xi')^{*}|\alpha(\xi')\rangle\langle\alpha(\xi')|\, d\xi$ with respect to A. In effect, the RHS formulation avoids the absurdity by requiring the state of a system to be a superposition with respect to each observable having a continuous spectrum.[26]

It was the lack of rigor of Dirac's delta function that led von Neumann to establish the notion of a separable Hilbert space as the principle structure of the standard formulation of QM in 1927. [27] One of the key steps taken by von Neumann in establishing that standard was his demonstration of the equivalence of the Schrödinger and Heisenberg formulations of QM without using Dirac's delta function (unlike earlier demonstrations by Dirac and Jordan).[28] A rigorous foundation for the delta function first became possible with the development of distribution theory in 1945 by Schwartz, who formulated a rigorous definition during 1950-1951. That inspired Gel'fand and collaborators during 1955-1959 to develop a new mathematical structure, the rigged Hilbert space, which later made possible a new formulation of quantum mechanics that was introduced independently by Böhm and Roberts in 1966[29]

Some have maintained that only separable Hilbert spaces are needed for quantum mechanics, as claimed in (Amerin et al. 1977). But, it has turned out that the rigged Hilbert space formalism makes quantum mechanics rigorously applicable to a substantial domain of physics for which the separable Hilbert space formalism either could not or could only approximately (and

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[26] If the observable has both a discrete part and a continuous part, then this requirement applies only if the decomposition of the state in terms of the eigenfunctions of the observable has a component in the continuous part of the spectrum.

[27] The notion of nonseparability mentioned here is not related to the other two that were discussed above (one for Hamiltonians, the other for quantum states). That notion is characterized in sections 32-33 of (Riesz and Sz.-Nagy 1955); other relevant sections are section 83 (their definition of an abstract Hilbert space) and section 101 (their example of a nonseparable Hilbert space). The rigged Hilbert space formalism does not involve the use of a nonseparable Hilbert space, but it does involve the use of a nonseparable space of a more general type—i.e., the space of distributions, which is the conjugate space of the chosen space of test functions.

[28] The matter is discussed more fully in Chapter 6 of (Jammer 1966), a superb, concise discussion of the history of the standardization of the formulation of quantum mechanics.

[29] See (Böhm and Gadella 1989) for references and more details.

awkwardly) describe such as scattering, decaying states, and irreversible processes (Böhm 1993), (Böhm and Gadella 1989). Despite its elegance and power, the rigged Hilbert space formalism is not well known. That may be explained, in part, as being due to its involving more advanced techniques of functional analysis (such as distribution theory, essentially self-adjoint and nuclear operators, nuclear Fréchet spaces, and countably-Hilbert spaces). Perhaps that situation will change as its promise continues to be made manifest. In any event, because the continuous case requires more powerful techniques in implementing the approach to finding quantum chaos characterized above, it will be discussed in a separate essay.

One other strategy that is worth mentioning is the construction of a simple model involving a nonseparable Hamiltonian with a continuous spectrum, perhaps some type of three-body scattering.[30] If the model is capable of generating quantum chaos in the sense under consideration, then it may be capable of generating classical chaos at the classical limit. Such a model might then serve to indicate what happens when a mesoscopic or macroscopic system continuously interacts with its environment, and thereby serve to vindicate the correspondence principle.

5. Concluding Remarks. Others have discussed open quantum systems in connection with the quantum chaos question. So, it is worth briefly characterizing the key differences between that research and the program presented above. Some researchers (Blümel and Esser 1994) have shown that quantum chaos can occur in open quantum system when the global system has a mixed classical-quantum description. In a brief note (Blümel and Esser 1995), they characterize this as a "type II" system, and indicate that the goal of quantum chaos is to prove the existence of chaos in a fully quantized system, a "type III" system; type I systems are the ones studied in the top-down approach. Blümel mentions other discussions of mixed systems, states that there is no fully quantized system that exhibits chaotic behavior, and then presents what he takes to be the first one that does so (Blümel 1994). That system consists of a spin-1/2 system that interacts with a spin-precession apparatus consisting of an infinite chain of magnetic field sections arranged spatially in a Fibonacci-like sequence. But, even if it is granted that the behavior of the spin-1/2 system is chaotic in such an environment, it seems impossible to say that this highly artificial contrivance gets to the root of what it is about quantum mechanics that enables it to make room for chaotic behavior in fully quantized systems; whereas, the project characterized above serves to expose the tap root. One other discussion of type III open systems that is worth mentioning. It involves describing the time evolution of an open system using the state diffusion equation (Spiller and Ralph 1994).

