Abstract Representations and Confirmation

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Abstract. Many philosophers would concede that mathematics contributes to the abstractness of some of our most successful scientific representations. Still, it is hard to know what this abstractness really comes to or how to make a link between abstractness and success. I start by explaining how mathematics can increase the abstractness of our representations by distinguishing two kinds of abstractness. First, there is an abstract representation that eschews causal content. Second, there are families of representations with a common mathematical core that is variously interpreted. The second part of the paper makes a connection between both kinds of abstractness and success by emphasizing confirmation. That is, I will argue that the mathematics contributes to the confirmation of these abstract scientific representations. This can happen in two ways which I label "direct" and "indirect". The contribution is direct when the mathematics facilitates the confirmation of an accurate representation, while the contribution is indirect when it helps the process of disconfirming an inaccurate representation. Establishing this conclusion helps to explain why mathematics is prevalent in some of our successful scientific theories, but I should emphasize that this is just one piece of a fairly daunting puzzle.

I. Modern science is incredibly successful when it comes to representing the world, and these representations seem to employ a lot of mathematics. Most philosophers of science seem uninterested in finding out if this correlation indicates any underlying relationship between the use of mathematics and the success of science. It is not clear why this is, but one problem with the topic of mathematics in science is that it has encouraged overly broad conclusions, as with Kant

> I maintain, however, that in every special doctrine of nature only so much science proper can be found as there is mathematics in it.

or Wigner

> The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve.

It is hard to know what to make of such sweeping claims or how they could be defended based on what we know about science and its success. By contrast, my approach will be to start with a series of questions that allow a variety of mutually supporting answers: what does mathematics contribute to the success of a given scientific representation? How does it make this contribution? What does this contribution presuppose? Here I aim to explore the answer that what mathematics contributes in many cases is abstractness. But, as we will see, there are two sorts of abstract contributions from mathematics. Once they are distinguished I will argue that both sorts of contribution relate to the confirmation of a scientific representation. If this sort of answer can be sustained, then we will see a link between mathematics and scientific knowledge. This is more modest than what it seems Kant or Wigner intended, but it might support a proposal like Daniel Bernoulli's: "There is no philosophy which is not founded upon knowledge of the phenomena, but to get any profit from this knowledge it is absolutely necessary to be a mathematician."[1] Despite its modesty, even this sort of epistemic answer would raise important questions about the source of the mathematical knowledge that is deployed in science and whether it might require some kind of a priori basis.

II. To start, let's clarify what I mean by "scientific representation". Much of the discussion of representation in science contrasts theories and models. Models are said to be autonomous from theories by one side, while the other side responds that theories just are collections of models (e.g. [17] vs. [9]). It is thus useful to have a generic term that embraces both theories and models, and this is how I use "scientific representation". Put briefly, a scientific representation is anything that has content, i.e. has conditions under which it is true of its target system and conditions under which it is false of its target

[1] Quoted in [13], p. 22, citing Truesdell.

system. There are important questions about how scientific representations get their content and where exactly they should be placed in our overall metaphysics. I will try to set these questions aside whenever possible so that we can focus on the specific case of a mathematical scientific representation.

There are two ways in which mathematics enters into the content of a given scientific representation. I call these "intrinsic" and "extrinsic". The basic idea is that the intrinsic mathematics is necessary to fix the content of the representation, while the extrinsic mathematics is not. To fix the distinction more clearly I present a simple example of a traffic model ([12], ch. 8). Suppose we have a system of N cars traveling down a single-lane road. The lead car's front bumper's position will be represented by the real-valued function of time $x_1(t)$, the second car's position by $x_2(t)$, and so on through $x_N(t)$. With L representing the length of each car, we impose the constraint that $x_i(t)+L < x_{i-1}(t)$, i.e. no car can be closer to the front of the car ahead of it than the length of the car. Violation of this constraint would imply a collision. Our representation also invokes a constant reaction time τ for each driver which is intuitively the time between when the driver sees a change on the road ahead and begins her accelerating/braking maneuver. Two qualitative assumptions can be used to motivate a representation of the braking force of each car after the lead car:

(a) The greater the difference in velocity between the two cars, the greater the braking force will be.

(b) The greater the distance between the two cars, the lesser the braking force will be.

This yields, for $1 < i \leq N$,

$$\ddot{x}_i(t+\tau) = \lambda \frac{\dot{x}_i(t) - \dot{x}_{i-1}(t)}{|x_i(t) - x_{i-1}(t)|} \tag{1}$$

where $\lambda = A/m$, m the mass of each car and A some constant. Appealing to our constraint allows us to simplify the equation further and integrating gives us the velocity for each car after the lead car in terms of its relative position at an earlier time: for $1 < i \leq N$,

$$\dot{x}_i(t+\tau) = \lambda \ln |x_i(t) - x_{i-1}(t)| + \alpha_i \tag{2}$$

where α_i is a constant. Through the specification of λ, τ, the α_i and some further choices, we could develop this representation into a complete specification of the trajectory of each car over some time interval.

