Quantum observables can be identified with vector fields on the sphere of normalized states. Consequently, the uncertainty relations for quantum observables become geometric statements. In the Letter the familiar uncertainty relation follows from the following stronger statement: Of all parallelograms with given sides the rectangle has the largest area.
space. This makes the interpretation particularly transparent and naturally leads one to a geometric uncertainty identity.

Let’s begin with the standard uncertainty relation for observables $\hat{A}, \hat{B}$:

$$\Delta A \Delta B \geq \frac{1}{2} \left| \langle \varphi, [\hat{A}, \hat{B}] \varphi \rangle \right|. \quad (5)$$

Here $\Delta A^2 = \langle \varphi, \hat{A}^2 \varphi \rangle - \langle \varphi, \hat{A} \varphi \rangle^2$ and similarly for $\Delta B^2$ and $\varphi$ is the state of the system under consideration. It is implicit in (5) that the state $\varphi$ is in the domain of all operators involved. As an immediate corollary of the relation one sees that, in general, the standard deviations $\Delta A, \Delta B$ of non-commuting observables cannot be made arbitrarily small at the same time (i.e., for the same state $\varphi$). This constitutes a version of the famous uncertainty principle of Heisenberg [14].

In light of the identification (1) of observables with vector fields on the sphere of states $S^{L_2} \subset L_2$, each term in (5) obtains a simple geometric interpretation. Namely, the equality

$$\hat{A} \equiv (\varphi, \hat{A} \varphi) = (-i \varphi, -i \hat{A} \varphi), \quad (6)$$

signifies that the expected value of an observable $\hat{A}$ in the state $\varphi$ is the projection of the vector $-i \hat{A} \varphi \in T_\varphi S^{L_2}$ on the vector $-i \varphi = -i \hat{A} \varphi \in T_\varphi S^{L_2}$, associated with the identity operator $I$. Because

$$(\varphi, \hat{A}^2 \varphi) = (\hat{A} \varphi, \hat{A} \varphi) = (-i \hat{A} \varphi, -i \hat{A} \varphi), \quad (7)$$

the term $(\varphi, \hat{A}^2 \varphi)$ is just the norm of the vector $-i \hat{A} \varphi$ squared. Note that the expected value $(\varphi, \hat{A} \varphi)$ of the operator $\hat{A} \equiv \hat{A} - \hat{A}I$ in the state $\varphi$ is zero. Therefore, the vector $-i \hat{A} \varphi = -i \hat{A} \varphi - (i \hat{A} \varphi)$, which is the component of $-i \hat{A} \varphi$ orthogonal to $i \varphi$ is orthogonal to the entire fibre $\{ \varphi \}$. Accordingly, the variance

$$\Delta A^2 = \langle \varphi, (\hat{A} - \hat{A}I)^2 \varphi \rangle = \langle \varphi, \hat{A}^2 \varphi \rangle = (-i \hat{A} \varphi, -i \hat{A} \varphi), \quad (8)$$

is the norm squared of the component $-i \hat{A} \varphi$. As discussed, the image of this vector under $d\tau$ can be identified with the vector itself. It follows that the norm of $-i \hat{A} \varphi$ in the Fubini-Study metric coincides with its norm in the Riemannian metric on $S^{L_2}$ (and in the original $L_2$-metric).

Consider the evolution equation

$$\frac{d \varphi_t}{d\tau} = -i \hat{A} \varphi_t \quad (9)$$

for the state $\varphi_t$ with the initial condition $\varphi_t|_{\tau=0} = \varphi$. By projecting both sides of this equation by $d\tau$, one obtains

$$\frac{d\{ \varphi_t \}}{d\tau} = -i \hat{A}_\perp \varphi_t, \quad (10)$$

The left hand side of (10) at $t = 0$ is the velocity of evolution of the projection of $\varphi_t$ at the point $\{ \varphi \} \in CP^{L_2}$.

