# Towards a geometrical understanding of the CPT theorem

Hilary Greaves

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**Abstract**

The CPT theorem of quantum field theory states that any relativistic (Lorentz-invariant) quantum field theory must also be invariant under CPT, the composition of charge conjugation, parity reversal and time reversal. This paper sketches a puzzle that seems to arise when one puts the existence of this sort of theorem alongside a standard way of thinking about symmetries, according to which *spacetime* symmetries (at any rate) are associated with features of the spacetime structure. The puzzle is, roughly, that the existence of a CPT theorem seems to show that it is not possible for a well-formulated theory that does not make use of a preferred frame or foliation to make use of a temporal orientation. Since a manifold with only a Lorentzian metric can be temporally orientable—capable of admitting a temporal orientation—this seems to be an odd sort of necessary connection between distinct existences. The paper then suggests a solution to the puzzle: it is suggested that the CPT theorem arises because temporal orientation is unlike other pieces of spacetime structure, in that one cannot represent it by a tensor field.

To avoid irrelevant technical details, the discussion is carried out in the setting of classical field theory, using a little-known classical analog of the CPT theorem.

## 1 Introduction

A story has it that in the early sixties, Richard Feynman was asked to give an evening talk to physics students at Caltech, explaining the basic idea of the

CPT theorem: the celebrated result in quantum field theory that states that any relativistic (i.e. Lorentz-invariant) quantum field theory must also be invariant under CPT, the composition of charge conjugation, parity reversal and time reversal. Feynman agreed to commit to doing this, commenting that if one cannot explain something to second year Caltech undergraduates then one does not understand it oneself. The story goes that Feynman spent a month or two trying to plan the talk, and then, in despair, cancelled the commitment.

Whether or not this story is true, its basic point is well taken: despite the importance of the CPT theorem in particle physics, the result itself is generally not well understood, even by those whose professional practice regularly appeals to it. It is often referred to as a 'remarkable result'. It seems worthwhile trying to attain a point of view from which the CPT theorem is not remarkable at all, but is, rather, precisely what one expects on elementary grounds. That is the aim of the project of which the present paper is a part.

More precisely, one can identify two positive sources of puzzlement:

- How can it come about that one symmetry (e.g. Lorentz invariance) entails another (e.g. CPT) *at all*?

- How can there be such an intimate relationship between *spatiotemporal* symmetries (Lorentz invariance, parity reversal, time reversal) on the one hand, and *charge conjugation*, not obviously a spatiotemporal notion at all, on the other?

This paper focusses on the first sort of puzzlement. I first sharpen the puzzle by suggesting that, according to a way of thinking about spacetime symmetries that is (for good reason) fairly common currency in the philosophy of physics community, there is a particular reason for thinking that Lorentz covariance should *not* be able to entail anything like CPT covariance. I then go on to offer a solution to the puzzle.

An outline of the paper is as follows.

Section 2 reviews the standard way of thinking about spacetime symmetries, well discussed by (in particular) Michael Friedman (1983) and John Earman (1989), that will give rise to the sharpened form of our puzzle and that will provide the framework for our discussion. The key point to be taken from this section, for the purposes of this paper, is that one generally expects to find a certain correspondence between the dynamical symmetries of a given spacetime theory, on the one hand, and the spacetime structure postulated by that theory, on the other. More precisely, we expect the following principle to hold: that the covariance group of a theory (when formulated non-generally-covariantly) should, for a well-formulated theory, be equal to the invariance group of the set of geometrical objects that are not represented explicitly in the coordinate-dependent, non-generally-covariant formulation in question. (Readers familiar with the standard framework in question can easily skip or skim this section.)

Section 3 suggests that, from this point of view, the existence of a CPT theorem is *prima facie* puzzling. The idea here will be that a CPT theorem seems to be telling us that it is not possible for a relativistic theory (that is, on our way of thinking, a theory that does not require the existence of a preferred frame or foliation) to make essential use of a temporal orientation.[1] Since a manifold with only a Lorentzian metric can be temporally orientable—capable of admitting a temporal orientation—this seems to be an odd sort of necessary connection between distinct existences; and, since there is no obstacle to theories' making essential use of *other* pieces of spacetime structure, such as a metric or a total orientation, we require an account of what makes temporal orientation special.

To anchor our discussion and to enable us to carry out its remainder in the simpler context of classical, rather than quantum, field theory, section 4 reviews a 'classical PT theorem' (originally stated by (Bell, 1955)), and discusses ways of formulating Lorentz-invariant, PT-violating theories by violating one or more of the auxiliary constraints required for that theorem. This discussion shows that (as we would expect) Lorentz-invariant, PT-violating theories are not ruled out *as a matter of logical or mathematical consistency*. However, at this stage we will still have a puzzle about how and why the auxiliary constraints suffice for the result.

Section 5 offers a solution to the puzzle: the key point is that temporal orientation is indeed (in a Lorentzian context) unlike many other pieces of spacetime structure, in that it cannot be represented by a Lorentz-invariant tensor field. Meanwhile, the 'auxiliary constraints' that we expect any 'reasonable' field theory to satisfy have the effect that pieces of spacetime structure that can be represented by such tensor fields can easily be made use of in the theory, but those that cannot be so represented *generally* (and emphperhaps universally) cannot be made use of.

[1]The discussion of this paper applies just as much to spatial orientation as to temporal orientation. We discuss the latter case for definiteness.

Section 6 considers the puzzle and its suggested resolution in the context of *Galilean*- (rather than Lorentz-) invariant field theories. The point here is that while we do have a 'Lorentzian CPT theorem'—a theorem stating that any *Lorentz*-invariant field theory must also be CPT invariant—we do not have a 'Galilean CPT theorem' (and there do exist Galilean-invariant, CPT-non-invariant theories). We can therefore perform a 'sanity check' on the discussion of this paper, by checking that the suggested explanation of how anything like a CPT theorem can come about in the Lorentzian case does not also suggest that we should expect to find a CPT theorem in the Galilean case. The result will be reassuring: there *is* a *Galilean*-invariant tensor field representing temporal orientation, and, indeed, by making use of the field in question we can easily construct examples of Galilean-invariant, non-CPT-invariant field theories.

Section 7 is the conclusion.

# 2 The connection between dynamical symmetries and spacetime structure

This section reviews a standard way of thinking about spacetime symmetries. This standard account provides the framework within which the existence of the CPT theorem is, I will suggest, *prima facie* puzzling. The review in this section is very much in the spirit of the discussions given by Friedman (1983, chapters 2 and 3) and Earman (1989, chapters 2 and 3). It may be skipped or skimmed by those familiar with the framework in question. (The only slightly idiosyncratic elements are the talk of 'special' rather than 'absolute' or 'kinematical' objects, and (relatedly) the terminology 'covariance$_Q$ group of a theory'; I indulge in this idiosyncrasy to avoid irrelevant complications concerning how, if at all, one might define 'absolute' or 'kinematical'; cf. footnote 3.)

**Spacetime theories.** Let $T$ be a spacetime theory. That is, $T$ is a theory whose intended models are structures of the form $\langle M, \Phi_1, \ldots, \Phi_n \rangle$, where $M$ is a differentiable manifold, and the $\Phi_i$ are geometrical objects on $M$.

Let us suppose that these structures 'explicitly represent all the structure that is presupposed by the theory'. On the intuitive level, this is supposed to require that, for any piece of spacetime structure, other than the topological and differential structure, that the theory requires, there is a field $\Phi$ representing that structure. (Topological and differential structure are exempted only because they are already given with the *manifold* (as opposed to: the point set) $M$.) For example, if the theory is supposed to be set in Minkowski spacetime, then one of the $\Phi_i$'s will be the Minkowski metric $g$. (We will formalize this condition below.)

**Proto-symmetries.** To discuss the symmetries of a theory $T$, we need first to regard the set $M_D$ of models of the theory ($D$ for 'dynamically allowed') as a subset, $M_D \subset M_K$, of a larger set $M_K$ of 'kinematically allowed structures'. We then consider maps from the set $M_K$ into itself; let us say that such a map is a *proto-symmetry* of the theory $T$ iff the map leaves the dynamically allowed subset $M_D$ invariant.

