Three Puzzles about Bohr's Correspondence Principle

Alisa Bokulich
Department of Philosophy
Boston University
abokulic@bu.edu

Abstract:

Niels Bohr's "correspondence principle" is typically believed to be the requirement that in the limit of large quantum numbers ($n\to\infty$) there is a statistical agreement between the quantum and classical frequencies. A closer reading of Bohr's writings on the correspondence principle, however, reveals that this interpretation is mistaken. Specifically, Bohr makes the following three puzzling claims: First, he claims that the correspondence principle applies to *small* quantum numbers as well as large (while the statistical agreement of frequencies is only for large n); second, he claims that the correspondence principle is a *law of quantum theory*; and third, Bohr argues that formal apparatus of matrix mechanics (the new quantum theory) can be thought of as a precise formulation of the correspondence principle. With further textual evidence, I offer an alternative interpretation of the correspondence principle in terms of what I call *Bohr's selection rule.* I conclude by showing how this new interpretation of the correspondence principle readily makes sense of Bohr's three puzzling claims.

1. Introduction

Regarding Niels Bohr's correspondence principle, the historian and philosopher of physics Max Jammer writes "[T]here was rarely in the history of physics a comprehensive theory which owed so much to one principle as quantum mechanics owed to Bohr's correspondence principle" (Jammer 1966, 118). While the importance of this principle is largely undisputed, a survey of the literature reveals that there is very little agreement or understanding concerning what, precisely, the correspondence principle is. In the physics literature the correspondence principle is almost ubiquitously used to mean the requirement that the predictions of quantum mechanics match the predictions of classical mechanics in domains, such as $n\to\infty$, for which classical mechanics is empirically adequate. Although Bohr might have agreed with this requirement, it is certainly not what he meant by the correspondence principle. Indeed when Bohr's student and collaborator Léon Rosenfeld suggested to Bohr that the correspondence principle was about the asymptotic agreement of quantum and classical predictions, Bohr emphatically protested and replied, "It is not the correspondence argument. The requirement that the quantum theory should go over to the classical description for low modes of frequency, is not at all a principle. It is an obvious requirement for the theory" (Rosenfeld 1973, p. 690).

In his highly influential book on the development of quantum mechanics Jammer takes the correspondence principle to be a relation between the kinematics of the electron and the properties of the emitted radiation. Like many interpreters of the correspondence principle he focuses primarily on the frequency relation. He writes,

> In the limit, therefore, the quantum-theoretic frequency ν_{qu} coincides with the classical mechanical frequency ν_{cl}. By demanding that this correspondence remain approximately valid also for moderate and small quantum numbers, Bohr generalized and modified into a principle what in the limit may be regarded formally as a theorem. (Jammer 1966, 111)

Thus the correspondence principle, interpreted as the frequency relation, applies by fiat to all quantum numbers and hence obtains the status of a "principle," even though it is an "approximate" relation that is only exact for large quantum numbers.

Yet a third interpretation of Bohr's correspondence principle has been defended by Olivier Darrigol. Instead of viewing the correspondence principle as a statement about quantum and classical frequencies, he interprets it as what might be called the *intensity correspondence*: The quantum transition probabilities between two stationary states separated by the number τ is proportional to the squared modulus of the classical amplitude of the τ^{th} harmonic vibration, which classically is a measure of the intensity: $A^n_{n-\tau} \propto \left|C_\tau(n)\right|^2$. Darrigol writes,

> Bohr assumed that, even for moderately excited states, the probability of a given quantum jump was approximately given by the intensity of the 'corresponding' harmonic component of the motion in the initial stationary state. This is what Bohr called 'the correspondence principle'. (Darrigol 1997, 550; see also Darrigol 1992, 126)

Strictly speaking this intensity correspondence is exact only in the limit of large quantum numbers, and cannot be extended to small quantum numbers.