Spiller and Ralph begin with a brief discussion of the Markovian approximation to the master equation which, as they note, is useful in many areas of physics. That equation describes the time evolution of a density operator. They then suggest that the density operator formalism is inappropriate for discussing chaos in open quantum systems on the grounds that "By definition, the density operator describes the evolution of an ensemble of systems." So, they switch to the state diffusion equation which describes the time evolution of a state vector and they regard as describing a component of the ensemble described by the master equation on the grounds that the same statistical predictions are given when the average is taken. They then use the state diffusion equation, which has proven useful in a variety of different contexts, to generate a Poincaré map for the quantum counterpart to a classical model known to exhibit chaos: the damped, driven oscillator model. The map for the quantum counterpart hints at the fractal like structure generated by classical model, and thereby constitutes motivation for future work along these lines.

Of course, it is by no means clear that a density operator represents an ensemble "by definition" since that assertion effectively ignores the crucial distinction between a proper and an improper mixture (d'Espagnat 1976).[31] That issue aside, there are other important differences between Spiller and Ralph's approach and mine that are worth mentioning. They use a Markovian approximation, whereas I do not resort to approximation techniques and do not assume randomness to get chaos. There are also important similarities. Neither approach has conclusive results so far, and both hint at promising developments to come.

It is also worth emphasizing here connections with other foundational issues associated with quantum mechanics. It was noted in the Section 3 the nonseparability of the state function underlies many of the key foundational issues including the measurement problem, the EPR paradox, violations of Bell's inequality, and quantum statistics. The approach to the quantum chaos question developed here emphasizes a related formal feature, nonseparable Hamiltonians. Some models involve nonseparable Hamiltonians in order to evolve tensor product states into nonseparable states, as in the measurement problem, others involve nonseparable states that cannot be evolved into or out of, meaning that certain nonseparable Hamiltonians are assumed not to exist (to preserve a symmetry). But, there is something novel and rather ironic about using nonseparability (of the Hamiltonian) to resolve a foundational problem, as opposed to generating another conundrum.

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[30] Two-body scattering is always separable in the center of mass coordinate system, meaning that it is dynamically equivalent to a one body system in an external potential.

[31] Another reason to regard that claim as too restrictive is that it conflicts with recent attempts to develop extensions of quantum mechanics in which the distinction between state vectors and density operators is lost; for example, see pp. 179-181 of (Prigogine 1980). The existence of a continuous spectrum is there identified as a necessary condition for that loss. More recently, Prigogine and his school have turned to the rigged Hilbert space formalism to further their development of an extension of quantum mechanics to the continuous domain to characterize irreversible and chaotic quantum phenomena involving large Poincaré systems; e.g., see (Antoniou and Prigogine 1993) and (Antoniou and Tasaki 1993).

The quantum chaos question has other ties to the measurement problem in addition to having this formal feature in common. The measurement problem has led some researchers to reject the Unitary Axiom, which says that the time evolution of all systems is unitary, or (equivalently) that all systems are governed by Schrödinger's equation at all times. If the Unitary Axiom precludes the existence of chaotic models for quantum mechanics, then that would support those who posit a non-unitary type of time evolution (which is often characterized as a "collapse of the wave-packet") to solve the measurement problem. Von Neumann (1955) holds that a quantum systems can undergo a non-unitary evolution during measurements. Cartwright (1983) holds that all systems can undergo a non-unitary process whether or not a measurement takes place, and Ghirardi et al. (1986) take this line further holding that the frequency of occurrence of such processes increasing with the size of the system. The preclusion thesis would support these positions in so far as it would serve to provide independent grounds for the existence of non-unitary time evolution. That is to say, the postulation of non-unitary processes would no longer be merely an ad hoc device for eliminating the measurement problem; conversely, a search for chaos in quantum mechanical models with collapse mechanisms becomes a much more attractive alternative.

