Constructible Models of Orthomodular Quantum Logics.

Piotr Wilczek

Abstract

Abstract: We continue in this article the abstract algebraic treatment of quantum sentential logics [39]. The Notions borrowed from the field of Model Theory and Abstract Algebraic Logic - *AAL* (i.e., consequence relation, variety, logical matrix, deductive filter, reduced product, ultraproduct, ultrapower, Frege relation, Leibniz congruence, Suszko congruence, Leibniz operator) are applied to quantum logics. We also proved several equivalences between state property systems (Jauch-Piron-Aerts line of investigations) and *AAL* treatment of quantum logics (corollary 18 and 19). We show that there exist the uniquely defined correspondence between state property system and consequence relation defined on quantum logics. We also signalize that a metalogical property - Lindenbaum property does not hold for the set of quantum logics.

Piotr WILCZEK, Department of Functional and Numerical Analysis, Institute of Mathematics, Poznan University of Technology, Piotrowo 3a, 60-965, Poznan, POLAND, edwil@mail.icpnet.pl

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1. Introduction

Quantum logics (just like classical logic) can be considered as a kind of propositional logic. A set of formulae of quantum sentential logics constitutes a complete formal description of physical systems. They describe the quantum entity in the terms of its actual and potential properties – or dually – in terms of its states [1].

The general idea of quantum logics is based on the isomorphism relation between the set of self-adjoint projection operators defined on a Hilbert space and the set of properties of physical system. The set of all self-adjoint projection operators defined on a Hilbert space form – in the algebraic terms – the orthomodular lattice. Above idea can be traced back to the work of von Neumann and G. Birkhoff [2]. In our considerations concerning the foundations of quantum mechanics, we will follow the approach developed by *Geneva-Brussels School* of quantum logic.

There exist two different and competitive ways of understanding the notion of logic. Historically speaking, the old style is to understand a logic as a set of valid formulae (these formulae are also forced to satisfy certain presupposed conditions, for instance the invariance under substitutions). In this case one can identify a logic $\mathcal{S}$ with a set of theorems [20]. The second manner of conceiving logic $\mathcal{S}$ is to define this concept as a consequence relation between sets of formulae and the formula denoted by $\vdash_{\mathcal{S}}$. In this case, a set of formulae is also forced to fulfill a set of certain specific conditions, for example the invariance under substitutions or finitarity. The consequences of the empty set of assumptions are called theorems and they constitute a logic in the old style. Above sketched second definition of logic is called *Tarski style* and belongs to the heritage of *Lvov-Warsaw School of Logic* [20]. This view constitutes the basis for the development of the so-called *Abstract Algebraic Logic* [21] . This kind of research is preferred especially by algebraically oriented logicians. In this paper, we follow this path of investigations.

Modern scientists – mainly theoretic physicists – are interested not only in one description of quantum (or cosmological) phenomena, but they are going to construct a whole set of possible models which correspond to possible pathway of the evolution of the investigated system. This model-theoretic approach is widespread among contemporary scientists and is advised by methodologists and philosophers of science [14, 15]. Basing on above hints concerning the qualitative face of investigations, one can get the complete knowledge indicating the possible ways of the evolution of the investigated physical system. Above methodological requirements prompted us to use the model-theoretic approach in the investigating of the realm of quantum logics.

This article tries to explore the models of quantum logics. In case of classical logic, and more popular and widespread non-classical logics (e.g., intuitionistic, modal and many-valued logics), the model-theoretic problems are well understood and deeply elaborated. However in the case of quantum logics, our knowledge concerning the possible models of this sentential logics is very poor [39]. This article is planned to bridge this gap.

Firstly, we define quantum sentential logic as an absolutely free algebra (section 2). We also define structural consequence operations on this algebra (section 2). The main results of this paper are included in section $3-7$ where we construct several models of quantum logics and give main theorems characterizing these models. The section 8 is devoted to concluding remarks.

2. Preliminary Remarks.

All algebras which are considered in this paper have the signature $\langle \mathbf{A}, \leq, \cap, \cup, \mathbf{0}, \mathbf{1} \rangle$ and are of similarity type $\langle 2, 2, 1, 0, 0 \rangle$. All abstract algebras, such as algebraic structures, are labeled with a set of boldface complexes of letters beginning with a capitalized Latin characters, e.g., $\mathbf{A}, \mathbf{B}, \mathbf{Fm}, ...$ and their universes by the corresponding lightface characters $A, B, Fm, ...$ All our classes of algebra are varieties (varieties of algebras are defined as an equationally definable classes of algebras closed under formation of Cartasian products, ultraproducts, subalgebras and homomorphic images [3]). The fact that the given class of algebras $\mathbf{K}$ is equationally definable means that there exists a set of equations Σ which are satisfied by all members of the class $\mathbf{K}$ [21].

Definition 1. An orthomodular lattice is an algebraic structure $\mathcal{U} = \langle \mathbf{A}, \leq, \cap, \cup, (.)', \mathbf{0}, \mathbf{1} \rangle$ if it satisfies the following conditions:

1) $\langle \mathbf{A}, \leq, \cap, \cup, \mathbf{0}, \mathbf{1} \rangle$ is a bounded lattice with the least element $\mathbf{0}$ and the greatest element $\mathbf{1}$.

2) $(.)'$ is a unary antitone and an idempotent operator called (orthocomplementation) on $\mathbf{A}$ which satisfies the following conditions:

$a)$ for any $x \in A, x'' = x$

$b)$ for any $x, y \in A$,if $x \leq y$ then $y' \leq x'$

$c)$ for any $x \in A, x \cap x' = 0$

3) for any orthomodular law.

We also supposed that all orthomodular lattices considered here are complete. When one removes the orthomodular law from the above definition, one gets the definition of ortholattice. All classes of algebras we mention here are varieties being subvarieties of **OL** (the variety of all ortholattices). One can symbolically depict the relation between algebraic structures which are mentioned in this paper as follows:

$$\mathbf{BA} \subseteq \mathbf{MOL} \subseteq \mathbf{OML} \subseteq \mathbf{OL}.$$

Above abbreviations mean: **BA** – the variety of all Boolean algebras, **MOL** – the variety of all modular ortholattices, **OML** – the variety of all orthomodular lattices, **OL** – the variety of all ortholattices.

Undoubtedly, one can define many other subvarieties of **OL**, but these algebraic structures are not mentioned here.

In our investigations, we work in the frame of binary orthologic introduced by Goldblatt [24]. The definition of binary orthologic corresponding to the **OL** variety can be found in our previous paper [39]. The reader can also find there the listed axiom schemes and inference rules for this logic. The definitions of orthomodular logics (OML) and the modular orthologic (MOL) are also included in [39].

In our investigation of different models of quantum propositional logics, we follow the path taken by algebraically oriented logicians. We use the definition of the sentential language as an absolutely free algebra [38, 40, 41, 39]. **Fm** denotes the algebra of formulae which is supposed to be absolutely free algebra of type **L** over a denumerable set of generators $Var = \{p, q, r, ...\}$. The set of free generators is identical with the infinite countable set of propositional variables. Inductive definition of formula describing quantum entities can be found in [39].

The algebra of terms **Fm** is endowed with finitely many finitary operations (in sentential language – connectives) $F_1, F_2, ..., F_n$. The structure $Fm = \langle \mathbf{Fm}, F_1, F_2, ..., F_n \rangle$ is called the algebra of formulae – or equivalently – the algebra of terms [40, 41, 21]. It was stated explicitly in our previous paper that the notion of quantum logic can be identified with the structural consequence operation [29, 39]. The concept of logic or – more generally – the concept of deductive system in the language of type **L** is defined as a pair $\mathcal{S} = \langle \mathbf{Fm}, \vdash_{\mathcal{S}} \rangle$ where **Fm** is the algebra of formulae of type **L**, and $\vdash_{\mathcal{S}}$ is a substitution-invariant consequence relation on $\mathbf{Fm}$. More precisely, the consequence relation is defined as a: $\vdash_{\mathcal{S}}$ $\mathcal{P}(Fm) \times Fm$ satisfying the formal conditions stated in [38, 40, 41, 39]. ($\mathcal{P}(Fm)$ denotes the power set of Fm). We also explicitly postulate that for every $X \subseteq Fm$ and every $\alpha \in Fm$, the subsequent equivalence holds:

$$X \vdash_{Cn} \alpha \text{ iff } \alpha \in Cn(X).$$

In our paper it is supposed that all considered logics are finite, i.e., structural consequence operations are finitary [39].