Arguably any interpretation of Bohr's correspondence principle must face the challenge of making sense of the following three puzzling claims: First, Bohr claims that the correspondence principle applies to all quantum numbers—not just the limit of large quantum numbers. This is puzzling insofar as the correspondence principle is typically interpreted as some sort of asymptotic relation. Second, Bohr claims that the correspondence principle should be understood as a *law of quantum theory*. This is surprising in light of the fact that the correspondence principle is typically interpreted as being no more than a heuristic analogy. Third, Bohr claims that the correspondence principle is preserved in the new matrix mechanics. Once again this is surprising since it is typically believed that the new mechanics marked a fundamental break with the sort of classically-based reasoning that characterized the correspondence principle.

In light of the difficulties in making sense of these puzzling claims, a fourth interpretation of Bohr's correspondence principle has become increasingly popular, and that is to argue that there is simply no such thing as "*the* correspondence principle." This approach has tempted even the most devoted interpreters of the correspondence principle. Jammer, for example, writes, "[Bohr's] numerous and often somewhat conflicting statements, made from 1920 to 1961, on the essence of the correspondence principle make it difficult, if not impossible, to ascribe to Bohr a clear-cut unvarying conception of the principle" (Jammer 1966, 117). Similarly Darrigol writes, "Confronted with this paradoxical appearance of Bohr's work, many physicists and historians have renounced the project of giving a rational account of it. In their opinion, Bohr's success owed much to an unusual tolerance for contradiction . . ." (Darrigol 1997, 546).

In what follows I shall adopt the unfashionable view that there is a consistent and coherent interpretation of Bohr's correspondence principle to be had.[1] Moreover I shall argue that these three puzzling claims, rather than being symptomatic of some sort of obscurantism or megalomania, are in fact straightforwardly true once the correspondence principle is properly understood.[2]

2. A Closer Reading of Bohr

As Bohr himself reports (Bohr 1922), the first germ of the correspondence principle can be found in his 1913 lecture "On the constitution of molecules and atoms," although the term does not appear in his writings until 1920.[3] In the years before Bohr adopted the expression "correspondence principle" he used the locution of tracing an analogy between classical and quantum mechanics. For example, in 1918 Bohr writes, "It seems possible to throw some light on the outstanding difficulties by trying to trace the analogy between the quantum theory and the ordinary theory of radiation as closely as

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[1] Let me emphasize that my project here is to elucidate *Bohr's own understanding* of the correspondence principle—not what that principle came to mean for the broader scientific community. For this latter interpretive project I recommend Duncan and Janssen (2007).
[2] This paper draws on Chapter 4, Section 2 of Bokulich (2008).
[3] For a comprehensive history of the correspondence principle see Rud Nielsen's introduction to Volume 3 of *Niels Bohr Collected Works* (Nielsen 1976).

possible" (Bohr 1918, p. 4; BCW 3, p. 70).[4] In his later writing, however, Bohr explicitly rejects this view that the correspondence principle can be thought of as an analogy between the two theories. He writes,

> In Q.o.L [Bohr 1918] this designation has not yet been used, but the substance of the principle is referred to there as a formal analogy between the quantum theory and the classical theory. Such expressions might cause misunderstanding, since in fact—as we shall see later on—this Correspondence Principle must be regarded purely as a law of the quantum theory, which can in no way diminish the contrast between the postulates and electrodynamic theory. (Bohr 1924, fn. p. 22)

The fact that Bohr refers to the correspondence principle as a *law* of quantum theory suggests, first, that he takes it to be a *universal* principle (not just applicable in a limited domain), and, second, that it is an essential part of quantum theory itself, not some sort of general methodological constraint coming from outside of quantum theory.

Recall that according to the old quantum theory, while it is assumed that the motion of an electron *within* a particular stationary state can still be described on the basis of classical mechanics, the radiation given off in a transition *between* stationary states (the "quantum jumps") cannot. The fundamental insight of Bohr's correspondence principle is that even these quantum transitions are determined in a surprising way by the classical description of the electron's motion.[5] In order to more clearly understand the substance of Bohr's correspondence principle as a rapprochement of the quantum and classical theories, it is helpful to first review more precisely how these theories differ.

