on the equivalence of fields of acceleration and gravitation

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The question of whether the same acceleration field that is found in a rigid uniformly rotating disc can annul a gravitational field is answered in the negative because their curvatures are different. There is an exact correspondence between a uniformly rotating disc and hyperbolic geometry of constant curvature, while, gravitational fields require non-constant, negative curvature. The connection between the two is the free-fall time; the former has constant density while the latter, constant mass. The distortion caused by motion is experienced in the hyperbolic world when the rim of the κ-disc is approached, where κ is the disc radius that determines the nature of the fields. Characteristic hyperbolic properties can thus be used to explain relativistic phenomena, like the angle defect in relation to the FitzGerald-Lorentz contraction, the electromagnetic Poincaré stress, aberration which violates of the laws of cosines and sines, gravitational frequency shifts and the bending of light near a massive object.

INTRODUCTION

The principle of equivalence asserts that, in some sense, a field of acceleration is equivalent to a gravitational field [6]. That is to say by transforming to an accelerated frame of reference the gravitational field can be made to disappear. As a consequence, a gravitational field can be mimicked by a field of acceleration, and both can be made to disappear by a transformation to a local inertial frame where material particles behave as if they were 'free' of gravitational or centrifugal forces. In fact, it has been claimed [26] that the seeming equivalence between a field of gravity and a non-inertial frame of motion, like a rotating reference frame, was behind Einstein's search for a geometrical theory of gravity. Drawing on the putative analogy between the properties of measuring rods and clocks on a rotating disc with gravity, Einstein writes [4]

> In the general theory of relativity space and time cannot be defined in such a way that differences in the spatial coordinates can be directly measured by the unit measuring rod, or differences in the time coordinate by a standard clock.

In this paper we will consider a very simple accelerated system, namely a rigid uniformly rotating system, and see how it compares with a gravitational field.

As a result of the deformations caused by an accelerated system of reference, the principle of equivalence had to undergo qualifications and restrictions. First, and foremost, the equivalence was a local one, at a single point in space [6]. Second, the deformations on the body and on the measuring sticks used in measurement must be small enough so that the notion of a rigid body retains a meaning [13]. Thus, an equivalence between an accelerated frame of reference, caused by a rigid uniformly rotating system, and a gravitational field should hold if the motion of the former is slow enough and the field created by the latter weak enough.

Deformations, whether large or small, cause deviations from Euclidean (e) geometry. If the measuring rods are contracted on a rotating disc in the direction of rotation we cannot expect the Pythagorean theorem to hold in its e-form for any inscribed right angle triangle. Thus, what might be locally Euclidean may very well deviate to other geometries when stresses are present. Not long after the discovery of the theory of special relativity it was noticed [18] that the e-triangle of velocities must be replaced by a Lobachevsky triangle for large velocities.

Distortions which require a geometry different than e-geometry also produce optical effects [18]. When light encounters a change in density of the medium, the rays bend in such a way that they minimize their propagation time between ray endpoints. Objects are not where they appear to be, but, are slightly displaced due to the curvature of the rays [8]. In an analogous way that a non-constant index of refraction relates the e-distance to the optical path length, a metric density relates the e-distance to the hyperbolic (h) distance [2]. The h-geometry that describes a uniformly rotating disc also describes the bending of light by a massive body when the transition is made from a

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constant to a non-constant surface of negative curvature. The transition occurs by relating the absolute constant of the h-geometry to a free-fall time. A uniformly rotating disc has constant density, while, in the deflection of light, the mass is constant. In freely falling frames, the laws of physics are locally the same as in inertial frames, so that this transformation does not introduce anything that would have an effect on non-inertial or gravitational forces. We will thus come to appreciate that all forms of accelerative motion are not equivalent.

It is well-known that by replacing the radius by i times the 'radius', formulas for circumferences of circles, areas and volumes are greater than their e-counterparts. We will compare the non-flat metrics of GR, that describe curved space and time, with the simpler h-arc length which is a product of the flat metric for e-space and a differential coefficient that resembles a varying index of refraction in an inhomogeneous medium. The latter approach has been used previously to derive the three celebrated tests of GR from a flat spatial metric [10], while now, the metric density, replacing the varying index of refraction, will be determined by the geometry alone. This will provide a measure of how well non-Euclidean geometries themselves account for physical relativistic phenomena.

The logarithmic definition of h-distance, involving the cross-ratio, is a generalization of the longitudinal Doppler shift for curved geodesics. Doppler shifts relate the times of signal and signal reception in two, or more, inertial frames. Moreover, we cannot expect the conservation of energy and angular momentum to be the same in h-space as it is in e-space. The correction terms will explain phenomena due to the distortion of objects, like the electromagnetic Poincaré stress, the gravitational shift of spectral lines, the bending of light in the vicinity of a massive object, the h-geodesics of the annual oscillation of a star's apparent position due to the earth's motion about the sun [28], and relate the h-triangle defect to FitzGerald-Lorentz (FL) rotation in terms of the angle of parallelism.

FERMAT PRINCIPLE OF LEAST TIME AND HYPERBOLIC GEOMETRY

Fermat's principle of least time states that light propagates between any two points in such a way as to minimize its travel time. Fermat knew that light travels more slowly in dense materials than in less dense materials, but he did not know whether or not light travels at a finite speed or infinitely fast. The index of refraction, η, takes into account the varying inhomogeneities through which light propagates. Over scales in which the earth appears as a flat surface $y=0$, η is a function only of the height y. The optical path length, λ, or the product of the propagation time and the velocity of light connecting two points, (x_1, y_1) and (x_2, y_2), in a plane extending above and normal to the surface, is

$$\lambda = \int_{x_1}^{x_2} \eta(y) \surd\left(1+y'^{\,2}\right)\, dx, \tag{1}$$

where the prime stands for differentiation with respect to x. Moreover, if we assume that the index of refraction decreases with height by taking the index of refraction as inversely proportional to its distance y from the x-axis, then Fermat's principle of least time, (1), becomes the Poincaré upper half-plane model of h-geometry. Poincarites [14], who inhabit such a space, find their rulers shrink as they approach the x-axis, thereby taking an infinite amount of time to reach it.

The geodesics look much different than straight lines connecting two points in e-geometry. The geodesics can be found from the condition that Fermat's principle (1) be an extremum. With $\eta(y) = 1/y$, the Euler equation for the extremality of (1) is

$$\frac{d}{dx}\frac{\partial \Lambda}{\partial y'} = \frac{\partial \Lambda}{\partial y}, \tag{2}$$

where $\Lambda = \surd(1+y'^{\,2})/y$ is the integrand of (1). The solution to the resulting differential equation

$$y'' + \left(1+y'^{\,2}\right)/y = 0,$$

is the family of circles

$$(x-a)^2 + y^2 = b^2,$$

where a and b are two constants of integration. These circles are centered on the x-axis, and since we are considering only the upper half-plane, $y>0$, the half-circumferences will be the geodesics of our space. As $x_1 \to x_2$ the geodesics straighten out into lines parallel to the y-axis.

Employing polar coordinates, the arc length, γ, between $(\bar{r}_1, \theta_1)$ and $(\bar{r}_2, \theta_2)$ cannot be less than [25]

$$h(\gamma) = \kappa \int_{\theta_1}^{\theta_2} \frac{\surd(\bar{r}'^{\,2} + r^2)}{\bar{r}\sin\theta} d\theta \geq \kappa \int_{\theta_1}^{\theta_2} \frac{d\theta}{\sin\theta} = \kappa \ln\left(\frac{\csc\theta_2 - \cot\theta_2}{\csc\theta_1 - \cot\theta_1}\right), \tag{3}$$

where κ, the radius of curvature, is the absolute constant of the h-geometry. Different h-geometries with different values of κ are not congruent [2]. In the limit as $\theta_2 \to \pi/2$, the angle θ_1 becomes the angle of parallelism

$$h(\gamma) = \kappa \ln \cot[\Pi(\gamma)/2]. \tag{4}$$

This is the celebrated Bolyai-Lobachevsky formula that expresses the angle of parallelism, Π, as a sole function of the h-arc length $h(\gamma)$, which is shown in Fig. 1 to be the shortest distance connecting the bounding parallels ℓ_1 and ℓ_2. The angle of parallelism enters in the analysis of the Terrell-Weinstein effect that is related to the FL contraction.

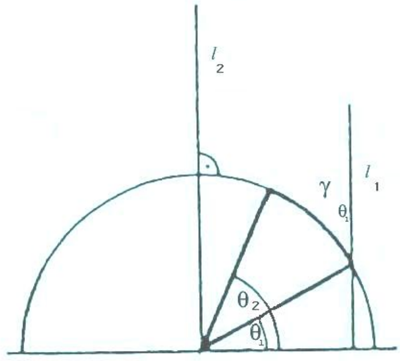

FIG. 1: The angle of parallelism between two bounding parallels connected by the geodesic curve γ.

Poincaré originally conceived of an unevenly heated slab for which the x-axis is infinitely cold. As the Poincarites approach the x-axis, the drop in temperature causes them and their rulers to contract. Had the Poincarites inhabited an elliptic (el) world, with constant positive curvature, they would have seen their rulers get longer, instead of shorter, as they move toward the boundary, like in stereographic projection. Now, Poincaré also considered a disc model of h-geometry in which an index of refraction varies as the surface of a shell from the center of a sphere, such as a variation of temperature proportional to the difference $(R^2 - \overline{r}^2)$, where $\overline{r}$ is the e-distance from the sphere's center and R, the radius of the sphere. If R is a star's radius its temperature at its surface would be infinitely cold, just like the x-axis in the upper half-plane model. By rescaling to a unit radius, the Poincaré unit disc model has a h-length of a curve γ given by

$$h(\gamma) = \kappa \int_\gamma \frac{\sqrt{(dx^2 + dy^2)}}{1 - x^2 - y^2}, \tag{5}$$

where the 'stereographic' inner product of the h-plane, $1 - x^2 - y^2$, plays the same role as the inverse of the index of refraction in (1). (5) is the metric that gives the interior of the unit disc its h-structure. It can be derived by mapping the entire half-plane into the unit disc by means of an inversion [14], but that will not concern us since it involves entering the complex plane.

