Percolation, Pretopology and Complex Systems Modeling

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Abstract

A complex system is generally regarded as being a network of elements in mutual interactions, which total behavior cannot be deduced from that of its parts and their properties. Thus, the study of a complex phenomenon requires a *holistic* approach considering the system in its totality. The aim of our approach is the design of a unifying theoretical framework to define complex systems according to their common properties and to tackle their modeling by generalizing percolation processes using pretopology theory.

keywords: Complex systems ; percolation ; pretopology; modeling; stochastic optimization.

1 Introduction

There is no unified definition of *complexity*, but recent efforts aim more and more at giving a generalization of this concept[1]. The problem arising from the notion of complexity is to know if it is of *ontological* or *epistemological* nature. We can note that complexity exists generally according to a specific model or a scientific field. Recently, a methodology of complex phenomena modeling, founded on a concept known as *hierarchical graphs* was developed in [12]. This promising modeling approach based on agents, makes it possible to take into account the various hierarchical levels in a system as well as the heterogeneous nature of the interactions. Nevertheless, the formal aspect in the study of complex systems remains fragmented. Our objective is to provide a work basis in order to specify some interesting research orientations and the adapted theoretical tools allowing the design of a general theory of complex systems.

2 Definitions and examples of complex systems

According to the NECSI (New England Complex Systems Institute)

> *Complex Systems is a new field of science studying how parts of a system give rise to the collective behavior of the system, and how the system interacts with its environment.*[2]

There also exists other definitions noting the fact that complex systems are often associated to the *Nonlinear Dynamic Systems* where, starting from an initial state, the system evolves more or less quickly to singular states depending on the value of certain control parameters. These states, qualitatively different, are on the temporal level either stationary, periodic or chaotic oscillating without period [11].

At the biological level, complexity is often associated the concept of emergence. A biological system is known as complex if it presents phenomena of emergence (i.e. it has a potential richness higher than that of the sum of its

[1]This is observable through the constitution of several laboratories and research institutes such as *New England Complex Systems Institute* and *Santa Fe Institute*.

[2]http://necsi.org

subsystems). This situation can be described by the following inequality :

$$H(x, y) > H(x) + H(y)$$

where x and y are two subsystems and H a mathematical function which expresses the potential "richness" of a system. This function is called in Statistical Physics the *entropy* of the system [14]. The concept of emergence is very present in the complex systems of any nature and is expressed through the appearance of a new property in the system. This shows the aspect more qualitative than quantitative of complexity. According to the philosopher Lucien Sève

> *Quality forms a unit with the some-thing determined by its limit : it cannot change without changing it, without making it different. The quantity is on the contrary "removed quality", in other words the indifferent change, the change which does not make differ from itself the something.* [15]

3 Our approach

In our approach, we define a complex system according to some common properties and behaviors of phenomena observed in different domains. We consider a system as a complex system if it exhibits an *emergence phenomenon*[3] occurring after a *phase transition*[4]. The phase transition phenomenon is obtained when a key parameter reaches a critical value called *transition threshold*. It is important to note that phase transition makes a link between complex systems and complexity theory. In fact, recent works [5] showed that the well known satisfiability problem 3-SAT[5] which belongs to the class of NP-complete problems exhibits a phase transition around a threshold α_c. This key parameter is the ratio between the number of clauses (i.e. constraints) and the number of literals (i.e variables). The transition is more abrupt as the number of literals is large (figure 1). The transition between satisfiability and unsatisfiability is accompanied by an important increase of the computational complexity to resolve the problem : finding a solution below the threshold and checking that there is no solution beyond it (figure 2).

We based our modeling and optimization of these systems, on a *systemic* approach. At the theoretical level this modeling can be carried out by using in particular *Percolation Theory*, which the french physicist Pierre-Gilles de Gennes [7] referred to as *concept unificateur* (unifying concept). Percolation theory was introduced in 1957 by John M. Hammersley and Simon R. Broadbent, to model the deterministic propagation of a fluid through a random medium. In formulating a stochastic process, which Hammersley called *percolation process*, to study such situations it was possible, for example, to answer the following question [8]:

> *Suppose we immerse a large porous stone in a bucket of water. What is the probability that the centre of the stone is wetted?*

Hammersley and Broadbent showed the existence of a critical value of the probability (or the density) of porous channels in the stone at which a fluid is able to pass trough the stone. This can be explained by the brutal appearance of a giant connected cluster[6] composed of unbounded open paths. We say that there is *percolation* in the system, when there is appearance of the giant (or infinite) cluster.

