THE CAUSAL STORY OF THE DOUBLE SLIT EXPERIMENT IN QUANTUM REAL NUMBERS

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Abstract. A causal story of the double slit experiment for a massive scalar particle is told using quantum real numbers as the numerical values of the position and momentum of the particle. The quantum real number interpretation postulates an independent physical reality for the quantum particle. It provides an ontology for the particle in which its qualities have numerical values even when they have not been measured. It satisfies experimental tests to the same degree of accuracy as the standard quantum theory because the standard expectation values are infinitesimal quantum real numbers. Questions, unanswerable in the standard theories, concerning the behaviour of single particles in the experiment are answered.

1. Introduction to quantum real numbers

In order to tell our causal story, we must temporarily replace the orthodox Copenhagen interpretation of quantum mechanics with a more general interpretation in which both the logic of propositions and the ring of real numbers[1] for the values of qualities are changed. Classical logic is replaced by intuitionistic logic and the standard real numbers are replaced by Dedekind real numbers in a sheaf[17]. In using Dedekind real numbers as values, we have consented to broaden the field of allowable real number systems in a way analogous to the broadening of metric geometry from Euclidean to Riemannian.[2]

The quantum real numbers (qrumbers[3]) scheme for Galilean invariant quantum mechanics maintains the concept of a massive particle satisfying Newton's law of motion, mass x acceleration = force, by changing the mathematical way of representing the values of the kinematic variables. This law, when the kinematic variables have qrumber values, holds for microscopic particles.[4]

[1]There are no "hidden variables" in the quantum real numbers interpretation, only "different numerical values" for the qualities.

[2]"It seems that the human mind has first to construct forms independently before we can find them in things. ..... the truth (is) that knowledge cannot spring from experience alone but only from the comparison of the inventions of the intellect with observed fact." A. Einstein[11]

[3]We will write qrumber as an abbreviation of quantum real number.

[4]The possibility of a complimentary principle between the type of mathematical structure S and its interpretation I is discussed in [18].

The qrumber description of the double slit experiment differs fundamentally from the standard description[16][13]:

(a) Microscopic entities possess qualities with definite qrumber values even in the absence of a specific macroscopic experimental arrangement. However the qrumber values exist to extents which may be limited by the experimental arrangement.

(b) Each particle that passes through the slits moves freely to the detecting screen where it is detected as a single particle. The "wave-like" interference pattern only emerges after many particles have been detected; it is produced by the ensemble of particles, not by a single particle. This means that Wheeler's "delayed choice" conundrum disappears, a quantum particle always behaves as a quantum particle.

(c) There is a less stringent form of complementarity. If a detector is placed immediately behind one of the slits it can determine if a particle passed through that slit and while that detector is kept in place, no interference pattern will appear.

(d)The system and apparatus interact during the measurement process to enable a standard real number to be registered.

The theory is realist in the sense that it postulates the existence of entities possessing properties corresponding to qualities such as the position, momentum or mass of a particle but does not identify the ontological quantitative values of these qualities with their observed numerical values. The ontological properties are related to the observed properties but are not identical with them. [5]

Mathematically, the qrumber interpretation uses the standard Hilbert space formalism of non-relativistic quantum mechanics; it represents qualities as self-adjoint operators on the Hilbert space that carries an irreducible projective unitary representation of the Galilean group and uses the standard quantum state space, $\mathcal{E}_{\mathcal{S}}$, as the base space of the topos of sheaves in which the qrumbers exist as Dedekind real numbers.[6]

A qrumber value for a quality carries an extent with it. If the quality is represented by the self-adjoint operator $\hat{A}$, its value is given by a continuous function, $a(\rho) = Tr\rho \cdot \hat{A}\ ;\ \rho \in U$, whose domain, U, is a open subset of $\mathcal{E}_{\mathcal{S}}$. The domain U is the extent and we call it the *ontological condition* of the system because U determines the qrumber value for every quality of the system.

The ontological and epistemological condition of a quantum system are differentiated as follows. If a system is prepared in an open set W of

[5]c.f. Bohr's severing of the "direct connection between observation properties and properties possessed by the independently existing object"[9].

[6]We use an O^* algebra $\mathcal{M}$ to represent the Lie algebra of an irreducible representation of the symmetry group.[8] The state space $\mathcal{E}_{\mathcal{S}}(\mathcal{M})$ is the Schwartzian subspace of the standard quantum mechanical state space on which unbounded operators define continuous functions. [2]

state space, then the *epistemological condition* of the system is a sieve $S(W)$ generated by W.[7] We will usually take $S(W)$ to be the family of all non-empty open subsets of W because for any open set $V \subset W$, the values of qualities defined to extent V will satisfy the experimental restrictions imposed on qualities defined to extent W. The *ontological condition* of the system is a non-empty open set $V \subset W$.