By the above, the norm of the right hand side at $t = 0$ is the uncertainty of $\hat{A}$ in the state $\varphi$:

$$\| -i \hat{A}_\perp \varphi \| = \Delta \hat{A}. \quad (11)$$

So the uncertainty $\Delta \hat{A}$ is equal to the speed of the state $\{ \varphi_t \}$ at the point $\{ \varphi \}$ under the evolution (9). In the case when $\hat{A}$ is equal to the Hamiltonian $\hat{h}$ of the system, one obtains the result of Ref.[8]: the energy uncertainty is the speed of evolution of the state in the projective space.

One concludes that the left hand side of (5) is the produce of norms of the projections of vectors $-i \hat{A} \varphi, -i \hat{B} \varphi$ onto $T_{\{ \varphi \}} CP^{L_2}$. In geometric terms, the left hand side is therefore the area $A_{\{ \varphi_t \}}$ of a rectangle with sides of lengths $\| -i \hat{A}_\perp \varphi \|, \| -i \hat{B}_\perp \varphi \|$. Let’s show that the right hand side of (5) can be estimated via the area of parallelogram formed by vectors $-i \hat{A}_\perp \varphi, -i \hat{B}_\perp \varphi$. For this note that $[\hat{A}, \hat{B}] = [\hat{A}_\perp, \hat{B}_\perp]$ and therefore,

$$(\varphi, [\hat{A}, \hat{B}] \varphi) = (\hat{A}_\perp \varphi, \hat{B}_\perp \varphi) - (\hat{B}_\perp \varphi, \hat{A}_\perp \varphi)$$

$$= 2i \text{Im}(\hat{A}_\perp \varphi, \hat{B}_\perp \varphi) = 2i \text{Im}(-i \hat{A}_\perp \varphi, -i \hat{B}_\perp \varphi). \quad (12)$$

The form $\text{Im}(\xi, \eta)$ is an anti-symmetric 2-form on vectors $\xi, \eta$. Let $\{ e_k \}$ be an orthonormal basis in $L_2$, such that $e_1 = -i \hat{A}_\perp \varphi$ and the vector $-i \hat{B}_\perp \varphi$ is in the linear envelop $C^2$ of the vectors $e_1, e_2$. Let $E_1 = e_1, E_2 = ie_1, E_3 = e_2, E_4 = ie_2, \ldots$ be the corresponding orthonormal basis in the realization $L_{R^4}$. Note that the linear envelope $R^4$ of the vectors $E_1, E_2, E_3, E_4$ is a subspace of the tangent space $T_{R_2} S^{L_2}$ and the Riemannian metric on the sphere yields the Euclidean metric on $R^4$. Let’s denote the realization of the vectors $\xi = -i \hat{A}_\perp \varphi, \eta = -i \hat{B}_\perp \varphi$ by $X$ and $Y$ and let’s denote the components of $X$ and $Y$ in the basis $\{ e_k \}$ by $x_k$ and $y_k$ respectively. Because $x_k = y_k = 0$ for $k > 4$, one has

$$\text{Im}(\xi, \eta) = \text{Im} \sum_k \xi_k \eta_k = (x_2 y_1 - x_1 y_2) + (x_4 y_3 - x_3 y_4), \quad (13)$$

and so the right hand side of (5) is equal to

$$\| (x_1 y_2 - x_2 y_1) + (x_3 y_4 - x_4 y_3) \|. \quad (14)$$

On the other hand, the area squared $A_{XY}^2$ of the parallelogram on vectors $X, Y$ is equal to

$$\| (x_1 y_2 - x_2 y_1)^2 + (x_1 y_3 - x_3 y_1)^2 + (x_1 y_4 - x_4 y_1)^2$$

$$+ (x_2 y_3 - x_3 y_2)^2 + (x_2 y_4 - x_4 y_2)^2 + (x_3 y_4 - x_4 y_3)^2 \|. \quad (15)$$

By the choice of $\{ E_k \}$, we have $x_2 = x_3 = x_4 = 0$. By comparing (14) and (15) one concludes that

$$A_{XY} \geq \frac{1}{2} \| \langle \varphi, [\hat{A}, \hat{B}] \varphi \rangle \|. \quad (16)$$

As a result, the obvious geometric inequality

$$A_{\{A_{XY} \}} \geq A_{XY}, \quad (17)$$
implies the uncertainty relation (5).

