Ghost World: A Context for Frege's Context Principle

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Beware the unexampled aphorism, however beguiling in appearance, for opposing parties can quote scripture to their own, divergent purposes. An excellent case in point is provided by Gottlob Frege's celebrated *context principle*:

> *We must never try to define the meaning of a word in isolation, but only as it is used in the context of a proposition*.[1]

"That sounds very nice," we should demur, "but can you supply some concrete 'fer instance' that makes clear what you have in mind?" Indeed, the context principle has been accorded the most extraordinary gamut of readings in the standard literature, in a fashion that draws misty uncertainty over the interpretation of virtually every other central contention within Frege's corpus. To encounter such a nugget of refractory murkiness is surprising, for he is not an author who otherwise courts obscurity.

Nonetheless, these obstacles are entirely Frege's fault, for the following reasons (for which the remainder of this essay will argue).

(i) As he wrote the Grundlagen, Frege had in mind a concrete family of practices, to be found in the "higher geometry" of his time, that he regarded as exemplifying the context principle as he intended it. This methodology, which was practiced in the Göttingen of his student days, is *not* one of "contextual definition" in the usual sense of the term, although allied to it in spirit (which explains why Frege's attitude towards "definitions in context" often seems elusive.

(ii) He initially planned to imitate these same policies in his own treatment of number, in a manner that sidesteps any need to cite a brute comprehension principle of an axiom V type.

(iii) However, in the course of writing sections §§66-8, he realized that a subtle hitch impeded total fulfilment of his ambitions, so he swiftly reverted, without adequate signal to the reader, to a workable form of comprehension principle that was also popular in the geometrical methodology of his time. This shift allowed him to defend a logical analysis of number, but the alteration in underlying policy leaves the context principle unillustrated in his own endeavors.

(iv) Because the glitch in question seems so minor, he continues to believe (at least for the entire period in which he worked upon the Grundlagen) that the *angel* of the context principle continues to hover over his endeavors as a patron saint, even though he actually resorted to cruder expedients to get the job accomplished.

(v) Because he fails to alert his readers to the geometrical precedents in sufficiently vivid terms and because those activities have now faded into the twilight of forgotten mathematical exploit, the modern reader is left with little clue as to how Frege originally expected to apply the context principle to numbers. Worse yet, the original paradigm of application that Frege had before his eyes is so bizarrely unexpected that few readers unfamiliar with nineteenth century geometrical practice will be likely to reconstruct its framework solely on the basis of Frege's own words. This loss of vital background leaves the modern student of the context principle wandering in the wilderness, with no geometrical compass home.

I have told a tale of the Grundlagen of roughly this narrative shape before, but I had some critical details muddled--specifically, I did not sufficiently distinguish the methodological approach pioneered by the German geometer Karl von Staudt from that introduced by his contemporary Julius Plücker, even though I discussed both men in those earlier writings. I saw my way to improvement only after reading an excellent essay on the context principle by Thomas Ricketts,[2] which is perhaps the best commentary written on our topic. Ricketts, however, makes no linkage between his stress on the manner in which "objecthood" and "range of variation" fit together and the geometrical practices recounted here. However, once these complementary strands of doctrine are brought into alignment, the Grundlagen's most mysterious passages can be supplied with satisfactory and concretely focused readings, able, in fact, to correct a few points where I regard Ricketts' unsupplemented account as strained. Such are the benefits of linking slogan with robust example.

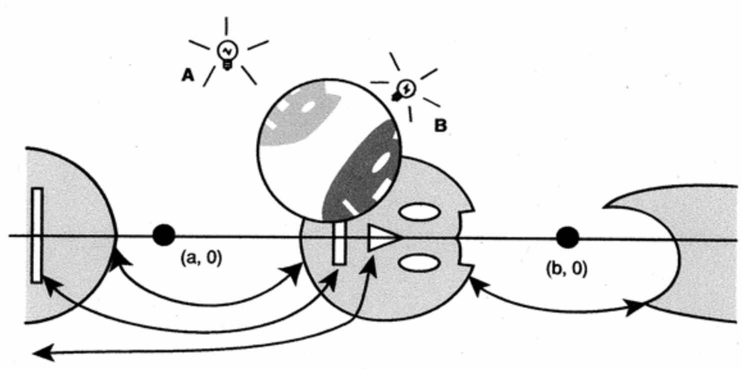
It turns out that the intended workings of the context principle can be best illustrated by examining the geometrical heritage from which Frege apparently extracted his methodological inclinations (and in whose terms much of his early mathematical work unfolded). As I explained more fully in my "Royal Road" paper,[3] the Grundlagen's rather odd discussion of how the *object* ***the direction of line L*** can be defined in terms of the *concept* ***--is parallel to L*** is actually plucked from prior geometrical discussions that seek to equip every line with a supplementary point lying at its infinitely faraway tip. Besides these additional *points at infinity*, geometrical practice also claims that many *imaginary points* sit upon a plane at coordinate locations such as ($-3\sqrt{-1}$, -1.5). The apologetic tales first offered, in the early nineteenth century, on behalf of these outlandish intrusions often sound rather mystical. Thus the important German geometer Jacob Steiner wrote:

> *In this manner one arrives, as it were, at the elements, which nature herself employs in order to endow figures with numberless properties with the utmost economy and simplicity; the organism, by means of which the most heterogeneous phenomena in the world of space are united one with another*[4]

Indeed, if we take such comments, along with those of his distinguished contemporary J.V. Poncelet, seriously, we are allowing stolid, venerable Euclidean geometry to be invaded by *ghosts*. Indeed, the great expositor Felix Klein later wrote:

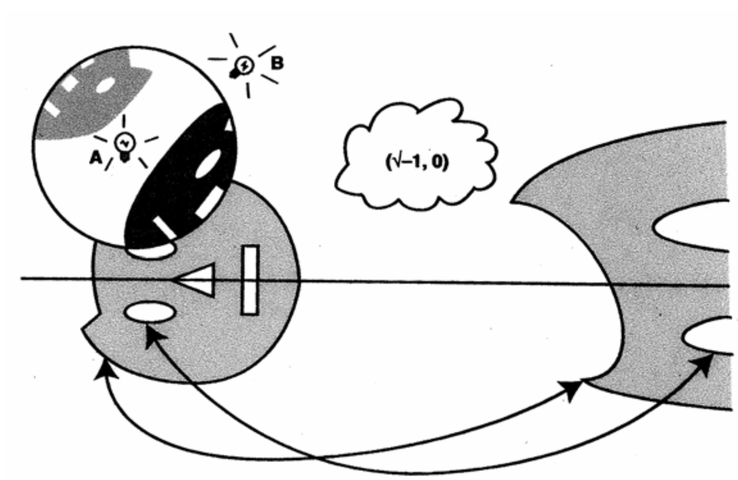
> *The older geometers, Poncelet and Steiner, were never clear on* [the exact rationale for geometry's extension elements]. *To Steiner, imaginary quantities were ghosts, which made their effect felt in some way from a higher world without our being able to gain a clear notion of their existence.*[5]

Because we are seeking a vivid context for Fregean phrases such as "recognition judgment," a quick sketch of how Poncelet and Steiner conceptualized the evidence for these novel apparitions will help set the scene (these details won't be critical in the sequel). The area of geometrical concern in which they worked investigates what happens to images under sequences of projection from one kind of screen to another. In the diagram, light bulb A projects the cat image painted on the left face of sphere **S** onto its opposite hemisphere. Light bulb B then transfers this doubled pattern as a shadow onto the plane **P** (we allow light to travel backwards from the image to the bulb, which is why a portion of the left hand cat face appears over the horizon to the right!). Despite all the stretching and compression that occurs, a good deal of abstract



pictorial structure remains preserved within the planar recasting. It turns out that its critical ingredients can be encapsulated in a remarkably simple way. Let us draw a conveniently placed x-axis **L** upon the plane. Two special points (a,0) and (b,0) can be located on **L** around which the corresponding parts (labeled as x and x') of the cat image must cluster in a very special way: viz. the ratio of the distance λ from (a,0) to the right-hand cat's chin over the distance μ of the matching left-hand chin x' with respect to (a,0) will equal the negative of this same ratio with respect to (b,0). And so on for every other alignment between the two cat portraits. images. This important form of pictorial correlation (a "harmonic" mapping between x and its mate x' along **L**) is called an *involution* and (a,0) and (b,0) represent the two *self-corresponding points* of the map (they also represent the images of the tangent points that light A makes with the sphere). In fact, the placements of (a,0) and (b,0) completely control all of the spatial relationships within our doubled image, for our "cross ratio" invariant can construct the placement of every pair of correspondent cat parts. More general planar points (a,c) and (b,d) likewise represent the "controlling points" of more complicated mapping situations of this same ilk. Thus far our account of geometrical fact remains entirely free of ghosts.

Notice that our two cat faces wind up oriented oppositely to one another on the plane. However, if the initiating light bulb A is moved *inside* the sphere **S**, we obtain a doubled planar shadow where the two cat heads now point in the same direction. Here their correlated parts match up along **L** in an *overlapping* rather than *nested pattern*; hence no self-corresponding points will be evident. However, if, following Steiner, we allow *imaginary points* such as (√-1 , 0) to sit upon **L**, we can again produce two simple "control centers" that completely fix all of the overlapping associations found in our new form of cat map. According to Steiner, such imaginary points represent hidden "elements, which nature herself employs in order to endow figures with numberless properties." "But we can't see any such points," we complain. "Ah, by their involutionary weaving you shall know them," replies Steiner, "look at the tidy ordering that persists in our new variety of cat map." Clearly, this is an argument in favor of *ghosts*, quite comparable to asseverations that we wouldn't be hearing chains dragged up the stairs if this joint weren't haunted.



Without delving further into more mathematics of this ilk, it is hard to convey the astonishing degree of unification amongst seemingly unrelated behaviors that blesses traditional geometry once these bizarre gambits are tolerated. Nor can I adequately survey the legions of allied extension elements that invaded other areas of mathematics within this same period. Some of the latter, it is important to note, proved ill-conceived in their later consequences and eventually required sheepish retraction..

What prompted our early nineteenth century heroes to argue in this mystic manner? If we write down, high school algebra style, the sundry equations for tangent lines, spheres and projections appearing in the light-bulb-outside-**S** case and solve them as required by inter-substitution, we can work our lengthy way to a simple expression for the shadow map in the form $Ax.x' + B(x + x') + C = 0$. Exactly the same schedule of calculations still go forward in light-bulb-inside-**S** circumstances except that lots of complex numbers will pop up in places where real values were obtained before (in particular, our imaginary self-correspondent points (a,0) and (b,0) will emerge as the roots of the associated $Ax^2 + 2Bx + C = 0$[6]). Fortunately, these complex intermediaries multiply out in the final step when our involution mapping formula is extracted. Clearly *something genuine* in the world of geometry must lie behind our complex-valued algebraic steps, Poncelet and Steiner argued, for reasoning cannot stagger its way repeatedly to just conclusions unless those steps are tacitly supported by genuine fact all the way along (a completely proper appraisal, I believe). However, claiming that *ghost lines and points* accordingly support our algebraic reasoning scarcely constitutes an adequate explanation of our successes and invites mathematical hotheads to populate mathematics with ill-conceived supplements willy-nilly .

