

# Typicality and the Approach to Equilibrium in Boltzmannian Statistical Mechanics

Roman Frigg  
London School of Economics  
r.p.frigg@lse.ac.uk

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## Abstract

Systems prepared in a non-equilibrium state approach, and eventually reach, equilibrium. Why do they do so? An important contemporary version of the Boltzmannian approach to statistical mechanics answers this question in terms of typicality. The problem with this approach is that it comes in different versions, which are, however, not recognised as such and not clearly distinguished. The aim of this paper is to identify three different versions of typicality-based explanations of thermodynamic-like behaviour and evaluate their respective success. My conclusion is that the first two are unsuccessful because they fail to take the system's dynamics into account. The third, however, is promising. I give a precise formulation of the proposal and present an argument in support of its central contention.

## 1 Introduction

Consider a gas confined to the left half of a container. Removing the dividing wall results in the gas spreading uniformly over the entire available space. It has approached equilibrium. Statistical mechanics (SM) aims to explain the approach to equilibrium in terms of the dynamical laws governing the individual molecules of which the gas is made up. What is it about molecules and their motions that leads them to spread out when the wall is removed? And why does this happen invariably? That is, why do we never observe gases staying in the left half even after the shutter has been removed?

An important contemporary version of the Boltzmannian approach to SM, originating in the work of Joel Lebowitz (1993a, 1993b), answers these questions in terms of the notion of *typicality*. Intuitively, something is typical if it happens in the ‘vast majority’ of cases: typical lottery tickets are blanks, typical olympic athletes are well trained, and in a typical series of a thousand coin tosses the ratio of the number of heads and the number of tails is approximately one. The aim of a typicality-based approach to SM is to show that approaching equilibrium is the typical behaviour of systems like gases.

This approach has grown increasingly popular in recent years (references will be given below). The problem with it is that it comes in different versions, which are, however, not recognised as such, much less clearly distinguished. The aim of this paper is to distinguish three different kinds of typicality-based explanation of the approach to equilibrium and evaluate their respective success. My conclusion will be that the first two are unsuccessful because they fail to take the system’s dynamics into account. The third, however, is promising. I give a precise formulation of the proposal and present the outline of a proof of its central contention.

## 2 Classical Boltzmannian SM

Consider a system consisting of  $n$  classical particles with three degrees of freedom each. The state of this system is specified by a point  $x$ , also referred to as the system’s microstate, in its  $6n$ -dimensional phase space  $\Gamma$ , which is endowed with the Lebesgue measure  $\mu_L$ . The dynamics of the system is governed by Hamilton’s equations of motion, which define a measure preserving flow  $\phi_t$  on  $\Gamma$ , meaning that for all times  $t$   $\phi_t : \Gamma \rightarrow \Gamma$  is a one-to-one mapping such that  $\mu(R) = \mu(\phi_t(R))$  for all measurable  $R \subseteq \Gamma$ . The system’s microstate at time  $t_0$  (its ‘initial condition’),  $x(t_0)$ , evolves into  $x(t) = \phi_t(x(t_0))$  at time  $t$ . In a Hamiltonian system energy is conserved and hence the motion of the system is confined to the  $6n - 1$  dimensional energy hypersurface  $\Gamma_E$ . The measure  $\mu_L$  can be restricted to  $\Gamma_E$ , which induces a natural invariant measure  $\mu$  on  $\Gamma_E$ .

To each macrostate  $M_i$  of the system,  $i = 1, \dots, m$ , there corresponds a macro-region  $\Gamma_{M_i}$  consisting of all  $x \in \Gamma_E$  for which the macroscopic variables assume the values characteristic for  $M_i$ . The  $\Gamma_{M_i}$  together form a partition of  $\Gamma_E$ , meaning that they do not overlap and jointly cover  $\Gamma_E$  up to measure

zero. The Boltzmann entropy of a macrostate  $M_i$  is defined as  $S_B(M_i) := k_B \log[\mu(\Gamma_{M_i})]$ , where  $k_B$  is the Boltzmann constant. Given this, we define the Boltzmann entropy of a *system* at time  $t$ ,  $S_B(t)$ , as the entropy of the system's macrostate at  $t$ :  $S_B(t) := S_B(M_{x(t)})$ , where  $x(t)$  is the system's microstate at  $t$  and  $M_{x(t)}$  is the macrostate corresponding to  $x(t)$ .