On the other hand, if the axiom gives rise to such models, then this would tend to support approaches to the measurement problem which accept this axiom, such as the ontological interpretation (Bohm and Hiley 1993), (Holland 1993) and (Cushing 1994); and decoherence models (Zeh 1970), (Zurek 1981, 1983) and (Joos and Zeh 1985). The manner in which the approach in this essay to the quantum chaos problem serves to complement the former is characterized in Section 3. It serves to complement the decoherence approach to the measurement problem, which maintains that the inescapable, continuous influence of the environment on the measuring apparatus plays a crucial role in serving to bring measurement processes to a close, in so far as it reinforces the thesis that the emergence of classical features is an important characteristic that is peculiar to open systems.

In my own research on interpretive issues associated with the measurement problem (Kronz 1991, 1992), I support interpretations that rejects the Unitary Axiom. It may seem odd that the opposing view appears to be supported in the research project under consideration. But, an objective determination of the extent to which unitary time evolution is compatible with chaotic behavior must be made, if the absence of chaotic models is to be explained as being due to the overly restrictive character of the axiom. If, on the other hand, chaotic models are compatible with this axiom, then it will be appropriate to reassess any commitment to a view that rejects it.

REFERENCES

Allegrini, P.; Bonci, L.; Grigolini, P.; Mannella, R.; Roncaglia, R. and Vitali, D. (1995), "Comment on 'Quantum Chaos in the Born-Oppenheimer Approximation'," *Physical Review Letters 74:* 1484.

Amrein, W.O.; Jauch, J.M. and Sinha, K.B. (1977), *Scattering Theory in Quantum Mechanics.* Reading, MA: W.A. Benjamin.

Antoniou, I.E. and Prigogine, I. (1993), "Intrinsic Irreversibility and Integrabilit[32]y of Dynamics," *Physica A 192:* 443-464.

Antoniou, I.E. and Tasaki, S. (1993), "Generalized Spectral Decompositions of Mixing Dynamical Systems," *International Journal of Quantum Chemistry 46:* 425-474.

Atmanspacher, H., and Scheingraber, (1987), "A Fundamental Link between System Theory and Statistical Mechanics," *Foundations of Physics 17:* 939-963.

Batterman, R.W. (1991), "Chaos, Quantization and the Correspondence Principle," *Synthese 89:* 189-227.

_________. (1992), "Quantum Chaos and Semiclassical Mechanics," *PSA 1992*, vol. 2, pp. 50-65.

Behara, M. (1990), *Additive and Nonadditive Measures of Entropy.* New York: Wiley.

Berry, M.V. (1987), "Quantum Chaology," in M.V. Berry, I.C. Percival and N.O. Weiss (eds.), *Dynamical Chaos.* Princeton University Press.

_________. (1989), "Quantum Chaology, Not Quantum Chaos," *Physica Scripta 40:* 335-336

_________. (1994), "Asymptotics, Singularities and the Reduction of Theories," D. Prawitz, B. Skyrms and D. Westerståhl (eds.), *Logic, Methodology and Philosophy of Science IX.* Elsevier Science.

Bell, J.S. (1966), "On the Problem of Hidden Variables in Quantum Mechanics," *Reviews of Modern Physics 38:* 447-452.

Blümel, R. (1994), "Quantum Chaos in the Born-Oppenheimer Approximation," *Physical Review Letters 73:* 428-431.

Blümel, R. and Esser, B. (1994), "Exponential Sensitivity and Chaos in Quantum Systems," *Physical Review Letters 72:* 3658-3661.

_________. (1995), Blümel and Esser's response to (Allegrini et al. 1995), *Physical Review Letters 74:* 1485.

Boccaletti, D. and Pucacco, G. (1996), *Theory of Orbits.* New York: Springer-Verlag.

Bohm, D. (1951), *Quantum Theory.* Englewood Cliffs, New Jersey: Prentice-Hall.