(2) and these additional components constitute our representation of a given traffic system. It should be clear how the mathematics that explicitly appears in (2) is intrinsic. This part of the mathematics is used to represent where the cars are over time. The intrinsic mathematics needs to be distinguished from the other extrinsic mathematics that was used to derive (2). In particular, (1) and the mathematics that gets us from (1) to (2) is extrinsic to (2). An account of how the traffic system has to be for (2) to be correct need not invoke the extrinsic mathematics of the representation. To see the value of this distinction, note that (2) could wind up being accurate even if (1) was not. Similarly, the main mediating link between (1) and (2) is

$$\frac{\dot{x}_i(t) - \dot{x}_{i-1}(t)}{|x_i(t) - x_{i-1}(t)|} = \frac{d}{dt} \ln |x_i(t) - x_{i-1}(t)| \tag{3}$$

when $x_i(t) < x_{i-1}(t)$. (3) does not enter into the content of the representation. Even when such mediating equations are false, it need not undermine the accuracy of the resulting representation.

With the intrinsic/extrinsic distinction in hand, we can turn to the two respects in which this representation fails to be abstract even though it makes crucial use of mathematics. First, its mathematical components have a *fixed* physical interpretation. x_i indicates the position of the i-th car, τ the reaction time, and so on. This fixed association between the elements of the mathematics and certain physical magnitudes is consistent with some degree of variation in the content of the representation from case to case. For example, we might use this very same representation to investigate a traffic case in Pennsylvania, and then use it to consider a traffic system in Indiana. But this degree of flexibility is quite minimal. What is more interesting, and what is lacking with this representation, is the possibility of taking x_i to stand for something else, e.g. the position of a gas molecule. If we abstract away from the fixed content of (2), what results is what I will call an *abstract varying* representation. Such a representation is really a family of representations within which each member of the family involves a more concrete specification of which parts of the mathematics are associated with which magnitudes. If we take this abstraction to its utmost, then what results is a wholly mathematical representation with no physical content. For example, we can discuss the linear harmonic oscillator model as a mathematical entity without reference to its various concrete interpretations.[2] It is important to

[2] Cf. the discussion of modeling in [22].

note that members of abstract varying representations need not totally agree in their mathematics, and that often a merely partial overlap is sufficient.

There is a second sense of abstractness, however, which should not be confused with the sense just introduced. Notice that in addition to having a fixed interpretation, (2) purports to represent the causal interactions of the cars in the traffic system. The causal content is reflected in the tracking of the position of each car over time and the coupling between the difference in relative position of two cars at t and the velocity of the lagging car at $t+\tau$. Such a representation satisfies the tests imposed for most accounts of causation. For example, its content supports the right kind of counterfactuals related to changing circumstances and it purports to capture the underlying mechanisms responsible for the phenomena.[3] The only thing missing from this representation are the microphysical details. But only the most restrictive notion of causal representation would view the inclusion of the microphysics as a necessary condition. The main point of what follows is to argue that there are important cases of non-causal representation where mathematics makes a crucial contribution. Anyone advocating even the strictest notion of causal representation will presumably agree that these are non-causal representations.

We can start with a causal representation and move to an *abstract acausal* representation if we remove the elements necessary for tracking the dynamics of the system as it evolves through time. Examples of the results of this process of abstraction are representations of the system at some specified equilibrium.[4] By jumping to how the system will look at this equilibrium we can gain crucial information about features of the system and we can study how it will behave if the equilibrium is disturbed. This is, in fact, one standard way of proceeding after one obtains (2). The reasons for this are not hard to appreciate once one notices how difficult it would really be to make any progress with (2). To begin with, there are many equations to work with. More importantly, to add realistic constraints would require a great deal of information about the system. For example, how are the cars initially arrayed, and what fixes the various α_i? Additional motivation for shifting to an equilibrium representation will become clear shortly. After that we will return to a discussion of the different sorts of contributions of the mathematics as we move from a fixed, causal representation to both abstract

[3] [24], [15].

[4] I should emphasize that this is just one kind of abstract acausal representation.

acausal and abstract varying representations.

III. We develop an acausal representation of a traffic system by defining an equilibrium point to be a case where the velocity of each car is the same and the distance between each car is the same:

$$\begin{aligned} \forall i, j \;\; \dot{x}_i &= \dot{x}_j \\ \forall i > 1 \;\; x_i(t) - x_{i-1}(t) &= d \end{aligned}$$

The choice made here is not ad hoc as a review of (1) shows that these system's cars, except the lead car, must have constant velocity. At equilibrium we have a well-defined density function ρ:

$$\rho(x_0, t) = \frac{\text{number of cars in } [x_0 - \epsilon, x_0 + \epsilon] \text{ at } t}{2\epsilon} \tag{4}$$

where $L \ll 2\epsilon \ll$ the length of the road. At equilibrium, ρ is independent of the choice of ϵ and becomes

$$\rho = \frac{1}{d + L} \tag{5}$$

We assume that $v(x, t)$, the velocity of cars at point x at time t, is a function just of the density $\rho(x, t)$. This allows us to distinguish two crucial equilibrium points for the system. First, there is the point of minimal velocity, which will occur at the maximum density $\rho_{max} = \frac{1}{L}$ when the distance d between the cars goes to zero. At this point, $v(\rho_{max}) = 0$ and the cars stand still. Second, there is a maximal velocity for which we label the density ρ_{crit}. We are interested in how the velocity varies as a function of the density as we move from $v_{max} = v(\rho_{crit})$ to $v(\rho_{max})$. Intuitively, as the density is increased, the velocity will decrease.