As we approach the rim of the disc, rulers begin to shrink, so, like in the half-plane model, it will take an infinite amount of time to reach the boundary. Introducing polar coordinates $(\overline{r}, \theta)$ in (5) gives the arc length at constant θ as

$$h(\overline{r}) := r = \kappa \int_0^{\overline{r}} \frac{d\overline{r}}{1 - \overline{r}^2} = \kappa \tanh^{-1} \overline{r} = \frac{\kappa}{2} \ln\left(\frac{1 + \overline{r}}{1 - \overline{r}}\right). \tag{6}$$

Straight lines are to e-geometry what the h-tangent is to h-geometry. As $\overline{r} \to 1$, the h-length becomes infinite. In terms of the c-disc, $\overline{r}$ becomes the relative velocity $\dot{\overline{r}}/c$, and (6) gives h-measure for the velocity, $\dot{r}$, in terms of the logarithm of the longitudinal Doppler shift. The absolute constant of h-geometry determines whether we are in configuration or velocity space, and we will have occasion to switch back and forth. The radii of the Poincaré discs will set the limitations imposed by relativity: Whereas in velocity space the disc will have radius c, the radius will be

c/ω in the configuration space of a uniformly rotating disc, where ω is the constant angular speed of rotation. The curvature of the space is negative and constant [cf (33) below]. The transition to non-constant curvature consists in replacing the angular velocity by the free-fall frequency, thereby transferring a system at constant density to one of constant mass. From this we conclude that it is either configuration space or velocity space which determines the metrical properties, and not space-time. Time enters in the magnification of the Beltrami coordinates in velocity space [11].

Physically, space and time are not separate entities, but, are united by the method of measurement: distances are determined by lapses in time [18]. That is, if t_1 and t_2 are the times of two events, $t_2 > t_1$, say the sending and receiving of light signals, Einstein synchronization gives the travel time as $\frac{1}{2}(t_1 + t_2)$, which is more of a convention than an inescapable fact [19], while the distance traveled is $\frac{1}{2}c(t_2 - t_1)$ [cf ref [11]]. According to Einstein, these definitions should be true in general, for "We can always regard an infinitesimally small region of the space-time continuum as Galilean. For such an infinitesimally small region there will be an inertial system relative to which we are to regard the laws of special relativity as valid" [5].

To determine the geodesics in the Poincaré κ-disc model, we write the infinitesimal square of the h-metric in polar coordinates as

$$dh^2 = E\, d\overline{r}^2 + G\, d\theta^2, \tag{7}$$

known as the Clairut, or orthogonal, representation, where the metric coefficients are [15]

$$E = \kappa^2/(1 - \overline{r}^2/\kappa^2)^2 \tag{8a}$$

$$G = \kappa^2\overline{r}^2/(1 - \overline{r}^2/\kappa^2)^2. \tag{8b}$$

Like Fermat's principle of least time, the equation for the geodesics can be derived from the condition that the metric (7) be an extremum. Assuming that $\overline{r}$ is the independent variable, setting $\Lambda = \surd(E + G\theta'^{\,2})$, where the prime stands for differentiation with respect to $\overline{r}$, and observing that θ is a cyclic coordinate so that its conjugate is constant, the condition for an extremum is

$$\frac{\partial\Lambda}{\partial\theta'} = \frac{G\theta'}{\surd(E + G\theta'^{\,2})} = \kappa\Delta = \text{const.},$$

where the slant Δ gives is the distance from the line to the origin. Solving for θ' gives

$$\theta' = \pm\frac{\kappa\Delta\surd E}{\surd G\surd[G - (\kappa\Delta)^2]}, \tag{9}$$

as the equation for the geodesic in the Clairut parameterization.

The equation of the geodesics (9) can further be decomposed into a radial equation [15],

$$\dot{\overline{r}} = \pm c\kappa\frac{\surd[G - (\kappa\Delta)^2]}{\surd(EG)}, \tag{10}$$

and an angular one,

$$\dot{\theta} = \frac{\kappa^2\Delta}{G}. \tag{11}$$

To integrate the equation of the geodesics, (9), which is explicitly given by

$$\theta' = \pm\Delta\frac{(1 - \overline{r}^2\kappa^2)/\overline{r}^2}{\surd\left[1 - \Delta^2(1 - \overline{r}^2/\kappa^2)^2/\overline{r}^2\right]}, \tag{12}$$

we observe that $dz = -\gamma\left(1 - \overline{r}^2/\kappa^2\right) d\overline{r}/\overline{r}^2$, where γ is an unknown parameter. Introducing this into (12) we will get

$$d\theta = \frac{dz}{\surd(1 - z^2)}, \tag{13}$$

if we set

$$\gamma := \frac{\Delta}{\surd(1 + \Delta^2/\kappa^2)}. \tag{14}$$

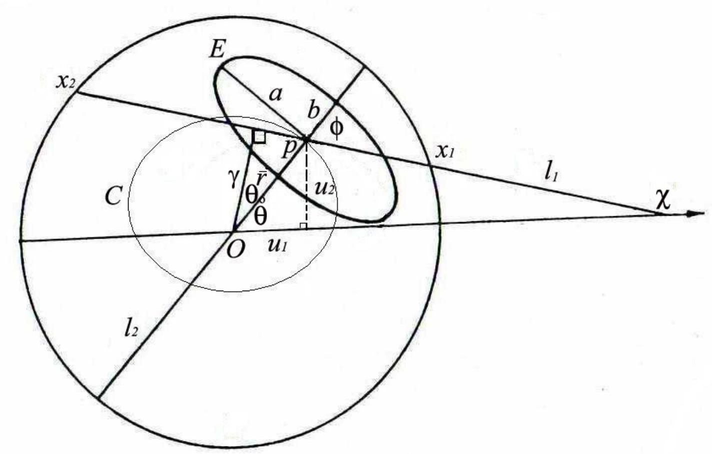

FIG. 2: Bow shaped geodesics which cut the unit disc orthogonally that occur in the limit of the vanishing of the aberration angle.

The integration of (13) is elementary and leads to

$$\overline{r}^2 + \overline{r}_0^2 - 2\overline{r}\overline{r}_0 \cos(\theta - \theta_0) = b^2, \tag{15}$$

which is the polar form for a circle of radius b centered at $(\overline{r}_0, \theta_0)$, where $\overline{r}_0^2 = b^2 + 1$. Since $\overline{r}_0 > 1$, the center of the circle will lie outside of the unit disc, which it cuts orthogonally at the rim $x^2 + y^2 = 1$, as shown in Fig. 2.

We may contrast this with the geodesics of the e-plane, where metric coefficients in (7) are $E = \kappa^2$ and $G = (\kappa\overline{r})^2$. The equation for the geodesics (9) becomes

$$\theta' = \pm\frac{\Delta}{\overline{r}\sqrt{(\overline{r}^2 - \Delta^2)}} = \pm\frac{d}{d\overline{r}}\left(\cos^{-1}\frac{\Delta}{\overline{r}}\right), \tag{16}$$

where $\overline{r} > \Delta$, is the distance of closest approach. The radius vector κ has completely disappeared meaning that it is no restriction on the motion. The integral of (16) is a straight line

$$\overline{r}\cos(\theta - \theta_0) = \Delta. \tag{17}$$

To the Poincarites living in the κ-disc their universe would appear infinite because their rulers shrink along with them as they approach the rim, κ. This can be seen by introducing the h-polar coordinates, $x = \kappa\tanh(r/\kappa)\cos\theta$ and $y = \kappa\tanh(r/\kappa)\sin\theta$ so that the h-metric (5) becomes

$$dh^2 = dr^2 + \kappa^2\sinh^2(r/\kappa)d\theta^2. \tag{18}$$

In this polar geodesic parameterization, $E = 1$ and $G > 0$, where $\sqrt{G}$ is the measure at which the radial geodesics are spreading out from the origin [15]. Because $\sinh x > x$ for all $x > 0$, the rate at which the geodesics spread out, $\kappa\sinh(r/\kappa)$, will be greater in the h-plane than the e-plane, where the rate of spreading is $\overline{r}$. Consequently, distances become larger, or equivalently, measuring sticks shrink as the rim is approached. This shrinkage causes the geodesics to bend in such a way that they cut the rim orthogonally. The e-parallel postulate, that if a point is not on given line then there is a unique line through this point that does not meet that line, is invalidated in the h-plane. In fact, there are an infinite number of geodesics that pass through any given point that do not meet another geodesic.

The geodesics still appear as straight lines to the Poincarites, whereas, to us Euclideans, they appear to be bent, and things vary in size depending on where we are in the unit disc. We then see things like the bending of light and the shifting of frequencies in a gravitational field.

THE ROTATING DISC

Consider a rotating κ-disc [17], where κ the the relativistic limit placed on the radius vector. At the center of the disc we have an inertial system which is described by e-geometry. It is well-known [13] that a clock located anywhere else on the disc will have a velocity $\overline{r}\omega$ relative to the inertial system, and its clock will be retarded by the amount

$$\tau = t\sqrt{(1 - \overline{r}^2\omega^2/c^2)}. \tag{19}$$

This fixes the absolute constant, or the disc radius, at $\kappa = c/\omega$. Now, it is argued [13] that any rod in motion should undergo a FL contraction. This means that any two points on the disc that are at a distance $\overline{r}$ from the center, say, $(\overline{r}, \theta)$ and $(\overline{r}, \theta + d\theta)$, and are connected by a measuring rod, is shortened with respect to the length of the rod in the inertial frame $d\overline{r}_0$ by an amount

$$\overline{r}\, d\theta = d\overline{r}_0\sqrt{(1 - \overline{r}^2/\kappa^2)}.$$

From the previous discussion, we expect the geodesics to be either bowed or straight lines through the origin. We will now give a geometric explanation of why successive Doppler shifts occur with rotations. In order to do so, we must determine the ratio of the h- to e-lengths.