Bohm, D. and Hiley, B.J. (1993), *The Undivided Universe.* New York: Routledge.

Böhm, A. (1993), "Gamow Vectors and the Arrow of Time," in P. Busch, P. Lahti, and P. Mittelstaedt (eds.), *Proceedings of the Symposium on the Foundations of Modern Physics.* World Scientific.

Böhm, A. and Gadella, M. (1989), *Dirac Kets, Gamow Vectors and Gel'fand Triplets.* New York: Springer-Verlag.

Cartwright, N. (1983), "How the Measurement Problem is an Artifact of the Mathematics," in *How the Laws of Physics Lie.* New York: Oxford University Press.

Chirikov, B.V. (1992), "The Problem of Quantum Chaos," in W. Dieter Heiss (ed.), *Chaos and Quantum Chaos.* New York: Springer-Verlag.

Connes, A.; Narnhofer, H. and Thirring, W. (1987) "Dynamical Entropy of C*-Algebras and von Neumann Algebras," *Communications in Mathematical Physics 112:* 691-719.

Cushing, J.T. (1994), *Quantum Mechanics: Historical Contingency and the Copenhagen Hegemony.* Chicago University Press.

Daróczy, Z. (1970), "Generalized Information Functions," *Information and Control 16:* 36-51.

Dewdney, C. and Malik, Z. (1996), "Measurement, Decoherence and Chaos in Quantum Pinball," *Physics Letters A:* 183-188.

Dirac, P.A.M. (1958), *The Principles of Quantum Mechanics.* Oxford University Press

---

32

Espagnat, B.d' (1976), *Conceptual Foundations of Quantum Mechanics.* Reading, MA: W.A. Benjamin.

Eckmann, J.P. and Ruelle, D. (1985), "Ergodic Theory of Chaos and Strange Attractors," *Reviews of Modern Physics 57:* 617-656.

Fano, U. (1957), "Description of States in Quantum Mechanics by Density Matrix and Operator Techniques," *Reviews of Modern Physics 29:* 74-93.

Flaschka, H.; Newell, A.C. and Tabor, M. (1991) "Integrability," in V.E. Zakharov (ed.), *What is Integrability?* New York: Springer-Verlag.

Ford, J. (1989), "What is Chaos That We Should Be Mindful of It?," in P.W. Davies (ed.), *The New Physics.* Cambridge University Press.

Ford, J. and Mantica, G. (1992), "Does Quantum Mechanics Obey the Correspondence Principle?," *American Journal of Physics 60:* 1086-1097.

Ford, J.; Mantica, G. and Ristow, G.H. (1990), "The Arnol'd Cat: Failure of the Correspondence Principle," in D. Campbell (ed.), *Chaos/Xaoc.* New York: The American Institute of Physics.

Ghirardi, G.C.; Rimini, A. and Weber, T. (1986), Unified Dynamics for Microscopic and Macroscopic Systems," *Physical Review D34:* 470-491.

Gisin, N. (1990), "Weinberg's Non-Linear Quantum Mechanics and Supraluminal Communications," *Physics Letters A 143:* 1-2.

Gutzwiller, M.C. (1990), "Quantum Chaos," *Scientific American 266:* 78-84.

Hilborn, R.C. (1994), *Chaos and Nonlinear Dynamics.* New York: Oxford University Press.

Helleman, R.H.G. (1980), "Self-Generated Chaotic Behavior in Nonlinear Mechanics," in E.G.D. Cohen (ed.), *Fundamental Problems in Statistical Mechanics V.* North Holland.

Henon, M. and Heiles, C (1964), "The Applicability of the Third Integral of Motion: Some Numerical Experiments," *Astronomical Journal 69:* 73-79

Holland, P.R. (1993), *The Quantum Theory of Motion.* Cambridge:Cambridge University Press.

Jammer, M. (1966), *The Conceptual Development of Quantum Mechanics.* New York: McGraw-Hill.

Jensen, R.V. (1992), "Quantum Chaos," *Nature 355:* 311-319.

Joos, E. and Zeh, H.D. (1985), "The Emergence of Classical Properties through Interaction with the Environment," *Zeitschrift für Physik B59:* 223-243.