Among the macrostates of a system two are of particular importance, the equilibrium state,  $M_{eq}$ , and the system's state at the beginning of the process,  $M_p$  (also referred to as the 'past state'). The latter is, by assumption, a low entropy state.<sup>1</sup> The idea now is that the behaviour of  $S_B(t)$  should mirror the behaviour of the thermodynamic entropy  $S_{TD}$ , at least approximately.<sup>2</sup> So we expect the Boltzmann entropy of a system initially prepared in  $M_p$  to increase more or less monotonically, reach its maximum fairly quickly, and then remain at or near the maximum for a long time. In other words, we expect the dynamics to be such that it carries the system's initial state  $x(t_0) \in \Gamma_{M_p}$  into  $\Gamma_{M_{eq}}$  reasonably quickly and then keeps it there for a long time. I refer to this as 'thermodynamic-like behaviour'.<sup>3</sup> The explanandum then is this: why does the system under investigation behave in a thermodynamic-like way?

The standard Boltzmannian response to this question is to introduce probabilities and argue that the values of these probabilities come out such that the system is overwhelmingly likely to evolve in a thermodynamic-like way.<sup>4</sup> Typicality approaches to SM eschew commitment to probabilities and offer a different kind of explanation: the system behaves in a thermodynamic-like way because it is typical for systems of this kind to behave in this way.<sup>5</sup>

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<sup>1</sup>If we study laboratory systems like the above-mentioned gas,  $M_p$  has low entropy by construction. If we take the universe as a whole to be the object of study, then that  $M_p$  be of low entropy is the subject matter of the so-called 'Past Hypothesis' (Albert 2000, 96).

<sup>2</sup>This 'mirroring' need not be perfect and occasional deviations of the Boltzmann entropy from its thermodynamic counterpart are no cause for concern (Callender 2001).

<sup>3</sup>This definition of thermodynamic-like behaviour is the one adopted by those writing on typicality; see, for instance Goldstein (2001, 43-44). Lavis (2005, 255) gives a somewhat different definition. These differences are inconsequential for what follows.

<sup>4</sup>A discussion of the different ways of introducing probabilities into the Boltzmannian framework can be found in Frigg (2009).

<sup>5</sup>This explanatory strategy is reminiscent of probabilistic explanations appealing to high probabilities, and hence is open to similar objections. For the sake of argument I set these worries aside and accept that something being typical has explanatory power.

Before turning to a discussion of this approach, an important technical result needs to be stated. Under certain conditions it is the case that  $\Gamma_{Meq}$  is vastly larger (with respect to  $\mu$ ) than any other macro-region. I refer to this matter of fact as the ‘dominance of the equilibrium macrostate’. This dominance is then often glossed as being equivalent that for large  $n$ ,  $\Gamma_E$  is *almost entirely* taken up by equilibrium microstates (Bricmont 1995, 146; Goldstein 2001, 45; Zanghì 2005, 191, 196).

Some caution is needed here. In certain systems non-equilibrium states can take up a substantial part of the phase space due to the degeneracy of below-equilibrium entropy values, and hence it not true that  $\Gamma_E$  is almost entirely filled with equilibrium states (Lavis 2005, 255-258; 2008, Sec. 2). However, it turns out that those non-equilibrium states that occupy most of the non-equilibrium area are close to equilibrium (in the sense of having close to equilibrium entropy values). We can then lump the equilibrium and these close-to-equilibrium states together and get an ‘equilibrium or almost equilibrium’ region, which indeed takes up most of  $\Gamma_E$ . The approach to equilibrium has then to be understood as the approach to this ‘equilibrium or almost equilibrium’ state, which is sufficient to give us thermodynamic-like behaviour in the sense introduced above.