By a model for quantum sentential logics we mean a couple $\mathcal{M} = \langle \mathbf{A}, F \rangle$ where $\mathbf{A}$ is an algebra of the same similarity type as the algebra of terms of a given propositional language, and F is a subset of the universe of the algebra $\mathbf{A}$, i.e., $F \subseteq \mathbf{A}$, and F is called the set of designated elements of $\mathcal{M}$. The structure $\mathcal{M} = \langle \mathbf{A}, F \rangle$ is termed logical matrix and can be understood as a semantical model of the given sentential logic. The notion of logical matrix is regarded as a fundamental notion of *Abstract Algebraic Logic* [38, 40, 41, 21]. Every logical matrix consists of an algebra which is homomorphic with the algebra of formulae of a considered propositional logic. Logical matrices adequate (see part 5) for quantum logics are formed of a variety of **OL** or **OML**. These varieties are considered as canonical classes of homomorphic algebras forming logical matrices. To every formula φ of the language of quantum logic, one can ascribe a unique interpretation in the algebra $\mathbf{A}$ which depends on the values in $\mathbf{A}$ that are assigned to variables of this formula [38, 40, 41].

Since $\mathbf{Fm}$ is absolutely free algebra freely generated by a set of variables (i.e., the set of free generators) and $\mathbf{A}$ is an algebra of the same similarity type as $\mathbf{Fm}$, then there exists a function $f : Var \to A$ and exactly one function $h^f : Fm \to A$ which is the extension of the function f, i.e., $h^f\ (p) = f(p)$ for each $p \in Var$. Above function is the homomorphism from the algebra of the terms into the algebra $\mathbf{A}$ constituting logical matrix $\mathcal{M} = \langle \mathbf{A}, F \rangle$ [40, 41, 21].

Using logical matrix as a basic tool in the algebraic treatment of logic, one can identify the interpretation of a given formula φ of Fm with $h(\varphi)$ where h is a homomorphism from $\mathbf{Fm}$ to $\mathbf{A}$ that maps each variable of φ into its algebraic counterpart, i.e., into its assigned value. If we represent a formula of quantum logic in the form $\varphi(x_0, x_1, ..., x_{n-1})$ in order to indicate that each of its variables occur in the list $x_0, x_1, ..., x_{n-1}$ then $\varphi^A(a_0, a_1, ..., a_{n-1})$ denotes the algebraic translation of this formula for a given homomorphism $h(\varphi)$ such that $h(x_i) = a_i$ for all $i < \omega$. Considering a quantum logic $\mathcal{S}$ in the language of the type $\mathbf{L}$, we can say that matrix $\mathcal{M} = \langle \mathbf{A}, F \rangle$ is a semantic model of $\mathcal{S}$ iff for every $h \in Hom_{\mathcal{S}}(\mathbf{Fm}, \mathbf{A})$ and every $\Gamma \cup \{\varphi\}$:

$$\text{If } h[\Gamma] \subseteq F \text{ and } \Gamma \vdash_S \varphi \text{ then } h(\varphi) \in F.$$

In this case, the set $\mathbf{F}$ is called a *deductive filter* of the logic $\mathcal{S}$ – or alternatively – *Sasaki deductive filter* of this logic [35, 38, 40, 41, 4]. By $h \in Hom_{\mathcal{S}}(\mathbf{Fm}, \mathbf{A})$ we mean a homomorphism from the algebra of formulae into the variety of algebra constituting logical matrix for quantum logics. For a given quantum logics one can define a whole set of Sasaki deductive filters. This set is partially ordered (by the set-theoretic relation of inclusion) and is denoted by $\mathcal{F}i_{\mathcal{S}}\mathbf{A}$. The class of logical models (i.e., logical matrices) for quantum propositional logics is denoted by $\mathbf{Mod}\mathcal{S}$ [21, 39].

As a starting point of our investigations in this paper we assume the corollaries included in [39]. The strong version of the consequence operation is determined by the class of models of quantum logics as follows [24]:

$$\Gamma \vdash_S \varphi \text{ iff } \forall \mathbf{A} \in \mathbf{OML}, \forall h \in Hom(\mathbf{Fm}, \mathbf{A}), \forall a \in A \text{ if } a \leq h(\beta), \forall \beta \in \Gamma \text{ then } a \leq h(\varphi).$$

Corollary 2. *The class of matrices:*

$$\mathbf{Mod}\mathcal{S} = \{\langle \mathbf{A}, [a) \rangle : \mathbf{A} \in \mathbf{Mod}\mathcal{S}, a \in A\}.$$

is a matrix semantics for the strong version of quantum logic. $[a)$ *is a principal filter of the form* $\{x \in A : x \geq a\}$ [24].

In this paper our attention will be focused mainly on above defined Sasaki deductive filters. If it is not stated otherwise F denotes Sasaki deductive filter of the form $[a) = \{x \in A : x \geq a\}$ [29, 39].

Corollary 3. *The class of matrices:*

$$\mathbf{Mod}\mathcal{S} = \{ \langle A, \{1\} \rangle : \mathbf{A} \in \mathbf{Mod}\mathcal{S}\}.$$

is a matrix semantics for the weak version of quantum logic [26, 29, 39].

The Sasaki deductive filters defined by these versions of quantum logics are one-element subsets of **OML**, i.e., $F = \{1\}$ [26]. This kind of Sasaki deductive filters will be mentioned only occasionally.

3. Simple Models for Quantum Sentential Logics.

By a simple model for quantum sentential logics we mean an ordered pair $\mathcal{M} = \langle \mathbf{A}, F \rangle$ where **A** denotes variety of algebra associated with algebra of formulae of quantum logics, i.e., the variety of **OL** or **OML**, and F denotes the Sasaki deductive filter of this algebra (also called a deductive filter). It was mentioned in the previous section that by a logic one can understand a structural consequence operation. This is a purely logical definition of a deductive system. Nevertheless, in case of quantum logics (identified with structural consequence operations defined on an algebra of formulae expressing properties of a given quantum system), there exists also the physical interpretation of such conceived notion of logic (i.e., the structural consequence operation).

In the realm of quantum mechanics, the rays of a Hilbert space are understood as a mathematical representation of (pure) states of a physical system [18]. In this paper we supposed that there exists a bijection between rays of the Hilbert space (formal representation of quantum entity) and the structural consequence operations defined on a corresponding orthomodular lattice (i.e., the lattice of the properties of quantum entity). Basing on the excellent paper of K. Engesser and D. M. Gabbay, [18] one can assume that the physical state of a quantum system can be understood as a *„state of provability”* or more adequately as a *„state of experimental provability”*. Such conceived correspondence between the logical notion of structural consequence operation and physical concept of state can be simply illustrated. Let x denote pure state, A and B denote two observables, for instance energy and momentum of an elementary particle. It is not supposed that observables must be *„sharp”* in x. It is said that a given observable with a value λ in the state x is sharp if a measurement yields the value λ with probability equal 1. Our considerations are conducted in the language containing atomic formulae which have the following meaning: $A = \lambda, B = \rho, ...$ It is supposed that observable A is not sharp in state x. By α we denote the proposition $A = \lambda$, and by β the proposition $B = \rho$. We measure A and the outcome is equal λ. If we end up our measurement (experiment), then the quantum entity is in a state y ($x \xrightarrow{A} y$) in which observable A is sharp (projection postulate of quantum mechanics). In the state y, observable B is sharp with value ρ (subsequent assumption). Shortly, it can be said *„if in the state x a measurement of A yields λ, then, after measurement, the system is in a state in which observable B is sharp with value ρ”* [18]. Symbolically it can be expressed: $\alpha \vdash_x \beta$.

The relation $\vdash_x$ is considered as a consequence relation since it has all formal properties of consequence operator (the whole example is borrowed from [18]).

4. Model-Theoretic Operations on Single Models.

As it was explained in the first part of this article, by a simple model for quantum sentential logics we mean an ordered pair $\mathcal{M} = \langle \mathbf{A}, F \rangle$ where $\mathbf{A}$ is a homomorphic algebra with regard to a quantum sentential language, and F is a Sasaki deductive filter of this algebra. Basing on the classical results from *Model Theory* obtained by Tarski, Malcev, Robinson, Łoś, Chang, Keisler and other ([38, 28, 36, 27, 7]) one can define several different constructible models adequate for quantum sentential logics and operation on them.