In a measurement process[4], the measuring apparatus only accepts systems whose qualities have values compatible with a given epistemological condition U. Systems whose ontological condition $V \subseteq U$ are accepted. There is an interaction between the system and the apparatus whose pointer quality has standard real number values. During the interaction the ontological condition V of the system is reduced to $V' \subset V$. The measurement of $\hat{A}$ is complete when $a(V')$ is approximated by a constant qrumber, $r \cdot 1(V')$, $r \in \mathbb{Q}$, to a prescribed level of accuracy[4]. Although there are no restrictions on the simultaneous qrumber values of canonically conjugate qualities, like position and momentum, in an ontological condition, the product of the accuracy levels of the measured values of position and momentum satisfies an inequality related to Heisenberg's uncertainty relation[4].

The formulae of standard quantum mechanics give local approximations to those of the qrumber interpretation.[8] Therefore any constant qrumber output can be made to satisfy experimental tests to the same degree of accuracy as the standard theory. The expectation values of standard quantum theory are infinitesimal qrumbers.[8] On the other hand, if U is a sufficiently small neighbourhood of a quantum state, ρ, then the standard real number, $Tr\rho\hat{A}$, gives a good constant qrumber approximation to $a(U)$.

The structure of the paper is as follows. We will first describe the double slit experiment and raise four questions about it that are not usually answered by standard quantum theories. Then the standard quantum mechanical descriptions are reviewed; we describe Feynman's rules for calculating the probability of going from the source to a detector, then express this result in the Schrödinger picture. Next we briefly discuss the classical particle picture before describing the approach that uses qrumbers. In the conclusion the four questions are answered in the quantum real number theory and some general conclusions are drawn. The appendix lists some mathematical properties of qrumbers that have been used.

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[7]A sieve $S(W)$ on an open set W is a family of open subsets of W with the property if $U \in S(W)$ and $V \subset U$ then $V \in S(W)$[17].

[8]Infinitesimal qrumbers are not qrumbers but are ideal quantities that arise as limits of qrumbers when their extents shrink to a point.[8]

2. The double slit experiment

The double-slit experiments with massive particles illustrate fundamental aspects of the behaviour of a quantum system. Feynmann went so far as to say that it was ".. a phenomenon which is impossible, absolutely impossible, to explain in any classical way, and which has in it the heart of quantum mechanics. In reality, it contains the only mystery."[12]

Consider a sequence of single scalar particles, all of mass $m > 0$ which, after being prepared at a source S, are sent to an opaque screen Σ_1 with two slits, I_+ and I_-, in it. Each particle that passes through the slits is subsequently detected and its position is recorded on the detector screen Σ_2 placed at is a large distance behind Σ_1. The source S is controlled so that it emits particles singly with a large enough time-interval between them that we are certain that there is only one particle in the apparatus at any time and that their arrival at Σ_2 can be recorded one at a time. After a large number of particles have been detected on Σ_2 the pattern of the spots recorded on Σ_2 is observed to be similar to the interference pattern of a wave that passed through the two slits.

Recall that when waves are used, the wavefront spreads out from the source S and produces secondary wave fronts at the slits I_+ and I_- in Σ_1. These secondary wavefronts spread so that when they reach the screen Σ_2 they largely overlap and produce an interference pattern with regions of destructive interference alternating with regions of constructive interference.

There are four questions concerning these outcomes that need to be explained:

A. Why are particles that have been prepared in a like manner detected at different spots on Σ_2?

B. How does the interference pattern emerge as the accumulated effect of many single particle events?

C. How can a single particle pass through both slits and subsequently interfer with itself?

D. Why are only individual particles detected on Σ_2?

2.1. **Feynman's rules.**

Feynman describes the experiment using three general principles that don't depend upon the way that the particle gets from S to a detector d on Σ_2[12], the slits appear only as markers that designate the different routes the particle could have taken. The rules are: (1) probabilities are given by the modulus squared of the probability amplitude, (2) when a process can be achieved in two different ways the total probability amplitude is the sum of the amplitudes of the alternatives and (3) the probability amplitude of a particular alternate is the product of the amplitude to go part way with the amplitude to go the rest of the way.

Writing the probability amplitude to go from a to b as $\langle b|a\rangle$, the probability amplitude to go from S to d via slit I_+ is $\langle d|+\rangle\langle +|S\rangle$ and the probability amplitude to go from S to d is A_{dS} where

$$A_{dS} = \langle d|+\rangle\langle +|S\rangle + \langle d|-\rangle\langle -|S\rangle \tag{1}$$

The probability of a particle going from S to d is $|A_{dS}|^2$. $\langle \pm|S\rangle = a_\pm \exp \imath\alpha_\pm$ if the probabilities of detection when only one slit is open given are $|\langle d|+\rangle|^2 = a_+^2$ or by $|\langle d|-\rangle|^2 = a_-^2$. Usually $a_+^2 + a_-^2 = 1$ on the assumption that the preparation is such that the probability of getting from S through the slits is 1. This assumption means that in the double slit experiment the preparation process continues up to the emergence of a particle from the slits in $\Sigma 1$. The double slit experiment is therefore an experiment to determine if a quantum particle can pass through two slits simultaneously. If a quantum particle can, the screens Σ_1 and Σ_2 can be set up so that an interference pattern will be observed on the screen Σ_2.