'Proto-symmetries' in this sense are very easy to come by: any theory with $N$ models has as many distinct proto-symmetries as there are bijections from an $N$-element set onto itself. Most proto-symmetries, of course, will be uninteresting; we will be interested only in those that are 'generated' in some particularly simple way.

**Groups and group actions.** A *group* is a pair $(G, \cdot)$, where $G$ is a set and $\cdot$ is a binary relation on $G$ satisfying certain formal conditions.[2] A group may be *abstract* (i.e., $G$ may be a set of 'bare points' having no features other than their the relation $\cdot$), or *concrete* (as, for example, if the elements of $G$ are transformations of some spacetime manifold, and the $\cdot$ operation is that of composition of maps).

An *action* of a group $(G, \cdot)$ on a set $S$ is a map $A : G \to \text{Perm}(S)$—that is, a map assigning to each element of $G$ a permutation of $S$—with the feature that for all $g_1, g_2 \in G$, $A(g_1)A(g_2) = A(g_1 \cdot g_2)$.

**Symmetries.** We wish formally to capture the sense in which particular groups of spacetime transformations—for example, the Lorentz or Galilei groups—may or may not 'be symmetries' of a given spacetime theory $T$.

[2]Namely: for all $g_1, g_2 \in G, g_1 \cdot g_2 \in G$ also (closure); for all $g_1, g_2, g_3 \in G, g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$ (associativity); there exists $e \in G$ such that for all $g \in G$, $g \cdot e = e \cdot g = g$ (existence of identity); for all $g \in G$, there exists $g^{-1} \in G$ such that $g \cdot g^{-1} = g^{-1} \cdot g = e$ (existence of inverses).

Neither abstract nor concrete groups are themselves maps from $M_K$ onto itself, so are not proto-symmetries. Instead, we will say that such a group $G$ is a symmetry group of the theory $T$ *relative to action $A$ of $G$ on $M_K$* iff $\forall g \in G, (A(g))(M_D) = M_D$. (There will often be some 'natural' action of $G$ on $M_K$ that is understood in the context, in which case we will suppress explicit statement of the intended relativization to that action, for the sake of brevity.

To go further, we need to specialise to the case of our theories of interest, and describe the 'natural' group actions that we intend to discuss for these particular theories.

Our particular interest is in *spacetime* symmetries. For any manifold $M$, we have the (concrete) group $\mathit{Diff}(M)$ of diffeomorphisms of $M$. We first note that, for any geometrical object $\Phi$ on a manifold $M$, there is a natural action of the diffeomorphism group $\mathit{Diff}(M)$ on $\Phi$: for example, if $\Phi_i$ is a vector field on $M$, the natural action of $h$ takes $\Phi_i$ to its push-forward $h_*\Phi_i$, while if $\Phi_i$ is a one-form then the natural action is the pull-back to $h^*\Phi_i$. (We will write $h * \Phi_j$ for the result of allowing $h$ to act in the natural way on $\Phi_j$, where the tensor nature of $\Phi_j$ is left unspecified. In the general case, the specification of this 'natural' action of the diffeomorphism group on $\Phi_j$ may be taken as part of the 'definition' of $\Phi_j$. Thus, for example, true tensors and so-called 'pseudo-tensors' are regarded as distinct types of geometrical object.)

We require an action of $\mathit{Diff}(M)$, not on individual geometric objects $\Phi_i$, but on *structures* $\langle M, \Phi_1, \ldots, \Phi_n \rangle$. Here we must proceed with some caution, because the most obvious way of having an action on the $\Phi_i$s induce an action on structures turns out not to be the one of current interest. That 'most obvious' way is to allow the diffeomorphism $h$ to act in the natural way on *each* of the geometrical objects in an arbitrary structure $m \in M_K$: that is, to consider the action $A : \mathit{Diff}(M) \to \mathit{Perm}(M_k)$ given by

$$\forall h \in \mathit{Diff}(M), A(h)(\langle M, \Phi_1, \ldots, \Phi_n \rangle) = \langle M, h * \Phi_1, \ldots, h * \Phi_n \rangle. \quad (1)$$

However, relative to *this* action $A$, if (as we are supposing) 'all the structure presupposed by the theory' is encoded in the $\Phi_i$s, *every* $h$ will be a symmetry of our theory $T$. (The condition that $M_D$ be 'diffeomorphism-invariant' in this sense is the promised formal expression of our assumption that the structures in question 'explicitly represent all the structure that is presupposed by the theory'.)

*This* sense of diffeomorphism-invariance is not our present interest, since we want to capture the special relationship of, say, the Poincaré group to relativistic electromagnetic theory, and the Galilei group to Newtonian gravitation theory. (One normally wants to say that Newtonian gravitation theory is Galilean-covariant and that Maxwell's equations are not, or that a theory counts as special relativistic just in case it is Poincaré-covariant; we seem to be losing an interesting and fruitful distinction if we have only the sense in which all theories that 'explicitly represent all the structure they presuppose' are diffeomorphism invariant.) To do this, we must find a *different* action of $\mathit{Diff}(M)$ on $M_K$, relative to which only some *proper subgroup* of $\mathit{Diff}(M)$ is a symmetry group of the theory.

The action we want is as follows. For a given theory $T$, we single out some subset $Q$ of the geometrical objects $\Phi_i$ as 'special'.[3] Having chosen our set $Q$, we then write candidate models of $T$ in the form $\langle M, S_1, \ldots, S_m, O_1, \ldots, O_n \rangle$, where the $S_i$ ('special' objects) are elements of $Q$ and the $O_i$ ('ordinary' objects) are not. We now allow the diffeomorphism $h$ to act only on the 'ordinary' objects $O_i \notin Q$. That is, we consider the action $A_Q : \mathit{Diff}(M) \rightarrow \mathit{Perm}(M_K)$ given by

$$\forall h \in \mathit{Diff}(M), A_Q(h)\left(\langle M, S_1, \ldots, S_m, O_1, \ldots, O_n \rangle\right) = \langle M, S_1, \ldots, S_m, h{*}O_1, \ldots, h{*}O_n \rangle. \tag{2}$$

We define the *covariance*$_Q$ *group of* $T$ to be the set of $h \in \mathit{Diff}(M)$ such that $h$ is a symmetry of T relative to the action $A_Q$.

**The connection between symmetries and spacetime structure.** The covariance$_Q$ group of a spacetime theory will, in general, be some proper subgroup of $\mathit{Diff}(M)$. But more can be said. Define the *invariance group* of a set $Q$ of geometrical objects as the group of diffeomorphisms $h \in \mathit{Diff}(M)$ such that (the natural action of) $h$ leaves each element of $Q$ invariant. Suppose it is the case that, for each model of our theory $T$, the invariance group of

[3] One way to go about branding objects 'special' is to look for some formal criterion that will pick some of them out, such as the Anderson-Friedman 'absoluteness' criterion (see, e.g., Friedman (1983, pp. 56-61). Another is to say that the 'special' ones are the 'kinematical' or 'geometrical' ones, and hope that we know what this means. An approach that is less ambitious, but that suffices for our present purposes—and that, therefore, we follow here—is to do without any such general criterion, and simply to specify some subset of the objects in a given theory on a case-by-case basis, putting a subscript on 'covariance' to indicate which set of objects we have chosen to treat as 'special'.

the set $(S_1, \ldots, S_m)$ of 'special' fields appearing in that model is the same. In this case, we can regard the invariance group of $Q$ as a property of the theory, rather than of a particular model of the theory. We then expect that, if our theory $T$ is 'well-formulated', *the covariance*$_Q$ *group of* $T$ *is equal to the invariance group of* $Q$.