Let $\mathcal{M} = \langle \mathbf{OML}, F \rangle$ and $\mathcal{N} = \langle \mathbf{OML}, G \rangle$ be similar matrices. Suppose that F and G are two Sasaki deductive filters. A mapping $h : \mathcal{M} \rightarrow \mathcal{N}$ is called a matrix homomorphism from $\mathcal{M}$ into $\mathcal{N}$, symbolically $h \in Hom_{\mathcal{S}}(\mathcal{M}, \mathcal{N})$, when:

$$\text{if } a \in OML \text{ and } a \in F \text{ then } h(a) \in G.$$

One-to-one matrix homomorphisms are called isomorphic embeddings. When an isomorphic embedding h is *onto*, then h is termed an isomorphism. If $h \in Hom_{\mathcal{S}}(\mathcal{M}, \mathcal{N})$ is onto, then $\mathcal{N}$ is called a matrix homomorphic image under h. We use the notation $\mathcal{M} \cong \mathcal{N}$ when matrices $\mathcal{M}$ and $\mathcal{N}$ are isomorphic.

Let $\mathcal{M} = \left\langle \mathbf{OML}^{\mathcal{M}}, F \right\rangle$ and $\mathcal{N} = \left\langle \mathbf{OML}^{\mathcal{N}}, G \right\rangle$ be similar matrices. $\mathcal{M}$ is said to be a *submatrix* (*submodel*) of $\mathcal{N}$ (in symbol $\mathcal{M} \subseteq \mathcal{N}$) if $\mathbf{OML}^{\mathcal{M}}$ is a subalgebra of $\mathbf{OML}^{\mathcal{N}}$ and $\mathcal{M} = \mathbf{OML}^{\mathcal{M}} \cap \mathcal{N}$.

Let $\mathcal{M}_i = \langle \mathbf{OML}_i, F_i \rangle$, $i \in I$, be a family of similar matrices. By the *direct product* of matrices $\mathcal{M}_i$, $i \in I$, we understand the matrix $\prod\limits_{i \in I} \mathcal{M}_i = \langle OML, F \rangle$ where $OML = \prod\limits_{i \in I} OML_i$ is the direct product of algebras , $i \in I$, and $F = \prod\limits_{i \in I} F_i$, i.e., is the direct product of Sasaki deductive filters. The elements of the set $\prod\limits_{i \in I} OML_i$ are denoted by $\langle f(i) : i \in I \rangle$, $\langle g(i) : i \in I \rangle$ if all matrices $\mathcal{M}_i$ are the same, then $\prod\limits_{i \in I} \mathcal{M}_i$ is called a *direct power* of $\mathcal{M}$. It is denoted by $\mathcal{M}^I$.

Considering classes of algebras and classes of logical matrices we may introduce the standard class operator symbols $\mathbb{I}$, $\mathbb{H}$, $\overleftarrow{\mathbb{H}}$, $\mathbb{S}$, $\mathbb{P}$, $\mathbb{P}_S$, $\mathbb{P}_U$, $\mathbb{P}_R, \mathbb{P}_{R_m}$. They means, respectively, for the formation of isomorphic and homomorphic images, homomorphic counterimages, subalgebras, direct and subdirect products, and ultraproducts. $\mathbb{P}_R$, $\mathbb{P}_{R_\sigma}$ stand for the reduced products and $\sigma-$reduced products, respectively, where σ is a regular cardinal number [34]. The class of all matrix/algebraic homomorphic counterimages of member of $\mathbf{K}$ (i.e., the class of algebras or logical matrices) is defined:

$$\mathcal{M} \in \overleftarrow{\mathbb{H}} (\mathbf{K}) \text{ iff there exists a matrix } \mathcal{N} \in \mathbf{K} \text{ and a matrix homomorphism } h : \mathcal{M} \to \mathcal{N}.$$

Additionally, the class operator $\mathbb{U} = \mathbb{U}_{Var}$ is defined:

$$\mathbb{U}(\mathbf{K}) = \{\mathbf{A} : \text{ every subalgebra of } \mathbf{A} \text{ generated by } \leq |Var| \text{ free generators belongs to } \mathbf{K}\}.$$

Definition 4. The class of **OML** algebras is termed $\mathbb{ISP}-$*class* if it is closed under $\mathbb{I}$, $\mathbb{S}$ and $\mathbb{P}$ [40, 41, 34].

$\mathbb{ISP}$-class is termed a $\mathbb{UISP}$-class if it is closed under $\mathbb{U}$. This is a quasivariety if it is closed under $\mathbb{P}_U$ and a variety if it is closed under $\mathbb{H}$. For every class **OML** of the orthomodular algebras it follows that:

$$\mathbb{ISP}(\mathbf{OML}) \subseteq \mathbb{UISP}(\mathbf{OML}) \subseteq \mathbb{ISPP}_U(\mathbf{OML}) \subseteq \mathbb{HSP}(\mathbf{OML})$$

These symbols stand for the smallest $\mathbb{ISP}$-class, the smallest $\mathbb{UISP}$-class, the smallest quasivariety and the smallest variety containing **OML**, respectively.

Applying the standard procedure of the models' construction, one can also define *reduced products* of elementary matrices. If $\{\mathcal{M}_i\}_{i\in I}$ is an indexed family of matrices of the same type, $\mathcal{M}_i = \langle OML_i, F_i\rangle$ and ∇ is a filter (proper filter) over the set of indexes I, then the reduced product of matrices $\{\mathcal{M}_i\}_{i\in I}$ modulo ∇ is denoted by

$$\prod_{i\in I}\mathcal{M}_i/\nabla$$

More precisely, the matrix $\prod\limits_{i\in I}\mathcal{M}_i$ is defined in the following manner. On the Cartesian product $\mathcal{C} = \prod\limits_{i\in I} OML_i$, we define the relation $=_\nabla$ of ∇-equivalence by the condition: for $f, g \in \mathcal{C}$, $f =_\nabla g$ iff $\{i \in I : f(i) = g(i)\} \in \nabla$. The relation of ∇-equivalence is a congruence of the algebra $\prod\limits_{i\in I} OML_i$. It follows from the definition that $\prod\limits_{i\in I}\mathcal{M}_i/\nabla = \langle OML_\nabla, F_\nabla\rangle$ where $OML_\nabla = \prod\limits_{i\in I} OML_i/\nabla$ and $F_\nabla = \prod\limits_{i\in I} F/\nabla$. The members of $\prod\limits_{i\in I}\mathcal{M}_i$ are denoted by f_∇, g_∇ or $\langle f(i) : i \in I\rangle_\nabla$, $\langle g(i) : i \in I\rangle_\nabla$. If $\mathcal{M}_i = \mathcal{M}$ for all $i \in I$, then the reduced product may be written $\prod\limits_{i\in I}\mathcal{M}/\nabla$ or simply $\mathcal{M}^I/\nabla$ and is called the reduced power of $\{\mathcal{M}_i\}_{i\in I}$ modulo ∇.

If filter ∇ is an *non-principal ultrafilter* over I, denoted by $\mathcal{U}$, then $\prod\limits_{i\in I} \mathcal{M}_i/\mathcal{U}$ is termed the ultraproduct of matrices $\{\mathcal{M}_i\}_{i\in I}$. If $\mathcal{M}_i = \mathcal{M}$ for all $i \in I$, then the ultraproduct may be written $\prod\limits_{i\in I}\mathcal{M}/\mathcal{U}$ or simply $\mathcal{M}^I/\mathcal{U}$. We suppose that this ultrafilter is non-principal and countably incomplete. The ultrafilter $\mathcal{U}$ defined on the set of natural numbers is termed *countably incomplete* if there is a sequence of elements of $\mathcal{U}$ satisfying for every $J \in \mathcal{U}$:

$$J_1 \supseteq J_2 \supseteq ...., \bigcap_{k=1}^{\infty} J_k = \varnothing.$$

Theorem 4 (cf. [40, 41]). For each standard consequence operation defined on the quantum sentential language, the class $Matr(C)$ is closed under $\mathbb{I}$, $\mathbb{S}$, $\mathbb{P}$, $\mathbb{H}$, $\mathbb{H}^C$, $\mathbb{P}_R$ and $\mathbb{P}_U$.