### 3 Typicality

Consider an element  $e$  of a set  $\Sigma$ . Typicality is a relational property of  $e$ , which  $e$  posses with respect to  $\Sigma$ , a property  $P$  and a measure  $\nu$ , often referred to as ‘typicality measure’. Roughly speaking,  $e$  is typical if most members of  $\Sigma$  have property  $P$  and  $e$  is one of them. More precisely, let  $\Pi$  be the subset of  $\Sigma$  consisting of all elements that have property  $P$ . Then the element  $e$  is typical iff  $e \in \Pi$  and  $\nu_{\Sigma}(\Pi) := \nu(\Pi)/\nu(\Sigma) \geq 1 - \varepsilon$ , where  $\varepsilon \geq 0$  is a small real number;  $\nu_{\Sigma}(\cdot)$  is referred to as the ‘measure conditional on  $\Sigma$ ’, or simply ‘conditional measure’.<sup>6</sup> Derivatively, one can then refer to  $\Pi$  as the ‘typical set’ and to those elements that possess property  $P$  (i.e. the members of  $\Pi$ ) as ‘typical elements’. Conversely, an element  $e$  is atypical iff it belongs to the complement of  $\Pi$ ,  $\Omega := \Sigma \setminus \Pi$ , in which case we refer to  $\Omega$  as the ‘atypical set’ and to its members as ‘atypical elements’.

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<sup>6</sup>This definition of typicality is adapted from Dürr (1998, Sec. 2), Lavis (2005, 258), Zanghì (2005, 185), and Volchan (2007, 805).

As an example consider the number  $\pi$ , which is typical with respect to the interval  $[0, 1]$ , the property ‘not being specifiable by a finite number of digits’ and the usual Lebesgue measure on the real numbers, because it is a theorem of number theory that the set of all numbers that have this property has measure one.

The element of interest in SM is a microstate  $x$ , and it is generally agreed that the relevant measure is the Lebesgue measure  $\mu$ . However, views diverge when it comes to specifying the relevant set  $\Sigma$  and relevant property  $P$ .

I now turn to a discussion of three different typicality-based accounts of SM that emerge from the writings of Goldstein, Lebowitz, and Zanghì. In conversation Goldstein and Zanghì have pointed out to me that they would not subscribe to Accounts 1 and 2 and that (something like) Account 3 is what they had intended. However, since the relevant papers can reasonably be read as proposing Accounts 1 and 2 it worth discussing them briefly to set the record straight (Sections 4 and 5) before turning to a detailed discussion of Account 3 (Section 6).

## 4 First Account

The first account sets out to explain the approach to equilibrium in terms of the dominance of the equilibrium macrostate. Zanghì explains:

‘reaching the equilibrium distribution in the course of the temporal evolution of a system is inevitable due to the fact that the overwhelming majority of microstates in the phase space have this distribution; a fact often not understood by the critics of Boltzmann [...]’ (Zanghì 2005, 196, my translation)

On this view, then, a system approaches equilibrium *simply because* the overwhelming majority of states in  $\Gamma_E$  are equilibrium microstates. If we now associate  $\Sigma$  with  $\Gamma_E$  and property  $P$  with ‘being an equilibrium state’ (and, as indicated above, regard microstates as elements of interest and use the Lebesgue measure  $\mu$  as typicality measure), the dominance of the equilibrium macrostate implies that equilibrium microstates are typical, and the view put forward in the above quote can be summarised as the claim that systems approach equilibrium because equilibrium microstates are typical and non-equilibrium microstates are atypical.

This explanation is unsuccessful. If a system is in an atypical microstate, it does not evolve into an equilibrium microstate *just because* the latter are typical. Typical states do not automatically attract trajectories.<sup>7</sup> In fact there are Hamiltonians – for instance the null Hamiltonian or a collection of uncoupled harmonic oscillators – that give rise to a dynamics that does not carry non-equilibrium states into equilibrium. To explain why non-equilibrium microstates eventually wind up in equilibrium the typicality of  $\Gamma_{Meq}$  is not enough and appeal has to be made to the system’s dynamics.