To support this expectation, we argue first that the invariance group of $Q$ is a subgroup of the covariance$_Q$ group of $T$, and then that the covariance$_Q$ group of $T$ is a subgroup of the invariance group of $Q$. (Similar arguments are given in Earman (*ibid.*, pp.46-7).)

The first claim—that the invariance group of $Q$ is a subgroup of the covariance$_Q$ group of $T$—follows trivially from the sense in which $T$ is diffeomorphism-invariant. (Since

$$\begin{aligned}&\langle M, S_1, \ldots, S_m, O_1, \ldots, O_n \rangle \in M_D \\ \Rightarrow &\langle M, h * S_1, \ldots, h * S_m, h * O_1, \ldots, h * O_n \rangle \in M_D,\end{aligned}$$

if in addition we have $h * S_i = S_i$ for $i = 1, \ldots, m$, it follows trivially that

$$\begin{aligned}&\langle M, S_1, \ldots, S_m, O_1, \ldots, O_n \rangle \in M_D \\ \Rightarrow &\langle M, S_1, \ldots, S_m, h * O_1, \ldots, h * O_n \rangle \in M_D;\end{aligned}$$

that is, that $A_Q(h)$ takes models to models.)

The second claim—that the covariance$_Q$ group of $T$ is a subgroup of the invariance group of $Q$—can arguably be defended, for suitable selections of the set $Q$, by an appeal to Ockham's Razor. Here it is important that the ('special') objects in $Q$ are not themselves 'directly observable' or 'given to us by a mechanical experiment': that their existence is, rather, inferred from empirical data that more directly gives us the 'ordinary' objects $O_i$. The basic idea is that, if we have a theory and a set $Q$ of 'special' objects such that the invariance group of $Q$ is a *proper* subset of the covariance$_Q$ group of $T$, then it ought to be possible to write down an alternative theory $T'$ that has the same empirical consequences as does $T$ as far as the $O_i$ are concerned, but that replaces $Q$ with a set $Q'$ whose invariance group is larger than that of $Q$; further, that this alternative theory $T'$ is more parsimonious than $T$. The claim then is that, *if* $T$ *is a 'well-formulated' theory* (i.e. if $T$ respects Ockham's Razor), the invariance group of $Q$ will be a subgroup of the covariance$_Q$ group of $T$.[4]

[4]In theories that 'have no absolute objects', such as general relativity (GR), the natural

**Non-generally covariant formulations of spacetime theories.** While it is often preferable, for the purposes of foundational discussions, to formulate theories in a coordinate-free framework, such a framework is often inconvenient for calculations, and is used in only a minority of the physics literature. It will therefore be useful to see how the abstract considerations above relate to coordinate-dependent formulations of theories. For simplicity, we will restrict our attention to spacetimes that are diffeomorphically $\mathbb{R}^4$ (i.e. for which there exists a global coordinate chart).

When formulating one's theory in a coordinate-dependent way, one faces a choice between two options. (The distinction between the two is analogous to the distinction between the two actions $A$ and $A_Q$ given above.

The first option is explicitly to take coordinate components of *all* the geometrical objects that appear in the coordinate-independent formulation. If one takes this first option, one arrives at a set of coordinate-dependent equations that picks out the intended class of models relative to an *arbitrary* coordinate system. Say that the *covariance group* of the theory is the group of transformations between those coordinate systems relative to which the given set of equations picks out the intended class of models; we thus have, in this first case, *Diff*$(M)$ as the covariance group.

The second, alternative, option is to represent some chosen subset $Q$ of one's geometrical objects *implicitly*: that is, to consider its coordinate components as functions of the coordinates, and to 'transform' them, when changing to any other coordinate system, by keeping the same function of the coordinates in the new frame. If one takes this second option, one arrives at a coordinate-dependent formulation that picks out the intended class of models only relative to a certain 'privileged' class of coordinate systems (the 'privileged' class being the class of coordinate systems in which the coordinate components of the implicit geometrical objects happen to be the same as their

move is to take the set $Q$ of 'special' objects to be the null set, in which case it is vacuously true that the invariance group of $Q$ is *Diff*$(M)$; that the covariance$_Q$ group of the theory is then also *Diff*$(M)$ follows from the 'cheap' sense in which $T$ is diffeomorphism-invariant. In this sense, the present account is consistent with the received wisdom that there is a *non*-trivial sense in which GR (but not SR) is diffeomorphism-invariant: GR (but not SR) is diffeomorphism-covariant$_Q$ for the choice of $Q$ that is 'most natural' given the theory. However, one can then ask in virtue of what it is 'natural' to take $Q$ to be null in such theories. The project of answering this question is of a piece with the (post-Friedman) project of supplying a criterion of 'absoluteness' according to which general relativity has no absolute objects, which, as noted above (footnote 3), lies outside the scope of this paper.

components in the original, defining, coordinate system). The covariance group of *this* set of equations will then, in general, be some proper subgroup of *Diff*$(M)$, and again we expect that the covariance group will be equal to the invariance group of the set $Q$ of objects that we chose to single out for special treatment.

**Example.** We illustrate the above abstract discussion using the example of special-relativistic electromagnetism. According to this theory, there is a flat Lorentzian metric $g_{ab}$, a tensor field $F_{ab}$ (the electromagnetic field) of type $(0,2)$, and a vector field $J^a$ (the charge-current density field). The equations relating these objects are

$$\begin{aligned} F^{ab}{}_{;b} &= -4\pi J^a, && (3) \\ F_{[ab;c]} &= 0, && (4) \end{aligned}$$

where indices are raised using the inverse $g^{ab}$ of the metric, and it is understood that the covariant derivative is the unique one that is compatible with the metric. *These* equations are generally covariant, in the following two (equivalent) senses:

**Coordinate-independent sense of general covariance.** If $\langle M, g_{ab}, F_{ab}, J^a \rangle$ satisfies (3) and (4), then so does $\langle M, h*g_{ab}, h*F_{ab}, h*J^a \rangle$, for any manifold diffeomorphism $h : M \to M$.

**Coordinate-dependent sense of general covariance.** In coordinate component form, the equations (3)–(4) become

$$\begin{aligned} F_{\mu\nu;\nu} &\equiv \frac{\partial F_{\mu\nu}}{\partial x^\nu} - \Gamma^\lambda{}_{\mu\nu} F_{\lambda\nu} - \Gamma^\lambda_{\nu\nu} F_{\mu\lambda} && (5) \\ &= J_\mu; && (6) \\ F_{[\mu\nu;\sigma]} &\equiv \frac{1}{3}(\frac{\partial F_{\mu\nu}}{\partial x^\sigma} - \Gamma^\lambda{}_{\mu\sigma} F_{\lambda\nu} - \Gamma^\lambda{}_{\nu\sigma} F_{\mu\lambda} && (7) \\ &\quad + \frac{\partial F_{\nu\sigma}}{\partial x^\mu} - \Gamma^\lambda_{\nu\mu} F_{\lambda\sigma} - \Gamma^\lambda_{\sigma\mu} F_{\nu\lambda} && (8) \\ &\quad + \frac{\partial F_{\sigma\mu}}{\partial x^\nu} - \Gamma^\lambda_{\sigma\nu} F_{\lambda\mu} - \Gamma^\lambda_{\mu\nu} F_{\sigma\lambda}) && (9) \\ &= 0. && (10) \end{aligned}$$

These equations pick out the same (i.e. the intended) class of models in *any* coordinate system $x : M \to \mathbb{R}^4$.

However, we can also identify a clear sense in which 'the symmetry group of classical electromagnetism' is the Poincaré group, rather than the full diffeomorphism group:

**Coordinate-independent sense of special covariance.** Let us single out the metric $g$ as 'special'. Then, we consider the action $A_{\{g\}} : \mathit{Diff}(M) \to \mathit{Perm}(M_K)$, given by

$$\forall h \in \mathit{Diff}(M), A_g(h)\left(\langle M, g_{ab}, F_{ab}, J^a\rangle\right) = \langle M, g_{ab}, h * F_{ab}, h * J^a\rangle. \quad (11)$$

For *arbitrary* $h$, we will not in general expect *this* transformation to take models to models. In general we will (instead) expect $h$-covariance$_g$ only when $h$ happens to leave $g$ invariant, since, in that case but in that case alone, the RHS of (11) is identical to $\langle M, h * g_{ab}, h * F_{ab}, h * J^a\rangle$. So now we have a nontrivial covariance$_g$ group, and it is precisely the group of transformations leaving the 'special' object $g$ invariant: that is, the Poincaré group.