Based on these assumptions and (2) we can find v as a function of ρ when $\rho > \rho_{crit}$:

$$v(\rho) = v_{max} \ln[\, \frac{\rho_{max}}{\rho} \,][\, \ln \frac{\rho_{max}}{\rho_{crit}} \,]^{-1} \tag{6}$$

Notice how the otherwise undetermined λ and α_i of (2) have been removed. To plot (6) we need only determine the density at which the maximum velocity is reached.

A second magnitude of interest is the flux at a given point, i.e. the number of cars that cross a given point per unit of time:

$$j(\rho) = \rho v(\rho) \tag{7}$$

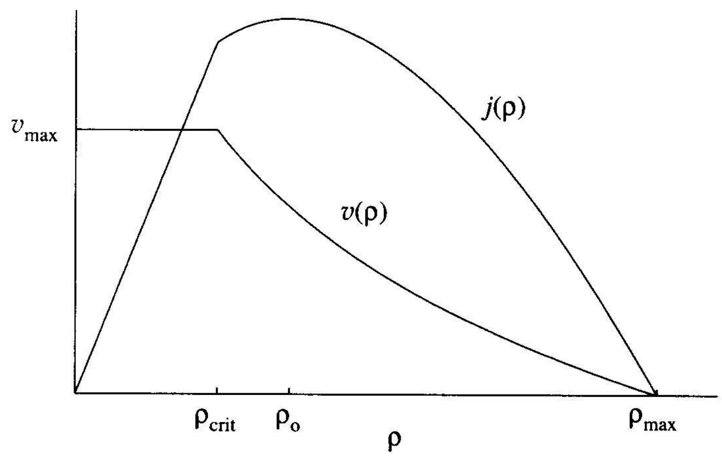

Figure 1: Velocity and Flux as Functions of Density ([12], p. 155)

Using (6), we can find the density ρ_0 with maximum flux by solving $\dot{j}(\rho) = 0$. This yields

$$\rho_0 = \frac{\rho_{max}}{e} \tag{8}$$

$e = 2.71828\ldots$ here is the base of the natural logarithm. See figure 1 for a graph of the density and flux.

How is the concrete causal representation (2) different from the abstract acausal representation (6), (8)? They clearly have different contents. The causal representation includes details about each car's position and velocity over time as well as information about the causal interactions between the cars. The acausal representation lacks this causal content or any other causal content. We can see this by noting that (6), (8) taken in isolation give us no information about how a particular system would change in counterfactual circumstances and the mechanisms responsible for the features of the system are not given. A second change pertains to the intrinsic mathematics of the acausal representation. If we consider the mathematical entities invoked by the two representations in their specification of how the system is, then we see that quite different constants and functions appear. There is, of course, some relationship between the two representations. We have linked them through some simple mathematical steps. Still, the functions ρ and v that appear in (6), (8) are not part of (2). We made various assumptions in defining these functions that could be questioned by someone who assented to (2). In general, abstract acausal representations involve mathematical entities and idealizing assumptions that outstrip the content of a causal representation. So, it is only in special cases that we could find a transparent relationship. In some cases scientists cannot even formulate a causal representation and work solely with an acausal representation.

IV. So far we have seen one way in which mathematics can contribute to a scientific representation. This is to allow the formulation of an abstract acausal representation from a concrete causal representation.[5] The mathematics here provides an extrinsic link between the two representations and permits the scientist to arrive at one representation from the other. This is really only the first step, though, in helping to explain what the mathematics contributes to the success of science. For as it stands, we have no account of what is so good about the equilibrium representation or why the scientist would be interested in it. Mathematics might allow the formulation of all sorts of acausal representations, but this is beside the point unless there is something better about these new representations. There are several different strategies that one could adopt here when it comes to championing equilibrium representations. I will pursue what seems to me to be a fairly uncontroversial and deflationary line.

One proposal, which I will not pursue here, is that acausal representations have distinct explanatory features over and above what a causal representation can offer. If mathematics contributes to the development of these more explanatory representations, and we understand why explanatory representations are desirable, then we have a simple solution to our puzzle about the contribution of mathematics, at least in these cases. This idea is explicit in [14], who argue that mathematics contributes to the explanatory power of their example involving phase spaces.[6] The main concern I have about an appeal to non-causal conceptions of explanation in the context of understanding the contribution of mathematics is that there is little agreement on what is responsible for the explanatory power in these cases or what exactly the mathematics is doing. So, while I do not want to deny these points about explanation and mathematics, I will explore what seems to me to be a more

[5] Batterman has argued for this conclusion for some time, but he may not wish to endorse the way I have drawn the causal/acausal distinction here. See [4], [3] and [6].

[6] The proposal is also consistent with an emphasis on special kinds of explanation in sciences like biology. Sober, for example, argues for a distinctive kind of equilibrium explanation ([19]), and Potochnik has delineated a notion of optimality explanation in models of natural selection ([18]).

tractable alternative.