Proof: see [40, 41]. □

In above theorem, the symbol $Matr(C)$ denotes the algebraic semantics for quantum logics. A good behaviour of a given logic – from semantical point of view – is often indicated by stipulation that this logic must satisfy the so-called *Czelakowski's theorem* ([40, 12]).

Theorem 5 ([12, 40]). *Let Cn be a standard consequence operation, and let $Cn = C_\mathcal{M}$ for some matrix semantics* **K**. *Then:*

$$Matr(C) = \overleftarrow{\mathbb{H}}\,\mathbb{HSP}_{R_\sigma}(\mathbf{K}).$$

Moreover, if Cn is finitary and $\sigma = \aleph_0$ then

$$Matr(C) = \mathbb{H}_C \mathbb{HSP}_R(\mathbf{K}) = \overleftarrow{\mathbb{H}}\, \mathbb{HSP}_U(\mathbf{K}).$$

Proof: see [40, 41]. □

5. Adequacy of single logical matrices for quantum logics.

As it was stated in the author's previous paper, all logical matrices constituting a model for quantum sentential logics determine not only the set of their own tautologies, but mainly the so-called matrix consequence operation – $C_{\mathcal{M}}$ [39].

For all logical matrices $\mathcal{M} = \langle \mathbf{OML}, F \rangle$ and for arbitrary $X \subseteq Fm$ and $\alpha \in Fm$, the operation $C_{\mathcal{M}}$ is defined:

$$\alpha \in C_{\mathcal{M}}(X) \leftrightarrow (h(X) \subseteq F \rightarrow h(\alpha) \in F) \text{ where } h \in Hom_{\mathcal{S}}(\mathbf{Fm}, \mathbf{OML}).$$

Such operator $C_{\mathcal{M}}$ can be understood as a structural consequence operation. Basing on above considerations, one can generalize the notion of $C_{\mathcal{M}}$ and introduce the operator $C_{\mathbf{K}}$. The symbol $\mathbf{K}$ denotes the class of matrices. The operator $C_{\mathbf{K}}$ is defined: for arbitrary $X \subseteq Fm$ and for arbitrary $\alpha \in Fm$ it is the case that:

$$\alpha \in C_{\mathbf{K}}(X) \text{ iff } \forall \mathcal{M} \in \mathbf{K}(\alpha \in C_{\mathcal{M}}(X)).$$

Above introduced operator $C_{\mathbf{K}}$ is named the consequence operator determined by the class $\mathbf{K}$ of matrices. The consequence operators $C_{\mathcal{M}}$ constitute the complete lattice. In the lattice-theoretic term, the operator $C_{\mathbf{K}}$ can be defined as follows:

$$C_{\mathbf{K}} = inf\{C_{\mathcal{M}} : \mathcal{M} \in \mathbf{K}\}.$$

Definition 6 ([27, 40, 41]). The class $\mathbf{K}$ of matrices is termed *adequate* for sentential calculus iff for arbitrary $X \subseteq Fm$ and $\alpha \in Fm$ subsequent conditions are satisfied:

$$\alpha \in Cn(X) \text{ iff for every } \alpha \in C_{\mathbf{K}}(X).$$

or shortly:

$$Cn = C_K$$

Definition 7 ([40, 41]). Logical matrix is termed *Cn-matrix* if for every set of formulae $X \subseteq Fm$ it follows that:

$$Cn(X) \subseteq C_{\mathcal{M}}(X).$$

Such matrix $\mathcal{M}$ is called Cn-matrix if the consequence operator determined by this matrix - $C_{\mathcal{M}}$ – is not weaker than consequence operator Cn. Symbolically:

$$Cn \leq C_{\mathcal{M}}.$$

In [39] several algebraic and semantical conditions were presented in the form of theorems so that the subsequent equality for quantum logic was satisfied:

$$Cn = C_{\mathcal{M}}$$

Such posed question concerning the sentential logics belongs to the core problems of *Abstract Algebraic Logic* and was studied from the early beginnings of this branch of logic. In modern terminology, above sketched problem can be expressed as follows: To give necessary and sufficient conditions (having synctactical and algebraic characters) which must be satisfied by a given logic $\langle \mathbf{Fm}, \vdash_{\mathcal{S}} \rangle$ in order to indicate a single matrix which is *strongly adequate* for this logic. A matrix is termed *strongly adequate* for a given logic if the following equality is satisfied:

$$Cn = C_{\mathcal{M}}.$$

The first investigations into the problem of the adequacy of logical matrices for quantum sentential logics were carried out by Malinowski [29]. Historically speaking, the problem of the existence of strongly adequate models for a given propositional logics can be regarded as a generalization of the problem of the so-called *weak adequacy* for logics. This topic constitutes the problem of finding a single matrix – for a given structural consequence operation Cn – so that the subsequent equality was satisfied :

$$Cn(\varnothing) = E(\mathcal{M}).$$

In this notation $E(\mathcal{M})$is a set of logical tautologies determined by a given logical matrices. Matrix which satisfied above equality is termed *weakly adequate* matrix for a given sentential logic. Every weakly adequate logical matrix for quantum sentential logic determines the set of tautologies of this logic. The formula which is satisfied in a matrix under the given homomorphism is denoted by $Sat_h(\mathcal{M})$.

In the case of quantum logic, subsequent equalities take place:

$$\alpha \in Sat_h(\mathcal{M}) \leftrightarrow h(\alpha) \in F.$$

$$Sat_h(\mathcal{M}) = h^{-1}(F).$$

The set of formulae which are satisfied under the homomorphism h is a counterimage of a set of designated values (in this case, the only designated value is **1**) with regards to this homomorphism. The tautologies of quantum logics are identified with a set of formulae which are satisfied for every valuations (i.e., for every homomorphisms) of sentential variables of the term algebra – **Fm**. Above set is designated by $E(\mathcal{M})$. The following equality takes place:

$$E(\mathcal{M}) = \bigcap_h Sat_h(\mathcal{M}) \text{ where } h \in Hom_{\mathcal{S}}(\mathbf{Fm}, \mathbf{OML}).$$

Basing on *Geneva-Brussels* approach to the foundation of quantum mechanics, the subsequent theorem can be deduced:

Theorem 8. *In the case of quantum sentential logics weakly adequate matrices (i.e., the sets of formulae determined by these matrices) can be identified with the so-called trivial question.*

Definition 9 ([31, 37]). *Trivial question* in the framework of *Geneva-Brussels* paradigm is identified with the following definite experimental procedure:"Do whatever you wish with the system and assign the response "yes"" [31].

Above experimental situation also encompasses doing nothing with the physical system. We can call this experimental procedure certain iff the physical entity exists. The trivial question is true always when we are certain of obtaining the positive answer (i.e., *„yes”*) were we to perform this question. The only condition – and indeed, *ontological (existential) condition* – of the trivial question is that we have a physical system to begin with.

Proof of the theorem 8. From the definition of weakly adequate matrices it follows that for all $\alpha \in Sat_h(M) \leftrightarrow h(\alpha) \in F$ for all $\alpha \in Sat_h(\mathcal{M}) \longleftrightarrow h(\alpha) \in F$ for all
$h \in Hom_{\mathcal{S}}(\mathbf{Fm}, \mathbf{OML})$. Now, consider two arbitrarily chosen Sasaki deductive filters determined by the strong version of quantum logic, i.e., they have the form:

$F_1 = [a_1) = \{x_1 \in OML : x_1 \geq a_1\}$ and $F_2 = [a_2) = \{x_2 \in OML : x_2 \geq a_2\}$ (Corollary 2). From these definitions of filters it is obvious that they must have *at least* one common element, i.e., top element of **OML**. This top element is identified with **1**. Hence, in order to be sure that a given formula is always true we choose such homomorphisms $h \in Hom_{\mathcal{S}}(\mathbf{Fm}, \mathbf{OML})$ that $h(\alpha) = \mathbf{1} \in F_i$ for arbitrary i. Such defined formula α is a trivial question in the sense of definition 9. □

Theorem 10. *There must exists at least one quantum entity.*

Proof : Above theorem belongs to the so-called ontological presuppositions of quantum logics. The tautologies (i.e., $Cn(\emptyset) = E(\mathcal{M})$) of classical logic are satisfied even in the empty domain. Since tautologies of quantum logics are satisfied under the presuppositions that there exists *at least* one quantum entity to answer the trivial question positively, i.e., $h(\alpha) \in F$. □

The problem of ontological assumptions of quantum logics will be discussed in full elsewhere.