As phase transition phenomenon, observed in percolation processes is a characteristic property of complex systems, percolation theory was successfully applied to describe and study the spreading of oil in water, the propagation of infectious diseases and fire forests. Currently, percolation theory appears in such fields as hydrology, fractal mathematics, statistical physics and economy. From a mathematical point of view, this theory is interesting because it exhibits relations with *random graph theory* dealing with topological properties of graphs. Our purpose is first to generalize percolation processes using *pretopology theory* [3] in order to formalize the concept of neighborhood. This can be achieved using the pretopological concepts of pseudoclosure and minimal closed subsets. It is a general formalism that expresses different types of connections that may exist between the components of a system. The second step will associate the study of dynamical aspects of complex system using percolation theory to the structural aspects using random graph theory (in particular free-scale and small-world models).

[3] Appearance of a new property in the system

[4] A brutal change in the state of the system producing the appearance of a new property.

[5] A 3-SAT problem is a satisfiability problem where each clause is composed of three literals.

[6] When we are dealing with an infinite size system, it is called *infinite cluster*.

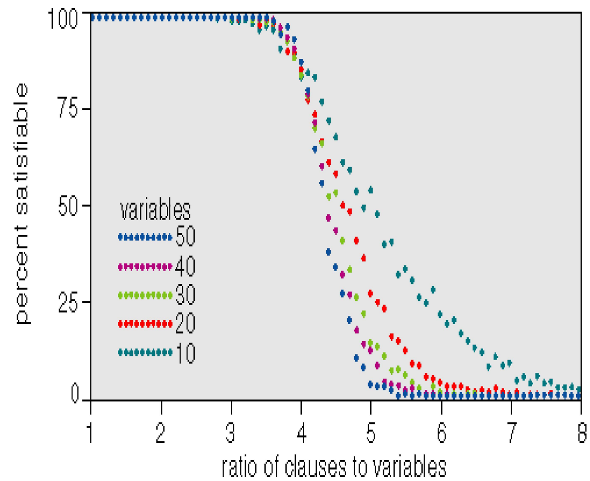

Figure 1: Phase transition of satisfiability probability in 3-SAT around α_c.

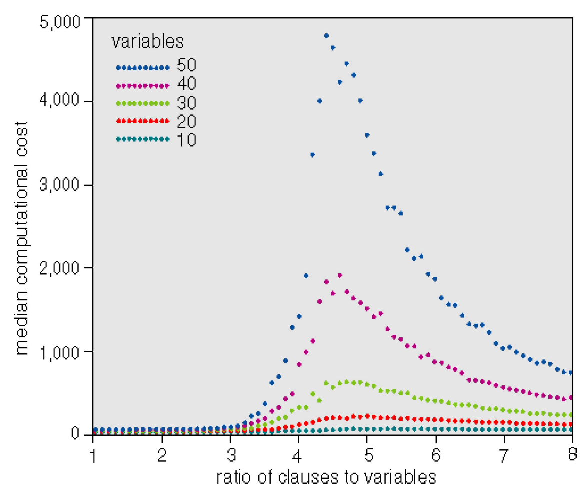

Figure 2: Brutal changes in the computational complexity of 3-SAT around α_c.

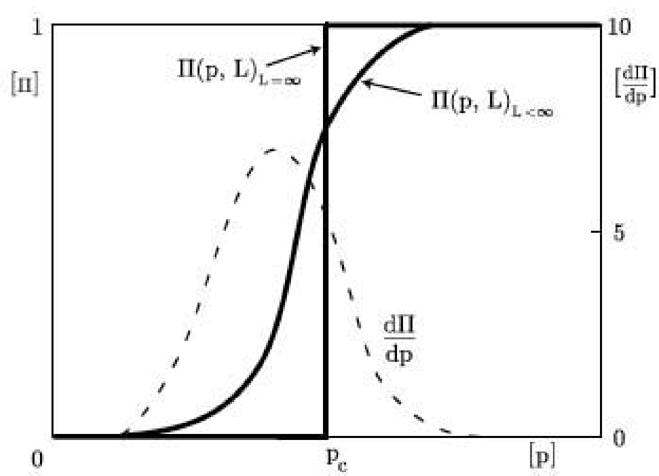

Figure 3: Phase transition around the threshold. Source: [16]

4 Conclusion

The approach presented in this abstract allowed us concrete applications in different fields. In [1] we proposed a model based on a site percolation model coupled with epidemic algorithms, in order to guarantee the connectivity of the mobile ad-hoc networks (MANET) and to ensure a good diffusion of messages, while minimizing the costs in terms of resources (energy, band-width...). In the air transportation context, we proposed in [4] a modeling of the ATC (*Air Traffic Control*) by combining of different percolation models to highlight the phase transition phenomenon related to the dynamics of the airspace congestion. We are also currently working with a team of psychologists and neuroscientists concerning cognitive and memorization processes. To conclude we want to emphasize on the importance of the design of a general theory of complex systems in one hand, and development of distributed simulation techniques in the other hand. A realistic understanding of a complex phenomenon needs not only good mathematical models, but also good simulation processes taking into account the parallel nature of the interactions in ”real systems”.

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