**Coordinate-dependent sense of special covariance.** If we choose a coordinate system in which the Christoffel symbols vanish (that is, an inertial coordinate system), then, the equations (5)–(10) reduce, respectively, to

$$\begin{aligned} \frac{\partial F_{\mu\nu}}{\partial x^\nu} &= J_\mu; & (12)\\ \frac{\partial F_{\mu\nu}}{\partial x^\sigma} + \frac{\partial F_{\nu\sigma}}{\partial x^\mu} + \frac{\partial F_{\sigma\mu}}{\partial x^\nu} &= 0. & (13)\end{aligned}$$

(Noting that

$$F = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}, \quad (14)$$

it is straightforward to see that these coincide with usual coordinate-dependent form of the Maxwell equations.)

We have gained notational simplicity, relative to (5)–(10), but now we must remember that our equations (12)–(13) pick out the intended class of models only relative to a privileged class of coordinate systems, which latter are related to one another by Poincaré transformations.

This concludes our review of the standard material within which our puzzle will appear. The key point of this section, for our purposes, is the following: there is an intimate relationship between the spacetime symmetries of a theory, on the one hand, and the spacetime structure postulated by that theory, on the other. Specifically, a 'well-formulated' theory fails to have a particular manifold diffeomorphism as one of its symmetries iff it postulates some piece of background (that is, 'special') structure that is not invariant under the diffeomorphism in question.

# 3 A puzzle about the CPT theorem.

We are now in a position to state our puzzle concerning the CPT theorem. This theorem states that, subject to some apparently innocuous auxiliary conditions, the following conditional must hold of any quantum field theory $T$:

> If $T$ is invariant under the restricted Lorentz group $L_+^\uparrow$, then $T$ is actually invariant under CPT.

I mentioned (in the introduction) that it is possible to decompose a general sense of puzzlement at this statement into two parts: one concerning how Lorentz invariance can entail another symmetry at all, and a second concerning how charge conjugation gets into an otherwise spatiotemporal picture. Since our present concern is with the first of these, let us 'pretend' (but justification for this move will be offered in the next section) that, instead of the $CPT$ theorem, we actually have a $PT$ theorem. Then we have (instead) the following statement:

> If $T$ is invariant under the restricted Lorentz group $L_+^\uparrow$, then $T$ is actually invariant under the whole of the proper Lorentz group $L_+$ (i.e. under the total-reflection component, as well as under the identity component).

In the light of the standard account of spacetime symmetries that we reviewed in section 2, this conditional is *prima facie* rather puzzling. Here is why. Suppose that we have a theory according to which there are, among other objects, a flat Lorentzian metric $g$, a total orientation $\epsilon$ and a temporal orientation $\tau$.[5]

Then, first, we note the invariance groups of three sets of objects we might choose to treat as 'special' in the sense of section 2:

| **Set $Q$ of special fields** | **Invariance group of $Q$ $\mathbf{S_k}$** |
|---|---|
| $\{g\}$ | $L$ (full Lorentz group) |
| $\{g, \epsilon\}$ | $L_+$ (proper Lorentz group) |
| $\{g, \epsilon, \tau\}$ | $L_+^\uparrow$ (restricted Lorentz group) |

The *restricted* Lorentz group, $L_+^\uparrow$, is the set of Lorentz (i.e. $g$-preserving) transformations that can be continuously connected to the identity: it includes all rotations, boosts and products thereof, but does not include parity or time reflection. The *proper* Lorentz group, $L_+$, is the set of all metric-preserving Lorentz transformations with determinant one, i.e. the union of $L_+^\uparrow$ with the set of all Lorentz transformations that reverse *both* spatial parity and time sense. Finally, the *full* Lorentz group, $L$, includes all metric-preserving transformations: those that reverse parity, time sense, both or neither. (See figure 1.)

Ignoring the first of the possibilities listed in the above table (i.e. that of treating $g$ alone as 'special'), we should then expect to be able to write down, not only a non-generally covariant theory whose covariance group is exactly $L_+$ (by treating $g$ and $\epsilon$ as 'special'), but also a non-generally covariant theory whose invariance group is exactly $L_+^\uparrow$ (by treating $g$, $\epsilon$ and $\tau$ as 'special'). A PT theorem, however, tells us that we cannot do the latter: that, subject to the (as yet unstated) auxiliary assumptions of our theorem, we cannot find theories that are invariant under precisely the restricted Lorentz group. It seems to be telling us, that is, that no theory that is 'nice' (in the sense of conforming to these auxiliary assumptions) can actually make use of a temporal orientation, over and above a flat metric and a total orientation.

[5]The total orientation is an object that determines, for any ordered quadruple consisting of one timelike and three linearly independent spacelike 4-vectors, whether that quadruple is 'right-handed' or 'left-handed'. It can be represented by a totally antisymmetric rank four tensor, $\epsilon_{abcd}$. The temporal orientation is an object that specifies in a continuous way, at each point $p$, which is the 'future' lobe of the lightcone in $T_pM$. *Its* possible representations will be considered in section 5.

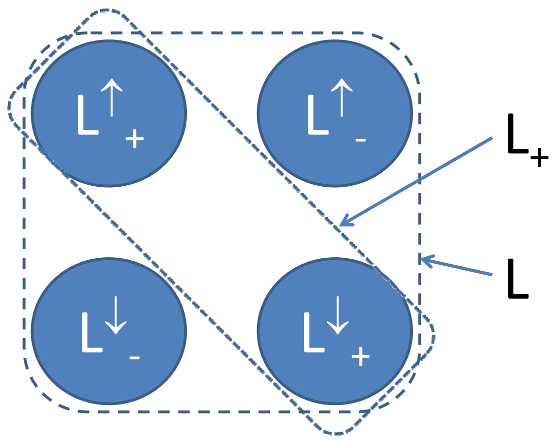


Figure 1: The (real) Lorentz group has four mutually disconnected components, labelled by $\uparrow$ or $\downarrow$ according to whether or not they reverse time sense, and by $+$ or $-$ according to whether or not they reverse total orientation (i.e. whether their determinant is $+1$ or $-1$). In this notation, $L^{\uparrow}_{+}$ is the 'restricted' Lorentz group; $L_{+} \equiv L^{\uparrow}_{+} \cup L^{\downarrow}_{+}$ is the 'proper' Lorentz group; $L \equiv L^{\uparrow}_{+} \cup L^{\uparrow}_{-} \cup L^{\downarrow}_{-} \cup L^{\downarrow}_{+}$ is the 'full' Lorentz group.

And now one might well wonder *why not*. Metric, temporal orientation and total orientation seem to be paradigm cases of distinct existences; it's odd to find such necessary connections between them. Or, to put the puzzle another way: where does this discrimination against temporal orientations come from? That is, what feature of temporal orientation can explain why, in the context of the existing objects $g$ and $\epsilon$, they are unusable in this way?

This is not a paradox, but it does seem to be a puzzle whose resolution is likely to be illuminating. In the next section, I give an explicit statement of the theorem that is the source of our puzzle, and in section 5 I offer a resolution.

# 4 A classical PT theorem

At the start of section 3, I promised some motivation for 'changing the subject' from CPT to PT.

The point here is the following. What we have, in the case of quantum field theory, is a theorem (a *mathematical* statement) asserting the invariance of a certain class of theories under a certain *mathematically specified* transformation. (For example, the the case of the Dirac field, it is the operator transformation

$$\hat{\psi}(t, \mathbf{x}) \mapsto i\gamma^5 \hat{\psi}(-t, -\mathbf{x}).) \tag{15}$$

We then give a name—usually 'CPT'—to the transformation thus specified, and then we call our theorem a 'CPT theorem'.