In this article it is only signalized that such model-theoretic constructions as reduced products and ultraproducts can be used to describe not only separated quantum entities but also entangled ones.

6. Pasting of Single Models of Quantum Logics.

Subsequent model-theoretic construction useful in the studying of quantum sentential logics is the so-called $\{0,1\}$-pasting (Bruns and Kalmbach 1971, 1972, Ptak and Pulmannova 1991, Miyazaki 2005)[5, 6, 33, 30].

Definition 11 ([30]). Let $\mathcal{A} = \langle \mathbf{A}, \leq, \cap, \cup, (.)', \mathbf{0}_{\mathcal{A}}, \mathbf{1}_{\mathcal{A}} \rangle$and $\mathcal{B} = \langle \mathbf{B}, \leq, \cap, \cup, \mathbf{0}_{\mathcal{B}}, \mathbf{1}_{\mathcal{B}} \rangle$ are two non-trivial orthomodular lattices. By $\{0,1\}$-pasting of these lattices one understands the structure $\mathcal{A}+\mathcal{B} = \langle (\mathbf{A} \cup \mathbf{B})/_{\equiv}, \leq, \cap, \cup, \mathbf{0}, \mathbf{1} \rangle$ where $\equiv$ is an equivalence relation defined $\equiv =_{df} \{(x, x) \mid x \in A \cup B\} \cup \{(\mathbf{0}_{\mathcal{A}}, \mathbf{0}_{\mathcal{B}}), (\mathbf{0}_{\mathcal{B}}, \mathbf{0}_{\mathcal{A}}), (\mathbf{1}_{\mathcal{A}}, \mathbf{1}_{\mathcal{B}}), (\mathbf{1}_{\mathcal{B}}, \mathbf{1}_{\mathcal{A}})\}$.The relation of order $\leq$ and other operations $\cap, \cup, (.)'$ are inherited from original orthomodular lattices $\mathcal{A}$ and $\mathcal{B}$.

In the literature one can find two alternative concepts naming this construction: $\{0,1\}$-pasting ([5, 6]) and the term *'horizontal sum'* ([33]). It is a well known facts, from the theory of orthomodular lattices that $\{0,1\}$-pasting of Boolean algebras is an orthomodular lattice [30]. The horizontal sum of finitely or infinitely many ortholattices or orthomodular lattices - $\sum \mathcal{A}_i$ – is defined in a similar way. Basically, for given orthomodular lattices $\mathcal{A}$ and $\mathcal{B}$, $\{0,1\}$-pasting $\mathcal{A}+\mathcal{B}$ is an orthomodular lattice where $\mathbf{0}_{\mathcal{A}}$ and $\mathbf{0}_{\mathcal{B}}$ are identical to the new smallest element $\mathbf{0}_{\mathcal{A}+\mathcal{B}}$ and $\mathbf{1}_{\mathcal{A}}$ and $\mathbf{1}_{\mathcal{B}}$ are identical to the new largest element $\mathbf{1}_{\mathcal{A}+\mathcal{B}}$. Other elements are the same as in the original lattices $\mathcal{A}$ and $\mathcal{B}$.

From the definition of variety it does not follow that variety must be closed with regards to the operation of $\{0,1\}$-pasting. In fact, there are varieties which are closed under this operation and varieties which are not closed with regards to horizontal sum.

Observation 12 ([30]). *The variety* **OML** *is closed under* $\{0,1\}$*-pasting. As it can be deduced from the figure below, neither variety* **MOL** *nor* **BA** *is closed under* $\{0,1\}$*-pasting.*

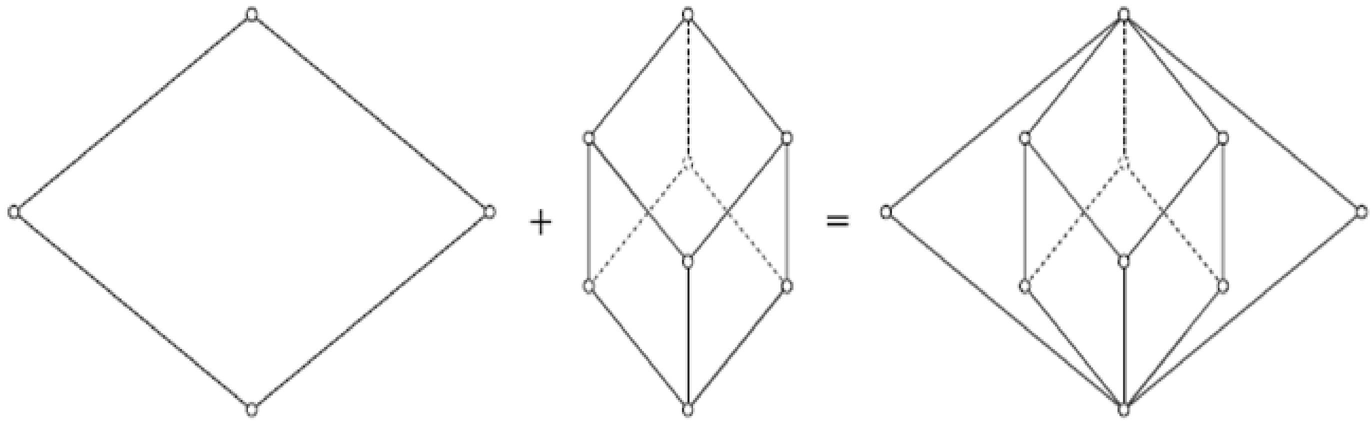

Figure 1. $\{0,1\}-$pasting of two Boolean algebras ([30]).

7. Congruences of Orthomodular Lattices and State Property System.

The set of all congruences of an algebra $\mathbf{A}$ is denoted by $Co\mathbf{A}$. Considering the set of all congruences of the orthomodular lattices $Co\mathbf{OML}$ it is stated that this set constitutes a distributive and Brouwerian lattices [8].

Definition 13 ([12, 22]). Let $\theta \in Co\mathbf{OML}$ and $F \subseteq OML$. It is defined that θ is *compatible* with F, symbolically $\theta compF$ when for all $a, b \in F$, if $\langle a, b\rangle \in \theta$ and $a \in F$ then $b \in F$. The set $Co\mathbf{OML}$ is a *distributive* and *Brouwerian lattice*.

The congruence θ is compatible with F iff F is a union of equivalence classes of θ [12, 21, 22]. Above relationship can be also expressed using the projection mapping π : $\mathbf{OML} \to \mathbf{OML}/\theta$, it is the case that $\theta compF$ iff $F = \pi^{-1}[G]$ for some $G \subseteq A/\theta$ [21, 22]. Congruences of an algebra $\mathbf{OML}$ compatible with F are also termed congruences of the matrix $\mathcal{M} = \langle \mathbf{OML}, F\rangle$ (or alternatively – the strict congruences of $\mathcal{M} = \langle \mathbf{OML}, F\rangle$). Above introduced the projection mapping π is canonical surjective homomorphism. When θ is compatible with F it can be assumed that:

$$F = \{a/\theta : a \in F\}.$$

The largest congruence of $\mathbf{OML}$ which is compatible with F can be always indicated. This congruence is called the *Leibniz congruence* of the matrix $\mathcal{M} = \langle \mathbf{OML}, F\rangle$ and is denoted by $\Omega_{\mathbf{OML}}F$ (the notion of the Leibniz congruence belongs to the field of Abstract Algebraic Logic and is also used when other algebras (logics) are considered) [12, 21, 22]. In the general case, the Leibniz congruence is denoted by $\Omega_{\mathbf{A}}F$. The congruences of the matrix constitute the *principal ideal* of the lattice $Co\mathbf{OML}$ generated by $\Omega_{\mathbf{OML}}F$. The matrix $\mathcal{M} = \langle \mathbf{OML}, F\rangle$ is called *reduced* – or *Leibniz reduced* – when its Leibniz congruence is the identity on **OML**, i.e., $\Omega_{\mathbf{OML}} = id$. For an arbitrary matrix $\mathcal{M} = \langle \mathbf{OML}, F\rangle$ its *reduction* is equivalent to its quotient by its Leibniz congruence, i.e., the matrix of the form $\mathcal{M}^* = \langle \mathbf{OML}/\Omega_{\mathbf{OML}}F, F/\Omega_{\mathbf{OML}}F\rangle$. The definition of the Leibniz congruence is absolutely independent of any logic (i.e., structural consequence operation). It is *intrinsic* to **OML** and F [12, 21, 22].