My proposal is that the abstract acausal representations are often easier to confirm than their concrete causal counterparts. As the mathematics allows the formulation of the acausal representation, the mathematics permits us to arrive at representations that are easier to confirm. The main reason for this is that the acausal representation imposes fewer conditions on its target systems and its target systems are a proper subset of the target systems of the causal representation. In this sense, I would argue that acausal representations typically have less content than their causal partners. This makes it easier to directly confirm such representations when they are accurate. Even when they are inaccurate, it is easier to find this out. This makes the scientific process of testing work better because it indicates to the scientist that she should look elsewhere for a better representation. As this is an indirect contribution to the process of testing, I call this a contribution to indirect confirmation.

To see this, notice that there is a barrier to confirming the causal representation that is absent for the acausal representation. The nature of this barrier depends on what sort of account of causation we prefer. Whatever it is, the account will impose more conditions on a causal representation than on an acausal representation. That is, a representation could track the trajectories of all the cars over time and still fail to correctly represent the causes of these trajectories. To take one example, most would require that causal claims support the right kind of counterfactuals. It follows that to correctly represent the causes operating in a traffic system the representation must also include information about how the system would change if a given car drove differently. This means that the confirmation of a causal representation must involve experiments on many traffic systems, and a determination of how changes in causes affect the traffic across these various systems. A related point of difference between the two representations is their scope. The equilibrium representation purports to represent the system only when it is in a very restricted kind of state, namely equilibrium. The causal representation aims to capture these cases, as well as all the others. A limited scope makes the equilibrium representation easier to confirm if we can set up systems in these equilibrium states and verify their behavior. But even confirming the causal representation for some cases would do little to assure us of its overall accuracy unless we somehow considered a meaningful sample of traffic systems.

The content of the causal representation is more complicated, then, than our associated acausal representation. It is this complexity which makes the causal representation harder to confirm. To the extent that these difficulties are absent in the case of an acausal representation, it will be easier to confirm. The mathematics makes this possible, and so we can conclude that here mathematics contributes to the success of science by allowing the confirmation of some scientific representations. If a scientist failed to know the relevant mathematics, and so could not formulate and work with the acausal representation, then she would not have access to this well-confirmed representation.

There is a temptation to go further than I have and take the success of the equilibrium representation as evidence of some new underlying metaphysics associated with the mathematics. A diagnosis of this tendency depends on taking seriously a controversial interpretation of the successful causal representations that we have at our disposal. Here it seems like the mathematics is making its contribution by tracking the genuine causal relations between the entities. On this general picture, successful mathematical representation involves reflecting mathematical aspects of a situation. If we then have a successful mathematical acausal representation, it seems like we should conclude that there are new mathematical aspects of the situation that are missed by the causal representation. For an equilibrium representation, we might then think that there are new "emergent" features of the system and start to investigate the metaphysical status of these features. On this more metaphysical approach, my focus on confirmation will seem too timid and to miss the heart of the matter. Mathematics guides us to a new kind of feature of systems, and not just to representations of ordinary features that we can now more easily confirm.

This metaphysical interpretation of at least some mathematical representations is offered by Franklin: "there are properties, such as symmetry, continuity, divisibility, increase, order, part and whole which are possessed by real things and are studie[d] directly by mathematics, resulting in necessary propositions about them" ([10], p. 17). Keeping this picture in mind is also one way to make sense of the debate surrounding Batterman's views on non-fundamental theories like classical thermodynamics. See, e.g., [4], [7] and [5]. Batterman argues that the need to employ various limit assumptions, like that the number of particles is infinite, shows that there are genuine features of the world that are missed by our so-called fundamental theories. These features, such as the existence of phase transitions, are said to be represented by the less fundamental theories because of their different mathematical character. Belot responds by refusing to assign the relevant mathematics any novel physical interpretation:

> The mathematics of the less fundamental theory is definable in terms of that of the more fundamental theory; so the requisite mathematical results can be proved by someone whose repertoire of interpreted physical theories includes only the latter; and it is far from obvious that the physical interpretation of such results requires that the mathematics of the less fundamental theory be given a physical interpretation ([7], p. 151).

Without taking a stand on this issue, it seems clear that abstract acausal representations can be successful without tracking new emergent features. In the traffic case, at least, this metaphysical approach is unwarranted. There is nothing more to the system than the cars and their motions. Shifting our representation from causes to equilibrium need not give us a representation of anything that is genuinely new in a metaphysical sense. Successful mathematical representation need not track the underlying metaphysics, and there are many cases where taking this metaphysical approach leads to a distorted understanding of why the representation is working so well. There may be cases where genuinely emergent properties arise, and where mathematical representations afford access to them. But simply using mathematics as we have done in the traffic case does not require this perspective.

V. So far we have emphasized only the causal vs. acausal distinction. But this is just one sense in which mathematics can contribute to successful abstract representations. I turn now to the other sense of "abstract": the contrast between a representation with a fixed interpretation and an abstract varying representation where the physical interpretation of the common mathematics varies within a family. The equilibrium traffic representation that we have developed can be extended into a more abstract representation of a system with a density function $\rho(x,t)$ and a flux function $j(x,t) = v(\rho)\rho(x,t)$, i.e. the velocity at a point is solely a function of the density. Assuming that the material in question is conserved and making various continuity assumptions, we can derive the conservation law

$$\rho_t + j_x = 0 \tag{9}$$

Here subscripts indicate partial differentiation. But j is solely a function of ρ, so we obtain

$$\rho_t + j'(\rho)\rho_x = 0 \tag{10}$$

Here the prime indicates differentiation with respect to ρ. Assuming that the initial density ρ_0 is given, we have what is called an initial value problem.