Names need not be arbitrary, and the name 'CPT' is not assigned arbitrarily in the present case: one finds, in quantum field theory textbooks, *arguments* for the claim that these transformations are equal to the composition of three transformations that are appropriately called 'charge conjugation', 'parity reversal' and 'time reversal' respectively. (In our example, the claim is that

$$\begin{array}{rcrl}
\hat{\psi}(t,\mathbf{x}) & \mapsto & \gamma^1\gamma^3\hat{\psi}(-t,\mathbf{x}) & \text{is appropriately called 'time reversal';} \\
\hat{\psi}(t,\mathbf{x}) & \mapsto & \gamma^0\hat{\psi}(t,-\mathbf{x}) & \text{is appropriately called 'parity reversal';} \\
\hat{\psi}(t,\mathbf{x}) & \mapsto & -i\gamma^2\hat{\psi}^*(t,\mathbf{x}) & \text{is appropriately called 'charge conjugation'.)}
\end{array}$$

However, these arguments do not amount to conclusive demonstrations that these transformations, and these alone, deserve the given names. The arguments depend on certain assumptions, such as that time reversal should take one-particle states to one-*particle* states, rather than to one-*antiparticle* states (equivalently: that it should take particle creation operators to *particle* creation operators, not to *antiparticle* creation operators; cf. (Peskin & Schroeder, 1995, p. 67)). Such assumptions *may* be very natural, but they are not forced on us by the mathematics alone.

Naturalness (furthermore) is relative to point of view. Here, two points of view are relevant: that a quantum field theory is an encoding of the phenomenology of particle physics, and that a quantum field theory is a quantization of a classical field theory. (Both are generally true, of course.) Emphasizing the former does indeed render assumptions such as that cited above very natural (time reversal, plausibly, should reverse the direction of motion *of a given type of entity*, but should not change that entity into something else). Emphasizing the latter, however, renders it more natural to approach the task of naming transformations as follows: start from a classical field theory, with assumptions about which *classical* transformations deserve the names 'time reversal' and 'parity reversal' already in place (never mind whence!); obtain a QFT by quantization; work out which transformations on QFT states and operators are induced by the already-named transformations on classical fields, and name the former accordingly. And the Bell/Feynman point is that when one carries out this *latter* project, with standard names for the classical transformations, *the transformation that is usually called 'TC' receives the name 'T'*. (Hence Feynman's oft-cited claim that 'time reversal turns particles into antiparticles'. The transformation *usually* called 'time reversal', as we noted in passing above, does not; Feynman is to be understood as advocating an alternative, and he thinks better motivated, naming convention.) It follows, other things being held fixed, that *the transformation that is standardly called 'CPT' receives the name 'PT'*. In this sense, and this sense alone, the so-called CPT theorem is 'really' a PT theorem.

(Question: Which approach names transformations in a way that is more appropriate, from a fundamental point of view?—If this question is 'which would we expect to reveal the true connections between symmetries and background spacetime structure?', the (perhaps disappointing) answer is 'neither'. Both particle phenomenology and classical field theory are *less* ontologically fundamental than quantum field theory. Fundamentally, we would like to take quantum field theory on its own terms, as a base from which particle physics and classical field theory should each emerge as good approximations in certain, quantum-field-theoretically specifiable, domains, and work out just in terms of the fundamental ontology of QFT (i.e. particle physics and classical field theory notwithstanding) which field transformations bear special relationships to which spacetime symmetries. Perhaps unfortunately, it is not clear how or whether this project can be carried out, since the 'ontology' (if such there be) of QFT is murky. If, however, our interest (as in the present paper) is in importing our better understanding of the mathematics of symmetries in classical field theory in order to understand the quantum-theoretic CPT theorem, then, clearly, it is the second approach that is of interest.)

In the present paper, we focus on a classical counterpart of the CPT theorem; this classical counterpart is, in the terms of section 2, naturally viewed as being a PT theorem. We do this because the classical case is simpler (hence, strategically, a good starting point for enhancing understanding), and we expect most features of our discussion of the classical PT theorem to apply equally well to the quantum 'CPT' theorem. A close examination of the quantum theorem itself, and of the relationship between the classical and quantum theorems, lies outside the scope of this paper.

## 4.1 Bell's theorem

The 'classical PT' result to be discussed is adapted from one given by John S Bell, in the surprisingly-little known paper cited above.

In outline, the result is as follows. We consider a classical theory given by a system of partial differential equations (PDEs) on a specified set of spacetime-tensor fields. Let $\Phi$ be the space of kinematically allowed fields. (In the general case, we may be dealing with a theory containing a number of interacting fields—scalar fields, tensor fields, etc—so, for a given theory, an element of $\Phi$ will be an ordered $m$-tuple of specified numbers of scalar fields, vector fields, rank 2 tensor fields, etc.) We note that any PDE can be expressed as the vanishing of some functional $F : \Phi \to \mathbb{R}^M$ of the fields. (That is, $F$ encodes the dynamics in the sense that: $\phi \in \Phi$ is dynamically allowed iff $F(\phi)$ is the zero map on $M$.) Let $A$ be the usual action of the proper Lorentz group $L_+ \subset \mathit{Diff}(M)$ on $\Phi$ (that is, each $l \in L_+$ acts on each component $\Phi_i$ of $\Phi$ via the push-forward/pull-back/etc, accordingly as $\Phi_i$ is a vector field/one-form/etc.) We *assume that $F$ is a local polynomial in the fields and their spacetime derivatives.* It can then be proved that, if the set $S$ of solutions of the equation $F(\phi) = 0$ is invariant (relative to this action $A$) under the restricted Lorentz group $L_+^\uparrow$, then $S$ is actually invariant (relative to $A$) under the tensor representation of the whole of the proper Lorentz group $L_+$ (i.e. including total reflections as well as rotations and boosts).

Summing this up, the claim is: Let $T$ be a theory according to which there are $n$ dynamical fields $\Phi_1, \ldots, \Phi_n$. Suppose that the following three conditions hold:

1. The dynamical fields are tensors (of arbitrary rank).

2. The dynamical equations are partial differential equations that are local polynomials in the fields and their spacetime derivatives.

3. The set $\mathcal{S}$ of solutions to the dynamical equations is invariant (relative to the natural action) under $L_+^\uparrow$.

Then, $\mathcal{S}$ is actually invariant (relative to the natural action) under all of $L_+ \left(\equiv L_+^\uparrow \cup PT(L_+^\uparrow)\right)$.

A more rigorous statement of the theorem behind this claim is given in Appendix A. A proof is given in (Greaves, n.d.).

## 4.2 Auxiliary constraints

We were careful, in section 3, to state our puzzle as arising from the fact that no 'nice' theory is invariant under precisely the restricted Lorentz group, rather than that *no theory whatsoever* has just that invariance group. 'Nice', here, means 'conforming to the auxiliary assumptions of the PT theorem' (i.e. the conditions (1) and (2) in section 4.1 above). It is worth highlighting, then, the fact that these 'auxiliary assumptions' play a crucial role in both the antecedent plausibility, and in the proof, of the PT theorem. There obviously do exist 'theories', in the minimal sense of 'classes of models', that are $L_+^\uparrow$-invariant but not $L_+$-invariant. (To generate one, we need only pick some particular scalar field on $\langle M, g\rangle$ that does not have any interesting symmetries, and take the set that results from closing under the action of the restricted Lorentz group.)

That the assumptions in question are *each* essential can be seen as follows.

In the classical theorem sketched above, our principal auxiliary constraint is a restriction on the dynamics: the dynamics must express the vanishing of all members of some particular set of polynomials in the coordinate components of the fields and their derivatives. There are two points here that are worthy of note. The first is that some equations, fairly plausible from the point of view of physics, are not 'polynomial' in the required sense.[6] It may be possible to weaken the assumptions of the theorem, so as to cover these cases also. The second point is that some 'dynamics' do not express the vanishing of any mathematically simple functional $F$ at all (polynomial or otherwise). One example of this phenomenon is given by the 'theory' sketched in the opening paragraph of this subsection; another is given by the theory **'Inc'** stated below (see footnote 10).