It isn't surprising that a variety of methodological incantations were quickly suggested for taming such ghostly antics into unthreatening domesticity. The most popular modern rationalization, dominant since its strong advocacy by David Hilbert <u>circa</u> 1900, claims that mathematicians enjoy an intrinsic right to study any "freely created structure" they wish as long as its framework can be tightly specified via categorical axiomatization. From this point of view, Steiner merely elected to switch his attention from the limited structure of traditional Euclidean geometry over to a richer realm he happens to like better (roughly, the idea is: it's okay to speak of ghosts as long as they're pretty and adequately axiomatized). This defense through formalism is less adequate than is commonly assumed,[7] but we needn't bother with those issues here. But we should observe that, historically, the subsequent popularity of Hilbert's recommendations quickly effaced the mathematician's memory of the early programs for geometrical rigorization that attracted Frege's own attention.

As I noted earlier, I failed to adequately distinguish two approaches to the extension elements in my earlier writings on the Grundlagen, although disentangling their ingredients helps greatly in appreciating the precise objectives that the context principle was designed to serve. In particular, I mistakenly assimilated the efforts of analytic geometers like Otto Stolz and Felix Klein,[8] working in the traditions of the German mathematician Julius Plücker, too hastily to the earlier approach of Karl von Staudt.

Let us remark, before we consider details of these schemes, that any attempt to rigorize the extension elements must somehow work with the evidential considerations to which Poncelet and Steiner hazily appealed. They had claimed to recognize the presence of their ghost points through witnessing their involutionary handiwork as it is arrayed along visible lines. An overlapping involution within a projected image is a "sign" that some imaginary point acts secretly behind the scenes. We should therefore expect that such visible mappings will prove central within any rigorization program for imaginary points, for they provide the *recognitional criteria* that allowed geometers to first detect the hidden presence of the latter.

In truth, the claim I have just articulated is too strong, for imaginary points display their geometric meddling in a wide variety of ways, just as imps both drag chains along staircases and rattle dishes in the pantry. The would-be rigorist can elect any of these symptoms of phantom activity to serve as the entry point for her improved treatment of Steiner-flavored geometry; no specific form of poltergeist presence enjoys any methodological privilege in this regard. Nonetheless, involutions induced along a line are commonly selected as the critical "recognitional criterion" utilized in most programs of rigorization.

Von Staudt's own approach is simple in plan, if not execution, although it relies upon a very strange ploy. Consider the *abstract objects* that naturally correspond to the *recognitional concepts* that allow Poncelet and Steiner to "see" their invisible points: if Poncelet espies a parcel of lines falling under the concept **--is parallel to L**, he "sees" a point at infinity; if Steiner witnesses an overlapping involution mapping acting on **L**, he "sees" an imaginary point hovering there.[9] Traditional logic allows that an ample zoo of concept-objects correspond to any such evaluative concept, in just the way that the abstract object **motherhood** answers to the evaluative concept **being a mother**. Since many of these traditional concept-objects arguably suffer fuzzy identity criteria, the least controversial critter within the whole menagerie is the trait's *extension*, viz., the set {x| x is a mother}. Here von Staudt's basic proposal is: consider the evaluations of setting that inspire Steiner and Poncelet to speak of ghost points and construct a salient concept-object based upon the traits central to their evaluations. Thus when Poncelet looks upon the packet of lines parallel to **L** , he says, "Gee, there must be an infinite point on **L** where all of these other lines will meet it. Here the salient concept before Poncelet's mind is **being parallel to L** and the phrase "the direction of **L"** provides an appropriate label for the abstract concept-object that derives from this evaluative trait. Identifying this "concept-object" with its extension equates **L**'s direction with {x| x is a line parallel to **L** }. With these abstract objects in hand, von Staudt proposes that **the direction of L** and **the involution (in either positive or negative senses) along L** can now serve as satisfactory surrogates for the desired ghost points.

I should might remark that, in my limited canvas, advocates of the von Staudt program often write rather nebulously with respect to the exact nature of the concept-objects to which they appeal. Thus Theodore Reye, in a very popular textbook of the period, glibly writes:

> *I would remind you at this point that the statements "parallel lines have the same direction" and "parallel lines contain the same infinitely distant point", mean exactly the same thing*.[10]

A more careful writer such as Frege is apt to select the concept's extension as its best concept-object surrogate, simply because we know exactly how they should be individuated. Through such a route, we quickly come into close proximity to the equivalence class techniques pioneered by Richard Dedekind, although, as I have argued elsewhere, the abstractionist philosophical thinking that leads him to this destination seems rather different in origin.[11]

As is well known, after §68 of the <u>Grundlagen</u>, Frege embraces the equivalence class ploy as his method for introducing numbers, although his discussion prior to this point reads entirely as if some more "contextual" form of account were being sought. Indeed, many readers have been startled by the fact that, in the course of apparently dispatching the last of several objections to a "contextual" treatment, Frege abruptly announces that the project isn't viable and that we should therefore "try another way"! It is my surmise that Frege had fully intended to introduce his numbers by the methodological route I shall now outline (which isn't exactly one of "contextual definition") but abandoned the strategy in favor of extensions after discovering the glitch recorded in §66. With respect to the context principle itself, it is scarcely evident that considerations of a strongly "consider the whole thought first" character play any vital contributing role to the unfolding of a von Staudt-like rigorization program, but it will prove central within the program I shall now outline. By reconstructing Frege's original scheme we can appreciate the concrete objectives for which the context principle was formulated.

As we noted, von Staudt's followers proposed that phrases like "the point at infinity on **L**" can be assigned abstract objects such as sets as their denotations.