One technique for approaching such an abstract problem involves what is known as the method of characteristics. Essentially, we start by assuming that a unique solution to the problem exists and isolate it using ρ_0. Applying the method delivers curves, known as the characteristic base curves, along which ρ is constant. The arrangement of these curves tells us whether or not there really is a unique solution. When this assumption breaks down, the base curves can still provide useful information on the evolution of the system. For example, they can be used to isolate where discontinuities in ρ known as shock waves develop. We consider a simple example of such an analysis so the insight it provides into such systems becomes clearer.

We start by making $j(\rho) = 4\rho(2-\rho)$ and imagine the initial density ρ_0 to be 1 for all $x \leq 1$, $1/2$ for $1 < x \leq 3$ and $3/2$ for $x > 3$. In the traffic case, this corresponds to the maximal density being 2, so to the left of $x = 1$ we have the cars spaced with one car length between them and in the other domains they are less densely spaced. We would expect a rightward flow of cars then, increasing the density. To say something more precise, we find the characteristic base curves along which the density remains constant as time increases. Their slope turns out to be

$$\begin{aligned} 0 &\text{ if } x \leq 1 \\ 4 &\text{ if } 1 < x \leq 3 \\ -4 &\text{ if } x > 3 \end{aligned}$$

The details of this derivation are in [12], ch. 9. If we plot a representative sample of these base curves emanating from the x-axis, as in figure 2, we notice two things. First, there is a gap at $x = 1$ where our method has given us no information about how the density will change as time increases. Second, there is a line coming from $x = 3$ where the base curves from two initial domains intersect, seemingly indicating that there will be two different densities at the same location. The former defect can be remedied in a physically intuitive wave by smoothing out the transition in the density at $x = 1$ from 1 to $1/2$. This produces a rarefaction wave where a fan of lines of gradually decreasing density are inserted into our original diagram. The second discontinuity is more serious, however, as it shows the existence of a shock wave where our representation of the changes in density over time breaks down. The representation can still be useful as we can trace how

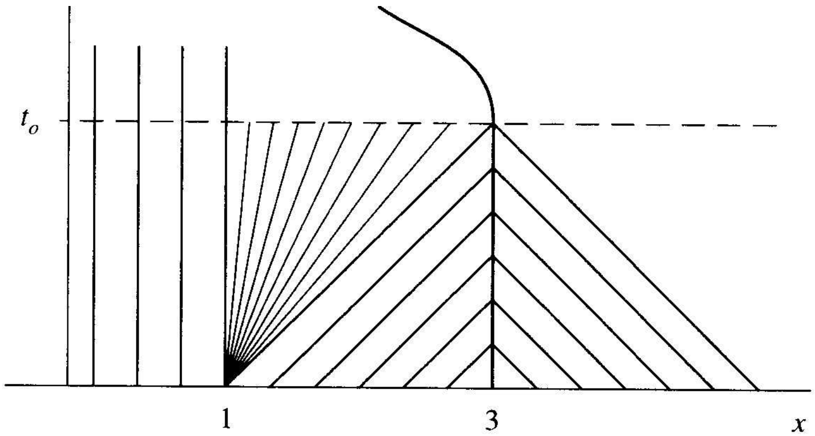

Figure 2: Characteristic Base Curves ([12], p. 187)

the density will develop on both sides of the shock and how the shock will propagate over time, e.g. how it will be affected by the development of the rarefaction wave.[7]

From our traffic perspective, we can consider what a driver would experience as she crossed the rarefaction wave and then the shock. At the beginning of the rarefaction wave the density decreases, i.e. the cars suddenly have more space between them. This leads the driver to accelerate. This density levels out, so the driver will then assume a steady speed. However, as she enters the left side of the shock, there is a dramatic increase in density. This requires sudden deceleration, presumably to a zero velocity associated with the maximal density. As she emerges from the shock regime on the right side, she finds a density below the maximal density, allowing her to accelerate. Her velocity after the shock is less than her original velocity as the density has increased. We see, then, that, whatever its artificial features, our simple example allows us to reproduce something that drivers are all too familiar with: a traffic jam produced by the volume of cars on the road.[8]

Now, even though we have used the traffic interpretation to guide our understanding of this representation, its mathematical character allows us to

[7]An explicit solution for this system is given at [12], p. 189.