Usually one takes $a_\pm = \sqrt{\frac{1}{2}}$, so that the probability of passage through either slit is 1/2, then, writing $\beta = (\alpha_+ - \alpha_-)$, we get $|A_{dS}|^2 =$

$$\frac{1}{2}(|\langle d|+\rangle|^2 + |\langle d|-\rangle|^2 + e^{\imath\beta}\langle -|d\rangle\langle d|+\rangle + e^{-\imath\beta}\langle +|d\rangle\langle d|-\rangle) \tag{2}$$

The interference is described by the last two terms. Note that although this formula does not answer the questions raised in points A,B,C and D, it does show that Feynman's three general principles suffice to describe the probability distribution on Σ_2 that is consistent with a single particle passing through both slits because if a single particle could only pass through one then the formula would only retain the first two terms and no interference pattern would result.

2.2. **Schrödinger picture.** The Hilbert space formula is obtained using wave functions $\psi_\pm$ that are unit vectors in the appropriate Hilbert space. Again on the assumption that the preparation phase of the double slit experiment continues up until a particle emerges from the double slits, the wave functions are assumed to satisfy Schrödinger's equation for free motion between the slits $I_\pm$ and a detector d on the screen Σ_2. The phases $\alpha_\pm$ are absorbed into the the rays of the wave functions $\psi_\pm$, which are assumed to be evaluated at the time when the particle reaches the detector d on the screen Σ_2. Initially ψ_+ started at the slit I_+ and ψ_- started at the slit I_-. Then, on assuming equal probabilities for the particle to start at either slit, we get the formula for the probability that a particle is detected at d,

$$|A_{dS}|^2 = \frac{1}{2}(\langle \psi_+, \hat{P}_d\psi_+\rangle + \langle \psi_-, \hat{P}_d\psi_-\rangle + \langle \psi_+, \hat{P}_d\psi_-\rangle + \langle \psi_-, \hat{P}_d\psi_+\rangle) \tag{3}$$

in which we have assumed that the detector d is such that $|d >< d| = \hat{P}_d$, a linear operator acting on the Hilbert space. In Feynman's notation the inner product of $\psi_\pm$ and $\hat{P}_d\psi_\pm$ equals the probability $|\langle d|\pm\rangle|^2$, that is, $\langle\psi_\pm, \hat{P}_d\psi_\pm)\rangle = |\langle d|\pm\rangle|^2$ and $\langle\psi_+, \hat{P}_d\psi_-\rangle = \langle +|d\rangle\langle d|-\rangle$

These formulae can be compared with the results of experimental measurements. There is a fundamental assumption that the situation of a quantum particle emerging from a slit can be accurately described by a single wave function. Even if we knew with infinite accuracy the position and momentum distributions of an emerging particle the answers to the Pauli problem[6] reveal that this information does not always determine a unique wave function. However if we assume that there is an initial wave function for each slit then the wave functions and operator $\hat{P}_d$ can be chosen so that this formula fits the detection patterns to an acceptable level of accuracy. Nevertheless it only gives an answer to the question whether the the probability distribution on Σ_2 was that of particles that could only pass through one slit at a time. It provides no answers to the questions A,B,C and D.

2.3. **Classical particle picture.** If the standard models of classical mechanics are used to describe the double slit experiment there is no interference pattern even when the values of the position and momentum variables are taken to be imprecise.

Consider a classical particle of positive mass which is prepared at a source S in such a way that it is equally likely that a particle will pass through I_+ as through I_-. That is, the particles are prepared with a range of initial momenta $\vec{p} \in M_0$ and initial position $\vec{x} \in R_0$ where the set of initial data, $M_0 \times R_0$ is large enough to contain subsets $M_+ \times R_+$ and $M_- \times R_-$ of initial data for particle trajectories that pass through I_+ or I_-. Clearly the subsets of initial momenta $M_\pm \subset M_0$ and of initial positions $R_\pm \subset R_0$ and while the initial positions may satisfy $R_+ \cap R_- \neq \emptyset$, usually the initial momenta will satisfy $M_+ \cap M_- = \emptyset$.

The ontology of a classical particle assumes that it is located in a connected region of space and therefore cannot simultaneously pass through the two slits which are located in two separated regions of space. Therefore a classical particle emerging from the double slit apparatus can only have passed through one slit and cannot produce an interference pattern when detected on the screen Σ_2. However each particle will be detected as an individual entity on Σ_2 because classical particles to only appear as discrete lumps located in space.

2.4. **Quantum real number picture.** The description using qrumbers is similar to the classical particle picture in that the particles are detected as discrete lumps, but differs from it because the cumulative effect of the detection of a large number of particles is an interference pattern. For the motion between the preparation and detection procedures, the dynamical equations, expressed in qrumber values of the particle's position and momentum, are deterministic and time-reversable.