## 5 Second Account

An different line of argument can be found in Lebowitz (1993a, 1993b, 1999) and Lebowitz & Goldstein (2004). This account differs from the above in that it focusses on the internal structure of the micro-regions  $\Gamma_{M_i}$  rather than the entire phase space:

‘By “typicality” we mean that for any  $[\Gamma_{M_i}]$  [...] the relative volume of the set of microstates  $[x]$  in  $[\Gamma_{M_i}]$  for which the second law is violated [...] goes to zero rapidly (exponentially) in the number of atoms and molecules in the system.’ (Goldstein & Lebowitz 2004, 57)<sup>8</sup>

This definition contains different elements that need to be distinguished. Let  $\Gamma_{M_i}^{(++)}$  be the subset of  $\Gamma_{M_i}$  containing all  $x$  that lie on trajectories that come into  $\Gamma_{M_i}$  from a macrostate of higher entropy and that leave  $\Gamma_{M_i}$  entering into a macrostate of higher entropy;  $\Gamma_{M_i}^{(+-)}$ ,  $\Gamma_{M_i}^{(-+)}$  and  $\Gamma_{M_i}^{(--)}$  are defined accordingly. These four subsets form a partition of  $\Gamma_{M_i}$ . Furthermore,  $\Gamma_{M_i}^{(+)} := \Gamma_{M_i}^{(++)} \cup \Gamma_{M_i}^{(-+)}$  and  $\Gamma_{M_i}^{(-)} := \Gamma_{M_i}^{(+-)} \cup \Gamma_{M_i}^{(--)}$  are the subsets of  $\Gamma_{M_i}$  that have a higher and lower entropy future respectively.

There is an interpretative question about how to understand the notion of a set of microstates in  $\Gamma_{M_i}$  violating the Second Law. A plausible reading takes these to be states that have an entropy decreasing future. Let us call this property  $D$ . Hence,  $x$  has  $D$  iff  $x \in \Gamma_{M_i}^{(-)}$ . Entropy decreasing states are atypical in  $\Gamma_{M_i}$  iff  $\mu_i(\Gamma_{M_i}^{(-)}) < \varepsilon$ , where  $\mu_i(\cdot) := \mu(\cdot)/\mu(\Gamma_{M_i})$ . Furthermore let

<sup>7</sup>Uffink (2007, 979-980) illustrates this with the example of a trajectory.

<sup>8</sup>Square brackets indicate that the original notation has been replaced by the notion used in this paper. I will use this convention throughout.

us say that a *system* is globally entropy decreasing, *GD*, iff entropy decreasing states microstates are atypical in every  $\Gamma_{M_i}$ . The claim made in the above quote then is tantamount to saying that a system with a reasonably large number of molecules is *GD*.

This claim needs to be qualified. The atypicality of  $x$  with property  $D$  in  $\Gamma_{M_i}$  trivially implies  $\mu(\Gamma_{M_i}^{+-}) < \varepsilon$ . Due to the time reversal invariance of the Hamiltonian dynamics we have  $\mu(\Gamma_{M_i}^{-+}) = \mu(\Gamma_{M_i}^{+-})$  and therefore  $\mu(\Gamma_{M_i}^{++}) < \varepsilon$ . Since  $\Gamma_{M_i}^{(+)} = \Gamma_{M_i}^{-+} \cup \Gamma_{M_i}^{++}$  we obtain  $\mu(\Gamma_{M_i}^{(+)}) > 1 - 2\varepsilon$ . Hence, even if  $x$  with property  $D$  are atypical in  $\Gamma_{M_i}$ , it is not the case that, as we would expect, most states in  $\Gamma_{M_i}$  behave thermodynamic-like since most states have a *higher* entropy *past*! But this is a familiar problem and remedy can be found in conditionalising on  $\Gamma_{M_p}$  (Albert 2000, Ch. 4).<sup>9</sup> So the correct requirement is that, at any time  $t$ , microstates violating the Second Law must be atypical in  $\Gamma_{M_i} \cap \phi_t(\Gamma_{M_p})$  rather than only  $\Gamma_{M_i}$ .

Do relevant systems meet this requirement? Immediately after the *passé* quoted above Goldstein & Lebowitz offer the following answer:

‘Boltzmann then argued that given this disparity in sizes of different  $M$ ’s, the time evolved  $[M_{x(t)}]$  will be such that  $[\mu(M_{x(t)})]$  and thus  $[S_B(t)]$  will *typically* increase in accord with the law.’ (2004, 57)

So the argument seems to be that the relevant condition must be true because the equilibrium state is much larger than other macrostates.