[8]See [11] for the claim that shock waves have been experimentally observed in traffic systems, based on [21]. The models employed there are different from the simple representation presented here.

interpret it in terms of very different kinds of physical systems. Consider, for example, the representation of shock waves in compressible fluids like air. Here we will have a partial overlap of mathematics, most importantly the changing density function ρ and the appearance of discontinuities that we can track as shock waves. To make the comparison with the traffic case as straightforward as possible we will imagine a fluid that is restricted to a thin pipe that is oriented in the x direction. The basic equations for fluid mechanics are the Navier-Stokes equations. These equations relate the velocity vector field to the pressure at a point p with reference to the density of the fluid ρ and additional forces such as gravity. The main simplification we make is to ignore the viscosity μ of the fluid. This means that the fluid elements do not resist internal circulation. This greatly simplifies the Navier-Stokes equations because it allows us to drop certain terms with second-order partial derivatives. Under this assumption, the Navier-Stokes equations reduce to the Euler equations for an ideal fluid. We will restrict our discussion to one-dimensional fluid flow and assume that the pressure p is a function of the density ρ (barotropic). This allows us to see the flow in the pipe as governed by the equations:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{11}$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0 \tag{12}$$

The engine driving the development of a shock wave in such a fluid is the interaction between the pressure, density and the speed at which a disturbance from an initial state will propagate through the fluid. It turns out that we can capture this speed c using the ratio of pressure to density: $c^2 = \frac{\gamma p}{\rho}$. γ is a constant that varies from fluid to fluid. If we think of a fluid like air at some initial pressure p_0 and density ρ_0, then $c_0 = \sqrt{\frac{\gamma p_0}{\rho_0}}$ is the speed at which a disturbed pressure and density wave would travel down the pipe. It is thus labeled the *speed of sound* for the air in that state because sound waves are instances of this kind of disturbance. Imagine, then, an initial density distribution that is analogous to our traffic case: 2 for all $x \leq 1$, 1 for $x > 1$. The air in the more dense region will expand to the right with some speed c_1. But this compressed air will have a higher local speed than the air just to the right of $x = 1$. As the air accumulates in the region around $x = 1$, more air comes in than goes out. This further compresses the air, making the local speed even higher. What develops, then, is a discontinuous front where the density drops dramatically. A shock wave appears in the compressible fluid case because the velocity of the disturbance is affected by the compression of the air.

Lin and Segel note that it remains "a remarkable fact" ([13], p. 549) that this discontinuity can be handled by adopting the policing of refraining from using this representation to understand what is going on within the shock wave while still taking seriously what is represented in the other regions. Conservation relations between the two sides of the shock tell us, for example, what is happening on either side of the shock, and in certain circumstances we can plot where the shock will travel over time. As Wilson has put it when discussing this sort of case, also with reference to the analogy with traffic jams, "if we can examine a situation from several sides and discern that some catastrophe is certain to occur, we needn't describe the complete details of that calamity in order to predict when it will occur and what its likely aftermath might be" ([23], p. 188). The procedure of selectively interpreting such representations, then, is not some strange abberation associated with some peculiar mathematics. To the contrary, it often provides our only means of representing phenomena of widespread scientific significance.

VI. I want to argue now that the way mathematics makes its contributions to abstract varying representations is also via a boost in potential confirmation. Still, the story here is quite different from what we saw for the case of abstract acausal cases. While both direct and indirect confirmation play a role, the way in which both sorts of confirmation arise is not due to relative amounts of content, but instead to a partial overlap of mathematical content. Abstract varying representations involve a great increase in content by linking together a family of more concrete representations via their shared mathematical structure. As we saw, the move to an abstract acausal representation produced a limited and impoverished content that targeted only selected aspects of the system in question. This might make it seem that considering an abstract varying representation would decrease our chances of arriving at a well-confirmed representation. This would be a mistake, however. While it is true that it is hard to confirm the entire family, when we have independently confirmed one member of the family that confirmation can contribute to the confirmation of another member of family in both a direct and indirect way.

To see how this can work in a best case scenario, imagine a scientist who knows the mathematics that we have reviewed for shock waves, has extensive experimental confirmation of its application for the case of fluids and has tentatively adopted the hypothesis that the same mathematics is part of an accurate representation of traffic jams. We have, then, a two member family of variously interpreted mathematical scientific representations, where one member of the family has accrued a high degree of confirmation through ordinary scientific testing.

The key claim I want to defend for such a case is that a small amount of experimental testing of the traffic representation should give it a larger boost in confirmation than if that representation was not mathematically linked to the successful fluid representation. This is because the independent confirmation of the way the mathematics is deployed for the fluids gives the scientist a template against which to judge the success of the traffic representation. Initially when using the notion of a discontinuity in density in the fluid case, the scientist would reasonably have some doubts about the appropriateness of using mathematics this way. She may worry that the mathematics will lead to absurd predictions or predictions that fail to track any genuine features of the fluids under investigation. This is not so much a concern about the mathematics itself, which I am taking for granted is confirmed independently by purely mathematical standards. The scientist who knows the mathematics still has a right to wonder if it is being deployed coherently for this kind of physical system. I claim that these doubts should be allayed by the success of the fluid representation at prediction and description of actual fluid flows. This success gives the scientist a reason to expect that a limited success with deploying the same mathematical apparatus in the traffic case will continue. Analogous doubts about the appropriateness of the mathematics for the traffic case should be more quickly put to rest based on the success with treating fluids. The more the mathematics of the two cases run in parallel, the more this benefit for confirmation will obtain.