From this point of view, we should naturally ask, “Which branch of knowledge assures us that objects of this ilk exist?” Most nineteenth century primers in logic include lengthy discussions of the nature of abstract entities such as **motherhood** and {x| x is a mother} and so it is natural to assume that the “science of logic” represents the repository from which a would-be rigorist might draw her “points at infinity” surrogates. This point of view accordingly mandates that "the science of logic," if rigorously formulated, must embrace primitive comprehension principles of the form

*For any given concept **C**, there is a concept-object #**C** to which it corresponds*.

And, of course, Frege's later axiom V is exactly of this type, although no mention of such principle appears in any of his logical work until after the Grundlagen.

I will now argue that Frege originally planned to evade employment of such a postulate by instead utilizing certain *rewriting techniques* that were familiar to him from his student days at Göttingen. A reexamination of these older practices will render the intended purposes of the context principle considerably more vivid than they appear within the unexampled pages of the Grundlagen’s “switching strategies in midstream” narrative.

The “rewriting” methodology I have in mind was developed by Otto Stolz and Felix Klein and self-consciously blends together elements of earlier geometrical thinking (such as von Staudt’s) in a very intriguing way. Let us begin with some mundane facts. If point (a,b) lies on a line **L**, (a,b) will satisfy an equation of the form y = mx + e. We can abbreviate this incidence claim notationally as [m, e]**I**(a,b), where the concept **[m, e]I--** evaluates (a,b) with respect to its lying upon (= is incident upon) the line **L** or not. Here we should think initially of "[m, e]**I**--" as a fused predicative unit. To take a particular example, "[2, -3]**I**(1, -1)" claims that the point (1, -1) lies on the line y = 2x - 3 (it does).

In point of fact, geometers such as Julius Plücker (who was Klein's early mentor) found it helpful to introduce so-called *homogeneous coordinates* that allow (1, a, b) and all of its multiples (viz. (3, 3a, 3b)) to qualify as equally representing the point (a, b) (the idea originally was inspired by considerations of centers of gravity). Employing this trick, our incidence claim looks mathematically prettier: viz., [2, -3]**I**(1, -1) becomes [-3, 2, -1]**I**(1, 1, -1) where any multiple of [-3, 2, -1] also qualifies as a representative of the same line. In doing so, no phrase with a 0 in its first slot of an (a,b,c) triple[12] is regarded as meaningful—that is, “[-3, 2, -1]**I**(0, 1, 2)” does not unpack into a sensible statement with respect to line/point incidence. Secretly, however, the plan is eventually to utilize the forbidden "(0, 1, 2)" (and all of its multiples) as a coordinate name for the point at infinity lying upon the line y = 2x – 3.

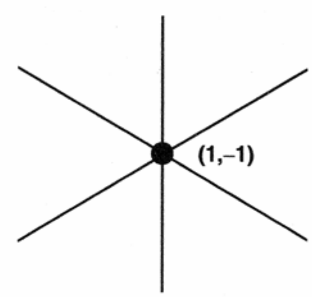
If we treat (1, a, b) as *variable* within our [e, m, -1]**I**(1, a, b) scheme, we can then talk about the *range of points* carved out along a line, viz. "For every point (1, a, b) satisfying [-3, 2, -1]**I**(1, x, y), there is another point (1, a', b') satisfying [-3, 2, -1]**I**(1, x, y) with which it stands in harmonic correlation relative to selected fixed points." When we think of the singular thought [-3, 2, -1]**I**(1, 1, -1) in this way, we conceive the *activity* of its conceptual evaluation as acting from left to right. Picture this as

$$\overset{\Rightarrow}{[-3, 2, -1]\mathbf{I}(1, 1, -1)}$$

However, Plücker suggested that this same thought can be analyzed with an opposite action, even if we don't naturally regard it in this way. That is, we picture its functional activity as

$$\overset{\Leftarrow}{[-3, 2, -1]\mathbf{I}(1, 1, -1)}$$

Parsed this way, the evaluative unit **--I**(1, 1, -1) carves out a packet of lines as its *left-hand value range*. Indeed, this totality of lines will comprise the *pencil* of lines that run through the Cartesian point (1, -1). Applying a quantifier to this newly emancipated slot, we can now express "dualized" general thoughts such as "For every line (e, m, -1) satisfying [x, y, -1]**I**(1, 1, -1), there is another line (e', m', -1) also satisfying [x, y, -1]**I**(1, 1, -1) in which it stands in harmonic correlation relative to selected fixed lines." In other words, the previously fused predicative subunit "[e, m, -1]" is now treated as a species of *name*, providing a *line coordinate* for the line **L** comparable to the homogeneous *point coordinates* (1, 1, -1) we have been employing to locate points on the plane.[13] By supplying appropriate geometrical meaning to the usual operations of addition and multiplication with respect to these new coordinates, equations can be readily articulated in line coordinate terms as well, supplying a *line equation* for a curve fully comparable to, although usually different in form, from its conventional expression as an equation with respect to points (such line equations carve out curves as the envelopes of their tangent lines rather than as loci of points).



Of course, lines are also geometrical objects already familiar to us, but with fairly minimal requirements on good behavior, we can harness the coefficients that appear in virtually any kind of evaluative equation as new "coordinate names" to designate "geometrical objects" that seem novel or even strange in their underlying conception. For example, in his "Lecture on the Geometry of Pairs of Points in the Plane,"[14] Frege employs our trick of reversing the direction of functional activity in a formula (he utilizes the line equation for a degenerate conic) in order to produce suitable coordinate names for *pairs of points* regarded as comprising *single, fused entities.* That is, Frege invites us to look upon a regular Euclidean plane and "see" it, not as decomposing into solitary points, but as instead fragmenting into a gaggle of point-partnerships bound irrevocably together over long distances (this remotely paired structuring is hard to visualize as it constitutes a four dimensional, non-Euclidean geometry). In other words, Frege claims that the usual collection {(9, -3), (-6, 7)} can be alternatively approached as a basic geometrical entity answering to five term homogeneous coordinates like <1, -2, -3.7, -9, 0>, in terms of which regular points like (9,-3) can be derivatively defined as a "locus of point pairs." Plainly, Frege embraces the astonishing contention of the Plücker school that such a methodology provides an acceptable scheme for sculpting a plane or space into base primitive elements other than regular points. Indeed, I believe that the context principle is intended to serve as a philosophical rationale to convince us that such direction-reversing recarvings of thoughts place genuine "objects" before our attention: point pairs as "just as good" qua "objects" as regular points, although they initially prove harder to "see."