It is also worth noting that the theorem claims invariance only under the usual *tensor representation of* $L_+$. Now, in the physics (as opposed to the mathematics) literature, one talks of 'how objects transform under PT' as part of the definition of those objects, rather than as part of the specification of which transformations on the set of fields one is making a claim about. In the physics-literature language, therefore, the restriction of our claim to tensor representations amounts to a substantive assumption about 'which types of fields' may be present in our theory: we are ruling out theories 'containing dynamical fields that transform as pseudotensors under PT'. If we are allowed PT-pseudotensors, then counterexamples to the claim of PT invariance are easy to come by. Here is one: let $\phi$ be a pseudoscalar field, and let the dynamical equation be

$$\phi = 1. \tag{16}$$

Under PT, a PT-pseudscalar $\phi$ transforms to $-\phi$; hence, PT does not take solutions of (16) to solutions.[7] (In mathematics-literature language: the equation is not invariant under 'PT-pseudotensor' representations of $L_+$.)

Be this as it may, there still seems to be something *prima facie* puzzling even about the restricted claim that all theories *within the stated class* obey the conditional 'if $L_+^\uparrow$-invariant then $L_+$-invariant' (relative to representations of $L_+$ within the stated class)—there is no connection yet apparent between the restrictions involved in the assumptions of the theorem on the one hand,

[6]For example, equations involving terms like $\sin(\nabla_a\phi\nabla^a\phi)$ or operators like $\sqrt{\nabla^2+m^2}$ are not 'polynomial'. I am grateful to Robert Geroch and Michael Kiessling (resp.) for pointing out these particular examples.

[7]Slightly less trivially, suppose that $\psi$ is a scalar field and $\chi$ a pseudoscalar under PT (i.e. $\chi \overset{e}{\longmapsto} \chi$ for $e \in L_+^\uparrow$, but $\chi \xrightarrow{PT} -\chi$). Then, the equation

$$\psi\chi - \psi = 0 \tag{17}$$

is $L_+^\uparrow$-invariant but not PT-invariant.

and the surprising ineffectiveness of temporal orientation on the other. This is the puzzle we wish to solve.

# 5 Resolution of the puzzle

Let us take stock. We started (section 2) by sketching a way of thinking about spacetime symmetries according to which the set of dynamical symmetries ought to coincide with the invariance group of a set of objects that we have (for some reason or none) decided to single out as 'special'. We then noted (section 3) that, on this way of thinking, a PT theorem seems to be an assertion that, subject to apparently innocuous auxiliary constraints, there is no theory that makes essential use of a temporal orientation, over and above a Lorentzian metric and a total orientation, and that this is puzzling. To ground the discussion, we then recalled (in section 4) an example of such a theorem, for the case of classical field theory. We now seek a more enlightened point of view: a point of view from which the existence of such theorems in certain cases is not puzzling at all, but is, rather, precisely to be expected, where and only where they in fact occur.

My suggestion is that the following observation lies at the heart of the otherwise puzzling nature of the CPT theorem: *there is no tensor field that represents temporal orientation and no more*, in the context of a flat Lorentzian metric and a total orientation.

The remainder of this section has two aims. The first is to explicate this observation—what exactly it means, and why it is true. The second is to explain how this helps to dissolve the puzzle. It will be easiest to tackle both of these aims simultaneously.

Intuitively, a temporal orientation on a (temporally orientable[8]) manifold $M$ is supposed to specify which temporal direction is 'the future'. Let $p$ be an arbitrary point in a temporally orientable manifold $M$ that is equipped with a Lorentzian metric $g$. Then, the tangent space $T_pM$ can be divided into timelike, spacelike and null vectors. Further, the set of *timelike* vectors in

[8]Definition: A manifold $M$ equipped with a Lorentzian metric $g$ is said to be *temporally orientable* iff there exists a continuous, nowhere-vanishing, timelike vector field on $M$. Heuristically: iff a manifold $M$ fails to be temporally orientable, then one can 'parallel-transport' a timelike vector $v$ at some point $p \in M$around the manifold, and return to the point $p$ with a vector $v'$ that points in the opposite temporal direction to $v$. In this case, it is not possible to make any continuous global specification of which temporal direction is 'the future'.

$T_pM$ has two disconnected components: these will be the 'past' and 'future' lobes of the lightcone at $p$ ('will be' rather than 'are', because until and unless we have a temporal orientation, neither lobe is distinguished as the 'future' one).

Now, we wish to represent temporal orientation by some geometric object on $M$. Here we have a choice: there are many structures on $M$ that would do the trick.

(Perhaps) the most obvious way of representing temporal orientation is by a map that assigns, to each point $p \in M$, one of the two lightcone lobes in $T_pM$ (and that does so in a continuous way, i.e. the assignments of lightcone lobes to neighboring points must be 'mutually consistent'). This is our first candidate way of representing temporal orientation.

But let us now recall the use we wish to make of our pieces of spacetime structure: we wish to formulate laws that relate other ('dynamical'/'matter') fields to them, so that, by treating the spacetime structures as 'special', we can restrict the covariance groups of non-generally-covariant formulations of our theories. We then note that, if, as seems to be usually the case, our physical laws take the form of *differential equations* coupling various geometrical objects to one another, then a 'map from spacetime points to lightcone lobes' is not an object we can easily work with. The point is that if $f$ is such a map, the idea of a 'differential equation for $f$' does not seem to make sense; $f$, that is, is not the right sort of object to appear in differential equations.[9]

This observation suggests a second possible way of representing temporal orientation. Instead of using a map from spacetime points to lightcone lobes, we could use a *continuous nonvanishing timelike vector field*, $t^a$, on $M$. (We can then pick out the 'future'-directed timelike vectors $v^a \in T_pM$ as those that have positive 'dot product' $g_{ab}v^a t^a$ with $t^a$ (relative to a convention according to which the metric has signature $(+,-,-,-)$ rather than

[9]A referee pointed out that in a theory in which *discontinuous* fields are allowed (for example, a theory of a one-particle GRW wavefunction defined on spacetime), differential equations could use this first representation of temporal orientation in requiring the derivatives of the field *in one temporal orientation but not the other* to satisfy some equation. Such theories are ruled out by *fiat* in the conditions for the theorem stated in appendix A (it is assumed there that the fields be *tensor fields*—i.e., among other things, that they be smooth). We then face the questions: is there some other proof that Lorentz-invariance entails PT-invariance for 'discontinuous field theories' of this sort; if so, what prevents theories of *this* sort from making use of a temporal orientation; if not, can we write down a counterexample? These questions warrant further investigation.

$(-,+,+,+)$).) This move solves the problem we faced when trying to make use of $f$: $t^a$, as a vector field, *is* an object of a type that we perfectly well know how to use in differential equations. However, we have now incurred a problem of a different sort: $t^a$ *is not restricted-Lorentz invariant.* That is, it is not the case that, $\forall l \in L_+^\uparrow, l * t^a = t^a$. The point here is that $t^a$ picks out more structure than we wanted to pick out: we wanted only to pick out a preferred lobe of the lightcone at each point, but a vector field picks out, in addition, a preferred *timelike vector* in the chosen lightcone lobe. The upshot of this is that when we combine our 'temporal orientation' $t^a$ with our existing pieces of structure $g_{ab}, \epsilon_{abcd}$, we do not have a set whose invariance group includes $L_+^\uparrow$: rather, the most we expect is the group of translations and rotations (if $g_{ab}$ is flat and $t^a$ is constant).