It is well known that the uncertainty relation (5) can be somewhat strengthened to take the form

$$\Delta A^2 \Delta B^2 \geq \frac{1}{4} \left( \left| \langle \varphi, [\hat{A}, \hat{B}] \varphi \rangle \right|^2 + \frac{1}{4} \left| \langle \varphi, [\hat{A} \perp, \hat{B} \perp] \varphi \rangle \right|^2 \right),$$

where \{\hat{A} \perp, \hat{B} \perp\} stands for the anticommutator of the operators \{\hat{A} \perp, \hat{B} \perp\}. Note that

$$\langle \varphi, [\hat{A} \perp, \hat{B} \perp] \varphi \rangle = (\hat{A} \perp \varphi, \hat{B} \perp \varphi) + (\hat{B} \perp \varphi, \hat{A} \perp \varphi) = 2 \text{Re}(\hat{A} \perp \varphi, \hat{B} \perp \varphi) = 2 \text{Re}(-i\hat{A} \perp \varphi, -i\hat{B} \perp \varphi).$$

So the second term on the right of (18) is simply the square of Riemannian inner product of vectors \(-i\hat{A} \perp \varphi, -i\hat{B} \perp \varphi\). With the help of (12) one can now identify the right hand side of (18) with \(|(-i\hat{A} \perp \varphi, -i\hat{B} \perp \varphi)|^2\). Using (11), one concludes that (18) is simply the Cauchy-Schwarz inequality

$$\| -i\hat{A} \perp \varphi \|^2 - i\| -i\hat{B} \perp \varphi \|^2 \geq |(-i\hat{A} \perp \varphi, -i\hat{B} \perp \varphi)|^2$$

for the vectors \(-i\hat{A} \perp \varphi, -i\hat{B} \perp \varphi\).

Recall that the left hand side of the uncertainty relations (5), (17), (18) is the product of lengths of vectors \(X, Y\). In particular, in the basis \(E_k\) one has:

$$\Delta A^2 \Delta B^2 = x_1^2(y_3^2 + y_4^2) + y_1^2(y_3^2 + y_4^2).$$

Note that the right hand sides of the uncertainty relations (5), (17) and (18) are formed by the terms of (21). In particular, these uncertainty relations follow from (21). Moreover, the right hand side of (21) is exactly the sum of the Riemannian inner product term squared

$$\left( \text{Re}(-i\hat{A} \perp \varphi, -i\hat{B} \perp \varphi) \right)^2 = G_{XY}(X, Y) = x_1^2y_1^2$$

and the area term squared \(A_{XY} = x_1^2(y_3^2 + y_4^2)\). It follows that the uncertainty relation can be written in the form of the “uncertainty identity”

$$\Delta A^2 \Delta B^2 = A_{XY} + G_{XY}(X, Y),$$

with \(X = -i\hat{A} \perp \varphi\) and \(Y = -i\hat{B} \perp \varphi\).

One concludes, once again, that \(A_{XY} = 0\) is a necessary condition for vanishing uncertainty \(\Delta A \Delta B\). This condition is satisfied when vectors \(-i\hat{A} \perp \varphi\) and \(-i\hat{B} \perp \varphi\) are linearly dependent over \(R\). Another necessary condition that follows from (22) is the condition of orthogonality of the vectors \(-i\hat{A} \perp \varphi\) and \(-i\hat{B} \perp \varphi\) in the Riemannian metric. The necessary and sufficient condition for \(\Delta A \Delta B = 0\) is the vanishing of both terms on the right hand side of (22). In particular, for bounded operators \(\hat{A}, \hat{B}\), the uncertainty \(\Delta A \Delta B\) vanishes if at least one of the vectors \(-i\hat{A} \perp \varphi, -i\hat{B} \perp \varphi\) vanishes. That is, if \(\varphi\) is an eigenstate of either \(\hat{A}\) or \(\hat{B}\). For example, for the Pauli matrices, \(\Delta \sigma_x \Delta \sigma_y = 0\) iff \(\varphi\) is an eigenstate of either \(\hat{\sigma}_x\) or \(\hat{\sigma}_y\).