Today, courtesy of the fashion in which Richard Dedekind and his school reshaped “algebra” in the late nineteenth century, we have become accustomed to constructing new forms of mathematical objects as equivalence classes of already accepted objects[15] and have probably lost sight of how novel—and even bizarre—the technique would have appeared when first introduced. Urging instead that a range of novel “objects” can be accepted as mathematically well defined once all of their sentential occurrences have been provided with crisp truth-values can easily appear, in the context of Frege’s time, like a less radical technique answering to the same necessities for enlarging mathematics’ dominions rigorously. From this point of view, it is imperative that we look upon Frege's peculiar “point pairs,” not in a modern manner as algebraically ordered pairs, but as primitive objects extracted from the geometrical woodwork by Plückerish techniques

From an *epistemological* point of view, Frege cheerfully concedes that our knowledge of the facts of geometry first comes to us in an intellectual presentation prejudiced in favor of points. But this genetic origin provides no reason, he thinks, why geometry, considered as an tidily organized science, might not find that alternative base elements work more effectively as a choice of primitive basis. As he observes in his review of Lange's book on The Law of Inertia:

> *I should like to subscribe to his statement "that elementary concepts are not the original data of a science," or as I would like to express it, that they must first be discovered by logical analysis.*[16]

Although Frege, in his geometrical endeavors, allows the pursuit of scientific organization to ignore, quite blatantly and radically, any decompositional responsibility to intuitive expectation, this crucial fact seems lost on the many commentators who assume contrariwise that Frege seeks a "conceptual analysis" of "number" that reports, in the mode of Anglo-American analytic philosophy, upon "the knowledge we must possess to prove competent in the use of 'number.'" But never was a thought further from his mind; Frege's loyalty to our "kindergarten" understandings reaches no farther than the weak fealty that geometrical rigorists owe to Poncelet and Steiner's original apologetics on behalf of their novel varieties of "point."

In sum, it is clearly Frege's belief that, although a "point pair" is not an object that we learn about from direct geometrical intuition, this newly extracted gizmo should be regarded as "every bit as real," on an ontological appraisal, as a regular geometrical point. The basic intent of the context principle was to argue for this rather startling form of ontological tolerance, as we shall soon see.

To appreciate that Frege's thinking is not idiosyncratic within his period, let us examine how Stolz and Klein propose to wed this vein of Plückerish decomposition to von Staudt's prior endeavors and thereby generate a novel rationale for the imaginary points and lines and their comrades at infinity. Let me first articulate the basic strategy in rough terms and then introduce a critical refinement. Expressed in homogenous terms, recall that as yet we have supplied no meaning to equations of the form "[m, n, r]**I**(0, b, c)." However, when we study the action of **--I**(1, 2, 1) in a thought like "[1,1,2]**I**(1, 2, 1)," we observe that a pencil of intersecting lines is carved out as its value range. The coordinates of each of these lines will be related to one another by linear transformations. It will then occur to us that we are missing certain packets of the same type, i.e., the arrays of lines that run *parallel* to one another, despite their similar formal relationships to one another. This observation suggests that we should try to *enlarge* (= redefine) our **--I**(a, b, c) family of evaluative concepts to some wider **I*** able to grade the missing bundles of parallels in the desired way. And lo and behold!, it happens that our previously meaningless syntactic combinations "--**I**(0, b, c)" can capture exactly the range of extended discriminations we want if we simply multiply left and right sides together in the same way as we would for "--**I**(1, 1, 2)"-like combinations. That is, the free variable formula [1, X, -Y]**I***(0, 1, 2) supplies us, after multiplication, with the line equation Y = 2X which is satisfied by all lines that satisfy equations of the form y = 2x + b (where the y-intercept b can assume any real value). Clearly every packet of parallels can be captured in these terms. Accordingly, it is easy to definitionally extend our old talk of "**I**" to a new "**I***" that also tolerates right-hand syntax of a "**I***(0, 1, 2)" mien. Of course, the fact that this works isn't a miracle; homogeneous coordinates were designed precisely to play this role.

At this point, "--**I***(0, 1, 2)" still constitutes a holistically fused unit. But, following our "context principle" liberality, we can reverse the functional activity of a singular thought such as [1, 2, -3]**I***(0, 1, 2) to a forward direction, as long as its new right hand range proves adequately well behaved (of which more in a moment). If so, our newly liberated "(0, 1, 2)" now qualifies as the "coordinate name" of some previously undisclosed geometrical object supplementary to our old-fashioned Euclidean points (let us dub this naturally extended value range POINTS). Voila! a suitable "point at infinity," prized from the woodwork of bland geometrical fact through no other means beyond the twin processes of explicitly defined conceptual enlargement and a Plückerish reorientation of functional activity. Thus we appear to have successfully crossed the bar that separates our original range of Euclidean points from the POINTS at infinity without evoking any brute existence postulate of an axiom V type along the way. By the same trick, we can install a "LINE at infinity" by assigning evaluative meaning to the previously meaningless
"[1, 0, 0]**I***--" evaluator**.** Stolz and Klein also showed that von Staudt's program for introducing the imaginary POINTS via involution-"objects" can be mimicked through a rewriting technique of roughly this ilk. That is, they will parse any claim of incidence with an imaginary POINT ("--**I***(-4, -2 √-1, 6)") in terms of a relationship to an overlapping involution installed along some affiliated real line.[17]

