Observable[1]

The term *observable* has become the standard name in quantum mechanics for what used to be called physical quantity or measurable quantity in classical physics. This term derives from *observable quantity* ("beobachtbare Grösse"), which was used by Werner Heisenberg in his groundbreaking work on matrix mechanics [1] to emphasize that the meaning of a physical quantity must be specified by means of an operational definition. Together with a → *state* an observable determines the probabilities of the possible outcomes of a measurement of that observable on the quantum system prepared in the given state. Conversely, observables are identified by the totalities of their measurement outcome probabilities. Examples of observables in quantum mechanics are position, velocity, momentum, angular momentum, spin, and energy.

In elementary quantum mechanics, the observables of a physical system are represented by, and identified with, → *selfadjoint operators* A acting in the Hilbert space $\mathcal{H}$ associated with the system. For any → *pure states* of the system, represented by a unit vector $\psi \in \mathcal{H}$, the probability $p^A_\psi(X)$ that a measurement of A leads to a result in a (Borel) set $X \subset \mathbb{R}$ is given by the inner product of ψ with $E^A(X)\psi$, that is, $p^A_\psi(X) = \langle\psi|E^A(X)\psi\rangle$; here $E^A(X)$ is the spectral → *projection* of A associated with the set X, and the map $X \mapsto E^A(X)$ is called the → *spectral measure* of A. The probability measures p^A_ψ, with ψ varying over all possible pure states of the system, determine the observable A. The expectation, or average $\int x\, dp^A_\psi(x)$, of the measurement outcome distribution of an observable A in a state ψ can be expressed as $\langle\psi|A\psi\rangle$ whenever ψ is in the domain of the operator A.

The statistical meaning of quantum observables was first recognized by Max Born [2] who proposed that, in the position representation, the absolute square $|\psi|^2$ of the 'wave function' ψ gives the probability density of observing a quantum object at a given point. This idea was systematically elaborated by John von Neumann [3] who formulated and proved the spectral theorem for selfadjoint (hypermaximal hermitian) operators and applied it to obtain the interpretation of expectations as statistical averages given above.

In his seminal paper on the uncertainty relations [4] Werner Heisenberg argued, among other things, that

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[1] In: *Compendium of Quantum Physics*, eds. F. Weinertg, K. Hentschel and D. Greenberger, Springer-Verlag, to appear.

> all concepts which can be used in classical theory for the description of a mechanical system can also be defined exactly for atomic processes in analogy to classical concepts.

This statement can be substantiated in precise form by virtue of the mathematical fact that for any value x in the → *spectrum* of a self-adjoint operator A and for each $\epsilon > 0$ there is a state ψ such that $p^A_\psi((x-\epsilon, x+\epsilon)) = 1$. In particular, if A has an eigenvalue a, that is, there is a state ψ such that $A\psi = a\psi$, then in such an eigenstate of A a measurement of A is certain to yield the value a. Such a situation is commonly described by saying that observable A has a *definite value* if the state of the system is an eigenstate of A. The generic situation in quantum mechanics, however, is that most observables have no definite value in any given pure state.

It is a basic feature of quantum mechanics that there are pairs of observables, such as position and momentum, which do not commute. This fact, which lies at the heart of the → *complementarity principle* and → *uncertainty principle*, reflects a fundamental limitation on the possibilities of assigning definite values to observables and to the possibilities of measurements in the quantum world. Indeed, observables A, B that do not commute do not share a complete system of eigenvectors, so that typically an eigenstate of (say) A will be a superposition of eigenstates of B. Moreover, according to a theorem due to von Neumann [5], observables A, B are jointly measurable, that is, they have a *joint observable* (see below), if and only if they commute.

New Item for cross-ref!

The idea of identifying an observable (with real values) with the totality of the outcome probabilities in a measurement does not single out spectral measures, but is exhausted by the wider class of (real) *positive operator (valued) measures*, or *semispectral measures*. A positive operator measure is a map $E : X \mapsto E(X)$ that assigns to every (Borel) subset X of $\mathbb{R}$ a → *positive operator* $E(X)$ in such a way that for every pure state ψ the map $X \mapsto p^E_\psi(X) := \langle\psi|E(X)\psi\rangle$ is a probability measure. This definition extends readily to cases where the measurement outcomes are represented as elements of $\mathbb{R}^n$ or more general sets. Excellent expositions of the definition and properties of positive operator measures can be found, e.g., in [8, 9].

Observables represented by positive operator measures which are not projection valued are referred to as *generalized observables*, or *unsharp observables*, while spectral measures and generally all projection valued measures are called *standard*, or *sharp observables*. Commonly used acronyms for positive operator measures are POVM or POM.

The generalized representation of observables as positive operator measures was discovered by several authors in the 1960s (e.g., [6, 7, 10, 11, 12, 13]) and has by now become a standard element of quantum mechanics. It has greatly advanced the mathematical coherence and conceptual clarity of the theory. For instance, the problem of the (approximate) joint measurability of noncommuting observables such as position and momentum and the relevance of the → *Heisenberg uncertainty relations* to this question is now fully understood.

Two (real) POMs E, F are jointly measurable if and only if there is a third POM, G, defined on the (Borel) subsets of $\mathbb{R}^2$, which has E and F as marginals, that is, $E(X) = G(X \times \mathbb{R})$ and $F(Y) = G(\mathbb{R} \times Y)$ for all (Borel) subsets X, Y of $\mathbb{R}$. This definition is an instance of Ludwig's notion of *coexistence*: a set of → *effects* is coexistent if it is contained in the range of some POM [12, 13]. Similarly, a collection of observables is coexistent if their ranges are contained in the ranges of some observable. For the coexistence or joint measurability of two unsharp observables E, F, their mutual commutativity is sufficient but not necessary. If one of the observables is sharp, then coexistence implies commutativity.

As two noncommuting standard observables are never jointly measurable, one can only try to approximate them (in a suitable sense) by unsharp observables which in turn may be jointly measurable. This turns out indeed to be possible as has been well demonstrated in the cases of position and momentum or spin components. [I PROPOSE to modify or cancel the next sentence: one can take any commutative sharp observable on $\mathbb{R}^2$ and declare it to be an approximate joint observable of position and momentum. What matters is that one insists in a "nontrivial" quality of theapproximation.] The price to be paid for the approximate joint measurability of two noncommuting observables is the unsharpness of the approximators.

Finally, the introduction of POMs has widely increased the applicability of quantum mechanics in the description of realistic experiments (see, e.g., [14, 15]), and POMs are now in full use also in the relatively new fields of quantum computation and information, see, e.g., [16, 17].

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Paul Busch
Department of Mathematics
University of York
York, England
Pekka Lahti
Department of Physics
University of Turku
Turku, Finland