Particles are prepared at a source S. Each particle then moves freely in qrumber space to the slits in Σ_1. We will only describe what happens after a particle has passed through the slits on Σ_1.

The particle's initial data on Σ_1 is determined by the open sets $W_+, W_-, W_m, W_{(+,-)}$, where $W_{(+,-)} = co(W_+ \cup W_-)$.[9] When the initial data is $(\vec{x}(U), \vec{p}(U))$ for $U \in \mathcal{O}(W_+)$[10] the particle had passed through the slit I_+, for $U \in \mathcal{O}(W_-)$ it had passed through the slit I_-, for $U \in \mathcal{O}(W_m)$ it had passed through both slits and for $U \in \mathcal{O}(W_{(+,-)})$ it had passed through at least one slit.

The slits are centred at points $z_\pm$ on Σ_1, have widths 2δ and separation $(z_+ - z_-) = 4\delta$ and are set up[11] so that an appropriate device placed directly behind either slit would record a standard real number, $z_\pm$, with an accuracy $\epsilon = \delta$,[12]

$$|z(W_\pm) - z_\pm| < \delta. \tag{4}$$

The qrumbers $z(W_{(+,-)})$ and $z(W_m)$ satisfy the inequalities

$$(z_- - \delta) < z(W_{(+,-)}) < (z_+ + \delta), \; z_- < z(W_m) < z_+. \tag{5}$$

which can be expressed as

$$|z(W_{(+,-)}) - z_m| < 3\delta, \; |z(W_m) - z_m| < 2\delta. \tag{6}$$

$z_m = (z_+ + z_-)/2$ is the z-coordinate of the mid-point of the slits.

After leaving Σ_1 the particle moves freely to Σ_2 along a trajectory in qrumber space. The separation between Σ_1 and Σ_2 is large enough to ensure that detectors on Σ_2 receive particles that have passed through both slits in Σ_1. If t is the time to go from Σ_1 to Σ_2 and μ is the mass of the particle, then for each open $U \subseteq W_m$,

$$z(t)(U) = z(U) + p_z(U)\frac{t}{\mu}. \tag{7}$$

whence

$$|z(t)(U) - p_z(U)\frac{t}{\mu} - z_m| = |z(U) - z_m| < 2\delta. \tag{8}$$

The detectors on Σ_2 are aligned along the z-axis with apertures $\{I_\alpha\}_{\alpha=1}^N$ where $I_\alpha = [z_1^\alpha, z_2^\alpha]$. If a particle with ontological condition U_k arrives at Σ_2 and $z_1^\alpha < z(t)(U_k) < z_2^\alpha$ then it passes into I_α.

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[9] Any state $\rho \in co(W_+ \cup W_-)$ is given by $\rho = \lambda\sigma_+ + (1-\lambda)\sigma_-$ for some $\lambda \in [0,1]$ and states $\sigma_\pm \in W_\pm$.

[10] $\mathcal{O}(V)$ denotes the set of all non-empty open subsets of V.

[11] See the appendix for the definition of an ϵ sharp collimator.

[12] Standard real numbers in the following expressions are constant qrumbers with extents determined by other qrumbers in the expression.

The α^{th} detector works by ϵ sharply realizing[13] a standard real number for the coordinate $\hat{Z}(t)$ of the particle. Let V_α be the largest convex open subset on which $z(t)(V_\alpha)$ is ϵ sharply realized on I_α and $\hat{P}_\alpha$ be the spectral projection operator $\hat{P}^{\hat{Z}(t)}(I_\alpha)$ of $\hat{Z}(t)$ on I_α, then[4] the probability of detecting a particle with initial ontological condition U_k in I_α is given by the qrumber, $\pi_\alpha(U_k)$, of $\hat{P}_\alpha$[14].

If $U_k \subset V_\alpha$ the particle will be detected by the α^{th}detector.

The argument follows from the conditions for ϵ sharp realization on V_α which imply that the quantum real number value $\pi_\alpha(V_\alpha)$ of $\hat{P}^{\hat{Z}(t)}(I_\alpha)$ satifies $|\pi_\alpha(V_\alpha) - 1(V_\alpha)| < \epsilon$[4]. Therefore, within ϵ, when $U_k \subset V_\alpha$, $\pi_\alpha(U_k) = 1$.

The particles are recorded by single detectors on Σ_2.

This again follows from the conditions for ϵ sharp realization. Suppose a particle, in the ontological condition U_k, is registered simultaneously in the disjoint intervals I_α and I_β then $U_k = U_k^\alpha \cup U_k^\beta$ where $\pi_\alpha(U_k^\alpha) = 1$ and $\pi_\beta(U_k^\beta) = 1$. The quantum real number values $z(U_k^\gamma); \gamma = \alpha, \beta$ can be separately ϵ sharp realized in the distinct slits, but to be registered as a single value in the two slits,

$$\frac{4((z^2(U_k)) - (z(U_k))^2)}{(|I_\alpha| + |I_\beta|)^2} \leq \epsilon. \tag{9}$$

for some small number ϵ. Now $z(U_k^\alpha \cup U_k^\beta) = z(U_k^\alpha) + z(U_k^\beta)$ and $z^2(U_k^\alpha \cup U_k^\beta) = z^2(U_k^\alpha) + z^2(U_k^\beta)$ so that a straightforward calculation shows the inequality cannot be satisfied unless only one slit is involved.

