Heisenberg Uncertainty Relation[1]

The term *Heisenberg uncertainty relation* is a name for not one but three distinct trade-off relations which are all formulated in a more or less intuitive and vague way in Heisenberg's seminal paper of 1927 [1]. These relations are expressions and quantifications of three fundamental limitations of the operational possibilities of preparing and measuring quantum mechanical systems which are stated here informally with reference to position and momentum as a paradigmatic example of canonically conjugate pairs of quantities:

(A) *It is impossible to prepare states in which position and momentum are simultaneously arbitrarily well localized. In every state, the probability distributions of these observables have widths that obey an uncertainty relation.*

(B) *It is impossible to make joint measurements of position and momentum. But it is possible to make approximate joint measurements of these observables, with inaccuracies that obey an uncertainty relation.*

(C) *It is impossible to measure position without disturbing momentum, and vice versa. The inaccuracy of the position measurement and the disturbance of the momentum distribution obey an uncertainty relation.*

Of these three statements, only (A) was immediately given a precise formulation. Heisenberg only proved $\Delta(Q,\varphi)\Delta(P,\varphi) = \hbar/2$ for the standard deviations of position Q and momentum P in a Gaussian state φ; this was successively generalized soon afterwards by Weyl, Kennard, Robertson and Schrödinger, and the most general form for two observables represented as selfadjoint operators A, B is given by

$$(1)\quad \Delta(A,T)^2\Delta(B,T)^2 \geq \tfrac{1}{4}|\langle[A,B]\rangle_T|^2 + \tfrac{1}{4}[\langle\{A,B\}_+\rangle_T - 2\langle A\rangle_T\langle B\rangle_T]^2.$$

Here the notation $\langle X\rangle_T := \mathrm{tr}[TX]$ is used for the expectation value of an operator X in a state T, and $\Delta(X,T)^2 := \langle X^2\rangle_T - \langle X\rangle_T^2$; further, $[A,B] = AB - BA$ and $\{A,B\}_+ = AB + BA$. Relation (1) holds for all states T for which all expectation values involved are well-defined and finite. For an account of the early formal and conceptual developments of the uncertainty relation the reader is referred to the monograph [9].

[1]In: *Compendium of Quantum Physics*, eds. F. Weinert, K. Hentschel and D. Greenberger, Springer-Verlag, to appear.

It should be noted that uncertainty relations can be formulated in terms of other measures of the widths of the relevant probability distributions; these are sometimes more stringent than the above, particularly in cases where the standard deviation is infinite or otherwise an inadequate representation of the width.

The uncertainty relation (1) is commonly called indeterminacy relation, reflecting the interpretation that this relation expresses an objective limitation on the definition of the values of noncommuting quantities and not just a limitation to accessing knowledge about these values. Successful tests of the uncertainty relation in single-slit and interferometric experiments with neutrons and recently with fullerenes have been reported in [2, 3, 4, 5].

The other two uncertainty relations, (B) and (C), have proved significantly harder to make precise and prove. Heisenberg only illustrated their validity by means of idealized thought experiments, such as the $\rightarrow$ γ-ray microscope experiment and the single- or $\rightarrow$ double-slit experiment. Other authors, notably Einstein, Margenau and Popper, proposed experiments which were intended to demonstrate that the uncertainty relations are only statistically relevant and have no bearing on the properties of the individual quantum system.

In recent quantum optics, a *which way* thought experiment was proposed in order to show that Niels Bohr's $\rightarrow$ *complementarity* principle is more fundamental than the uncertainty relation (C) [6]. A polemic debate arose about this question [7]. Finally, a *which way* experiment with single atoms showed that for the "complementary" observables D (path distinguishability) and V (visibility of interference fringes) a duality relation holds, which is indeed a generalized type (A) uncertainty relation [7, 8, 11]. Hence, a debate on (C) could be settled in terms of (A).

A general proof of both (B) and (C) (assuming that these relations are in fact valid) requires the development of a theory of approximate joint measurements ($\rightarrow$ *observable*) of noncommuting observables, which has become possible on the basis of the generalized notion of an $\rightarrow$ *observable* represented as a positive operator measure (POM) and the corresponding extended measurement theory. The quality of the approximation of one observable by means of another can be assessed and quantified by comparing the associated probability distributions. Similarly, the disturbance of one observable, B, due to the measurement of another one, A, can be quantified by a comparison of the probability distributions of B immediately before and after the measurement of A. In the case of position and momentum, the theory of approximate joint measurements is well developed and has led to rigorous formulations of trade-off relations in the spirit of (B) and (C). The conceptual development that has led to this result is reviewed in [10]. Work on obtaining formalizations of (B) and (C) for general pairs of noncommuting observables is still under way.

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