The class of reduced matrix models of a logic $\mathcal{S}$ is symbolized by $\mathbf{Mod}^*\mathcal{S}$. The class of algebraic reducts of the reduced models of $\mathcal{S}$ i.e., the class of algebras that is associated with a logic $\mathcal{S}$ is denoted by $\mathbf{OML}^*\mathcal{S}$ (in a general case : $\mathbf{A}\lg^* \mathcal{S}$) [12, 21, 22]. Formally, the class $\mathbf{OML}^*\mathcal{S}$ can be defined as follows:

$$\mathbf{OML}^*\mathcal{S} = \{\mathbf{OML} : \exists F \in Fi_{\mathbf{S}}\mathbf{OML} \text{ and } \Omega_{\mathbf{OML}}F = id\ \}\,.$$

The subsequent useful tool to study logical matrices for quantum logics is a *Frege relation* of a matrix $\mathcal{M} = \langle \mathbf{OML}, F\rangle$ relative to the logic $\mathcal{S}$ [12, 21, 22]. This relation is denoted by $\Lambda_{\mathcal{S}}^{\mathbf{OML}}F$ and is defined on **OML** by:

$$\Lambda_{\mathcal{S}}^{\mathbf{OML}}F = \{\langle a,b\rangle \in OML \times OML : \forall G \in \mathcal{F}i_{\mathcal{S}}\mathbf{OML}, F \subseteq G \rightarrow (a \in G \leftrightarrow b \in G)\}\,.$$

Above relation means that $\langle a,b\rangle \in \Lambda_{\mathcal{S}}^{\mathbf{OML}}$ iff a and b belong to the same Sasaki deductive filter of an algebra **OML** which include F. Alternatively, it can be expressed:

$$\langle a,b\rangle \in \Lambda_{\mathcal{S}}^{\mathbf{OML}}F \text{ iff } Fi_{\mathcal{S}}^{\mathbf{OML}}(F \cup \{a\}) = Fi_{\mathcal{S}}^{\mathbf{OML}}(F \cup \{b\}).$$

One can also introduce the notion of the *Suszko congruence* of the matrix $\mathcal{M} = \langle \mathbf{OML}, F\rangle$ relative to $\mathcal{S}$ – it is the largest congruence included in $\Lambda_{\mathcal{S}}^{\mathbf{OML}}F$. The Suszko congruence is denoted by $\overset{\sim}{\Omega}{}_{\mathcal{S}}^{\mathbf{OML}}$. Formally, the Suszko congruence for every Sasaki deductive filter F on the algebra **OML** is defined:

$$\overset{\sim}{\Omega}{}_{\mathcal{S}}^{\mathbf{OML}} F = \bigcap\ \{\Omega_{\mathbf{OML}}G : G \text{ is a Sasaki deductive filter of } \mathbf{OML} \text{ and }\ F \subseteq G\}\,.$$

Contrary to the Leibniz congruence, the notion of Suszko congruence *is not intrinsic* to **OML** and F but it depends on the whole logic $\mathcal{S}$, i.e., all Sasaki deductive filters embracing a given F [12, 13, 21, 22]. Explicitely, it can be expressed that the Suszko congruence relative to a logic $\mathcal{S}$ of a matrix $\langle \mathbf{OML}, F\rangle \in \mathbf{Mod}\mathcal{S}$ depends not only on the Sasaki deductive filter F but also on the whole family of these filters which include F:

$$[F)_{\mathcal{S}} = \{G \in \mathcal{F}i_{\mathcal{S}}\mathbf{OML} : F \subseteq G\}\,.$$

This collection of the Sasaki deductive filters is a closure system (or closed-set system) on the universe of **OML**. The Suszko congruence can be conceived as a function of the family of models for the quantum logics, i.e., $\{\langle \mathbf{OML}, G\rangle : G \in [F)_{\mathcal{S}}\}$,or equivalently of the pair $\langle \mathbf{OML}, [F)_{\mathcal{S}}\rangle$ [21, 22].

In the operational approach to algebraic logic, the notion of Leibniz operator is introduced [12, 21, 22]. The Leibniz operator Ω is a function which assigns to each Sasaki deductive filter the largest congruence θ of the term algebra compatible with F. Compatibility of the largest congruence of **OML** with arbitrary Sasaki deductive filter is defined as a congruence of **OML** such that for all $a \in OML$ we have:

$$\text{either } a/\theta \subseteq F \text{ or } (a/\theta) \cap F = \varnothing.$$

.

If two elements $a, b \in OML$ are orthogonal (written $a \perp b$), i.e., $a \leq b'$, then they can not simultaneously belong to the same Sasaki deductive filter. Alternatively, there does not exist deductive filter (i.e., $\mathcal{S}-$theory) which embraces these two elements.

Theorem 14. *Let $a, b \in OML$. It follows that:*

if $a \perp b$ then $\neg\exists F$ such that $a, b \in F$ where F is an arbitrary Sasaki deductive filter.

In words, there does not exist the deductive filter i.e., the logical theory, which simultaneously realizes two orthogonal properties.

Proof: Simple, from definitions. *If* $a \perp b$ then there does not exist congruence relation θ such that $\langle a, b\rangle \in \theta$ and θ would be compatible with F. $\square$

In the considerations of the foundation of quantum mechanics the problem of the orthogonal states arises. In the operational approach to orthogonality relation developed mainly in the *Geneva-Brussels School* the following definition can be formulated ([1]):

Definition 15 [37]. Two quantum states $p, q \in \Sigma$ are *orthogonal*, written $p \perp q$, if there exists a definite experimental project α such that α is certain for p and impossible for q.

In the seminal works of Aerts it was stated explicitly that the complete description of the quantum particle (the quantum entity) can be identified with the so-called *state property system*, i.e., the ordered triple $(\Sigma, \mathcal{L}, \xi)$ [1]. In this notation, Σ denotes the set of states, $\mathcal{L}$ is a set of properties (the so-called property lattice) and ξ is a function from Σ to $\mathcal{P}(\mathcal{L})$. Conceptually, a state is an abstract name for a singular realization of the particular physical system [31]. Equivalently, a state (or more precisely a state of provability or a state of experimental provability, cf. section 3 of this article) can be identified with the consequence operation defined on the property lattice. Above equivalence can be deduced from the fact that according to the *Geneva-Brussels School*, a state is a dual notion with regards to the concept of property. To each state p we can associate the family $\xi(p)$ of all of its actual properties, and conversely, to each property a we can associate the family $\kappa(a)$ of all states in which this property is actual. In order to formulate above sketched duality, one can introduce the mapping $\xi : \Sigma \longrightarrow \mathcal{P}(\mathcal{L})$. The set of all properties which are actual in a given state $p \in \Sigma$ are denoted by $\xi\ (p) \in \mathcal{P}(\mathcal{L})$. Dually, if $(\Sigma, \mathcal{L}, \xi)$ is a state property system, then its *Cartan map* is the mapping $\kappa : \mathcal{L} \to \mathcal{P}(\Sigma)$ defined ([1]):

$$\kappa : \mathcal{L} \to \mathcal{P}(\Sigma) : a \to \kappa(a) = \{p \in \Sigma \mid a \in \xi(p)\}\,.$$

Theorem 16. *If the property lattice $\mathcal{L}$ is atomistic and orthomodular, then to each state $p \in \Sigma$ one can attribute the unique Sasaki deductive filter (of above lattice). The mapping between the definite state p and F defined on the property lattice is injective.*

Proof and comments : Basing on the mapping $\xi : \Sigma \longrightarrow \mathcal{P}(\mathcal{L})$ in the *Geneva-Brussels School* notation, it can be deduced that a given state $p \in \Sigma$ is identified with the unique subset of property lattice, i.e., with the set of all properties which are actual in this state - $\xi(p)$. In our terminology, the set of all actual properties (or more precisely – their algebraic counterparts in orthomodular lattices) is identified with the Sasaki deductive filter defined on $\mathcal{L}$. Hence, for the definite state $p \in \Sigma$ there exists the unique Sasaki deductive filter of $\mathcal{L}$. Above defined correspondence is not surjective since not all possible Sasaki deductive filters must be realized by the considered quantum entity T. The injectivity of this correspondence derives form the fact that there does not exist two non-equivalent states p_1, p_2 such that $p_1 \neq p_2$ and $\xi(p_1) = \xi(p_2)$. $\square$