The interference pattern on Σ_2 results from the accumulated effect of many spots on Σ_2 coming from different particles whose initial ontological conditions cover W_m.

This argument depends on the experimentalist's inability to perfectly control the initial ontological conditions so that when many particles have been recorded the initial ontological conditions form an open cover of W_m. The function $\pi_\alpha : W_m \to [0,1]$ is continuous so that its restriction to any open set $U \subset W_m$ is a continuous function $\pi_\alpha|_U : U \to [0,1]$. We can collate the probability functions $\{\pi_\alpha|_U, U \in \mathcal{O}(W_m)\}$ to reform the probability function π_α on W_m because they agree on the overlaps. Therefore the probability of a particle being detected in the α^{th} slit is given by the qrumber $\pi_\alpha(W_m)$.

The qrumber probability $\pi_\alpha(W_m)$ is related to the standard quantum mechanical formula for the probability of detection at the α^{th} detector. Firstly some notation, if $\rho_\pm = |\psi_\pm\rangle\langle\psi_\pm|$, where $\psi_\pm$ are approximate eigenvectors for $\hat{Z}$ with eigenvalues $z_\pm$, then $Tr\rho_\pm\hat{Z} = z_\pm$. If $\psi_m = \frac{1}{\sqrt{2}}(\psi_+ + \psi_-)$ and $\rho_m = |\psi_m\rangle\langle\psi_m|$ then $Tr\rho_m\hat{Z} = z_m$. Therefore we can find a standard real number $\eta > 0$ such that $Tr|\rho - \rho_m| < \eta$, $\forall \rho \in W_m$.

[13]See the appendix for the definition, it is the same as ϵ sharply collimating.

[14]The formula $\pi_\alpha(\rho) = Tr\rho\hat{P}_\alpha$ defines a continuous function on W_m.

The standard quantum mechanical formula for the probability of detection at the α^{th} detector is obtained as an approximation to $\pi_\alpha(W_m)$.

From the definition of W_m we know that $|z(W_m) - z_m| < 2\delta$, where 2δ is both the width of the slits and their separation.

Each $\rho \in W_m$ satisfies $Tr|\rho - \rho_m| < \eta$. But $\hat{P}_\alpha$ is a bounded operator of norm 1, therefore, $|Tr(\rho\hat{P}_\alpha) - Tr(\rho_m\hat{P}_\alpha)| < \eta$, $\forall \rho \in W_m$.

That is, $|\pi_\alpha(W_m) - Tr(\rho_m\hat{P}_\alpha)| < \eta$. But $Tr\rho_m\hat{P}_\alpha =$

$$\frac{1}{2}(\langle\psi_+, \hat{P}_\alpha\psi_+\rangle + \langle\psi_-, \hat{P}_\alpha\psi_-\rangle + \langle\psi_+, \hat{P}_\alpha\psi_-\rangle + \langle\psi_-, \hat{P}_\alpha\psi_+\rangle) \tag{10}$$

the standard quantum mechanical expression for the probability that a particle is detected at the α^{th} detector that is usually interpreted as displaying a wave-like behaviour of the quantum particle. To within the margin of error η the same pattern is obtained in the qrumber model after a large number of particles with definite qrumber trajectories have been detected.

Let $\delta \to 0$, then $z(W_m) \to z_m$ and $\eta \to 0$ so that $W_m \to \{\rho_m\}$ whence $\pi_\alpha(W_m) \to Tr\rho_m\hat{P}_\alpha$. This limiting process, in which both the width of the slits and the distance between them go to zero, has as its "limit" an infinitesimal qrumber, $Tr\rho_m\hat{P}_\alpha$, not a qrumber. The standard quantum mechanical result $Tr\rho_m\hat{P}_\alpha$ defines an infinitesimal qrumber[8]. This result encapsulates the relation between the qrumber and the standard interpretations.

3. Conclusions

The qrumber interpretation provides answers to some questions that have no answers in the standard interpretation.

A. Why can particles that have been prepared in a like manner be detected at different spots on Σ_2?

Because like preparation refers to the initial epistemological condition but different particles can have different initial ontological conditions contained in the initial epistemological condition. Different initial ontological condition determine different trajectories in qrumber space.

B. How does the interference pattern emerge as the accumulated effect of many single particle events?

Because the probability function is obtained as the collation of all the probability functions for individual particles in all ontological conditions that are compatible with the original epistemological condition.

C. How can a single particle pass through both slits and subsequently interfer with itself?

Because locality in qrumber space is different from locality in classical real number space. A particle in a ontological condition $U = U_+ \cup U_-$ with $U_+ \cap U_- = \emptyset$ is located in qrumber space at the point with coordinates $\vec{x}(U)$ even though there are disjoint regions $R_{\pm} \subset \mathbb{R}^3$ with $\vec{x}(U_+) \in R_+$ and $\vec{x}(U_-) \in R_-$.