[4] Bohr later refers to this paper as "Q. o. L.", an abbreviation for the title "On the Quantum Theory of Line Spectra."

[5] As it shall become clearer below, it is the distance of the jump (the change in quantum number)—not the time at which the jump occurs—that is determined by the classical motion. I shall argue that the more familiar asymptotic agreement of quantum and classical frequencies for large quantum numbers is in fact a consequence of Bohr's correspondence principle, but not the correspondence principle itself.

Following Bohr, it is easiest to present the correspondence principle in the context of a simplified model of the atom as a one-dimensional system, where the electron is undergoing simply periodic motion. Classically the trajectory of the electron is given by $x(t)$, which is the solution to Newton's equation of motion, and is periodic, which means it simply retraces its steps over and over again with a frequency, ω, known as the fundamental frequency. Because the motion is periodic, the position of the electron can be represented by a Fourier series in accordance with Equation (1):[6]

$$x(t) = C_1 \cos \omega t + C_2 \cos 2\omega t + C_3 \cos 3\omega t + ... \qquad (1)$$

Each of these terms in the sum is known as a harmonic, and the τ^{th} harmonic is given in terms of the time t, an amplitude C_τ, and a frequency ω_τ, which is an integer multiple of the fundamental frequency, $\omega_\tau = \tau\omega$ (these multiples of the fundamental frequency are referred to as the "overtones"). According to classical electrodynamics, the frequencies of the radiation emitted by this atom should just be given by the frequencies in the harmonics of the motion: ω, 2ω, 3ω, etc; hence the spectrum of this classical atom should be a series of discrete evenly spaced lines.[7]

According to Bohr's old quantum theory, by contrast, the radiation is not a result of the accelerated motion of the electron in its orbit, but rather of the electron jumping from one stationary state to another; and rather than giving off all of the harmonic "overtones" together, only a single frequency, ν, is emitted, and the value of that

[6] A Fourier series, recall, is a way of representing any periodic function $F(x)$ in terms of a weighted sum of sinusoidal components (e.g., sines and cosines).
[7] In presenting the physics behind Bohr's correspondence principle, I have benefited from an excellent article by Fedak and Prentis (2002). As Fedak and Prentis (2002) explain, because the electron is radiating energy away, its motion will not be strictly periodic, but rather a spiral to the nucleus; if, however, the initial energy of the electron is large compared to energy being lost, then this loss can be neglected and the motion is well-approximated as being periodic, and hence its spectrum will be discrete (p. 333).

frequency is given by the Bohr-Einstein frequency condition $\nu = \left(E_{n'} - E_{n''}\right)/h$. The spectral lines are built up by a whole ensemble of atoms undergoing transitions between different stationary states, and these spectral lines, though they exhibit a pattern of regularity, are not evenly spaced—except in the limit of large quantum numbers.

It was by investigating this limit of large quantum numbers that Bohr uncovered the remarkable fact that, despite these striking differences between the quantum and classical theories, there is nonetheless a deep relation between the quantum frequency, ν, and the harmonic components of the classical motion. More specifically, Bohr considers the radiation that is emitted in the transition between two stationary states labeled by quantum numbers, *n'* and *n''*, in the case where these quantum numbers are large compared to the difference between their values. Looking ahead, we can label the difference between the *n'*th stationary state and the *n''*th stationary state by τ (e.g., if the electron jumps to the nearest stationary state, τ = 1; if it jumps two stationary states away, τ = 2; and so on).[8] In this high *n* limit, Bohr discovered that the frequency of radiation, $\nu_{n' \to n''}$, emitted in a quantum jump of difference τ from state *n'* to *n''* is equal to the frequency in the τth harmonic of the classical motion in the *n'* stationary state, in accordance with Equation (2):

$$\nu_{n' \to n''} = \tau\omega \text{, where } n' - n'' = \tau. \qquad (2)$$

[8] Although classically τ specifies a particular harmonic component of the classical motion and quantum mechanically τ specifies the change in the quantum number in a particular jump, the fact that these physically different 'τ' are numerically equal is the deep insight of the correspondence principle (Fedak and Prentis 2002, p. 335).