Corollary 17. *Any state p of the quantum entity T can be uniquely represented as a particular Sasaki deductive filter F defined on $\mathcal{L}$. Equivalently, any state p can be identified with a particular Sasaki deductive filter F defined on a term algebra. It can be expressed by the following equality:*

$$\xi(p) = F \subseteq OML.$$

Let us recall that a $\mathcal{S}$*-theory* of quantum logic is an arbitrary set of sentences describing the quantum entity of the fixed language. If this set is closed under a consequence operation C, i.e., if $X = C(X)$, or equivalently if $X = C(Y)$ for some Y, then X is called a $\mathcal{S}$-theory of C [38, 39]. In equivalent terminology, $C(X)$ is also called a deductive system or, simply, a system of C. $C(X)$ is the least $\mathcal{S}$-theory of C that contain X and $C(\varnothing)$ and is the system of all logically provable or - equivalently speaking - logically valid sentences of C. It can be stated that :

$$\text{if } \varphi \in C(X) \text{ then } \varphi \in X.$$

One can say that the *deductively closed set* $C(X)$ is termed a $\mathcal{S}$*-theory.* The set of all $\mathcal{S}$-theories of a given quantum logic is denoted by $\mathbf{Th}\mathcal{S}$. This set of all $\mathcal{S}$-theories defined on one given logic is ordered by set-theoretic inclusion and it constitutes a complete lattice $\mathbf{Th}\mathcal{S}=\left\langle \mathbf{Th}\mathcal{S}, \bigcap, \bigcup \right\rangle$ [38, 12, 11, 39]. Considering an algebraic semantics for this quantum logic (i.e., logical matrices - $\mathcal{M} = \langle \mathbf{OML}, F\rangle$) it can be stated that any $\mathcal{S}$-*theory* has its algebraic counterpart in the form of Sasaki deductive filters $\{F_i\}_{i\in I}$ defined on **OML**, i.e., property lattice $\mathcal{L}$. The different $\mathcal{S}$-theories correspond to different Sasaki deductive filters. Hence, if F has the form $[a) = \{x \in OML : x \geq a\}$ (corollary 2) then it corresponds to one $\mathcal{S}$-theory $C(X)$ defined in an orthomodular quantum logic $\mathcal{S}$. The set of all Sasaki deductive filters is denoted by $\mathcal{F}i_{\mathcal{S}}\mathbf{OML}$ and if these filters have the form $[a)$ then their set constitute a complete lattice. If $X \subseteq OML$ then one can always indicates the least Sasaki deductive filter of **OML** which contains X. This filter is *generated* by X and is denoted by $\mathcal{F}i_{\mathcal{S}}^{\mathbf{OML}}(X)$. The largest $\mathcal{S}$-theory is the set Fm of all formulae - and dually - the smallest $\mathcal{S}$-theory is the set of all $\mathcal{S}$-theorems (i.e., the formulae φ such that $\vdash_{\mathcal{S}}$, where $\vdash_{\mathcal{S}}$means that $\varnothing \vdash_{\mathcal{S}} \varphi$). For any two $\mathcal{S}$-theories $\mathcal{T}, \mathcal{S}$ we have $\mathcal{T} \cup \mathcal{S} = \bigcap \{\mathcal{R} \in \mathbf{Th}\mathcal{S} : \mathcal{T} \cup \mathcal{S} \subseteq \mathcal{R}\}$. So it is possible to define a deductive system as the pair $\langle Fm, \mathbf{Th}\mathcal{S}\rangle$.

Corollary 18. *The lattice of all $\mathcal{S}$-theories $\mathbf{Th}\mathcal{S} = \left\langle \mathbf{Th}\mathcal{S}, \bigcap, \bigcup \right\rangle$ defined on* **Fm** *(i.e., on the term algebra describing quantum entity) is isomorphic to the lattice of all Sasaki deductive filters $\mathcal{F}i_{\mathcal{S}}\mathbf{OML}$.*

Summing up above considerations (sections 3 *and* 7*) one can claim that every quantum state (p) can be identified with the particular set of its actual properties i.e., $\xi(p)$. These sets of actual properties are proper subsets of the property lattice $\mathcal{L}$. By an identification of $\mathcal{L}$ with an algebraic semantics for orthomodular quantum logics, i.e., OML, one can deduce that above proper subsets of $\mathcal{L}$ are exactly the Sasaki deductive filters of an algebra constituting above mentioned algebraic semantics for these logics. The deductive filters in Abstract Algebraic Logic (AAL) correspond to the deductively closed sets named $\mathcal{S}-$theories. Every deductively closed set, i.e., every $\mathcal{S}-$theory, corresponds to a quantum consequence operation C defined on a term algebra of quantum logics $\mathcal{S}$ deeply studied in [24, 18, 29, 39]. An algebraic treatment of logical systems gives a general and uniform understanding of the deductive relationship between different terms and between sets of these terms [38, 28, 40, 41, 19, 21, 12, 39].*

Hence, there exist the well-defined one-to-one correspondence between the different consequence operations defined on **OML** *(which determine the Sasaki deductive filters - corollary* 1 *and* 2*) and the different $\mathcal{S}-$theories (i.e., different deductively closed sets on* **OML***). Any structural consequence operation can be understood as a separate sentential logic. It brings about that one can say that there exist plenty of quantum logics on the same* **OML***. Any quantum logic is identified with a separate deductive Sasaki filter. A consequence operation C defined on* **OML** *is additionally termed a structural consequence operation if C also satisfies the following condition:*

$$e(C(X)) \subseteq C(e(X)) \text{ for } X \subseteq Fm.$$

Here, e denotes any substitution in the universe of the term algebra $\mathbf{Fm}$. *From a purely algebraic point of view a substitution in quantum logic can be regarded as a function:*

$$e : Var \rightarrow Fm.$$

Based on above fact and assuming that the algebra of terms is the free algebra this function e can be extended to an endomorphism:

$$h^e : Fm \rightarrow Fm.$$

Or under assumption that there exist the set of homomorphisms $h : Fm \rightarrow OML$ one can introduce a composition of two functions namely $h \circ e$ which is defined:

$$h \circ e : Fm \rightarrow OML.$$

In the author's opinion the process of identification of a single quantum state with one consequence operation on **OML**- *or alternatively - with one Sasaki deductive filter (or with one $\mathcal{S}-$theory) is a fundamental concept bringing together the logical notion of provability or deducibility with the physical notion of a quantum state. One can see that there exist the uniquely determined one-to-one correspondence between the Geneva-Brussel approach to the foundation of quantum theory and the above algebraic treatment of quantum sentential logic. In a common opinion the notion of logic understood as a structural consequence operation is the most important logical concept. Regarding logic as a structural consequence operation is the contribution of Lvov-Warsaw school of logic and initiates the development of the so-called abstract algebraic logic (AAL) and model theory of propositional logic.*

One can states two following theorems:

Corollary 18. *A logical matrix (i.e.,a logical model) constituting of* **OML** *and of Sasaki deductive filter F, i.e., $\mathcal{M} = \langle \mathbf{OML}, F \rangle$, can be understood as a particular realization of one quantum state $p \in \Sigma$.*

Corollary 19. *Following conditions are equivalent:*

a) there is a one-to-one correspondence between the set of all quantum states Σ which are allowed for one quantum entity T and the family of all logical matrices $\mathcal{M}_i = \langle \mathbf{OML}, F_i \rangle$ (where $i \in I$) *adequate for quantum logic describing this entity.*

b) the family of all Sasaki deductive filters $\{F_i\}_{i \in I}$ is in a one-to-one correspondence with the set of all quantum states Σ which are allowed for this quantum entity T.

c) the set of all theories of quantum logic $\mathcal{S}$ denoted by $\boldsymbol{Th}\mathcal{S}$ is in a one-to-one correspondence with the set of all quantum states Σ which are allowed for this quantum entity T.

d) the set of all theories of quantum logic $\mathcal{S}$ denoted by $\mathbf{Th}\mathcal{S}$ is in a one-to-one correspondence with the closed set system defined on the property lattice $\mathcal{L}$.

e) if there exist a sequence (finite or infinite) of quantum states $p_1, p_2, p_3, \ldots$ describing the evolution of quantum entity then there exist the corresponding sequence of logical matrices (i.e., the models) $\mathcal{M}_1 = \langle \mathbf{OML}, F_1 \rangle$, $\mathcal{M}_2 = \langle \mathbf{OML}, F_2 \rangle$, $\mathcal{M}_3 = \langle \mathbf{OML}, F_3 \rangle$, ...which differ by their Sasaki deductive filters.

f) the lattices $\mathcal{F}i_{\mathcal{S}}\mathbf{OML}$ and $\mathbf{Th}\mathcal{S} = \left\langle \mathbf{Th}\mathcal{S}, \bigcap, \bigcup \right\rangle$ are isomorphic.