*This* observation suggests a *third* possible way of representing temporal orientation: rather than a *single* (continuous nowhere-vanishing timelike) vector field, we could take an *equivalence class* of such vector fields (where $s^a \sim t^a$ iff $g_{ab}s^a t^b > 0$). But now we are back to our original problem: an *equivalence class* of vector fields, as opposed to a particular vector field, is not the right sort of object to appear in a single partial differential equation.[10]

More generally: suppose we convince ourselves that the geometric objects we can make use of, in equations that satisfy the restrictions we have laid down, are just those that can be represented by tensor fields.[11] Then, we

[10]Such an equivalence class of vector fields *can*, of course, be used to generate a *set* of differential equations. Here is a non-PT-invariant theory that makes use of this idea: Take the temporal orientation $\tau$ to be the set of all nowhere vanishing, future-directed timelike vector fields. Let there be (besides the temporal orientation, total orientation and metric) a single scalar field $\phi$. Say that $\phi$ is dynamically allowed iff the following condition holds:

**(Inc)** There exists at least one vector field $v^a \in \tau$ such that, at every spacetime point $p \in \mathbb{R}^4$, $v^a \nabla_a \phi > 0$.

(This theory is cooked up to say, in a restricted-Lorentz-invariant way, '$\phi$ increases towards the future', and hence not to be $PT$-invariant.)

This example shows that the restrictions on the dynamics that appear in the premises of the theorem include restrictions on the 'logical form' of the dynamics: it's crucial to the theorem that the sort of existential quantification that's going on in this example is disallowed.

[11]It is not entirely clear that this is true. For example, the covariant derivative is usually thought of as a map from tensor fields of type $(n,m)$ to tensor fields of type $(n,m+1)$, and not itself as a tensor field; and yet it can be used in PDEs. This suggests that *perhaps* the present discussion must be extended to some class of geometric objects that is wider than the class of tensor fields. However, it is also true that the covariant derivative *can*

can avail ourselves of the following mathematical fact: *any tensor that is invariant under the restricted Lorentz group is invariant under the whole of the proper Lorentz group.*[12] Meanwhile, it is clear that no PT-invariant tensor (hence, no $L_+$-invariant tensor) can represent temporal (or spatial) orientation.

# 6 Galilean-invariant field theories

We now wish to perform a sanity check on the suggestion of section 5, by considering *Galilean*-invariant (as opposed to Lorentz-invariant) field theories.

The point here is that we do not have a CPT theorem for Galilean-invariant quantum field theory (see, e.g., (Levy-Leblond, 1967)). Correspondingly, the following hypothesis concerning *classical* Galilean-invariant field theories is false:

**Galilean PT hypothesis.** If $T$ is a classical field theory containing tensor

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be represented by a tensor field (viz. the metric—since the covariant derivative operator is uniquely determined by the metric), so perhaps not.

A second point in this vein is that I am ignoring the issue of density weight. When one writes "tensor" rather than "tensor density of weight $n$", one normally implies that the object under discussion has density weight zero. I do *not* intend this implication. Density weight is irrelevant for present purposes, since all the transformations under consideration have determinant unity. (For an explanation of the concept of density weight, see, e.g., (Anderson, 1967, pp. 23–5).)

[12] Proof: Let $T_{a_1,\ldots,a_n}$ be an $L_+^\uparrow$-invariant tensor. Let $(lT)_{a_1,\ldots,a_n}$ be the result of acting on $T$ with an arbitrary Lorentz transformation $l$ (so, $(lT)_{a_1,\ldots,a_n} = l^{b_1}{}_{a_1} \ldots l^{b_n}{}_{a_n} T_{b_1\ldots b_n}$). Note that each component of $lT$ is polynomial in $l$ (in the sense that, relative to an arbitrary coordinate system for spacetime and for each point $p \in M$, each component of $lT(p)$ is a polynomial in the matrix elements of $l$).

Let $T_Z^L$ be the Zariski topology on $L$, and $T_Z^{\mathbb{R}}$ the Zariski topology on $\mathbb{R}$. (The Zariski topology, on a given space on which a notion of polynomial function is defined, is that according to which the closed sets are just those that consist of the zeros of some polynomial.) Recall that the *closure* of a set $S$ is the smallest closed set containing $S$. We claim that, in the topology $T_Z^L$, the closure of $L_+^\uparrow$ is $L_+$. (The proof of this claim forms the bulk of the proof of the classical CPT theorem discussed in section 4.1.) But also, any polynomial on $L$ is continuous in the topologies $T_Z^L, T_Z^{\mathbb{R}}$. Hence, if a polynomial on $L$ vanishes on $L_+^\uparrow$ it must also vanish on the whole of the $T_Z^L$-closure of $L_+^\uparrow$, namely, the whole of $L_+$. Applying this argument to the polynomials given by the components of the $(lT - T)(p)$ yields the desired result.

(Thanks to Joaquin Teruji Thomas for pointing out this proof.)

> fields, whose dynamics are polynomial in the fields and their derivatives, and if in addition $T$ is invariant under the restricted Galilean group $G_+^\uparrow$ (relative to the usual tensor representation), then $T$ is PT-invariant.

Therefore, if our suggested explanation of the possibility of a Lorentzian PT theorem is on the mark, it had better not be the case that the analogous statement is also true in the Galilean case. That is, it had better not also be true that there is no way of representing temporal orientation against a background of *Galilean* spacetime structure, without 'picking out more structure than we want'. That is, in this case, the object that represents temporal orientation had better be invariant under the restricted Galilean group.

At first sight, things look worrying. One of the points we met in the Lorentzian case was that a vector field picked out a timelike direction, as well as a privileged direction of time. But privileged timelike directions are no more acceptable in the Galilean than in the Lorentzian setting.

Fortunately for our suggested explanation, however, it does not, in fact, also go through in the Galilean case. There is no Galilean-invariant *vector* field, but there *is* a Galilean-invariant *one-form* (corresponding to the fact that, in Galilean spacetime, there is no preferred timelike direction, but there is a priveleged notion of simultaneity). In this section we explain this point, and we use it to develop a counterexample to the Galilean PT hypothesis.

## 6.1 Temporal orientation in Galilean spacetime

One encodes the structure of Minkowski spacetime using a flat Lorentzian metric $g$; elements of the (full) Lorentz group are then transformations leaving $g$ invariant. Things are less simple in the Galilean case: there is no 'single geometric object' that will encode, in a single shot, all of the structure of Galilean spacetime.

Let us first get clear about what the structure is that we are trying to encode, over and above topological and differential structure. To model the Galilean case, we want our spacetime to possess a natural foliation into a family of three-dimensional hypersurfaces, the preferred simultaneity slices. We want each simultaneity slice to be equipped with a Euclidean spatial metric. We want there to be a fact, for any two points of spacetime, about what is the (absolute value of the) temporal distance between them. And we want there to be a fact, for any timelike curve, about whether or not it is ‘straight’ (i.e. is an inertial trajectory). *Iff* we want to endow our Galilean spacetime with a temporal orientation, then we also want there to be a privileged total ordering on the set of simultaneity slices.

One way of encoding the aspects of this structure, aside from temporal metric and temporal orientation, is as follows (here I largely follow Friedman (1983, pp. 71–92), who sets this approach out in far more detail). We start, as in the Lorentzian case, with a four-dimensional differentiable manifold $M$. The affine structure (i.e. the set of facts about which lines in the spacetime are ‘straight’) is encoded by a connection $\Gamma$. The Euclidean spatial metrics are encoded by a rank 2 tensor field $h^{ab}$.

We now face the question of how to encode the temporal metric and/or temporal orientation. Suppose first that we wish to encode temporal metric *without* picking out a preferred temporal orientation. This can be done by means of a symmetric tensor field of type $(0, 2)$ (satisfying certain restrictions; cf. Earman (1989, pp. 30–1); in Earman’s notation, the tensor field in question is $h_{ij}$). This object will tell us the temporal distance between any two time-slices, but will not tell us which is to the future of which. Second, though, suppose that we do wish to encode a temporal orientation, in addition to a temporal metric. Then, we can use a one-form, $t_a$ (this can be thought of as the exterior derivative, $t_a := (dt)_a$, of a global time function $t$ that respects the simultaneity structure in the sense that the surfaces of constant $t$ are the simultaneity surfaces). This represents temporal metric and temporal orientation at once, in the natural way: if $v^a$ is a timelike vector, then $|t_a v^a|$ is the temporal length of that vector, and the sign of $t_a v^a$ tells us whether $v^a$ is future- or past-directed. And $t_a$ can be chosen to be invariant under the restricted Galilean group $G_+^{\uparrow}$, so we have not picked out more structure than we wished to encode.