Our discussion so far has focused squarely on the confirmation of already existing representations. As a result, I have not said anything about how the existence of mathematical similarities contributes to the formulation or discovery of new scientific representations. Like explanation, the role of mathematics in the discovery of new representations is a large and controversial topic ([20]). I would suggest, though, that the epistemic benefits of having families of variously interpreted representations can at least help us to understand why they are desirable. That is, a scientist might hope to formulate or discover a mathematically linked series of representations of different physical systems because if she found such a family, then this would contribute significantly to the success of those representations. This point is reinforced once we grant the difficulty of actually working with mathematical scientific representations. Extracting solutions to systems of equations is not a trivial task, and so it is more or less a waste of time to formulate such systems if the scientist knows of no way to solve them or to at least approximate a solution. This implies that it is a sensible heuristic on scientific discovery for a scientist, other things being equal, to aim to deploy a mathematical theory that is well understood and that has a track record of success.

This point does not entail that there is any good reason, prior to experimentation, to expect physical systems made up of different things to be accurately represented by similar mathematical structures. Instead, our case indicates how rare and exceptional even a partial overlap is. There are any number of disanalogies between the traffic system and the compressible fluid system that have been ignored in an attempt to highlight the salient mathematical similarity. It is not too hard to imagine the genesis of an inaccurate representation of a traffic system based on an attempt to transfer some of the success from the fluid case to a new domain. Suppose, for example, that a scientist knows all the relevant mathematics and has obtained good empirical success treating compressible fluids along the lines discussed above. In particular, she has learned how to handle the genesis and development of shock waves in her representations and in her fluids, and as a result has made a number of successful predictions. It is tempting, then, to take the entire mathematical representation of shock waves in fluids and transfer it over to the traffic case based on the following two similarities. First, traffic systems, like compressible fluids, have varying density. Second, both fluids and traffic systems are constituted out of individual objects that it seems can be fruitfully ignored in an idealization as a continuous medium. The scientist might try to use her understanding of the role of variation in the speed of sound to somehow represent how traffic shock waves develop. Based on our discussion, we can see that this is a mistake because the underlying reason for the evolution of the traffic system has nothing to do with this. Instead, it is because we assumed that velocity was a certain kind of function of density that we were able to derive (10). The velocity here was the velocity of the cars, not anything like the velocity of a disturbance in pressure and density. We see, then, how an error could be committed by this kind of erroneous transfer of mathematics. It is a delicate matter to decide which mathematics to transfer over to a new case and how its interpretation should be adjusted based on the new subject matter.

Such a turn of events can still lead to a case indirect confirmation, though, when the flawed representation is experimentally tested and found to be in error. Indirect confirmation occurs when an inaccurate representation is discovered to be inaccurate. While there is nothing about families of variously interpreted representations per se that makes them any easier to test, I want to argue that when a representation in the family is found to be inaccurate the scientist has more information about how to proceed than if the representation was not in the family. Essentially, the family of mathematically related representations gives the scientist a framework within which to probe for the source of the inaccuracies. Two possibilities are especially likely. First, each member of the family may be inaccurate because there is some genuine underlying mathematical similarity to all the systems, but the current family of representations has missed this in some respects. A scientist can determine whether or not this is the case by testing other members of the family for the features that were determined to be inaccurate for one member. If all the members are found to be inaccurate in the same way, then the scientist knows where to adjust her family of representations. A new term may be needed for the equations, for example, or perhaps a magnitude that was treated as a constant should be treated as a variable. The second kind of scenario is what I had in mind when describing the overextension of the fluid representation to the traffic case. This is where one member of the representation is accurate, but a second representation is inaccurate in some respects, even though it does ultimately agree with the original representation in other respects. In this case the failure of the second representation can help to pinpoint precisely how the second kind of system differs from the first representation. As a result, a failure of a prediction can alert the scientist to some new and perhaps unappreciated features of the second system, here the traffic system. She might then investigate what is responsible for the difference between the two systems. This is a limited tool for discovering new accurate representations based on the partial failure of a mathematical analogy. A scientist can exploit her understanding of the mathematics to help formulate new proposed representations that have a better chance of being accurate.

In summary, then, the epistemic benefits of varying interpretations of some mathematical scientific representation fall into both the direct and indirect categories. When the members of the family turn out to be accurate, the mathematical similarities make it easier to confirm the accuracy of the representations. Even when some parts of the reinterpretation wind up being inaccurate, the scientist can use the mathematical links to more easily diagnose the source of the failure, and even to help in the quest for the formulation of a more accurate representation of the system in question. It is important see how this story is different from what we saw in the abstract acausal case. There it was the elimination of the causal content that made it easier to obtain the confirmation. Here it is the variation in the interpretation of the mathematical components of the representation that is doing the work. The two steps could of course work together and reinforce one another. This is actually what has happened here. Neither the traffic representation nor the fluid representation purport to represent the causes of the phenomena in question. The whole apparatus of shock waves, in fact, seems to sit poorly with a causal interpretation of either representation. Nevertheless, they are successful representations of their respective domains. It seems that the absence of causal content in these sorts of cases has led some to think that acausal content is what mathematics contributes in all cases. But I hope to have shown that there is a distinct contribution from the mathematics when we vary the interpretation of the representation beyond what we get from abstracting from causes.