D. Why are only individual particles detected on Σ_2?

Assuming that at any time the apparatus contains only one particle, it could pass into two or more different detectors. However such an event can only be registered as if individual particles had been ϵ sharply registered separately in each detector.

3.1. **General conclusions.** The qrumber interpretation of quantum mechanics is realist because it builds an independent reality for quantum objects with properties that have values. It agrees with classical realism in that the numerical values of qualities such as the position, momentum and energy of a quantum particle represent the properties actually possessed by the real particle. It differs from classical reality because the numerical values are qrumbers so that although the real particle causes the observed phenomena, the numerical values of the observed quantities need not be uniquely determinable from the numerical values of the properties of the particle.

Bohr's observation[9] that the standard quantum theory does not give a description of an independently existing reality even though it successfully predicts relations between observed phenomena is supported in the qrumber interpretation. The qrumber interpretation provides a layer of quantum reality which fits in between the observed phenomena. Pictorially this layer lies under the layer of classical reality and touches it at isolated points.[15]

In the qrumber interpretation, the structure of the physical qualities of the system, whose general characteristic is quantity, is expressed in qrumbers and the laws that govern relations between the physical qualities are expressed as equations relating the qrumber values of the qualities. There is no difficulty with this formalism until in a measurement process a quantum system interacts with the measurement apparatus whose qualities either have standard real number values or have qrumber values that closely approximate standard real numbers. However the interaction between the system and apparatus can always be described by a qrumber equation because standard real number values can always be expressed as constant qrumbers globally defined on the state space of the system.

A quantum system always has an ontological condition and hence always has qrumber values for all its qualities. An experimenter determines only an epistemological condition that contains a variety of ontological conditions all compatible with the experimental parameters. The qrumber interpretation accepts that we do not have complete

[15]Because the standard real numbers form a sub-sheaf of the sheaf of Dedekind real numbers.

knowledge of a quantum system. We cannot infer the ontological condition of a system from observations but we can put numerical bounds on the qrumber values that its qualities may possess.

Standard rational numbers are recorded by observers and every standard rational number defines a constant qrumber. Measurement consists in finding a constant qrumber that approximates the qrumber value of a quality. It is solved by interactions between the system and apparatus that change the ontological condition of system so that the changed qrumber value can be approximated to a prescribe level of accuracy. Since the numerical outcomes are only determined with limited accuracy, tests of "empirical adequacy" cannot prove the truth of the theory but only show that, at some level of precision, it is not false.

The mathematical relation between the standard quantum mechanical and qrumber formalisms is best seen through the language of sheaf theory. The quantum mechanical expectation values at a state are discrete points on the stalk (or fibre) of the sheaf over that state and a qrumber over an open set of states is a cross-section of a bundle of stalks over the open set. The expectation values from the standard quantum formalism determine infinitesimal qrumbers, which when continuously tied together make up a qrumber. Since every infinitesimal qrumber is an ideal limit of qrumbers, standard quantum mechanical formulae appear as infinitesimal approximations to the qrumber formulae.

The causal story of the double slit experiment that we have related removes much of the mystery that Feynmann referred to. The causality is deterministic in the sense used in classical mechanics.[13] Other foundational problems of quantum mechanics are being studied using the qrumbers interpretation. We have began the study of the conditions of separability and locality for composite systems. Massive particles with spin are described in the qrumbers theory by using appropriate irreducible representations of the Galilean group. We have sketched an argument that shows that identical massive particles in the EPR-Bohm-Bell experiment have zero qrumber distance between them, that is, the two particles are close to each other in qrumber space.[5]

Finally we point out that the question of which real numbers should be used for the values of quantities has rarely been raised mainly because the set theoretical definition of the real numbers does not allow any variation. The intuitionistic logic of the Dedekind real numbers in a topos of sheaves on a topological space opens up many possibilities. [16] However it has been shown that Dedekind real numbers in a topos of sheaves on a topological space have enough structure to be used as the values of qualities[20], for we can even develop integral and differential calculus with them.

[16]For example, if a qrumber b is not invertible then $b = 0$ but $b \neq 0$ does not imply that b is invertible, so qrumbers form only a residue field[15].

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5. Appendix

In this appendix we list some properties of the qrumbers theory that we have used in this paper and refer to where proofs can be found.

(i)Quantum real numbers For a quantum system whose qualities are represented by self-adjoint operators in a concrete operator algebra $\mathcal{M}$. The qrumbers are sections of $\mathbb{R}_{\mathrm{D}}(\mathcal{E}_{\mathcal{S}}(\mathcal{M}))$ where $\mathcal{E}_{\mathcal{S}}(\mathcal{M})$ is the state space of the algebra $\mathcal{M}$. The logic of $\mathbb{R}_{\mathrm{D}}(\mathcal{E}_{\mathcal{S}}(\mathcal{M}))$ is intuitionistic.