This is unconvincing. The disparity of sizes of macro-regions is, of course, compatible with being *GD*, but the latter does not follow from the former. Whether macro-regions have the above internal structure depends on the system’s phase flow  $\phi_t$  and every attempt to answer this question without even mentioning the system’s dynamics is doomed to failure right from the start (and this is true both of the qualified and the unqualified version of the claim).

## 6 Third Account

As we have seen, the basic problem with the two accounts discussed so far is that they attempt to explain the approach to equilibrium without reference

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<sup>9</sup>Notice that an attempt do define  $D$  in terms of  $\Gamma_{M_i}^{(-)} \cup \Gamma_{M_i}^{(++)}$  rather than only  $\Gamma_{M_i}^{(-)}$  leads to a contradiction.

to the system’s dynamics. The third account, which emerges from a passage in Goldstein’s (2001), rectifies this problem:

‘ $[\Gamma_E]$  consists almost entirely of phase points in the equilibrium macrostate  $[\Gamma_{M_{eq}}]$ , with ridiculously few exceptions whose totality has volume of order  $10^{-10^{20}}$  relative to that of  $[\Gamma_E]$ . For a non-equilibrium phase point  $[x]$  of energy  $E$ , the Hamiltonian dynamics governing the motion  $[x(t)]$  would have to be ridiculously special to avoid reasonably quickly carrying  $[x(t)]$  into  $[\Gamma_{M_{eq}}]$  and keeping it there for an extremely long time – unless, of course,  $[x]$  itself were ridiculously special.’ (Goldstein 2001, 43-44)

This is an interesting claim, but one that stands in need of clarification. A reasonable reading of this passage seems to be that an argument involving three different typicality claims is made:

*Premise 1:* The system’s macrostate structure is such that equilibrium states are typical in  $\Gamma_E$  in the sense introduced in Section 4.

*Premise 2:* The system’s Hamiltonian is typical in the class of all Hamiltonians.

*Conclusion:* Initial conditions lying on trajectories showing thermodynamic-like behaviour are typical in  $\Gamma_{M_p}$  with respect to  $\mu_p(\cdot) := \mu(\cdot) / \mu(\Gamma_{M_p})$ .

Let us refer to this as the ‘T-Argument’. Premise 1 is familiar from Section 4 and is taken for granted here. Premise 2 and the conclusion are restatements in the language of typicality of the claims that the Hamiltonian of the system and the initial condition be not ‘ridiculously special’.

The T-Argument, if sound, gives us the sought-after explanation of the approach to equilibrium in terms of typicality. But before we can address the question of soundness we need to make Premise 2 more precise. What does it mean for a system’s Hamiltonian to be typical in the class of all Hamiltonians? More specifically, what is the typicality measure and what is the relevant property  $P$ ?

Let us begin with the first question. The problem is that function spaces do not come equipped with normalised measures that can plausibly be used to capture the intuitive idea that some sets of functions are typical while others are atypical. A natural way around this difficulty is to replace the measure theoretic notion of typicality introduced in Section 3 by a topological one



based on Baire categories for an introduction). Sets can be of two kinds: meagre (first Baire category) or nonmeagre (second Baire category). Loosely speaking, a meagre set is the ‘topological counterpart’ of a set of measure zero in measure theory, and a nonmeagre set is the ‘counterpart’ of a set of non-zero measure. Given this, it is natural to say that meagre sets are atypical and nonmeagre sets are typical. I call this notion of typicality ‘t-typicality’ (‘t’ for ‘topological’) and, to avoid confusion, from now on refer to the notion of typicality introduced in Section 3 as ‘m-typicality’ in order to make it explicit that it is a measure theoretic notion.

Unfortunately there is no straightforward answer to the question about the property  $P$ . But a more promising line of argument emerges from Dürr’s (1998) and Maudlin’s (2007) discussion of typicality in the so-called Galton Board, a triangular arrangement of nails on an infinitely long vertical board. Balls are fed into the board from the top and then move down the board. Every time a ball collides with a nail it moves either to the right (R) or to the left (L). If we follow a ball’s trajectory and take down whether it moves to the left or to the right every time it hits a nail, we obtain a string of R’s and L’s that looks as random as one that has been generated by a coin toss: the Galton Board seems to exhibit random behaviour. Why is this? Dürr’s and Maudlin’s answer is that the Board appears random because random looking trajectories are typical in the sense that the set of those initial conditions that give rise to non-random looking trajectories has measure zero in the set of all possible initial conditions, and this is so because the board’s dynamics is chaotic (Dürr 1998, Sec. 2).