## 6.2 Counterexample to the Galilean PT hypothesis

The above suggestion for encoding temporal orientation in Galilean spacetime, via the one-form $t_a$, can easily be used to generate a counterexample to the ‘Galilean PT hypothesis’ above.

Here is one such counterexample: Suppose we have a theory containing a scalar field $\phi$ and vector field $v^a$, whose dynamics are given in generally covariant form by single equation

$$t_a v^a = h^{ab} \phi_{;a;b}. \tag{18}$$

Here, $t_a, h^{ab}$ are understood as, respectively, the temporal structure and Euclidean spatial metric structure outlined above for Galilean spacetime.

Suppose now that we treat $t_a$ and $h^{ab}$, and in addition the flat connection $\Gamma$, as 'special'. Then, we have a privileged class of coordinate systems: the inertial frames in which $t$ increases towards the future. In these coordinate systems, the dynamics is given by the non-generally-covariant equation

$$v^0 = \nabla^2\phi. \tag{19}$$

Under a restricted Galilean transformation, both $v^0$ and $\nabla^2\phi$ are invariant. However, under PT, $\nabla^2\phi$ is invariant while $v^0$ flips sign. Hence, PT in general does not take solutions to solutions, while restricted Galilean transformations do. So this theory constitutes a counterexample to the Galilean PT hypothesis.

This completes our 'sanity check': there is no Galilean CPT theorem and, as we therefore hoped, the explanation of the CPT theorem that we offered in the Lorentzian case does not go through in the Galilean case.

# 7 Conclusions

The existence of a PT theorem (such as that discussed in this paper) is *prima facie* puzzling, since it seems to show that a reasonable theory cannot make use of a temporal orientation, over and above a flat Lorentzian metric and total orientation, without also using extra, 'unwanted' structure such as a preferred frame. One might well wonder where this discrimination against temporal orientations comes from. This paper has suggested that temporal orientation in a relativistic context indeed is special, as pieces of spacetime structure go: unlike the metric and total orientation, temporal orientation cannot be represented by a tensor field. Meanwhile, we seem to be committed to constraining principles on our physical theories (for example, constraints on the types of PDEs theories may use), such that structure that cannot be encoded via tensor fields (or 'similar') cannot essentially be used. This dissolves the puzzle.

The discussion above was carried out in the classical context, using a 'classical PT theorem'. However, the hope is that the same sort of line of thought can be used to illuminate the CPT theorem in quantum field theory. The idea underwriting this hope is that, from the point of view of classical field theory (as opposed to particle phenomenology), the operation usually called

'CPT' is in fact more naturally regarded as a PT-reversing operation, so that the 'CPT' theorem is also, properly understood, a PT theorem; furthermore, we hope that precisely analogous proofs can be given for the classical- and quantum-theoretic PT theorems. (Both these suggestions are also made by Bell (1955).) One then wonders whether similar lines of thought can also illuminate the relationship between parity violation and 'CP' violation, the possibility of CPT violation, and so on.

Several open questions remain. The most pressing is perhaps the following. We have restricted attention thus far to *tensor* field theories. But this is not, of course, the most general type of field theory of physical interest, or for which we have (quantum-mechanically) a CPT theorem. In particular, the above treatment has said nothing about *spinor* field theories. In fact, preliminary investigation suggests that the case of spinors is rather complicated: one *can* construct a Lorentz-invariant temporal orientation in a spinor field theory (see, e.g., (Wald, 1984, p. 352)); thus, according to the argument of this paper, one should expect to be able to formulate a Lorentz-invariant, but PT-violating, spinorial field theory; *if the spinor fields are taken to commute* then indeed one can; but, mysteriously (from the geometrical point of view), requiring spinors to anticommute blocks the counterexamples. The further investigation of these matters is a future project.

Merton College, Oxford OX1 4JD
hilary.greaves@philosophy.ox.ac.uk

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# A Classical PT theorem

A more rigorous statement of the theorem referred to in section 4.1 follows. A proof is given in (Greaves, n.d.).

**Theorem 1** (Classical PT theorem for polynomial systems of real tensor fields.)**.** *Let $M$ be a differentiable manifold that is diffeomorphic to $\mathbb{R}^4$.*

*Let $\Phi$ be a space of $n$-tuples of tensor fields of specified types on $M$. That is, suppose there is a fixed set of integers $n_i, m_i : i = 1, \ldots, n$ such that each $\phi \in \Phi$ is an $n$-tuple of the form $(\phi_1, \ldots, \phi_n)$, where, for each $i$, $\phi_i$ is a tensor field of type $(n_i, m_i)$.*

*Let $\eta$ be a flat Lorentzian metric on $M$. Let $L$ be the group of manifold diffeomorphisms $l : M \to M$ leaving $\eta$ invariant (i.e., $L$ is the Lorentz group). Let $L_+^\uparrow, L_-^\downarrow, L_-^\uparrow, L_+^\downarrow$ be the connected subsets of $L$ that reverse neither time sense nor parity, time sense but not parity, parity but not time sense, and both time sense and parity respectively. For arbitrary $l \in L$ and $\phi \in \Phi$, let $l\phi$ be the $n$-tuple of fields obtained by allowing $l$ to act in the natural way on each element of $\phi$.*

*Let $F : \Phi \to \mathbb{R}^M$ be a functional that is polynomial in the fields and their derivatives. That is, suppose there exists a chart $x : M \to \mathbb{R}^4$, non-negative integers $p, q$ and real coefficients $\{a_{m_1,\ldots,m_n} \in \mathbb{R} : m_1, \ldots, m_n = 0, \ldots, p\}$ such that for all $\phi \in \Phi$,*

$$F(\phi) = \sum_{m_1,\ldots,m_n=0}^{p} a_{m_1,\ldots,m_q}(\psi_1)^{m_1}(\psi_2)^{m_2}\ldots(\psi_q)^{m_q}, \tag{20}$$

*where each $\psi_j$ is a specified partial coordinate derivative (possibly zeroth order) of a specified one of the coordinate components $(\phi_i)^{\mu_1\ldots\mu_{n_i}}{}_{\nu_1\ldots\nu_{m_i}}$, and multiplication is defined pointwise in the obvious way.*

*Let $S \subseteq \Phi$ be given by*

$$S := \{\phi \in \Phi : F(\phi) = 0\} \tag{21}$$

*[the intended interpretation being that $S$ is the set of solutions to the partial differential equation expressed by the condition $F = 0$].*

*Suppose that $S$ is invariant under $L_+^\uparrow$, i.e. for any $\phi \in S$ and $l \in L_+^\uparrow$, $l\phi \in S$ also. Then, $S$ is invariant under all of $L_+$.*

# References


Anderson, J. L. (1967). *Principles of relativity physics.* New York/London: Academic Press.

Bell, J. S. (1955). Time reversal in field theory. *Proceedings of the Royal Society of London A*, *231*(1187), 479-95.

Earman, J. (1989). *World enough and space-time.* Cambridge, MA: MIT Press.

Friedman, M. (1983). *Foundations of space-time theories.* Princeton: Princeton University Press.

Greaves, H. (n.d.). *A classical CPT theorem.* (Forthcoming.)

Levy-Leblond, J.-M. (1967). Galilean quantum field theories and a ghostless Lee model. *Communications in mathematical physics*, *4*, 157-76.

Peskin, M., & Schroeder, D. (1995). *Introduction to quantum field theory.* Westview Press.

Wald, R. (1984). *General relativity.* Chicago: University of Chicago Press.