It is this equality between the quantum frequency and one component of the classical frequency, in the limit of large quantum numbers, that was the "clue" that led Bohr to what he calls the "law" of the correspondence principle.

In Bohr's own words, the correspondence principle can be characterized as follows:

> [T]he possibility of the occurrence of a transition, accompanied by radiation, between two states of a multiply periodic system, of quantum numbers for example $n_1{}',...,n_u{}'$ and $n_1{}'',...,n_u{}''$, is conditioned by the presence of certain harmonic components in the expression given by . . . [the Fourier series expansion of the classical electron motion]. The frequencies $\tau_1\omega_1 + ... + \tau_u\omega_u$ of these harmonic components are given by the following equation
>
> $$\tau_1 = n_1{}' - n_1{}'',...,\tau_u = n_u{}' - n_u{}''. \qquad (3)$$
>
> We, therefore, call these the "corresponding" harmonic components in the motion, and the substance of the above statement we designate as the "Correspondence Principle." (Bohr 1924, p. 22; BCW 3, p. 479)

I want to argue that, as we see in this passage, Bohr's correspondence principle is not the asymptotic agreement of quantum and classical frequencies, but rather what we might call "*Bohr's selection rule*." This selection rule states that the transition from a stationary state *n*' to another stationary state *n*'' whose separation is τ is allowed *if and only if* there exists a τ^{th} harmonic in the classical motion of the electron in the stationary state; if there is no τ^{th} harmonic in the classical motion, then transitions between stationary states whose separation is τ are not allowed quantum mechanically.[9] The essence of Bohr's correspondence principle is depicted in the figure given here.

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[9] Although many authors have contributed greatly to our understanding of the correspondence principle (CP), none of them quite define the CP in this way. Scott Tanona, for example, takes the CP to be a connection between the atomic *spectrum* (radiation) and the classical orbital mechanics (rather than the *transitions* and the orbital mechanics) (Tanona 2002, p. 60). Darrigol takes the CP to be a connection between the *amplitude* of the harmonic components and the *probabilities* of the transitions (Darrigol 1992, p. 126). I want to argue instead that what these authors are identifying are *correspondences* which are

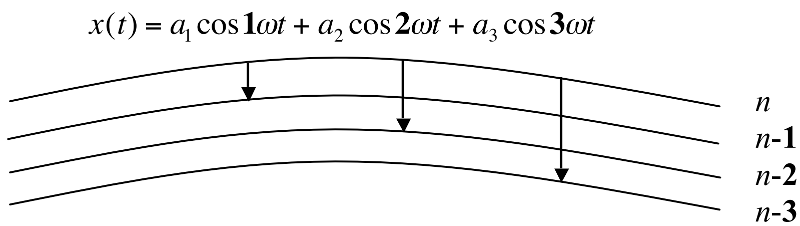

Fig. 1: A classical periodic orbit, $x(t)$, can be represented as a Fourier sum of "harmonics" which are integer multiples of the fundamental frequency, ω, representing the periodicity of the motion. The correspondence principle is Bohr's insight that each allowed transition between stationary states corresponds to one harmonic component of the classical motion. (Redrawn, with permission, from Fig. 3 of Fedak and Prentis 2002, copyright 2002, American Association of Physics Teachers.)