Translating this idea into the context of SM suggests that the relevant property  $P$  is being chaotic. This sounds *prima facie* plausible, and, most importantly, would make Premise 2 true. Markus & Meyer prove the following theorem:

Completely integrable Hamiltonians are meagre in the space of all normalised and infinitely differentiable Hamiltonians on a compact symplectic manifold. (1974, 13)<sup>10</sup>

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<sup>10</sup>Two Hamiltonians that only differ by a constant are considered equivalent, and an equivalence class of Hamiltonians is called a ‘normalised Hamiltonian’. This is because in practical calculations any Hamiltonian of this class can be chosen as a representative since they all yield the same flow (Markus & Meyer 1974, 11).

This implies that non-integrable Hamiltonians are nonmeagre, which is tantamount to saying that the class of chaotic Hamiltonians is nonmeagre,<sup>11</sup> and hence t-typical.<sup>12</sup>

The question now is whether the T-Argument valid, i.e. whether the conclusion follows from the premises. This turns out to be a thorny issue. There is an entire class of systems that are chaotic but whose phase space is full of invariant curves, namely so-called KAM systems (Argyris, Faust & Haase 1994, Ch. 4). Naturally one would expect these curves to divide  $\Gamma_E$  into a set of closed volumes bounded by the invariant curves, which would prevent the system from approaching equilibrium (invariant curves are ubiquitous in KAM systems and so it would be highly unlikely that  $\Gamma_{M_p}$  and  $\Gamma_{M_{eq}}$  would not be separated by one). In systems with two degrees of freedom this is exactly what happens: the two dimensional invariant surfaces divide the three dimensional energy hypersurface in disconnected parts. Fortunately the situation is better for systems with  $f > 2$  degrees of freedom. The energy hypersurface has  $2f - 1$  dimensions, and for another surface to divide it into two disconnected parts it must have  $2f - 2$  dimensions. But the invariant KAM tori are  $f$ -dimensional, and since  $2f - 2 > f$  for all  $f > 2$  invariant KAM surfaces do not divide  $\Gamma_E$  into separate parts; the invariant surfaces are a bit like lines in a three-dimensional Euclidean space. So the trajectories can, in principle, wander around relatively unhindered and without being ‘sandwiched’ between invariant surfaces. This process is known as *Arnold Diffusion*. It has first been proven analytically to exist in a particular example, and there is now numerical evidence that it can also be found in other systems. In such systems the chaotic parts of  $\Gamma_E$  are connected and form a single net, the so-called *Arnold Web*, which permeates the entire phase space in the sense that a trajectory moving on the web will eventually visit almost every finite region of  $\Gamma_E$ .<sup>13</sup>

This looks like what we need, but unfortunately some difficulties arise on the finishing line. These can be circumvented only at the cost of accepting three conjectures, which are only supported by plausibility arguments.

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<sup>11</sup>I here follow common practice and assume that non-integrability implies chaos, at least on some region of the phase space. However, to the best of my knowledge there is no strict mathematical proof of this.

<sup>12</sup>Mathematicians refer to t-typical Hamiltonians as ‘generic’.

<sup>13</sup>In fact, Ott (1993, 257) and Lichtenberg & Liebermann (1992, 61) say that the system visits every finite region of  $\Gamma_E$ , but this seems to be too strong.

First, there is no proof for the existence of Arnold webs in *all* nonintegrable systems with  $f > 2$ . The good news is that so far there are no known examples where this is not the case, and so we can venture the conjecture that all nonintegrable systems with  $f > 2$  have Arnold Webs (Conjecture 1).<sup>14</sup>