Consider two variable points, u and v, on the interval (x_1, x_2). The h-distance between u and v is given by the cross-ratio [2]

$$h(u,v) = \frac{\kappa}{2}\ln\left[\frac{e(x_1,u)}{e(x_1,v)}\cdot\frac{e(x_2,v)}{e(x_2,u)}\right] = \frac{\kappa}{2}\ln\left[1 + \frac{e(u,v)}{e(v,x_1)}\right] + \frac{\kappa}{2}\ln\left[1 + \frac{e(u,v)}{e(u,x_2)}\right],$$

where $e(u,v)$ is the e-distance between u and v, and $e(x_1,x_2) = e(x_1,v) + e(v,x_2)$. Since we will let u and v tend to a common limit, p, we can expand the logarithms in series and retain only the lowest order to obtain, in the limit, the metric density [2]

$$\lim_{u,v\to p}\frac{h(u,v)}{e(u,v)} = \frac{\kappa}{2}\left[\frac{1}{e(p,x_1)} + \frac{1}{e(p,x_2)}\right] =: \Xi(p), \tag{20}$$

which is the inverse of the harmonic mean of the two distances.

In respect to Fermat's principle of least time, (1), the metric density (20) can be associated with a non-constant index of refraction for it converts the e-distance, de, into an h-distance, dh. Just as the index of refraction varies with height, causing light to arch its path upwards in order to minimize its propagation time between given endpoints, acceleration, in general, creates distortion causing objects to vary in size and not be where they seem to be [8].

The distance from the origin O to the point p in the κ-disc is $\overline{r} < \kappa$ in Fig. 3.

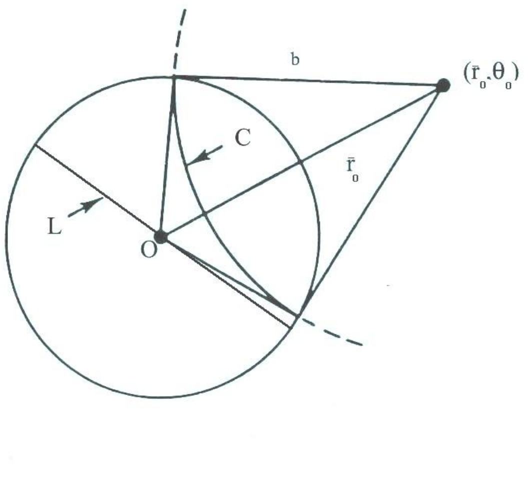

FIG. 3: Geometric characterization of the metric density.

In order to obtain an explicit expression for the metric density, (20), we use two elementary facts about circles: (i) All chords passing through an interior fixed point are divided into two parts whose lengths have a constant product

$$e(p,x_1)e(p,x_2) = (1 + \overline{r}/\kappa)(1 - \overline{r}/\kappa) = 1 - \overline{r}^2/\kappa^2,$$

and (ii) the length of a chord is twice the square root of the squares of the difference between the radius and the perpendicular distance from the center to the chord,

$$e(p,x_1)+e(p,x_2)=e(x_1,x_2)=2\surd(1-\frac{\overline{r}^2}{\kappa^2}\sin^2\phi).$$

Combining these two geometrical facts gives

$$\Xi(u_1,u_2)=\kappa\frac{\surd\left[1-(\overline{r}^2/\kappa^2)\sin^2\phi\right]}{1-\overline{r}^2/\kappa^2}, \tag{21}$$

where ϕ is the angle formed by the intersection of lines ℓ_1 and ℓ_2. The polar coordinates are

$$u_1=(\overline{r}/\kappa)\cos\theta, \qquad u_2=(\overline{r}/\kappa)\sin\theta \tag{22}$$

in either velocity or configuration space, where the radius of curvature κ has the values c and c/ω, respectively. Whereas lengths are relative in e-geometry, and angles are absolute, the relation between lengths and angles in h-geometry makes lengths, as well as angles, absolute.

Denoting χ as the angle of inclination of the tangent line ℓ_1 we have

$$\tan\chi=\frac{du_2}{du_1}=\frac{\overline{r}'\sin\theta+\overline{r}\cos\theta}{\overline{r}'\cos\theta-\overline{r}\sin\theta}. \tag{23}$$

Moreover, since $\chi=\theta+\pi-\phi$, we have

$$\tan\chi=\frac{\tan\theta-\tan\phi}{1+\tan\theta\ \tan\phi}. \tag{24}$$

Equating the two expressions (23) and (24) we find

$$\tan\phi=-\overline{r}/\overline{r}'=-\overline{r}\theta', \tag{25a}$$

$$\sin\phi=\frac{u_1du_2-u_2du_1}{\overline{r}\surd(du_1^2+du_2^2)}=\frac{\overline{r}\theta'}{\surd(1+\overline{r}^2\theta'^{\,2})}, \tag{25b}$$

$$\cos\phi=\frac{u_1du_1+u_2du_2}{\overline{r}\surd(du_1^2+du_2^2)}=\frac{1}{\surd(1+\overline{r}^2\theta'^{\,2})}. \tag{25c}$$

(25d)

Consequently, [1]

$$dh^2(u_1,u_2)=\kappa^2\frac{du_1^2+du_2^2-(u_1du_2-u_2du_1)^2}{(1-u_1^2-u_2^2)^2}=\frac{d\overline{r}^2+\overline{r}^2d\theta^2(1-\overline{r}^2)}{(1-\overline{r}^2)^2}=Ed\overline{r}^2+Gd\theta^2, \tag{26}$$

is the square of the h-line element expressed in terms of u_1 and u_2, and polar coordinates, $\overline{r}$ and θ. The first two terms in the numerator of (26) is twice the e-kinetic energy

$$2T=\dot{\overline{r}}^2+\overline{r}^2\dot{\theta}^2,$$

while the last term in the numerator is the square of the e-angular momentum

$$L_e=\overline{r}^2\dot{\theta}. \tag{27}$$

But, the denominators in (26), which represent the stereographic inner product, will cause the e-conservation laws of energy and angular momentum to be modified. It will not due to write off these modifications as 'ambiguities' in the notion of a 'radius vector' in geometries other than Euclidean [13]. They are, in fact, very real and unambiguous.

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[1] GR proposes a metric [13, eqn (7), p. 224]

$$dh=\surd\left[d\overline{r}^2+\frac{\overline{r}^2d\theta^2}{(1-\overline{r}^2\omega^2/c^2)}\right],$$

where the first term, at $\theta=$ const., does not integrate to give the h-measure of distance, namely (6), but, rather, gives its e-measure, $\overline{r}$.

For uniform radial motion $\theta = \text{const.}$, and (26) reduces to

$$dh = \Xi\, d\overline{r} = \frac{d\overline{r}}{(1-\overline{r}^2/\kappa^2)}, \tag{28}$$

whose integral is (6), where $\overline{r} \to \overline{r}/\kappa$. The e-measure of the h-distance r is thus

$$\overline{r} = \kappa \tanh(r/\kappa). \tag{29}$$

Alternatively, for uniform circular motion, $\overline{r} = \text{const.}$, and (26) reduces to

$$dh = \frac{\overline{r} d\theta}{\sqrt{(1-\overline{r}^2/\kappa^2)}}. \tag{30}$$

Integrating over a period gives

$$h = \frac{2\pi\overline{r}}{\sqrt{(1-\overline{r}^2/\kappa^2)}} > 2\pi\overline{r}, \tag{31}$$

and shows that the length of a h-circle of radius $\sinh(r/\kappa)$ is greater than that of an e-circle having an radius $\overline{r}(=\kappa\tanh(r/\kappa))$. This is supported by considering the circumference of a h-circle with center O and radius (29) in Fig. 3, which is determined by observing that every point p on this circle $\phi = \pi/2$. Thus, $\Xi = \kappa/\sqrt{(1-\overline{r}^2/\kappa^2)} = \kappa\cosh(r/\kappa)$, and this value multiplied by $2\pi\overline{r}/\kappa = 2\pi\tanh(r/\kappa)$, gives the h-circumference $2\pi\kappa\sinh(r/\kappa)$ [2].

The metric coefficients in (26)

$$E = \kappa^2/(1-\overline{r}^2/\kappa^2)^2 \tag{32a}$$

$$G = \kappa^2\overline{r}^2/(1-\overline{r}^2/\kappa^2) \tag{32b}$$

fix the constant, Gaussian curvature at [15]

$$K = -\frac{1}{2\sqrt{(EG)}}\frac{d}{d\overline{r}}\left(\frac{G_{\overline{r}}}{\sqrt{(EG)}}\right) = -1/\kappa^2. \tag{33}$$

They also determine the geodesics from (9). In contrast to (12), the equation of the geodesics is

$$\theta' = \pm\frac{\Delta}{\overline{r}^2\sqrt{\left[1-(\Delta/\overline{r})^2(1-\overline{r}^2/\kappa^2)\right]}}. \tag{34}$$

The geodesics are straight lines,

$$\overline{r}\cos(\theta-\theta_0) = \gamma, \tag{35}$$

where θ_0 is the angle that the perpendicular to the line $\overline{r} = \gamma$ makes with the polar axis, as shown in Fig. 3. We already know that the geodesics must pass through the origin, and these have an inclination $\theta = \theta_0$ with respect to the polar axis. In physical terms, there is nothing to counter the centrifugal force that would allow for the formation of a closed orbit.