Koopman, B.O. (1931), "Hamiltonian Systems and Transformations in Hilbert Space" in *Proceedings of the National Academy of Science 17:* 315-318.

Kronz, F.M. (1992), "The Projection Postulate and the Time-Energy Uncertainty Relation." *Philosophy of Science 59:* 1-15.

_________. (1991), "Quantum Entanglement and Nonideal Measurements: A Critique of Margenau's Objections to the Projection Postulate." *Synthese 89:* 229-251.

Lunsford, G. and Ford, J. (1972) "On the Stability of Periodic Orbits for Nonlinear Oscillator Systems in Regions Exhibiting Stochastic Behavior," *Journal of Mathematical Physics 13:* 700-705.

Mariz, A.M. (1992), "On the Irreversible Nature of Tsallis and Renyi Entropies," *Physics Letters A 165:* 409-411.

Merzbacher, E. (1970), *Quantum Mechanics,* 2nd edition. New York: Wiley.

Messiah, A. (1961), *Quantum Mechanics,* Vol. I. New York: Wiley.

Park, J.L. (1968), "Nature of Quantum States," *American Journal of Physics 36:* 211-226.

Parmenter, R.H. and Valentine, R.W. (1995), "Deterministic Chaos and the Causal Interpretation of Quantum Mechanics," *Physics Letters A 201:* 1-8.

Polchinski, J. (1991), "Weinberg's Nonlinear Quantum Mechanics and the Einstein-Podolsky-Rosen Paradox," *Physical Review Letters 66:* 397-400.

Ramshaw, J. (1993), "Irreversibility and Generalized Entropies," *Physics Letters A175:* 171-172.

_________. (1992), "H-Theorems for the Tsallis and Renyi Entropies," *Physics Letters A 175:* 169-170.

Reichl, L.E. (1980), *A Modern Course in Statistical Physics.* Austin: University of Texas Press.

Renyi, A. (1960), "On Measures of Entropy and Information," in *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*. University CA Press.

Riesz, F. and Sz.-Nagy, B. (1955), *Functional Analysis.* New York: Frederick Unger Publishing Co. (republished by Dover Publications in 1990).
Spiller, T.P. and Ralph, J.F. (1995), "The Emergence of Chaos in an Open Quantum System," *Physics Letters A 194:* 235-240.
Tabor, M. (1989), *Chaos and Integrability in Nonlinear Dynamics.* New York: Wiley.
Tsallis, C. (1988), "Possible Generalization of Boltzmann-Gibbs Statistics," *Journal of Statistical Physics 52:* 479-487.
Von Neumann, J. (1927), "Thermodynamik Quantenmechanischer Gesamtheiten," *Gesellschaft der Wissenschaften zu Göttingen 11:* 273-291.
_________. *Mathematical Foundations of Quantum Mechanics,* a translation of the 1932 German edition by R.T. Beyer. Princeton: Princeton University Press.
Wehrl, A. (1991), "Information Theoretical Aspects of Quantum Mechanical Entropy," in H. Atmanspacher and H Scheingraber (eds.), *Information Dynamics.* New York: Plenum.
_________. (1978), "General Properties of Entropy," *Reviews of Modern Physics 50:* 221-260.
Weinberg, S. (1989), "Testing Quantum Mechanics," *Annals of Physics 194:* 336-386.
Whittaker, E.T. (1944), *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies.* Cambridge University Press.
Wigner, E. (1932), "On the Quantum Correction for Thermodynamic Equilibrium," *Physical Review 40:* 749-759.
Zeh, H.D. (1970), "On the Interpretation of Measurement in Quantum Mechanics," *Foundations of Physics 1:* 69-76.
Zurek, W.H. (1981) "Pointer Basis of Quantum Apparatus: Into What Mixture Does the Wave Packet Collapse?," *Physical Review D24:* 1516-25.
_________. (1983) "Information Transfer in Quantum Measurements: Irreversibility and Amplification," in P. Meystre and M.O. Scully (eds.), *Quantum Optics, Experimental Gravity, and Measurement Theory*. New York: Plenum Press.