Second, there is a question about the relative measure in  $\Gamma_E$  occupied by Arnold Webs. It could in principle be that these webs are of measure zero, or else only fill a small part of the phase space. If this were the case, it would be unlikely that m-typical initial conditions would come to lie on trajectories that wander around randomly (and therefore wind up in  $\Gamma_{Meq}$ ), which would undercut the conclusion in the T-Argument. However, numerical simulations on simple systems have shown that the relative measure occupied by invariant KAM curves decreases as  $f$  increases (Earman & Re-dei 1996, 70). Furthermore, Sklar (1993, 175) observes that there are good numerical reasons to think that large systems are ‘at least ergodic-like’ on the ‘overwhelmingly largest part’ of the accessible part of the phase space, and Vranas (1998, 695-698) gathers a welter of numerical evidence for the conclusion that many systems of interest in SM are  $\varepsilon$ -ergodic, i.e. ergodic on nearly the entire energy hypersurface. This suggests that it may well be the case that Arnold webs not only have finite measure, but that they in fact fill most of  $\Gamma_E$  (Conjecture 2).

Third, in order to explain thermodynamic-like behaviour we need to know how how much time the system spends in different parts of the phase space. Again, little is proven rigorously, but Ott (1993, 257) suggests that system is ergodic on the Arnold Web (Conjecture 3). The numerical evidence just mentioned supports this conjecture.

If we assume that these three assumptions are correct, then the T-Argument is sound. By Premise 2 the system is chaotic, and by Conjecture 1 it has an Arnold Web, which, by Conjecture 2, fills most of the energy surface and hence most of  $\Gamma_{Mp}$ . Therefore points on the Web are m-typical in  $\Gamma_{Mp}$ . By Conjecture 3, these points wander around ergocially on the Web and hence approach  $\Gamma_{Meq}$  fairly soon and stay there for a long time (where ‘fairly soon’ means that the time taken to arrive at equilibrium is much shorter than the time spent in equilibrium) because, by Premise 1, non-equilibrium states occupy a much smaller volume than equilibrium states.

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<sup>14</sup>Or if not *all* nonintegrable systems have Arnold Webs, then those that don’t should be so few that the class of those with Arnold Webs is still nonmeagre.

To put this argument on secure footing, more would have to be said about the three conjectures. This is an extremely difficult task, and so it is worth asking whether there is not a simpler way to arrive at the same conclusion. I will now discuss a plausible way of doing so and show that it is a blind alley. Hence there is no way around trying to make progress on the conjectures.

The new line of argument departs from the observation that we might have chosen too liberal a notion of chaos. In fact, there is a great deal of controversy over the correct characterisation of chaos (Smith 1998, Ch. 10), and so we might say that KAM systems show the wrong kind of chaotic behaviour: they exhibit ‘local chaos’, meaning that the dynamics is chaotic only on parts of the phase space. What we need, so the argument goes, is that the relevant systems show ‘global chaos’, which disqualifies KAM systems.

The question then becomes how to characterise global chaos. Commonly this is done either of two ways, a topological and a measure-theoretic one. The former always involves ergodicity, and is therefore untenable: Markus & Meyer (1974, 14) also prove that in the space of all normalised and infinitely differentiable Hamiltonians on a compact symplectic manifold the class of ergodic Hamiltonians is meagre, and hence strongly chaotic systems are t-atypical. Topological definitions of chaos (the best know of which is Devaney’s) always involve topologically transitivity, the condition that for any two open regions  $A$  and  $B$  in  $\Gamma_E$ , there is a trajectory initiating in  $A$  that eventually visits  $B$ . But this condition does not fit the bill: while it is at least plausible that topological chaos is a sufficient condition for the approach to equilibrium, it requires a revision of  $P$  that seems to render Premise 2 false. As Markus & Meyer (*ibid.*, 1) point out, ergodic systems are meagre because generic systems have invariant surfaces preventing the trajectory from accessing the entire phase energy hypersurface. But a system that cannot access certain regions of  $\Gamma_E$  not only fails to be ergodic; it also fails to be topologically transitive. So topologically transitive systems must be meagre too, and hence also fail to be t-typical. For this reason shifting attention to global chaos is a dead end.

## 7 Conclusion

I have distinguished three different accounts of how typicality is used to explain thermodynamic-like behaviour. I have argued that while the first two fail, the third is promising and I have sketched a proof. The proof

rests on three conjectures which need to be further substantiated to put the argument on secure footing. Furthermore, an argument needs to be given that m-typicality and t-typicality have explanatory power from the point of view of physics.

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