Using above sketched formalism the theorem characterizing the orthogonal quantum states can be formulated (cf. definition 15):

Theorem 20. *If two quantum states $p, q \in \Sigma$ are orthogonal (i.e., $p \perp q$) then two Sasaki deductive filters F_p and F_q which correspond to these states have at most one common element. It means that their intersection constitute of a one-element set. This one-element set is* **1** *- the top element of* **OML** *lattice. Formally,*

$$F_p \cap F_q = \{\mathbf{1}\}.$$

Proof: Two Sasaki deductive filters which correspond to the orthogonal quantum states have the form $F_p = [a_p) = \{x_p \in OML : a_p \leq x_p \leq \mathbf{1}_p\}$ and $F_q = [a_q) = \{x_q \in OML : a_q \leq x_q \leq \mathbf{1}_q\}$ where $\mathbf{1}_p$ and $\mathbf{1}_q$ are the maximal elements of these filters. Basing the Zorn lemma it is obvious that $\mathbf{1}_p = \mathbf{1}_q$. It means that these two quantum states answer in the same manner to definite experimental project consisting only of a trivial question. $\square$

The orthogonality relation is symmetric and antireflexive [1, 37]:

$$\text{If } p \perp q \text{ then } q \perp p \text{ and } p \neq q.$$

It can be easily observed that a physical condition of symmetricity of this relation can be translated into the language of quantum logics in the form of a *filter distributivity property* of these deductive systems. The filter distributivity property is a metalogical property deeply investigated in *AAL* [12, 21, 22].

Proposition 21. *If the orthogonality relation between two different states is symmetric then two Sasaki deductive filters corresponding to these states commutes. It can be alternatively stated that the lattice of all Sasaki deductive filters ($\mathcal{F}i_{\mathcal{S}}\mathbf{OML}$) which can be defined on the same* **OML** *is distributive. The logic with this property is termed a filter-distributive logic.*

The class of the filter-distributive logics is very wide and includes also all orthomodular quantum logics. This property is shared by all those logics which have a disjunction. The fact that the lattice $\mathcal{F}i_{\mathcal{S}}\mathbf{OML}$ is distributive has its purely algebraic counterpart in the observation that **OML** has the property of congruence-distributivity.

The Lindenbaum property states that any semantically consistent set of terms admits a semantically consistent complete extension [23]. It is well known that a $\mathcal{S}-$theory in *AAL* can be equivalently considered as a set of non-contradictory formulae. A complete $\mathcal{S}-$theory is characterized by the following formal condition:

$$\forall\beta(\beta \in \mathcal{T} \text{ or } \neg\beta \in \mathcal{T}).$$

The *Lindenbaum property* asserts that any $\mathcal{S}-$theory can be extended to a complete, maximal non-contradictory $\mathcal{S}-$theory. Formally:

$$\mathcal{T} \subseteq \mathcal{T}_{\max}.$$

In [23] it was proved that the orthomodular quantum logics does not satisfies the Lindenbaum property. It means that there does not exist a complete, maximal $\mathcal{S}-$theory formulated in the quantum logic language. Here we give an alternative proof for this fact.

Basing on the previous corollaries it is known that any $\mathcal{S}-$theory can be equivalently represented as a Sasaki deductive filters defined on **OML**. Supposing that any formula of quantum logic is uniquely represented as a single element of **OML**, formally:

$$h(\varphi) = a \in OML \text{ where } h \in Hom_{\mathcal{S}}(\mathbf{Fm}, \mathbf{OML}).$$

then a complete, maximal $\mathcal{S}-$theory of quantum logic is represented as a Sasaki deductive filter which is an ultrafilter.

Claim 22. In the case of the orthomodular quantum logics the Lindenbaum property is equivalent to the Ultrafilter lemma.

The Ultrafilter lemma asserts that every filter on a set X can be extended to some ultrafilter on X.

Basing on the claim 22 it can be deduced that a complete, maximal $\mathcal{S}-$theory on **OML** is represented by a *Sasaki deductive ultrafilter* on **OML**.

Theorem 23. *In the orthomodular quantum logics the Lindenbaum property does not hold - or equivalently - there does not exist the Sasaki deductive ultrafilters on* **OML**.

Proof and comments: From the definition of an ultrafilter $\mathcal{U}$ defined on a set X it follows that if A is a subset of X then either A or $X \backslash A$ is an element of $\mathcal{U}$. In the language of AAL it is equal to the fact that for any formula φ, φ or $\neg\varphi$ has its algebraic counterpart, i.e., a or a', belonging to $\mathcal{U}$ defined on **OML**. Formally, if $\mathcal{S}-$theory is complete and maximal then

$$\forall\varphi(\varphi \in \mathcal{T} \text{ or } \neg\varphi \in \mathcal{T}).$$

Algebraically such $\mathcal{S}-$theory corresponds to an ultrafilter $\mathcal{U}$ defined on **OML**. Suppose that if $h(\varphi) = a$ where $h \in Hom_{\mathcal{S}}(\mathbf{Fm}, \mathbf{OML})$ then

$$a \in OML \text{ or } a' \in OML.$$

From the theorem 14 concerning the orthogonal properties we know that if $a \perp b$ are two orthogonal properties represented by two elements $a, b \in OML$ then there does not exist the Sasaki deductive filter embracing these two elements (a and a' are trivially orthogonal). Now choose such $a, b \in OML$ which are in the relation of non-trivial orthogonality, i.e., $a \perp b$ and $a' \neq b$ then there does not exist the Sasaki deductive ultrafilter embracing these two orthogonal elements, i.e.,

$$\neg\, \mathcal{U} \text{ such that if } a \perp b \text{ then } a, b \in \mathcal{U} \text{ where } a' \neq b.$$

Undoubtedly, if a and b are trivially orthogonal, i.e., $a' = b$, then it is trivially true that there does not exist such $\mathcal{U}$ that $a, a' \in \mathcal{U}$. □

Subsequent property which can be expressed in the language of AAL applied to quantum logics is the property of equivalent quantum states:

Definition 24. We call states $p, q \in \Sigma$ *equivalent* and denote them by $p \approx q$ iff $\xi(p) = \xi(q)$.

In the AAL treatment we know that any quantum state may be identified with a single Sasaki deductive filter (corollary **19**), i.e., $\xi(p) = F_p \subseteq OML$.

Hence, if we suppose that two states are equivalent, i.e., $\xi(p) = \xi(q)$, then their Sasaki deductive filters must be equivalent, i.e., $F_p = F_q$. From the fact that a deductive filter correspond to a deductively closed set of formulae we obtain the following relationship:

$$\text{if } p \approx q \text{ then } C(X_p) = C(X_q).$$

Above equality means that two set of terms $X_p, X_q \in Fm$ describing the properties of quantum entity are equivalent with respect to a given logic C if their closures, i.e., $C(X_p)$ and $C(X_q)$ are equal.

8. Concluding Remarks.

Contrary to other non-classical logics (just like many-valued logics, modal logics and intuitionistic logics) quantum logic is not well elaborated from the view point of Abstract Algebraic Logic (AAL). This article constitutes a author's second attempt to applying a machinery of AAL and Model Theory to quantum logic and inference rules encountered in this logic [39]. We also shown that there exist a one-to-one correspondence between approach based on the notion of state property system (Jauch-Piron-Aerts line of investigations) and our attitudes which use the sophisticated tools derived from two core branches of modern mathematical logic – AAL and Model Theory.

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Piotr WILCZEK, Department of Functional and Numerical Analysis, Institute of Mathematics, Poznań University of Technology, ul. Piotrowo 3a, 60-965 Poznań, POLAND, edwil@mail.icpnet.pl