Once we have begun in this way, we must take great care in insuring, through a schedule of explicit definitions, that other old geometrical notions (such as "—intersects ...") rigorously extend to liberalized replacements ("--INTERSECTS...") able to ratify the geometrical behaviors we expect to witness within our expanded dominions. Each of these definitional rewritings must be precisely specified and we must never assume, in Poncelet or Steiner's offhanded manner, that such extended relationships continue to hold within our POINT/LINE domain without proof. In this regard, Stolz and Klein successfully imitate the methodological rigor that von Staudt brought to his own work, without needing to rely upon abstract intermediaries such as sets as artificial surrogates for the additional POINTS et al. True: both forms of treatments (von Staudt's and Stolz/Klein's) are founded in the same *recognitional criteria* (parallels and involutions) that inspired Steiner and Poncelet to make their original leaps of geometrical faith, but, in adhering to strict standards of definitional introduction throughout, both forms of rigorist program manage to convert loose inspiration into sober scientific construction. We are thus assured that the extension elements of nineteenth century geometry will never engender the inconsistencies that often bedeviled other forms of mathematical enlargements that had been explicated only by loose hand waving of the sort supplied by Poncelet and Steiner (some of Riemann's great work became subject to these humilations, for example).

Here, then, is the original purpose of the context principle as I reconstruct it. Introduce by definitional extension a brace of new names and predicative expressions, in a manner such that a *well behaved range of syntactic surrogates* for the desired imaginary and infinitely distant points appear. Then argue philosophically that, since the newly introduced names behave exactly like those for accepted forms of object, these new specimens should qualify, in the same fashion as Frege's point-pairs, as "just as good," ontologically, as regular points. Here our Plückerish capacity to reverse the predicative activity within a complete thought is integral to this contention. But if such tenets are accepted, we require no axiom V to install our new points—or natural numbers,--upon the mathematical stage.

However, to carry this philosophical program through, we must ensure that our techniques genuinely assign a truth-value to every grammatically accepted claim and that our newly introduced name-like syntax will behave, from a logical point of view, completely like a range of true names. But this success can be guaranteed, Frege thinks, if we approach our rewriting task in a proscribed order (in these methodological demands Frege goes beyond anything I find in Klein—a point to which I'll revert in a moment). In particular, our program should always commence by articulating the standards we will follow in deciding, given some circumstance involving several imaginary POINTS, how many of them are actually at issue (being able to count the number of POINTS of intersection of two curves is critical to the success of our extended geometry). In this instance, the concern proves rather tricky because quite dissimilar names "(-4, -2 √-1, 6)" and "(-2√ -1, 1 , 3√ -1)" need to designate the same point (they differ merely by a complex multiple). Since Stolz and Klein tie these names to the evaluation of involution maps upon nearby lines, it is necessary to check that our "common multiple" requirement reflects answers to a well-defined equivalence relation amongst these maps, as Stolz and Klein verify. Following Frege, we can set this demand for coherence in the following format.[18] For ordinary points, we already know that

(a, b, c) = (a', b', c') *if and only if there is a* λ *such that* $a = \lambda a'$, $b = \lambda b'$, *etc*.

This "common multiple" behavior, it turns out, naturally corresponds to the way that a certain λ -1/ λ pairing lays itself out as a nested involution along the line leading from the origin through (a, b, c), with the latter serving as one of the involution's self-correspondent points. In other words, for ordinary points a geometrical criterion of the following type holds:

(*) (a, b, c) = (a', b', c') *if and only if both represent the same sense of a coincident involution along exactly the same induced line.*

Stolz and Klein establish the existence of an extended equivalence relation **E** with respect to their imaginary POINT names:

(**) (a, b, c) **E** (a', b', c') *if and only if both represent the same sense of an coincident involution along exactly the same induced line.*

(although the recipe for finding the induced line in question is more elaborate and the involution will now prove overlapping). Once again, the "common multiple" identification of our imaginary POINT names rests upon a natural λ -1/ λ behavior along their associated real line. This affinity shows that the relation **E** behaves as a natural analog to old "=" with respect to our new dominion of POINTS. Finding the proper **E** to extend "=" looks as if it provides a precise key for rigorizing Steiner's old forms of fuzzy pleading on behalf of his "ghosts."

As I understand his terminology, Frege considers (**) to represent the basic *recognition judgment* that, through conceptually extending the old identity criterion (*) in a natural manner, supplies a suitable bridge to an extended relationship (call it =*) that can serve as a suitably behaved identity surrogate over our expanded POINT dominion. In other words, the contents of (*) and (**) supply the ingredients of a definition by cases for the new identity notion "=*" required within our expanded language. Once this surrogate has been installed upon a suitable footing, Frege asks that the other new predicates we shall need (e.g., "--- INTERSECTS __") should be systematically introduced by explicit definition, verifying as we proceed that Leibniz' law with respect to our basal "=*" remains obeyed. As Frege articulates this plan with respect to the points at infinity:

> *The meaning of any other type of assertion about* [points at infinity] *would have first of all to be defined and in defining it we can make it a rule always to see that it must remain possible to substitute for the* [infinite point on any line] *the* [infinite point on] *any line parallel to it.*[19]

Here I have substituted "point at infinity" for Frege's pallid "direction" to render its methodological substance more vivid.