There is close link between the benefits of families of variously interpreted representations and the unification of similar representations by mathematical means. The power to unify, of course, is one feature that some have placed at the heart of an account of scientific explanation. This is a different sort of criticism of causal explanation than we noted earlier by those who emphasize equilibrium or other stability notions of explanation. A much-discussed case of mathematical explanation in science is Baker's periodic cicada example.[9] Here scientists asked why the life-cycle of a family of species of cicadas was prime. The explanation offered was in terms of the mathematical fact that prime cycles minimize intersections with competing species or predators. In our terms, we have a family of representations of different species of cicadas and we deploy the mathematical concept of primeness in each. This permits scientists to represent similarities between systems that seem to have unifying and explanatory benefits. This is, of course, consistent with my emphasis on confirmation, but I have not followed Baker and others by focusing on explanation because, as with the acausal case, it remains unclear what the source of the explanatory power really is. Still, if this explanatory power can be further clarified in the acausal and varying interpretation cases, there is the

[9] See [8], [16], [2] and [1].

prospect of grounding an important inferential principle for scientific realists, namely inference to the best explanation. For if we can independently argue for the epistemic benefits of these sorts of mathematical scientific representations, and see that they tend to coincide with certain explanatory virtues, then we can present an argument for inferring the correctness of an explanatory representation. Much more work would have to be done to forge these links, but I hope this paper has made a useful first step.

VII. Let us conclude by returning to our original claim: mathematics contributes to the abstractness of some of our most successful scientific representations. First, I claimed that there are two sorts of abstract contributions that mathematics can bring. These were labeled as abstract acausal and abstract varying. Second, I argued that both sorts of contributions bring with them associated boosts in either direct or indirect confirmation. Acausal representations are often easier to confirm than their causal counterparts. Varying representations that are grouped together by their mathematical similarities provide a link between representations where support that has accrued to one can be used to investigate another member of the group. We see, then, that we can start to understand why mathematics is so prevalent in successful scientific representations. This success is cashed out partly in terms of the empirical confirmation of the representations. We can make sense of how mathematics helps in science without sliding into a dubious metaphysical interpretation according to which the mathematics must be tracking otherwise inaccessible aspects of the physical world.

References

[1] Alan Baker. Mathematical explanation in science. *British Journal for the Philosophy of Science*, forthcoming.

[2] Alan Baker. Are there genuine mathematical explanations of physical phenomena? *Mind*, 114:223–238, 2005.

[3] Robert Batterman. Asymptotics and the role of minimal models. *British Journal for the Philosophy of Science*, 53:21–38, 2002.

[4] Robert Batterman. *The Devil in the Details: Asymptotic Reasoning in Explanation, Reduction, and Emergence*. Oxford University Press, 2002.

[5] Robert Batterman. Response to Belot's "Whose devil? Which details?". *Philosophy of Science*, 72:154–163, 2005.

[6] Robert Batterman. On the explanatory role of mathematics in the empirical sciences. http://philsci-archive.pitt.edu/archive/00004115/, 2008.

[7] Gordon Belot. Whose devil? Which details? *Philosophy of Science*, 72:128–153, 2005.

[8] Mark Colyvan. Mathematics and aesthetic considerations in science. *Mind*, 111:69–74, 2002.

[9] Newton da Costa and Steven French. *Science and Partial Truth: A Unitary Approach to Models and Scientific Reasoning*. Oxford University Press, 2003.

[10] James Franklin. Mathematical necessity and reality. *Australasian Journal of Philosophy*, 67:11–17, 1989.

[11] Max Glaskin. Shockwave traffic jam recreated for first time. NewScientist.com news service, March 4, 2008.

[12] R. Illner, C. S. Bohun, S. McCollum, and T. van Roode. *Mathematical Modelling: A Case Studies Approach*. American Mathematical Society, 2005.

[13] C. C. Lin and Lee A. Segel. *Mathematics Applied to Deterministic Problems in the Natural Sciences*. SIAM, 1988.

[14] Aidan Lyon and Mark Colyvan. The explanatory power of phase spaces. *Philosophia Mathematica*, 16:227–243, 2008.

[15] Peter Machamer, Lindley Darden, and Carl F. Craver. Thinking about mechanisms. *Philosophy of Science*, 67:1–25, 2000.

[16] Joseph Melia. Response to Colyvan. *Mind*, 111:75–79, 2002.

[17] Mary S. Morgan and Margaret Morrison, editors. *Models as Mediators: Perspectives on Natural and Social Science*. Cambridge University Press, 1999.

[18] Angela Potochnik. Optimality modeling and explanatory generality. *Philosophy of Science (Proceedings)*, 74:680–691, 2007.

[19] Elliott Sober. Equilibrium explanation. *Philosophical Studies*, 43:201–210, 1983.

[20] Mark Steiner. *The Applicability of Mathematics as a Philosophical Problem*. Harvard University Press, 1998.

[21] Yuki Sugiyama, Minoru Fukui, Macoto Kikuchi, and et. al. Traffic jams without bottlenecks – experimental evidence for the physical mechanism of the formation of a jam. *New Journal of Physics*, 10:033001, 2008.

[22] Michael Weisberg. Who is a modeler? *British Journal for the Philosophy of Science*, 58:207–233, 2007.

[23] Mark Wilson. *Wandering Significance: An Essay on Conceptual Behavior*. Oxford University Press, 2006.

[24] James Woodward. *Making Things Happen: A Theory of Causal Explanation*. Oxford University Press, 2003.