Assume now that \([\hat{A}, \hat{B}] = cI\), where \(c\) is a number. Recall that \(A_{XY} \geq \frac{1}{4} \left( |\langle \varphi, [\hat{A}, \hat{B}] \varphi \rangle|^2 + \frac{1}{4} |\langle \varphi, [\hat{A} \perp, \hat{B} \perp] \varphi \rangle|^2 \right)\) and so the first term on the right of (22) is at least \(|c|^2/2\). Therefore, the uncertainty \(\Delta A \Delta B\) is at least \(|c|^2/2\). This minimal value of the uncertainty can only be achieved if \(A_{XY} = |c|^2/2\) and \(G_{XY}(X, Y) = 0\). Recall that in the basis \(E_k\) one has

$$A_{XY} = x_1^2 \left( y_3^2 + y_4^2 \right) + \frac{1}{2} \left( \left| \langle \varphi, [\hat{A}, \hat{B}] \varphi \rangle \right|^2 \right) = |x_1y_2|^2.$$  

Therefore, to achieve the minimal value one must have

$$y_3^2 + y_4^2 = 0.$$  

It follows that \(-i\hat{B} \varphi = \lambda \left(-i\hat{A} \varphi\right)\) for some complex \(\lambda\). The condition \(G_{XY}(X, Y) = 0\) reads in the basis \(E_k\) as \(x_1y_1 = 0\). It follows that \(y_1\) must be zero, which means that the constant \(\lambda\) is purely imaginary. In particular, for the momentum and position operators \(\hat{p}\) and \(\hat{x}\) these conditions yield Gaussian states for which \(\Delta p \Delta x = h/2\).

Note that the terms on the right of (22) can be written as \(|X|^2 |Y|^2 \sin^2 \theta\) and \(|X|^2 |Y|^2 \cos^2 \theta\), where \(\theta\) is the angle between the vectors \(X\) and \(Y\). In particular, when \(\theta = 0\) the uncertainty comes from the inner product term \(G_{XY}(X, Y)\) only and when \(\theta = \pi/2\), the uncertainty is due to the area term. By replacing \(\hat{B}\) with a real linear combination of the operators \(\hat{A}, \hat{B}\), one can change \(\theta\) in any desirable way while preserving the uncertainty \(\Delta A \Delta B\).

The standard uncertainty relations (5), (18), the derived geometric uncertainty relation (17) and the uncertainty identity (22) are mathematical statements. The mystery of the uncertainty principle lies not so much in these statements, but rather in a physical interpretation of operators and states entering the statements. So, what is the significance of the provided derivation in this respect?

The quantum evolution of a system yields a path on the sphere of states. The projection \(\pi: S^{2L_2} \rightarrow CP^{L_2}\) gives then a path on the projective space \(CP^{L_2}\) of physical states. As advocated in Refs.[10]-[12], the evolution of state along the manifolds \(S^{2L_2}\) and \(CP^{L_2}\) should be treated as a fundamental physical process, rather than just a way of describing changes in probability distributions of measured quantities. As shown in Ref.[12], by choosing an appropriate Riemannian metric on the sphere \(S^{2L_2}\), one can ensure that the Schrödinger path of the state is a geodesic on the sphere. Moreover, at least in the finite dimensional spaces of states, the process of collapse can be also modeled by a geodesic motion of the state in the metric perturbed by the measuring device. The Born rule for probability of collapse can be derived from simple additional assumptions (see Ref.[12]).

One is faced then with a new point of view on quantum mechanics that makes that theory quite similar to Einstein’s general relativity, but considered on a manifold of states rather than on space-time. The approach turns out to be fruitful in explaining various paradoxical results in quantum theory via the geometry of the manifold of states. Moreover, the formalism allows one to naturally
embed the physics of macroscopic particles on the classical Riemannian space into the theory (see Ref.[10]). In light of this, the provided geometric derivation of the uncertainty relation and the uncertainty identity seems to be another piece of the puzzle falling into place.