It is easy to see how the introduction of the various species of number can be accommodated to the same scheme: we let "#C" be the name we carve out when we Plückerize the thought that "the objects bearing concept C can be mapped 1-to-1 to the objects bearing concept D." The celebrated "Hume's principle" ("#C = #D if and only if C maps 1-to-1 to D") then can serve as the natural "recognition judgment" we should employ to extend old "=" into the enlarged dominion of natural number. And it is plain that he plans to approach the real and complex numbers in this same vein as well, for he writes in this regard:

> *Everything will in the end come down to the search for a judgment-content which can be transformed into an identity whose sides precisely are the new numbers.*[20]

As I understand his intent, he conceives the judgment on the right side of (**) as a kind of box, such that if we rattle its contents properly (= associate its "(a, b, c)" and "(a', b', c')" pieces to suitable ranges of variation), we can be left with a relational "__**E**..." in its middle so relieved of specific content (the bulk having shifted into the (a, b, c) and (a', b', c') ends of our carton) that this unloaded **E** can adequately play =*'s role with respect to our new objects.

However, let us now examine the subtle glitch that derails all of these pretty plans. Let us attempt to assemble our old "=" and "**E**" pieces into a formal definition by cases for "=*". We obtain:

(a, b, c) =* (a', b', c') *if and only if*

(i) (a, b, c) and (a', b', c') *are old objects and* (a, b, c) = (a', b', c') **or**

(ii) (a, b, c) and (a', b', c') *are new points and both represent the same sense of a coincident involution along exactly the same induced line.*

Note that no clause is required to monitor mixtures of new and old objects because our identity claim will always prove false in such circumstances. Unfortunately, our proposal plainly does *not* satisfy the strict demands upon definitional practice we have set ourselves, for our defiens utilizes vocabulary that does not appear in the old language of unvarnished Euclidean geometry. In fact, neither of the classifiers "old object" or "new point" belong to that vocabulary. Nor, for the same reason, can we permissibly employ terms like "(0,1,2)" as meaningful, for the whole purpose of our definitional endeavors is to *settle a meaning* upon such phrases; we cannot presume that such meanings already lie in place. But without the forbidden vocabulary, we lack a *meaningful pivot* around which the two parts of our definition by cases can turn, for we need a criterion to decide to whether "=*" is to be applied in manner (i) or in manner (ii). But the natural concepts we would like to appoint to this role are "is a new point" and "is a point at infinity" but proper restrictions upon definitional practice forbid both of them to us at this stage.

This, essentially, is the objection that Frege poses for himself in §§66 and 67. There are many passages in the later parts of the <u>Grundlagen</u> that suggest that he harbored a persistent hope that this obstacle might be eventually evaded and that numbers might be successfully introduced without recourse to sets. However, Frege plainly never found a way to do this and he was forced to rely upon the crutch of extensions to complete the <u>Grundlagen</u>.

What has gone wrong? The nub of the problem apparently lies in the fact that, in concentrating initially upon simple identity judgments, we lose our hold over the *fuller geometrical contexts* that motivated the postulation of extension elements in the first place. Recall how Steiner-type thinking conjured up several imaginary points in our discussion of the cat map: we appealed to the qualities of visible involutions allegedly controlled by their unseen influence. As long as we can appeal to the visible clutter that surrounds our ghost points, we will be able to separate the two pivotal clauses in our "definition by cases" by reference to the behaviors of old-fashioned Euclidean objects. For example, consider the claim "Every line and every circle always intersect in four points" (importantly true in our extended framework although plainly false in a strict Euclidean format). Given a specific choice of line L and circle C, then our expanded treatment of "INTERSECTION POINT" can be divided into the correct subcases by appeal to the Euclidean characteristics of L and C: if they intersect in a Euclidian sense, then "POINT common to L and C" is to interpreted as old-fashioned "point" but if L and C fail to intersect in a conventional sense, "POINT common to L and C" is to be parsed, in Stolz/Klein fashion, as "L and C induce the same overlapping involutions along the coordinate axes." In short, once a larger swatch of context is made available to us than is provided with respect to Frege's narrow and out of context identity claims, the critical claims of our new geometry can be easily rewritten in Euclidean terms without circularity. Wider context allows us to eschew forbidden vocabulary such as "is a new point" in favor of entirely kosher Euclidean notions.

However, if we follow this liberalized policy in justifying the key assertions of extended geometry, we will be approaching our task *contextually* in Bertrand Russell's manner: pertinent discourse with respect to "imaginary points" et al. gets rewritten in terms of wider scope articulations that speak only of regular points, parallels and involutionary relationships. If this ploy is adopted, our new surrogate for extended identity "=*" will almost surely fail to obey the regular laws of identity (e.g., "the present king of France = the present king of France" comes out false). But then our "context principle" rationale for claiming that our newly hewn POINTS qualify as "just as good ontologically" as regular points utterly collapses, for our "POINTS" manifest various forms of oddball logical behavior never witnessed in real "objects" (which is one reason why Russell regarded his own contextually introduced "descriptions" as designating *logical fictions*). Such ontological foibles needn't prove an embarrassment insofar as Stolz and Klein are concerned, for they are not motivated by any philosophical concern to establish the imaginary points as "self-subsisting objects." For their purposes, it is enough to settle all discourse of POINTS upon firm truth-values. However, for Frege's more philosophical enterprises, to admit POINTS or numbers merely as “logical fictions” is tantamount to utter defeat. Such are the reasons why I earlier declared that the context principle was never intended as apologetics for “contextual definition” in the Russellian sense illustrated here, but only for a more narrowly proscribed—and, most likely, unworkable—scheme of definitional rewriting.