We now inquire into the physical meaning of ϕ whose tangent is related to the equation of the geodesics according to (25a). The equations of aberration [1]

$$u\cos\phi' = \frac{u_1\cos\phi - u_2}{1-u_1\cdot u_2\cos\phi} \tag{36a}$$

$$u\sin\phi' = \frac{u_1\sin\phi\sqrt{(1-u_2^2)}}{1-u_1\cdot u_2\cos\phi}, \tag{36b}$$

can be used to derive the most general composition law of velocities. Squaring (36a) and (36b), and adding result in

$$u^2 = \frac{(\vec{u}_1-\vec{u}_2)^2-(\vec{u}_1\times\vec{u}_2)^2}{(1-\vec{u}_1\cdot\vec{u}_2)^2} = \frac{u_1^2+u_2^2-2u_1\cdot u_2\cos\phi-(u_1\cdot u_2\sin\phi)^2}{(1-u_1\cdot u_2\cos\phi)^2}. \tag{37}$$

Dividing (36b) by (36a) gives

$$\tan\phi' = \frac{u_1 \sin\phi\sqrt{(1-u_2^2)}}{u_1\cos\phi - u_2}. \tag{38}$$

Whereas the composition law of velocities, (37), invalidates the law of cosines

$$u^2 = u_1^2 + u_2^2 - 2u_1 \cdot u_2 \cos\phi, \tag{39}$$

(38) invalidates the law of sines. We can appreciate this by considering the phenomenon of stellar aberration, where $u_1 = c$ and u_2 equals the earth's velocity. Equation (38) becomes

$$\tan\phi' = \frac{\sin\phi\sqrt{(1-\beta^2)}}{\cos\phi - \beta}. \tag{40}$$

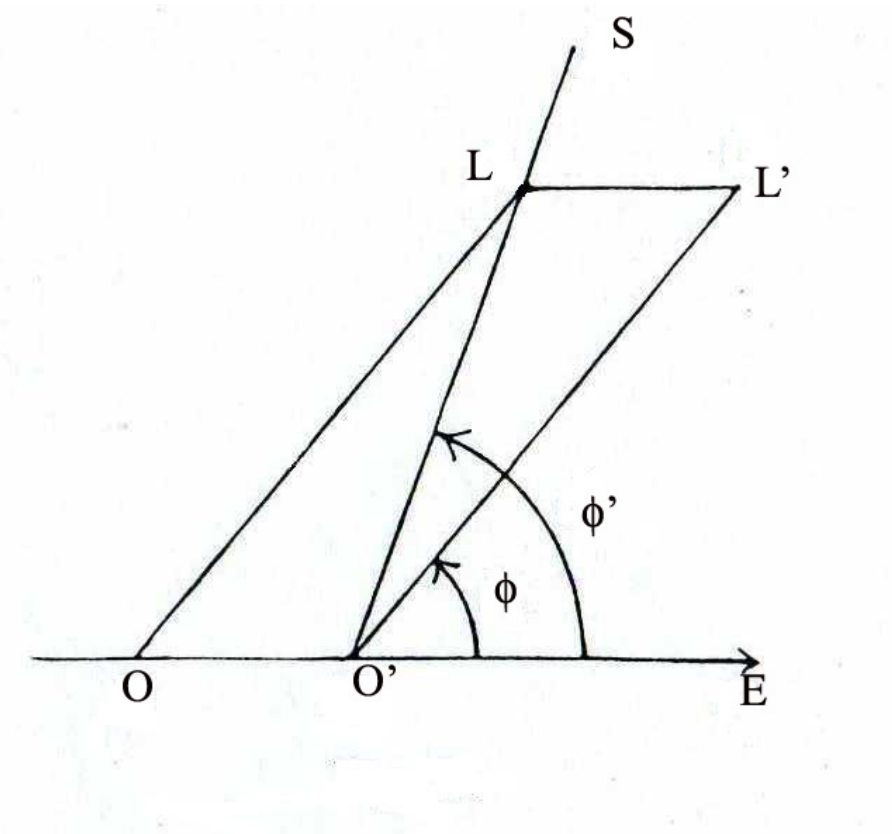

FIG. 4: Geometrical set-up for aberration.

In Fig. 4, L represents the telescope's lens, and O the eye of the observer at the moment a light ray arrives at L from a star S. OE indicates the direction in which the earth is orbiting about the sun. In time τ that it takes the light ray to pass through the telescope, the earth will have moved a distance $u_2\tau$ to position O'. The distance between the lens and the new position of the earth is $c\tau$. If the earth were stationary then the telescope would be pointed along $O'L$, but, because of the earth's motion, it is pointed in direction OL. Drawing $O'L'$ and LL' completes the parallelogram. Denote $\angle LO'E$ by ϕ' and $\angle L'O'E$ by ϕ. The difference $(\phi' - \phi)$ is attributed to aberration [22], which displaces the star's actual position toward the direction OE in the plane $SO'E$.

The law of sines for $\triangle LO'L'$ is

$$\frac{\sin\angle LO'L'}{LL'} = \frac{\sin\angle LL'O'}{LO'}. \tag{41}$$

Introducing the facts that $LL' = OO' = u_2\tau$, and $LO' = c\tau$, we get

$$\sin(\phi' - \phi) = \beta\phi. \tag{42}$$

Since the earth's relative velocity, $\beta := u_2/c = 10^{-4}$, is small, we may replace the sine by its argument to get the formula

$$\Delta\phi := \phi' - \phi = \beta \sin\phi, \tag{43}$$

which is commonly used to calculate aberration [22], where β is called the constant of aberration. This e-approximation is equivalent to approximating the square-root in (38) by unity, and neglecting terms of higher power than first in β so that (38) will reduce to

$$\tan\phi' \simeq \tan\phi\,(1 + \beta\sec\phi)\,.$$

Then performing the expansion of the trigonometric functions to first-order in the difference $\Delta\phi$ results in (43).

The violations of the laws of cosines and sines, (39) and (41), mean that the addition law of velocities cannot be represented as a triangle in the flat e-plane, but, rather, as a distorted triangle on the surface of a pseudosphere, a surface of revolution with constant negative curvature resembling a bugle surface [15]. Although a bugle has a rim, so that occupies only a finite region of the h-plane, it displays the h-axiom that for any given line ℓ and point p not on ℓ, there are at least two lines through p that do not intersect ℓ. The angle defect of the triangle is a direct consequence of this axiom, and the area of the triangle is proportional to its defect. Furthermore, the parallelogram rule for the addition of velocities in Newtonian kinematics is no longer valid, and we are left only with the triangle rule [21]. Thus, (38) can be considered as the relativistic generalization of the law of aberration that would be applicable to relativistic velocities.

The plane of the orbit that the sun traces out in a year is called the ecliptic plane. The great circle in which this plane intersects the celestial sphere, at whose center the earth is found, is called the ecliptic. If the fixed star is at the pole of the ecliptic, $\phi = \pi/2$ all along the earth's orbit. The aberrational orbit will be a circle about the pole of the ecliptic with radius β. This corresponds to the circle C with center O and e-radius $\overline{r}$ in Fig. 3. For every point p on the locus of points at a given distance from the center, $\phi = \pi/2$, and $\Xi = \kappa/\surd(1 - \overline{r}^2/\kappa^2) = \kappa\cosh(r/\kappa)$. Multiplying this by the e-circumference, $2\pi\overline{r}/\kappa = 2\pi\tanh(r/\kappa)$ gives the circumference of the h-circle, $2\pi\kappa\sinh(r/\kappa)$, a result which we found earlier.

For stars that lie in the ecliptic, ϕ varies between $\pm\pi/2$ and 0 [24]. An h-motion which takes O to p transforms a circle C centered at O into an ellipse E centered at p, shown in Fig. 3, that has a semi-minor axis, $b = 1/\Xi = (1 - \overline{r}^2/\kappa^2)/\kappa$ at $\phi = \pi/2$, and a semi-major axis, $a = 1/\Xi = \surd(1 - \overline{r}^2/\kappa^2)/\kappa$ at $\pi/2$. The e-area the ellipse is $\pi(1 - \overline{r}^2/\kappa^2)^{3/2}/\kappa^2 = (\pi/\kappa^2)\mathrm{sech}^3(r/\kappa)$ [2].

The equation for the radial velocity can be determined from the h-conservation of angular momentum [cf eqn (11)]

$$L_h = \frac{\overline{r}^2\dot{\theta}}{1 - \overline{r}^2/\kappa^2} = \text{const.}, \tag{44}$$

and (9). Introducing (44) into the (34), where we write $\theta' = \dot{\theta}/\dot{\overline{r}}$, gives [cf eqn (10)]

$$\dot{\overline{r}} = \pm c\left(1 - \frac{\overline{r}^2}{\kappa^2}\right)\surd\left[1 - \frac{\Delta^2}{\overline{r}^2}\left(1 - \frac{\overline{r}^2}{\kappa^2}\right)\right]. \tag{45}$$

As (44) clearly shows, it is necessary to modify the definition of the angular momentum in curved space. Møller [13] claims that the notion of a 'radius vector' in the definition of angular momentum can only be defined unambiguously in the e-plane, and assumes that $(1 - \overline{r}^2\omega^2/c^2)^{-1}$ is a small correction to the angular momentum, so that there is only a slight violation of the conservation law of angular momentum. On the contrary, (44) is to be considered as the h-conservation law for angular momentum.

GR agrees with (44), but not with (45). The radial equation proposed by GR,

$$\dot{\overline{r}} = \pm c\surd\left[1 - \frac{L_h^2}{c^2\overline{r}^2}\left(1 - \frac{\overline{r}^2\omega^2}{c^2}\right)\right], \tag{46}$$

differs from (45) by the absence of the first factor, which is related to the correction term in the expression for the h-angular momentum. Thus, according to GR, the equation for the geodesic is

$$\theta' = \pm\frac{\Delta(1 - \overline{r}^2/\kappa^2)/r^2}{\surd[1 - (\Delta/\overline{r})^2(1 - \overline{r}^2/\kappa^2)]}. \tag{47}$$

This equation is identical to (12) and leads to bowed geodesics whose centers lie outside the disc. The bending of the geodesic is attributed to the rotation of the disc [7]. However, it is the first term in the radial equation (45) that cancels the correction term to the e-angular momentum in (44) in the equation for the geodesic (34). Hence, there are no bowed geodesics. Notwithstanding this difference, both radial equations identify the slant parameter Δ with the h-measure of the angular momentum, (44). If we had, instead, identified the slant parameter with the e-angular momentum, (27), the radial equation would have been

$$\dot{\overline{r}} = \pm c\surd\left[1 - \frac{L_e^2}{c^2\overline{r}^2}\left(1 - \overline{r}^2\omega^2/c^2\right)^{-1}\right], \tag{48}$$

which is obviously incorrect because the correction term would have the opposite sign when the denominator is developed as a series in powers of $(\overline{r}\omega/c)^2$.