By the construction of the topology on $\mathcal{E}_{\mathcal{S}}(\mathcal{M})$, for any self-adjoint operator $A \in \mathcal{M}$ the function $a(\rho) = Tr\rho\hat{A}$ is a globally defined continuous function and therefore determines a global section of $\mathbb{R}_{\mathrm{D}}(\mathcal{E}_{\mathcal{S}}(\mathcal{M}))$. The functions $a(W)$, determined by the self-adjoint operator $\hat{A}$ with domains given by open subsets $W \subset \mathcal{E}_{\mathcal{S}}(\mathcal{M})$, are interpreted as the numerical values of the physical quantity that is represented by the self-adjoint operator $\hat{A} \in \mathcal{M}$. Real numbers of this form are a proper sub-sheaf $\mathbb{A}(\mathcal{E}_{\mathcal{S}}(\mathcal{M}))$ called the sheaf of *locally linear functions.*[2]

Every qrumber is a continuous function of some locally linear functions in $\mathbb{A}(\mathcal{E}_{\mathcal{S}}(\mathcal{M}))$[2], whose sections, the functions a, are *locally linear qrumbers.*

(ii) The quantum real number equations of motion. For the position and momentum of a single massive particle they are given by Hamilton's equations of motion, therefore a quantum particle will follow a trajectory in qrumber space [2]. If the interaction potential function is smooth enough, Heisenberg's operator equations of motion when averaged over certain open sets in state space approximate closely the Hamiltonian equations for the qrumbers defined on those open sets [2]. This shows that, locally in $\mathcal{E}_{\mathcal{S}}(\mathcal{M})$, averages of Heisenberg's equations of motion can give good approximations to the qrumber equations of motion.

(iii) The measurement process. It has three stages:

(1) *Preparing the system* which produces an initial open set W_0, the epistemological condition of the system.

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(2) *A filtering process* which separates different outcomes, algebraically it defines a grating [23] $G = \{E_1, ..., E_k\}$ of maps $\mathcal{O}(\mathcal{E}_{\mathcal{S}}(\pi)) \to \mathcal{O}(\mathcal{E}_{\mathcal{S}}(\pi))$ with $E_j(W_0) = W_j \subset W_0$; $W_i \cap W_j = \emptyset, W_i \neq W_j$; $\cup_i W_i \subset W_0$.

ϵ sharp collimation/realization. If the quality being measured is $\hat{Z}$ and the grating is defined by a family of disjoint slits $\{I_k\}_{k=i}^{N}$, then W_j is such that the qrumber $z(W_j)$ is ϵ sharp collimated for the slit $I_j = [a_j, b_j]$. This means that

$$a_j < z(W_j) < b_j \tag{11}$$

$$\frac{4((z(W_j))^2 - (z)^2(W_j))}{(b_j \; - \; a_j)^2} \leq \; \epsilon. \tag{12}$$

Then $z(W_j) \approx z_j(W_j)$ with an error $k(\epsilon) \to 0$ as $\epsilon \to 0$, where z_j is the qrumber for $\hat{P}_j\hat{Z}\hat{P}_j$, the von Neumann transform of $\hat{Z}$ when $\hat{P}_j$ is the spectral projection operator $\hat{P}^Z(I_j)$ of $\hat{Z}$ for the interval I_j. The standard real number ϵ is controlled by the experimenter.

(3) *Realizing the outcome as a standard real number.* e.g. for $\hat{Z}$ with $I_j = [a_j, b_j]$. If $U_j \in \mathcal{O}(W_j)$ then $a_j < z(U_j) < b_j$ and $\frac{4((z(U_j))^2-(z)^2(U_j))}{(b_j \; - \; a_j)^2} \leq \epsilon$, therefore, when $(b_j - a_j)$ and ϵ are small, $z(U_j)$ is well approximated by $c_j 1(U)$, where $a_j < c_j < b_j$, is an (approximate) eigenvalue of $\hat{Z}$ in the (continuous) discrete spectrum of $\hat{Z}$.

Moreover on U_j the qrumber value of any quality $\hat{A}$ is $a(U_j) \approx a_j(U_j)$, the value of the quality $\hat{P}_j\hat{A}\hat{P}_j$. The approximation gets better as $\epsilon \to 0$.

The "collapse of the wavefunction" rule is an infinitesimal approximation to the qrumber result. If initially the particle is in $V = \Lambda(\rho_0; \delta) = \{\rho | Tr\big|(\rho \; - \; \rho_0)\big| \; < \; \delta\}$ and $\hat{Z}$ is measured on U_j so that its value $z(U_j \cap V)$ is ϵ-realized in an interval I_j. Then any quality $\hat{B}$ has a value $b(U_j \cap V)$ given approximately by $(Tr\rho_0' \cdot \hat{B}) \cdot 1(U_j \cap V)$, where $\rho_0' = \frac{\hat{P}_j \cdot \rho_0 \cdot \hat{P}_j}{Tr(\rho_0 \cdot P_j)}$. on the assumption that $\hat{P}_j \cdot \hat{\rho}_0 \neq \hat{0}$.[4] Infinitesimally, as $\delta \to 0$ we get the the standard collapse result.