What is the physical interpretation of quantum uncertainty in the the new geometrical setting? The answer depends on the one’s definition of the uncertainty. Here are some possible definitions together with their geometric interpretation.

(a) Note first of all that the set of eigenstates of two non-commuting observables $\hat{A}, \hat{B}$ form two non-identical (often, non-overlapping) subsets $S_A, S_B$ of the sphere of states. If the intersection $S_A \cap S_B$ is empty, the state cannot belong to both of them at once. If the state is close in the Riemannian metric to one of these subsets, it cannot be arbitrarily close to the other one, hence, the uncertainty principle.

Mathematically, the principle can be formulated in this case via the triangle inequality on the sphere of states.

By projecting on $CP^{L-2}$, one obtains a similar inequality for physical states.

In such an interpretation the uncertainty of an observable $\hat{A}$ is the distance from the state to the set of eigenstates of $\hat{A}$ in the Riemannian metric. The uncertainty relation (23) shows that for two observables with no common eigenvectors the state cannot be made arbitrarily close to both $S_A$ and $S_B$ at once. For example, for spin states $\varphi$ of a non-relativistic electron one has $d(\varphi, \{S_a\}) + d(\{\varphi\}, \{S_a\}) \geq \frac{\pi}{2}$.

(b) More commonly, the uncertainty of an observable $\hat{A}$ in state $\varphi$ is defined as the standard deviation $\Delta A$. Recall that $\Delta A$ is the norm of the velocity vector $-i\hat{A}_\perp \varphi$ of the evolution $\frac{d\varphi}{dt} = -i\hat{A}_\perp \varphi$. The velocity vector vanishes at the eigenstates (and only at the eigenstates) of the operator $\hat{A}$. Therefore, the uncertainty $\Delta A$ vanishes only at the eigenstates as well.

Note that in the case of the space $CP^1$ of spin states of a non-relativistic electron, the standard deviation $\Delta A$ of any observable $\hat{A}$ with $-i\hat{A} \in su(2)$ can be identified with the distance $d(\{\varphi\}, \{S_a\})$ between the state and the set of eigenstates of $\hat{A}$ (see Ref.[12]). In other words, the speed of evolution of the state in $CP^1$ is proportional to the distance $d(\{\varphi\}, \{S_a\})$. In this particular case the definitions (a) and (b) coincide.

(c) The uncertainty can be understood as the product $\Delta A \Delta B$ of standard deviations of two observables for a system in a given state $\varphi$ (or, in some cases, as the infimum of the set of such products for all possible states). Suppose that the velocity vectors $-i\hat{A}_\perp \varphi, -i\hat{B}_\perp \varphi$, considered as vectors in the real space $L_2$, are linearly independent. Then the area of the parallelogram based on these vectors vanishes. In this case the right hand side of the geometric uncertainty relation (17) also vanishes. This provides one with a simple geometrical necessary condition for vanishing $\Delta A \Delta B$.

A related and most common understanding of quantum uncertainty is based on the standard uncertainty relation (5). This relation is often used to identify quantum uncertainty in the sense (γ) with non-commutativity of quantum observables under consideration. Note however that according to (18), the lower bound of the product of standard deviations of two commuting observables on a given set of states may be positive [15]. Conversely, even if two observables do not commute, they could still have a common eigenvector so that the standard deviations of both observables on this vector would vanish. In other words, the non-commutativity of observables $\hat{A}, \hat{B}$ is neither necessary nor sufficient for a nontrivial uncertainty relation.

It is a pleasure to thank Malcolm Forster for numerous discussions that helped shaping this paper.

[14] In the original Heisenberg formulation of the principle the issue of simultaneous measurements of observables is central. The mathematically derived uncertainty relations are not about such measurements, but rather about the standard deviations (or the like measures) of the observables measured separately on the same state.
[15] Of course, the lower bound of the product of standard deviations of two commuting observables over the entire sphere of states is zero. However, it is an unnecessary limitation to consider this case only.