To calculate the h-distance between points $\overline{r}_1$ and $\overline{r}_2$,

$$h(\overline{r}_1, \overline{r}_2) = \kappa\int_{\overline{r}_1}^{\overline{r}_2} \frac{\surd\left[d\overline{r}^2 + \overline{r}^2 d\theta^2(1 - \overline{r}^2/\kappa^2)\right]}{(1 - \overline{r}^2/\kappa^2)}, \tag{49}$$

we introduce the 'effective' centrifugal potential,

$$V(\overline{r}) = \frac{L_h^2}{2\overline{r}^2}\left(1 - \frac{\overline{r}^2}{\kappa^2}\right), \tag{50}$$

where relativistic effects are accounted for in the second term. Squaring (45) gives the h-energy conservation law,

$$\dot{r}^2 + 2V(\overline{r}) = c^2, \tag{51}$$

where we have used the fact that $dr = d\overline{r}/(1 - \overline{r}^2/\kappa^2)$. This is the origin of the first factor in (45). *The h-measures of radius and angular momentum appear in the energy conservation law, and not their e-measures.*

In terms of the effective potential, we can write the h-distance (49) as the logarithm of the cross-ratio

$$h(\overline{r}_1, \overline{r}_2) = \frac{\kappa}{2}\ln\left[\frac{\kappa + \overline{r}_2\surd(1 - 2V(\overline{r}_2)/c^2)}{\kappa - \overline{r}_2\surd(1 - 2V(\overline{r}_2)/c^2)} \cdot \frac{\kappa - \overline{r}_1\surd(1 - 2V(\overline{r}_1)/c^2)}{\kappa + \overline{r}_1\surd(1 - 2V(\overline{r}_1)/c^2)}\right].$$

For large radial velocities, or equivalently, at low angular momentum, the distance between points $\overline{r}_1$ and $\overline{r}_2$ becomes

$$h(\overline{r}_1, \overline{r}_2) = \frac{\kappa}{2}\ln R, \tag{52}$$

where R is the cross-ratio [2]:

$$R = \left(\frac{\kappa + \overline{r}_2}{\kappa - \overline{r}_2}\right) \cdot \left(\frac{\kappa - \overline{r}_1}{\kappa + \overline{r}_1}\right),$$

between the ordered points $(\kappa, \overline{r}_2, \overline{r}_1, -\kappa)$. The h-distance (52) vanishes when $\overline{r}_1 = \overline{r}_2$ and tends to infinity when either $\overline{r}_2 \uparrow \kappa$ or $\overline{r}_1 \downarrow -\kappa$. To the Poincarities, it would seem like the distance to the rim is infinitely far away.

If the uniform acceleration is that due to gravity, this will fix the radius of curvature as

$$\kappa = c\surd(3/4\pi G\varrho) \tag{53}$$

for a mass of constant density ϱ, and G is the Newtonian gravitational constant. The ratio, κ/c appears as a free-fall time. Freely-falling objects know no restrictions placed on their velocities, like the restriction to regions of the rotating disc where $\overline{r} < c/\omega$. The frequency $\omega = \surd(4\pi G\varrho)$ is the minimum frequency which an object must rotate in order to avoid gravitational collapse. A constant free-fall time is thus compatible with a uniformly rotating system, which, as we have shown, is described by a h-metric with constant curvature. This choice of the absolute constant sets the Gaussian curvature (33) directly proportional to the constant mass density, viz., $K = -1/\kappa^2 = -4\pi G\varrho/3c^2$. In the h-space of constant negative curvature, a gravitational potential cannot be appended to the energy conservation law (51), as a separate entity. However, if mass, rather than density, is constant, the curvature will no longer be constant so that gravitational acceleration will not be equivalent to a uniformly rotating disc. Although the curvature will no longer be constant, it will show that gravitational effects enter, not through a potential in energy conservation, but, rather by the specification of the absolute constant κ that determines the point at infinity, or the ideal point, where two parallel lines intersect. We will come back to this case in the section on h-geometries of non-constant curvature.

FITZGERALD-LORENTZ CONTRACTION THROUGH THE TRIANGLE DEFECT

The fact that an h-triangle must be fit onto a pseudosphere in h-space, rather than lying flatly in the e-plane, causes an angle defect where the sum of the angles of a triangle is less than two right angles. Moreover, since the angles of a h-triangle determines the sides, we can expect the defect to shorten the length of one of its sides. And since the h-triangle is in velocity space we can expect that the angle defect will be related to the FL contraction, or rather, rotation.

In the early days of relativity, Ehrenfest [3] arrived at the paradoxical conclusion that the circumference of a rotating disc should be shorter than $2\pi\overline{r}$ due to the FL-contraction. Einstein [26] came to the opposite conclusion that the circumference had to be longer than $2\pi\overline{r}$ claiming that it was necessary to keep the longitudinal FL-contraction as distinct from the shortening of the tangential components of the measuring rods on the disc by a factor of $\surd(1-\overline{r}^2\omega^2/c^2)$. And because you need more tangential measuring rods than when the disc is at rest, its circumference should be increased by the factor $1/\surd(1-\overline{r}^2\omega^2/c^2)$. He concludes that a "rigid disc must break up if it is set into motion, on account of the Lorentz contraction of the tangential fibres and the non-contraction of the radial ones" [26]. Einstein's argument, that the measuring rods contract so that more are needed to measure the circumference of the disc, doesn't answer the question of why the object they are measuring doesn't also contract. And the contraction should be greater the faster the disc rotates.

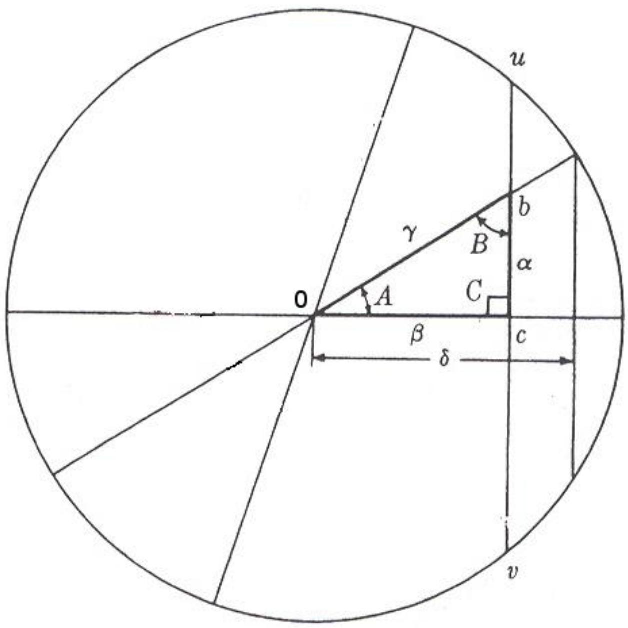

FIG. 5: Hyperbolic right triangle inscribed in a unit disc.

Consider a unit-disc in velocity space with a right triangle inscribed in it, as shown in Fig. 5. The e-measures of the sides are $\bar{\beta}=\dot{\overline{r}}/c$ and $\bar{\alpha}=\overline{r}\dot{\theta}/c$, and the hypotenuse $\bar{\gamma}=\overline{r}\omega/c<1$, which ensures that the rotating system may be represented by a uniformly rotating 'material' disc [13]. The angle A at the center of the disc will not be distorted so that it will obey e-geometry. Thus, its e-measure $\bar{A}$ will coincide with its h-measure A

$$\cos A=\cos\bar{A}=\bar{\beta}/\bar{\gamma}=\bar{\beta}/\bar{\gamma}=\tanh\beta/\tanh\gamma. \tag{54}$$

However, because the angle B is non-central, its e-measure, $\bar{B}$, and h-measure, B, will not coincide. The logarithm of the cross-ratio is the h-distance

$$\alpha=\frac{c}{2}\ln R=\frac{c}{2}\ln\left[\frac{e(c,u)}{e(b,u)}\cdot\frac{e(b,v)}{e(c,v)}\right]=\frac{c}{2}\ln\left[\frac{\surd(1-\dot{\overline{r}}^2/c^2)}{\surd(1-\dot{\overline{r}}^2/c^2)-\dot{\alpha}}\cdot\frac{\surd(1-\dot{\overline{r}}^2/c^2)+\bar{\alpha}}{\surd(1-\dot{\overline{r}}^2/c^2)}\right]. \tag{55}$$

Exponentiating both sides and solving for $\bar{\alpha}$ we find

$$\bar{\alpha}=\tanh(\alpha/c)\cdot\operatorname{sech}(\dot{r}/c), \tag{56}$$

which together with

$$\bar{\beta} = \tanh\beta \tag{57}$$

are the set of Beltrami coordinates. In terms of these coordinates the metric form is

$$dh^2 = c^2\left(\cosh^2\alpha\, d\beta^2 + d\alpha^2\right).$$

Moreover, the e-Pythagorean theorem, $\bar{\gamma}^2 = \bar{\alpha}^2 + \bar{\beta}^2$, asserting that $\tanh^2\gamma = \tanh^2\alpha \cdot \text{sech}^2\beta + \tanh^2\beta$, gives way to the h-Pythagorean theorem

$$\cosh\gamma = \cosh\alpha \cdot \cosh\beta. \tag{58}$$

This shows that the Lobachevsky straight line in h-space, $\tanh\alpha$, has been shortened by the amount, $\text{sech}\beta$, which is the FL contraction factor.