(iv)Quantum real number probability has a frequency definition from which the Born probability rule can be deduced. We assume the ergodic hypothesis and that if the particle, in the ontological condition U, ϵ sharply realizes a standard real number for $\hat{Z}$ in an interval $]a, b[$, then the probability that $a < z(U) < b$ is greater than $1 - \epsilon$.[4] Then the probability that a particle, in an ontological condition V, passes through a slit I is the quantum real number $\pi^Z(V)$ for the spectral projection operator, $\hat{P}^Z(I)$, of $\hat{Z}$ on I. If $V = \mathcal{N}(\rho_0; \epsilon) = \{\rho | Tr\big|(\rho - \rho_0)\big| \; < \; \epsilon\}$ then to within ϵ the probability of passing through I equals $Tr\hat{\rho}_0\hat{P}^Z(I)$. Born's rule follows when $\hat{P}^Z(I) = \hat{P}_{\phi_1}$ and $\hat{\rho}_0 = \hat{P}_{\psi_0}$.[4]

References

[1] M. Adelman and J. V. Corbett, Applied Categorical Structures **3**, 79 (1995).

[2] M.Adelman and J.V. Corbett: "Quantum Mechanics as an Intuitionistic form of Classical Mechanics" Proceedings of the Centre Mathematics and its Applications, pp15-29, ANU, Canberra (2001).

[3] J.L.Bell, *A Primer of Infinitesimal Analysis*, pp117-118 (Cambridge University Press,1998)

[4] J.V.Corbett and T Durt, *Quantum Mechanics re-interpreted in qr-numbers*, (2004) preprint.

[5] J.V.Corbett and T Durt, *Quantum Mechanics as space-time theory*, (2005) preprint.

[6] J.V.Corbett, *The Pauli Problem,state reconstruction and quantum real numbers*, Reports on Mathematical Physics, **57**,53-68, (2006) .

[7] J.V.Corbett, *The classical Limit of Qrumber trajectories*, (2006) preprint.

[8] J.V.Corbett, *The complete quantum real numbers picture*, (2006) in preparation.

[9] J.T.Cushing, *Foundational Problems in and Methodological Lessons from Quantum Field Theory*, p29 *Philosophical Foundations of Quantum Field Theory* eds. H.R.Brown and R.Harré, Oxford University Press, Oxford (1990).

[10] R. Dedekind, "Was sind und was sollen die Zahlen",Braunschweig, 1887, SIII.

[11] A. Einstein, "Johannes Kepler", Frankfurter Zeitung ,9 November,1930, on the 300th anniversary of Kepler's death, republished in "Ideas and Opinions", Crown Publishers, Inc., (1954), pp 256-259.

[12] R.P.Feynman, R.B.Leighton, and M.Sands,*Feynman Lectures on Physics*, vol 3(addison-Wesley,Reading,MA,1964), chap 9.1

[13] D. Home, *Conceptual Foundations of Quantum Physics*, Chapter 2.3, pp 78-83, Plenum Press, New York and London, (1997)

[14] A. Inoue,*Tomita-Takesaki Theory in Algebras of Unbounded Operators* Lecture Notes in Mathematics; 1699, Springer (1999), K. Schmudgen, *Unbounded Operator Algebras and Representation Theory* Operator Theory: Vol. 37, Birkhauser Verlag (1990).

[15] P.T. Johnstone, *Topos Theory*, Academic Press, New York, (1977).

[16] A.J. Leggett, in *The Lesson of Quantum Theory*(J.de Boer, E. Dal and O. Ulfbeck,eds.), Elsevier Science Publishers B.V., Amsterdam, (1986) pp 35-57.

[17] S. MacLane and I. Moerdijk, *Sheaves in Geometry and Logic* (Springer–Verlag, New York, 1994).

[18] I. Moerdijk and G.E. Reyes, *Models for Smooth Infinitesimal Analysis* (Springer–Verlag, New York, 1991).

[19] Editorial, "The double-slit experiment", Physics World, September (2002) extended article May (2003) on physicsweb.

[20] L. N. Stout, Cahiers Top. et Geom. Diff. **XVII**, 295 (1976); C. Mulvey, "Intuitionistic Algebra and Representation of Rings" in Memoirs of the AMS **148** (1974), W. Lawvere, "Quantifiers as Sheaves", Actes Congres Intern. Math. **1**, (1970)..

[21] R.G. Swan, *The Theory of Sheaves* (University of Chicago Press, Chicago and London, 1964) §3.2, pp29-31.

[22] J. V. von Neumann: *Mathematische grundlagen der Quanten-Mechanik* , Springer-Verlag, Berlin (1932). English translation: *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, Princeton, 1955.

[23] H. Weyl: *Philosophy of Mathematics and Natural Science* Appendix C, pp255-257, Princeton University Press, Princeton (1949).