The alternative von Staudt plan for introducing the extension points via identification with sets (or some other variety of “concept-object”) suffers no comparable lapse because its point surrogates are certified to exist by primitive logical principle. Confronted with his §66 glitch, Frege apparently elected to revert to this older definitional tactic in order to finish the Grundlagen with a properly framed account of number as "self-subsisting." In so doing, he was forced to abandon his plans for reaching that objective without brute appeal to a comprehension principle of axiom V type. Since the Stolz/Klein approach comes tantalizingly close to satisfying his original intentions, we can sympathize with his apparent hope that he will someday discover the clever trick that can patch over the §66 obstacle and supply an introduction of number closer to his original context principle expectations, despite the fact that, at present, he must utilize the expedient of extensions in order to finish his book. And this story, I submit, provides the critical background that explains why the discussion throughout the Grundlagen proves so persistently mystifying: why it is that Frege continually hints at some radical approach to ontological issues that never seems exemplified within his own proposals.

In our own time, a number of authors such as Crispin Wright and Bob Hale have suggested that “recognition judgments” like Hume’s Principle should be accepted as a wholly adequate foundation upon which the introduction of abstract objects can be founded. The most striking aspect of these proposals is that they seem insufficiently concerned, in my limited canvas, with the methodological dangers to which incomplete hand waving specifications of a Poncelet/Steiner ilk are prone. Frege’s own labors, in contrast (as well as those of von Staudt, Stolz and Klein), were motivated by an ambition to ward off such disasters through strict methodological procedures. To our nineteenth century investigators, partial specifications like an unsupplemented Hume’s principle seemed apt to open a Pandora’s box of problems later on. Indeed, I believe that if we must tolerate the pathways of the ill-defined in mathematics, it would be wiser to embrace the darker vision of “conceptual growth” sketched by Ernst Cassirer, for he recognized better the potential disorder that may lie ahead than do the Pollyannas of modern neo-logicism.[21]

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[1] Gottlob Frege, Grundlagen, translated as The Foundations of Arithmetic, J.L. Austin (trans.), Basil Blackwell (Oxford: 1959), p. 116.

[2] "Concepts, Objects and the Context Principle, " forthcoming in Ricketts, ed., The Cambridge Companion to Frege. My main divergence from Ricketts is that he regards subsentential decompositions extracted via range assignment as fixed for all time, whereas I see the technique as authorizing crab-like enlargements within a subject matter such as geometry, carving out ever larger domains in stages as we go. Rickett's treatment thus supplies an accurate and very insightful portrait of Frege's views on logical segmentation, but misses the possibility of the dynamic processes of definitional extension discussed here, necessary, I think, to appreciating how the context principle is directly relevant to Frege's anticipated construction of number, rather than merely serving as a static thesis with respect to logic.

[3] "Frege: The Royal Road from Geometry" in William Demopolous (ed.), Frege's Philosophy of Mathematics, Harvard (Cambridge: 1995). See also my "Frege's Mathematical Setting," to appear in the Ricketts collection cited above.

[4] R.E. Moritz, Memorabilia Mathematica , Dover,(New York: 1958), p. 315.

[5] Felix Klein, Elementary Mathematics from an Advanced Standpoint: Geometry, E.D. Hedrick and C.A. Noble (trans.), Dover (New York:), p. 120.

[6] Cf. R.M. Winger, An Introduction to Projective Geometry, Dover (New York: 1962), p. 167.

[7] E.g., by Ernest Nagel, "The Formation of Modern Conceptions of Formal Logic in the Development of Geometry" in Teleology Revisited, Columbia University Press (New York: 1971). Most ontological issues in mathematics need to be settled by appeal to the existence postulates of modern set theory, not through appeal to formalism alone. Despite its tendentious philosophical opinions, Nagel's essay supplies an excellent survey of the historical background to which I allude.

[8] Stolz, "Die Geometrische Bedeutung der Complexen Elemente in der Analytischen Geometrie," Mathematische Annalen, vol. 4, 1871. I have mainly followed Klein's presentation in Chapter V of his Elementary Mathematics here.

[9] Actually he sees them in conjugate pairs, which raises complications of "sense" that I will ignore here.

[10] Lectures on the Geometry of Position, T.F. Holgate (trans.), MacMillan (New York: 1898), p. 20.

[11] I outlined my reasons for claiming this in "To Err is Humean," Philosophia Mathematica, Vol. 7 (1999). However, the present parsing of Frege's §66 glitch greatly improves upon the account I provided there.

[12] Likewise, b and c cannot both be 0 in [a,b,c].

[13] Line coordinates can also be expressed in inhomogeneous forms as well: [e , m].

[14] In Brian McGuiness (ed)., Collected Papers, Blackwell (Oxford: 1984). Translated by Hans Kaal.

[15] I like Michael Spivak's remark: "The precise definition...uses the same trick that mathematicians always use when they want two things that are not equal to be equal." A Comprehensive Introduction to Differential Geometry, Vol. 1, Publish or Perish (Boston: 1976), p. I-12.

[16] "On the Law of Inertia," Hans Kaal (trans.) in Collected Papers, p. 135.

[17] I refer the interested reader to Klein, op cit., for the details. Strictly speaking, I should write "sensed involution" here.

[18] For a regular point P, the familiar model of the projective plane as lines through the origin within a three dimensional space supplies a real line **L** on which a natural representative for P sits on the plane $z = 1$. In this format, the $(a, b, c) = (\lambda a, \lambda b, \lambda c)$ behavior corresponds to a λ-$1/\lambda$ involution around P on **L** (which can then projected into the $z = 1$ plane). If P is imaginary, Klein separates its values into real and imaginary components and frames a real line **L** between the derivative points P' and P". No matter which $(\lambda a, \lambda b, \lambda c)$ representative for P we choose, the Klein division will select the same real **L**, with the λ-$1/\lambda$ pairings now correlating with an overlapping involution along its length.

[19] Grundlagen, p. 77. It is worth observing that Frege seems to recommend an identical procedure in the Grundgesetze's peculiar §10.

[20] Grundlagen, pp. 114-5.

[21] I should like to thank the editors, Anil Gupta, Jeremy Heis and the members of my Frege seminar for helpful advice.