Now, the cosine of the h-measure of the angle B, $\cos B = \tanh(\alpha/c)/\tanh(\gamma/c)$, or the ratio of the adjacent to the hypotenuse, will be related to the cosine of its e-measure by

$$\cos\bar{B} = \frac{\bar{\alpha}}{\bar{\gamma}} = \frac{\tanh(\alpha/c)\text{sech}(\beta)}{\tanh(\gamma)} = \cos B\ \text{sech}(\beta) = \cos B\ \surd(1-\bar{\beta}^2). \tag{59}$$

Consequently, $\cos B > \cos\bar{B}$, and since the cosine decreases monotonically on the open interval $(0,\pi)$ it follows that $B < \bar{B}$. Since $\bar{B} = \pi/2 - A$, the sum of the angles of a h-triangle will be less than π, which is the well-known angle defect. And since the angles of a h-triangle determine its sides, the side $\bar{\alpha}$ will appear smaller than its h-measure, $\tanh\alpha$, by an amount given by the FL contraction factor, $\surd(1-\beta^2)$. *The angle defect in h-triangles is the geometrical origin of the FL contraction, not of the configuration length, but, rather the angular velocity $\bar{r}\dot{\theta}$ of an object that is on a disc rotating at angular velocity $\bar{r}\omega$.*

HYPERBOLIC NATURE OF THE ELECTROMAGNETIC FIELD AND THE POINCARÉ STRESS

H-geometry also applies to Maxwell's equations, and, in this section, we will show how it can be used to calculate the Poincaré stress that make the assumption of charge conservation on the surface of the electron superfluous.

Consider a charge moving in the x-direction at a constant speed $c\beta$. The law of transformation of the electromagnetic fields, E and H, are

$$E'_x = E_x \qquad H'_x = H_x \tag{60a}$$
$$E'_y = \Gamma\left(E_y - \beta H_z\right) \qquad H'_y = \Gamma\left(H_y + \beta E_z\right) \tag{60b}$$
$$E'_z = \Gamma\left(E_z + \beta H_y\right) \qquad H'_z = \Gamma\left(H_z - \beta E_y\right), \tag{60c}$$

where the Lorentz factor $\Gamma = 1/\surd(1-\beta^2)$. Furthermore, consider the prime inertial system to be at rest in the xy plane.

According to h-geometry, the sides of a triangle may be expressed in terms of the angles of the triangle. Consequently, the first two transformation laws, (60a) and (60b) can be stated as

$$\cos A = \cos\bar{A} = \bar{\beta}/\bar{\gamma} = \tanh\beta/\tanh\gamma \tag{61a}$$
$$\cos\bar{B} = \bar{\alpha}/\bar{\gamma} = \frac{\text{sech}\beta\tanh\alpha}{\tanh\gamma} = \text{sech}\beta\cdot\cos B \tag{61b}$$

where the latter follows from the fact that $H'_z = 0$, and it is an expression of the h-Pythagorean theorem (58).

The components of the force are obtained by multiplying their projections in the x, y, and z planes by the factors 1, Γ^{-1}, and Γ^{-1}, respectively [12]. In the xy-plane the force components will be given by

$$F_x = \frac{e}{4\pi a^2}E_x = \frac{e}{4\pi a^2}\cos\bar{A} \tag{62a}$$
$$F_y = \frac{e}{4\pi a^2}\left(E_y - \beta H_z\right) = \frac{e}{4\pi a^2}E'_y/\Gamma = \frac{e}{4\pi a^2}\cos B\cdot\text{sech}\beta, \tag{62b}$$

where a is the radius of the sphere of charge e. The magnitude of the force is

$$F = \surd(F_x^2 + F_y^2) = \frac{e}{4\pi a^2}\surd(\cos^2\bar{A} + \cos^2\bar{B}). \tag{63}$$

Without realizing that

$$\cos^2\bar{A} + \cos^2\bar{B} = \frac{1 - \text{sech}^2\beta + (1 - \text{sech}^2\alpha)\text{sech}^2\beta}{\tanh^2\gamma} = 1,$$

which follows directly from the h-Pythagorean theorem, (58), Page and Adams [16] invent a charge conservation per unit area on the surface of the electron, $\rho d\sigma = \rho' d\sigma'$, where $\rho' = e/4\pi a^2$, and the surface elements are supposedly related by

$$d\sigma = d\sigma'\surd(\cos^2\bar{A} + \cos^2\bar{B}).$$

Then, solving for the unknown charge density ρ, they eliminate the square-root in (63). Since the electromagnetic field vanishes inside the electron, the stress acting on the surface,

$$\mathcal{S} = \tfrac{1}{2}F^2 = \frac{e^2}{32\pi^2 a^4},$$

is the well-known Poincaré stress that was needed to reduce the 4/3 factor in the expression for the energy of an electron to unity, and led to the conclusion that the mass of an electron is not totally electromagnetic in origin. Therefore, *there is no need to invoke charge conservation on the surface of the electron when it is realized that the surface element is in the h-, and not the e-, plane.*

TERRELL-WEINSTEIN EFFECT AND THE ANGLE OF PARALLELISM

If we want to determine the size of a rod traveling at a velocity $\dot{\bar{r}}$, we have to take into account that the photons we observe emanating from the ends of the rod will arrive at different times. Terrell [27] showed that one can interpret what is usually viewed as a FL contraction as a distortion due to the rotation of the rod. Weinstein [30] claimed, about the same time, that the length of a rod can appear infinite which he claimed cannot be due to a mere rotation. Here, we will show it to be due a phenomenon analogous to stellar parallax, and involves the angle of parallelism in h-geometry.

In the limiting case we have the e-distance $\cos A = \dot{\bar{r}}/c$ (see Fig. 5, since the maximum length of hypotenuse of the inscribed right-triangle in a unit disc is 1). According to the definition of the angle of parallelism, (4), the h-measure of the velocity, $\dot{r}$, whose e-value satisfies $\dot{\bar{r}} < c$, is

$$\dot{\bar{r}}/c = \tfrac{1}{2}\ln\left(\frac{1+\dot{\bar{r}}/c}{1-\dot{\bar{r}}/c}\right) = \tfrac{1}{2}\ln\left(\frac{1+\cos A}{1-\cos A}\right) = \tfrac{1}{2}\left(\frac{1+\cos A}{\sin A}\right)^2 = \ln\cot(A/2), \tag{64}$$

where we have used a half-angle trigonometric formula in writing down the third equality. Exponentiating both sides of (64) results in the Bolyai-Lobachevsky formula [cf. eqn (4)]

$$\cot\left[\Pi(\dot{r})/2\right] = e^{\dot{r}/c}, \tag{65}$$

where $A = \Pi(\dot{r})$ is the angle of parallelism.

Consider a rod moving with velocity $\dot{\bar{r}}$ along the $\bar{r}$ axis. Light from the trailing and leading edges must travel over different distances, and, hence arrive a different times. Suppose the distance covered by photons emanating from the trailing edge is d_1, while that from the leading edge d_2. Their respective times are d_1/c and d_2/c. The observer that sees light at time t will have emanated from the trailing and leading edges at $t - d_1/c$ and $t - d_2/c$, respectively, because of the finite propagation of light. The Lorentz transformations for the space coordinates will then be

$$\begin{aligned}\bar{r}_1' &= \Gamma\left[\bar{r}_1 - \dot{\bar{r}}(t - d_1/c)\right]\\ \bar{r}_2' &= \Gamma\left[\bar{r}_2 - \dot{\bar{r}}(t - d_2/c)\right],\end{aligned}$$

where $\Gamma = 1/\surd(1 - \dot{\bar{r}}^2/c^2)$.

Their difference provides a relation between the lengths of the rod in the system traveling at the velocity $\dot{\overline{r}}$, $\ell' = \overline{r}_2' - \overline{r}_1'$, and that measured in the coordinate system at rest, $\ell = \overline{r}_2 - \overline{r}_1$, which is

$$\ell' = \Gamma \left[\ell + (\dot{\overline{r}}/c)(d_2 - d_1) \right].$$

Now, the difference in distances traveled by the photons from the leading and trailing edges is just the length of the rod in the system at rest, and so that

$$\ell' = \ell \left(\frac{1 + \dot{\overline{r}}/c}{1 - \dot{\overline{r}}/c} \right)^{1/2}, \tag{67}$$

if the rod is approaching the stationary observer. For a rod receding from the observer, the signs in the numerator and denominator must be exchangeded because $\dot{\overline{r}} \to -\dot{\overline{r}}$. We could have arrived at (67) directly by observing that in addition to the usual Doppler effect there is a time dilatation between observers located on the moving and stationary frames.

Setting $\dot{\overline{r}}/c$ equal to the cosine of the angle of parallelism in the Doppler expression (67) gives

$$\ell'/\ell = \cot\left[\Pi(\dot{r})/2\right] = e^{\dot{r}/c}, \tag{68}$$

if the rod is approaching, while

$$\ell'/\ell = \tan\left[\Pi(\dot{r})/2\right] = e^{-\dot{r}/c}, \tag{69}$$

if it is receding. In general, the angle of parallelism Π must be greater than A, and the larger the relative velocity $\dot{\overline{r}}/c$ the smaller will be the angle A. Thus, we would expect to see a large expansion of the object as it approaches us, and a corresponding large contraction as it recedes from us. These are the conclusions that a single observer would make, and not those of two observers, as in usual explanation of the FL contraction. Weinstein came to same conclusions by plotting the exponent of the h-arc tangent, rather than the tangent, because he did not go to the limit where $\dot{\overline{r}}/c = \cos A$, which then defines the angle of parallelism.

However, unlike the astronomical phenomenon of parallax, where the space constant κ is so large and the parallax angle so small as to thwart all attempts to date at measuring a positive defect, the distortions predicted by (68) and (69) are actually more dramatic, precisely because of the finite speed of light. Since

$$\pi - (\pi/2 + A + B) < \pi/2 - A = \Omega,$$

the defect is smaller than the complementary angle to A, known as the parallax angle in astronomy, Ω. The astronomical approximation,

$$\tan\left(\pi/4 - \Pi/2\right) = \tan\Omega/2 = \frac{1 - \tan\Pi/2}{1 + \tan\Pi/2} > r/2\kappa,$$

which uses the smallness of $r/\kappa = -\ln\tan(\Pi/2)$ to give an upper bound of $2\tan\Omega/2$, will not work here because $\tan(\Pi/2)$ can be quite small for values of $\dot{\overline{r}}/c$ of the order unity that make the angle A quite small. Moreover, since $A(=\pi/2-\Omega) \leq \Pi$, or $\Omega > \pi/2 - \Pi$, may place a larger lower bound on the parallax angle. Since Ω is the upper bound of the defect, the latter may stand a greater chance of being measured. Although no astronomical lower bound for the parallelax angle has to date been found, the non-Euclidean nature of light rays may be easier to access because the finite velocity of light is not a constraint on the h-measure of the velocity.

HYPERBOLIC GEOMETRIES WITH NON-CONSTANT CURVATURE

a matter of curvature

Geometries which are both homogeneous and isotropic have constant (Gaussian) curvature. Gaussian curvature is a measure of a surface's intrinsic geometry, or the invariance of a surface to bending without stretching. There are three distinct simply connected isotropic geometries in any dimension: Euclidean with zero curvature, elliptic with positive curvature, and hyperbolic with negative curvature. Homogeneity implies that there is at least one isometry that takes one point to another so that the points appear to be indistinguishable. Isotropy implies that space is isotropic so that all directions appear the same.

Up until now we have considered h-geometry with constant negative curvature. However, homogeneity and isotropy are very strong conditions which are seldom met with in GR. The fact that the inertial mass of a rotating system can be handled within the confines of constant negative curvature while gravitational mass cannot leads to the belief that the fields of acceleration of uniform rotation and gravitation are not equivalent.

To transform the exterior solution of the Schwarzschild metric into the interior one [20], possessing constant (negative) curvature, it is necessary is to assume that the mass is not independent of the radius $\overline{r}$. Then for objects in which the density ϱ is essentially uniform, $M=(4\pi/3)\varrho\overline{r}^3$, and introducing this into the Schwarzschild metric, renders it equivalent to the h-metric (26) with constant Gaussian curvature, (33), where the absolute constant is given by (53). Gaussian curvature appears here as a relativistic effect, vanishing in the nonrelativistic limit as $c\rightarrow\infty$. Flatness cannot only be achieved in the limit of a vanishing density, but, also, in the case where relativistic effects become negligible.

However, if the mass M is independent of $\overline{r}$, the relativistic correction term will be given by

$$\overline{r}^2/\kappa^2=\alpha/\overline{r}, \tag{70}$$

where $\alpha:=GM/c^2$, is no longer constant. One effect of this change is that the effective potential, (50), will not be everywhere convex, but only for $\overline{r}>2\alpha$, the so-called Schwarzschild radius. Another effect is the change in the dependence of the rotational velocity on frequency: For constant curvature, the rotational velocity increases as the square-root of the radius, while, for non-constant curvature, it decreases as the square-root of the radius.

The two cases of constant density and constant mass are also distinguishable by the different slopes of the curve of the velocity of rotation of galaxies as a function of their distance from the galatic center. For distances less than $\overline{r}_c=2\times10^4$ light years the curve rises with a constant positive slope. That means, if the centrifugal and gravitational forces just balance one another, the rotational velocity is proportional to the density, which remains essentially uniform. For distances greater than this value, the curve slopes downward, where the rotational velocity is now proportional to the inverse square-root of the distance from the galatic center. This implies that the galatic mass is confined to a region whose volume has a radius less than $\overline{r}_c$, for once outside this volume it appears that the mass is independent of the radius, which is the Schwarzschild case. The crucial point is that although these two situations are physical different they both have the same zero curvature as predicted by the GR criterion that the contracted Ricci tensor vanish. Thus, it would appear, that the Ricci tensor is not related to the presence of matter which casts doubt on the validity of the Einstein equations. Moreover, it would also place in doubt the equivalence principle since centrifugal forces, corresponding to a h-space of constant negative curvature, are not equivalent to gravitational forces, that would be described by the Schwarzschild metric.

We will discuss three examples of non-constant negative curvature: The red-shift of spectral lines, the bending of light, and the transformation from the exterior to the interior solution of the Schwarzschild metric.

gravitational shift of spectral lines

By the principle of equivalence, it is immaterial of whether a clock is retarded by its non-central location on a rotating disc, or whether it is in a gravitational field. Consequently, we can replace (19) by

$$\tau=t\surd(1-\alpha/\overline{r}). \tag{71}$$

As we have mentioned, this assumes the validity of a Newtonian free-fall time, and the replacement of constant density with constant mass M. This transformation also implies the transition from a h-metric of constant to non-constant curvature. Under this transformation, the metric (26) transforms into

$$dh=\frac{\surd\left[\dot{\overline{r}}^2+(\overline{r}\dot{\theta})^2(1-\alpha/\overline{r})\right]}{(1-\alpha/\overline{r})}dt,$$

For weak gravitation fields, (71) can be written as

$$\frac{\nu-\nu_0}{\nu_0}\simeq-\alpha/2\overline{r}, \tag{72}$$

where ν_0 is the proper frequency of the light emitting source and ν is the received frequency.

bending of light by a massive object

The transformation from constant density to one of constant mass necessitates replacing the metric coefficients (32a) and (32b) by

$$E = \kappa^2/(1-\alpha/\overline{r})^2 \tag{73a}$$

$$G = \kappa^2\overline{r}^2/(1-\alpha/\overline{r}), \tag{73b}$$

respectively. The Gaussian curvature (33)

$$K = -\frac{\alpha}{\overline{r}^3}\left(1-\tfrac{3}{4}\frac{\alpha}{\overline{r}}\right) \tag{74}$$

will be negative provided $\overline{r} > \frac{3}{4}\alpha$. Although this distance is less than the Schwarzschild radius, it means that the singularity cannot be approached without a change in the sign of curvature. The coexistence of *el*- and *h*-spaces depending on the distance from the singularity does seem surprising.

The equation for the radial velocity

$$\dot{\overline{r}}^2 = c^2\left(1-\frac{\alpha}{\overline{r}}\right)^2\left[1-\frac{\Delta^2}{\overline{r}^2}\left(1-\frac{\alpha}{\overline{r}}\right)\right], \tag{75}$$

is obtained from (45), where $\Delta := L_h/c$ is the distance of closest approach. In the case of the deflection of light by a massive object, the last term in (75) represents a quadrupolar interaction [10]. With the change of variable, $\overline{r} = 1/\rho$, and the transformation from constant to non-constant curvature implied by (70), the equation for the geodesic (34) becomes

$$\left(\frac{d\rho}{d\theta}\right)^2 = (1-\alpha\rho)^2\left[\frac{1}{\Delta^2}-\rho^2(1-\alpha\rho)\right]. \tag{76}$$

Neglecting terms in α because of their smallness, we get a straight path for a ray coming in from infinity in the direction $\theta = 0$. However, retaining terms linear in α, and substituting $\sigma = \Delta\rho\surd(1-\alpha\rho)$, we find

$$\theta = \int_0^1\left(\frac{1+\alpha\sigma\Delta}{1-\alpha\sigma/\Delta}\right)\frac{d\sigma}{\surd(1-\sigma^2)} \simeq \int_0^1(1+2\alpha\sigma/\Delta)\frac{d\sigma}{\surd(1-\sigma^2)} = \sin^{-1}\sigma - \frac{2\alpha}{\Delta}\surd(1-\sigma^2)+\frac{2\alpha}{\Delta}, \tag{77}$$

which is to be evaluated at $\sigma = 1$, or $\rho_m = \Delta$, the distance of closest approach. The value of the angle at this distance is $\theta_m = \pi/2+2\alpha/\Delta$ [13, eqn (51) p. 354], which gives the identical expression found by Møller, $\psi = 2(\theta_m-\pi/2) = 2\alpha/\Delta$, since his $\alpha = 2GM/c^2$ is twice our value.

Although the result formally agrees with Møller's procedure there is one fundamental difference: Møller [13, eqn above (45) p. 353] uses the substitution

$$\dot{\overline{r}} = \frac{d\overline{r}}{d\theta}\dot{\theta} = -\frac{\dot{\theta}}{\rho^2}\frac{d\rho}{d\theta} = -L_e\frac{d\rho}{d\theta},$$

and proceeds to confuse L_e with L_h in his equation for the radial velocity, (46). Hence, Møller cannot perform the integration since his distance of closest approach is not a constant. The correction term is linear in α

$$\Delta = L_e/c = (L_h/c)(1-\alpha/\overline{r}), \tag{78}$$

which is the same order as the deflection of light that he finds. Møller then goes on to show that the same result can be obtained by adding two separate deflections, one caused by a varying velocity of light, $c\surd(1-\alpha/\overline{r})$ instead of c, and the other due the "non-Euclidean character of the spatial geometry." These two effects are hardly additive; rather, we have shown that it is due to the fact the spatial geometry is hyperbolic with non-constant curvature, and an absolute constant that depends on the constant mass of the deflecting object.

transformation from the exterior to the interior Schwarzschild metric

The transition from a system of constant mass to one of constant density is exemplified by the exterior and interior solutions to the Schwarzschild metric. Schwarzschild [20] sought his metric in the form

$$ds^2 = e^{\lambda}d\overline{r}^2+\overline{r}^2d\sigma^2-e^{\nu}c^2dt^2, \tag{79}$$

where λ and ν are unknown functions of the radial coordinate r, and

$$d\sigma^2 = d\theta^2 + \sin^2\theta d\phi^2.$$

The unknowns are determined by Einstein's requirement that the contracted Ricci tensor

$$R_{ik} = \frac{\partial \Gamma^l_{il}}{\partial x^k} - \frac{\partial \Gamma^l_{ik}}{\partial x^l} + \Gamma^r_{il}\Gamma^l_{kr} - \Gamma^r_{ik}\Gamma^l_{rl}, \tag{80}$$

vanish. The Christoffel symbols,

$$\Gamma^i_{kl} = \tfrac{1}{2}\frac{1}{g_{ii}}\left(\delta_{ik}\frac{\partial g_{ii}}{\partial x^l} + \delta_{il}\frac{\partial g_{ii}}{\partial x^k} - \delta_{kl}\frac{\partial g_{kk}}{\partial x^i}\right),$$

make it clear that only coordinate dependent elements of the metric tensor will influence the value of the contracted Ricci tensor (80), and Einstein's condition for the absence of mass, (84).

Einstein built his 'general' theory on the invariance of the space-time interval between two events, which can be considered as a natural generalization of the Minkowski four-dimensional invariance of the *h*-line element [4]. First, this would be applicable to inertial systems, and second the criterion would not necessarily have any affect upon the Ricci tensor, since the squares of space or time intervals could enter with constant metric coefficients.

Multiplying the Schwarzschild line element (79) through by $e^{-\nu}$, and observing that the square of the time interval will not have any effect on the contracted Ricci tensor, we get the spatial part of the metric as

$$d\Sigma^2 = e^{\lambda-\nu}d\overline{r}^2 + e^{-\nu}\overline{r}^2 d\sigma^2. \tag{81}$$

The components of the metric tensor are

$$g_{11} = e^{\lambda-\nu} \qquad \text{and} \qquad g_{22} = \overline{r}^2 e^{-\nu}. \tag{82}$$

Hence, the non-vanishing Christoffel symbols are

$$\Gamma^1_{11} = \tfrac{1}{2}(\lambda' - \nu'), \Gamma^2_{12} = \frac{1}{\overline{r}} - \tfrac{1}{2}\nu', \Gamma^1_{22} = (\tfrac{1}{2}\overline{r}\nu' - 1)\,\overline{r}e^{-\lambda}.$$

These terms give the non-vanishing components

$$R_{11} = \tfrac{1}{4}\nu'\lambda' - \frac{1}{2\overline{r}}(\nu' + \lambda') - \tfrac{1}{2}\nu''$$
$$R_{22} = \left\{\tfrac{1}{4}\overline{r}^2\nu'\lambda' - \tfrac{1}{2}\overline{r}(\lambda' + \nu') - \tfrac{1}{2}\overline{r}^2\nu''\right\}e^{-\lambda}$$

of the contracted Ricci tensor, (80). However, what does vanish *identically* is the difference

$$R_{ii} - \tfrac{1}{2}Rg_{ii} = 0, \tag{84}$$

for $i = 1, 2$, where the scalar Ricci curvature is given by

$$R = g^{11}R_{11} + g^{22}R_{22} = \left\{\tfrac{1}{2}\nu'\lambda' - \frac{1}{\overline{r}}(\nu' + \lambda') - \nu''\right\}.$$

Consequently, there is nothing invariant about any generalization of the Minkowskian line element, (79), nor is there any validity to Einstein's criterion that (80) vanish in the absence of mass. Expression (84) vanishes independently of the absence, or presence, of mass, and is independent of any explicit form for the parameters λ and ν.

The time-invariant metric form (81) becomes the Lobachevskian spatial metric (18) when $\overline{r}$ and r are related by (29). The coordinates r and σ (which in the plane $\theta = \pi/2$ reduces to ϕ) are known as 'semi-geodesic' coordinates. In the neighborhood of any point of the pseudosphere the metric form in semi-geodesic coordinates is of the form (18).

Taking the differential of (29) gives the invariant *h*-line element, (18), as

$$d\Sigma^2 = \frac{d\overline{r}^2}{(1 - \overline{r}^2/\kappa^2)^2} + \frac{\overline{r}^2 d\sigma^2}{1 - \overline{r}^2/\kappa^2}. \tag{85}$$

Moreover, the foregoing analysis shows that $g_{11} = 1/g_{00}$, and this rules out Schwarzschild's inner solution for constants other than $A = 0$ and $B = 1$.

The geometry of a surface $\overline{r} = \overline{r}_1 =$ constant is the same as a sphere with e-radius $\overline{r} < \kappa$, which is also a h-sphere of radius [cf eqn (6)]

$$r_1 = \kappa \int_0^{\overline{r}_1/\kappa} \frac{dx}{1-x^2} = \kappa \tanh^{-1}(\overline{r}_1/\kappa). \tag{86}$$

As the e-distance $\overline{r}$ approaches the rim κ, the h-distance r increases without limit. Rulers shrink as they approach the rim so that the radius $\overline{r}_1$ actually has infinite h-length. In contrast, Schwarzschild's line element (79), where $e^{-\lambda} = 1 - \overline{r}^2/\kappa^2$ for the inner solution, gives

$$r_1 = \int_0^{\overline{r}_1/\kappa} \frac{dx}{\surd(1-x^2)} = \kappa \sin^{-1}(\overline{r}_1/\kappa) \tag{87}$$

for the radius. Replacing h-functions by circular ones changes h-trigonometry into el-trigonometry [2]. Since $\sinh(r/\kappa)$ is the radius of a h-circle, $\sin(r/\kappa)$ will be the radius of an el-circle. This is confirmed by the fact that as $\overline{r}_1 \rightarrow \kappa$, $r_1 \rightarrow \frac{1}{2}\pi\kappa$.

The volume of the hyperbolic sphere of radius r_1 is

$$V_1(r_1) = \kappa^3 \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^{\overline{r}_1/\kappa} \frac{x^2 dx}{(1-x^2)^2} = 2\pi\kappa^3 \left[\frac{\overline{r}_1/\kappa}{1-\overline{r}_1^2/\kappa^2} - \tanh^{-1}(\overline{r}_1/\kappa)\right] = 2\pi\kappa^3 \left\{\sinh(r_1/\kappa)\cosh(r_1/\kappa) - \frac{r_1}{\kappa}\right\},$$

which is infinite as r_1 increases beyond bound. This would be representative of a truly 'open' universe. On the contrary, Schwarzschild's interior solution gives the volume enclosed by radius r_1 as

$$V_1(r_1) = \kappa^3 \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^{\overline{r}_1/\kappa} \frac{x^2 dx}{\surd(1-x^2)} = 2\pi\kappa^3 \left[\sin^{-1}(\overline{r}_1/\kappa) - \frac{\overline{r}_1}{\kappa}\surd(1-\overline{r}_1^2/\kappa^2)\right] = 2\pi\kappa^3 \left\{\frac{r_1}{\kappa} - \sin(r_1/\kappa)\cos(r_1/\kappa)\right\},$$

in el-space [2], which has a maximum $\pi^2\kappa^3$ as $\overline{r}_1 \rightarrow \kappa$! We thus come to the unescapable conclusion that h-space can coexist with el-space, depending on the distance from the singularity, and what was negative curvature for Schwarzschild's exterior solution becomes positive curvature in the interior solution.

CONCLUSION

Almost a century has past since Ehrenfest [3] came up with his paradox that the circumference of a rotating disc undergoes FL-contraction while the radius of the disc remains invariant. How then, he asked, can the circumference decrease if the radius remains constant? The term 'paradox' was coined by Varićak [29], who established the connection between the Poincaré composition law for velocities and the trigonometric formulas for h-functions.[2] Over the years a large literature developed with two diametrically opposite views: Those who argue that the space geometry is Euclidean, while, others like Einstein, who contend that it is non-Euclidean [7]. Recourse was taken to GR to resolve the dispute [9]. However, GR gives an inaccurate expression for the spatial line-element: While it gives the correct increase in the circumference of the rotating disc at constant radius, it gives the e-, and not the h-, expression for the radius at constant angle. An e-expression for the radius and an h-expression for the circumference of the disc are totally incompatible. Although entire monographs have been dedicated to the problem [17], no one seems to have applied h-geometry to either the rotating disc, or the corresponding gravitational problem, which according to the principle of equivalence, should be equivalent to it.[3]

We have shown that a rigid, uniformly rotating disc fits h-geometry exactly with an absolute constant inversely proportional to the angular velocity of the rotating disc. When this constant angular velocity is set equal to the free-fall frequency with which the object must rotate in order to avoid gravitational collapse, and the total mass, rather

[2] A year or so earlier, Sommerfeld [23] established the same connections, but was unaware that these formula imply h-geometry.

[3] Grøn [7] integrates the geodesic equation (12) for the bowed geodesics, and implies that there are also straight geodesics through the axis of the disc, but was unaware that these geodesics belong to h-geometry.

than the density, is assumed to be independent of the radial coordinate, the angular equation becomes identical with that predicted by GR, both predicting a modification of the conservation of angular momentum. However, the radial equations do not coincide, and GR does not give the correct h-plane 'stereographic' inner product scale factor. In determining the angle of the deflection of light rays in the neighborhood of massive objects, the absence of the first factor in (45) is compensated by assuming that the e-expression for the angular momentum is valid [13]. However, it is the h-expression for the angular momentum, (44), which is a constant of the motion. Although the acceleration fields of a uniformly rotating disc and gravitation both conform to h-geometry the former and latter possess constant and non-constant curvature, respectively. We have also highlighted the lack of coincidence of the Gaussian and Ricci predictions regarding curvature.

The cross-ratio, whose logarithm is a measure of h-distance, is the generalization of the longitudinal Doppler effect to bowed geodesics in h-velocity space. The metric density relating e- and h-distances is shown to be analogous to a non-constant index of refraction. The form of the index of refraction is determined from the geometry alone. Compounding Doppler effects gives the most general law for the composition of velocities comprised of 'boosts' and rotations. The physical significance of rotations can be understood in terms of aberration. Just as the law of cosines and sines are invalidated so, too, the conservation laws of energy and angular momentum must be modified in h-space. Distortion caused by acceleration is accountable for the angle defect in h-geometry and leads immediately to the FL-contraction, which appears like a rotation when the angle of parallelism is introduced.

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