It is worth taking a brief detour from textual exegesis here to illustrate Bohr's selection rule by considering the following simplified example taken from Fedak and Prentis (2002, p. 337). Suppose that the solution to Newton's equation, $F = m\ddot{x}$, and the quantum condition $\oint p dx = nh$ is[10]

$$x(t,n) = n\cos\sqrt{n}t + \sqrt{n}\cos 3\sqrt{n}t, \qquad (4)$$

which is the Fourier decomposition of the classical periodic motion of the electron in an allowed stationary state n. For this stationary state n, the fundamental frequency (i.e., periodicity of the electron motion) is $\omega = \sqrt{n}$. Note that there are only the first (τ=1) and third (τ=3) harmonics present in the classical motion. According to Bohr's selection rule, this means that there can only be quantum jumps between stationary states that are one or

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consequences of, or applications of, the CP, but not the CP itself. Robert Batterman, who in particular has rightly emphasized that the CP is an *explanation* or *justification* for the asymptotic agreement (not an asymptotic agreement itself) seems to be somewhere in between Tanona and myself in identifying the CP as a correspondence between "radiative processes" and periodic motions of the electron (Batterman 1991, p. 203).

[10] One substitutes the Fourier series representation of the solution $x(t)$ to Newton's equation into the quantum condition to obtain a "quantized" Fourier series representation of the solution $x(t,n)$ of the following form: $x(t,n) = \sum_{k=1}^{\infty} C_k(n) \cos k\omega(n)t$.

three stationary states apart. So, for example, there can be transitions from the $n = 100$ stationary state to the $n = 99$ or $n = 97$ stationary states; but there cannot be transitions from the $n = 100$ stationary state to the $n = 98$ stationary state, because there is no second harmonic in the classical electron orbit.

My claim, then, is that the correspondence principle is the following statement: Each allowed quantum transition is determined by the presence of a "corresponding" harmonic component in the electron's classical motion; if a harmonic is missing from the classical motion, then that quantum transition is not allowed. As Bohr notes, the correspondence principle provides

> an immediate interpretation of the apparent capriciousness, involved in the application of the principle of combination of spectral lines, which consists in the circumstance, that only a small part of the spectral lines, which might be anticipated from an unrestricted application of this [Rydberg-Ritz combination] principle, are actually observed in the experiments. (Bohr 1921b unpublished; BCW 4, p. 150)

In addition to explaining the capriciousness of the spectral lines, the correspondence principle also leads to an explanation of several "correspondences," or what Bohr sometimes calls "applications of the correspondence principle" (in German, "*Anwendung des Korrespondenzprinzips"*) (Bohr 1921a, xii; BCW 3, p. 331).[11] One of these applications of the correspondence principle is the statistical asymptotic agreement between the quantum radiation frequency and the classical frequency of the corresponding harmonic component, that was discussed earlier. Another application of the correspondence principle that Bohr often discusses is the correspondence between the

[11] It is perhaps helpful to note that when Bohr is in his "context of discovery" mode, he talks about these asymptotic correspondences as "clues" to the CP, whereas when he is in his "context of justification" mode he refers to them as "applications."

intensities of the spectral lines, which are given by transition probabilities, and the *amplitude* of the corresponding harmonic in the classical motion.

Classically the intensity of a frequency $\tau\omega$ depends on the amplitude C_τ. Quantum mechanically, however, the intensity depends on how probable a particular transition is between two stationary states; that is, more photons will be given off at a frequency given by highly probable transitions between stationary states and hence that spectral line will be brighter, whereas if a transition between two stationary states is unlikely to occur, then fewer photons will be given off at that frequency and that spectral line will be fainter. As Bohr notes, the quantum mechanical account depends on using the notion of probabilities for transitions, which Einstein introduced in his theory of heat radiation, and this "raises the serious question of whether we must rest content with statements of probabilities for individual processes. As matters stand at present, we are so far from being able to give a real description of these processes that we may well assume that Einstein's mode of treatment may actually be the most appropriate" (Bohr 1922 unpublished lecture; BCW 4, p. 348). As an aside, this is somewhat of an ironic twist, given the popular understanding of the Einstein-Bohr debate, insofar as it is Bohr who is uncomfortable with Einstein's introduction of probabilities into quantum mechanics! Despite these striking physical differences between the classical and quantum intensities, Bohr notes that there is, nonetheless, another direct asymptotic correspondence:

> [A] relation, as that just proved for the frequencies, will, in the limit of large *n*, hold also for the intensities of the different lines in the spectrum. Since now on ordinary electrodynamics the intensities of the radiations corresponding to different values of τ are directly determined from the coefficients C_τ in [$x(t) = \sum C_\tau \cos 2\pi(\tau\omega t + c_\tau)$] we must therefore expect that for large values of *n* these coefficients will on the quantum theory determine the probability of

spontaneous transition from a given stationary state for which $n = n'$ to a neighboring state for which $n = n'' = n' - \tau$. (Bohr 1918, p. 15; BCW 3, p. 81)

In other words, in the limit of large *n* the probability of a transition between two stationary states separated by τ is given by the (square of the) amplitude of the τ[th] harmonic component of the classical motion.[12] Thus in the limit of large *n* the amplitudes of the harmonic components of the electron's classical orbit can be used to calculate the intensities of the spectral lines.[13] Bohr notes that these correspondences can also be used to determine the polarization of the photon emitted in the transition.[14]

It is important to recognize, however, that none of these particular correspondences, which, in the limit of large *n*, allow for a direct calculation of various quantum quantities from the classical harmonic components, are themselves the correspondence principle. Rather they can be used alternatively as inductive evidence for the correspondence principle, or, once the correspondence principle is established, as applications or consequences of the correspondence principle.[15] The correspondence principle is a more general relation underlying these various particular correspondences.

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[12] More precisely the quantum transition probability $P_{n'\to n''}$ is given by

$$P_{n'\to n''} = \frac{e^2\tau^2\omega^3(n)}{12\pi\varepsilon_0\hbar c}\left(C_\tau(n)\right)^2$$

where *e* is the charge of the electron, ε_0 the permittivity of free space, and *c* the speed of light. For further details see Fedak and Prentis (2002, p. 335).

[13] Note that for both the frequencies and intensities, one can only speak of a *statistical* asymptotic agreement, since in a quantum transition only one photon is emitted with one frequency.

[14] For example, Bohr notes "Further as regards the state of polarisation of the radiation corresponding to the various transitions we shall in general expect an elliptical polarization in accordance with the fact, that in the general case the constituent harmonic vibrations of a multiple-periodic motion possess an elliptical character . . . " (Bohr 1921b, unpublished lecture; BCW 4, p. 150).

[15] Although Darrigol's interpretation differs slightly from the one I am arguing for here, he also notes these inductive and deductive uses of the correspondence principle when he writes, "The precise expression and scope of the CP depended on the assumptions made about the electronic motion. Whenever this motion was a priori determined, the "correspondence" aided in *deducing* properties of emitted radiation. In the opposite case, characteristics of the electronic motion could be *induced* from the observed atomic spectra.

3. Solving the Three Puzzles

Once again, Bohr's correspondence principle states that to each allowed quantum transition there is a corresponding harmonic component in the electron's classical motion. This interpretation of the correspondence principle as Bohr's selection rule, rather than any sort of statistical asymptotic agreement involving frequencies or intensities, allows us to make sense of the three *prima facie* puzzling claims introduced earlier. These were, first, that the correspondence principle applies to *small n* as well as large *n* (i.e., it is not just an asymptotic relation); second, that the correspondence principle is a *law of quantum theory*; and third, that the essence of the correspondence principle survives in the new matrix mechanics.

For those who interpret the correspondence principle as the asymptotic agreement between quantum and classical frequencies for large *n*, there has always been the troubling fact that Bohr claims the correspondence principle also applies to small *n*. We see this, for example, in Bohr's discussion of the well-known red and green spectral lines of the Balmer series in the visible part of the hydrogen spectrum.

This ambiguity made the CP a very flexible tool that was able to draw the most from the permanent inflow of empirical data" (Darrigol 1992, p. 83).

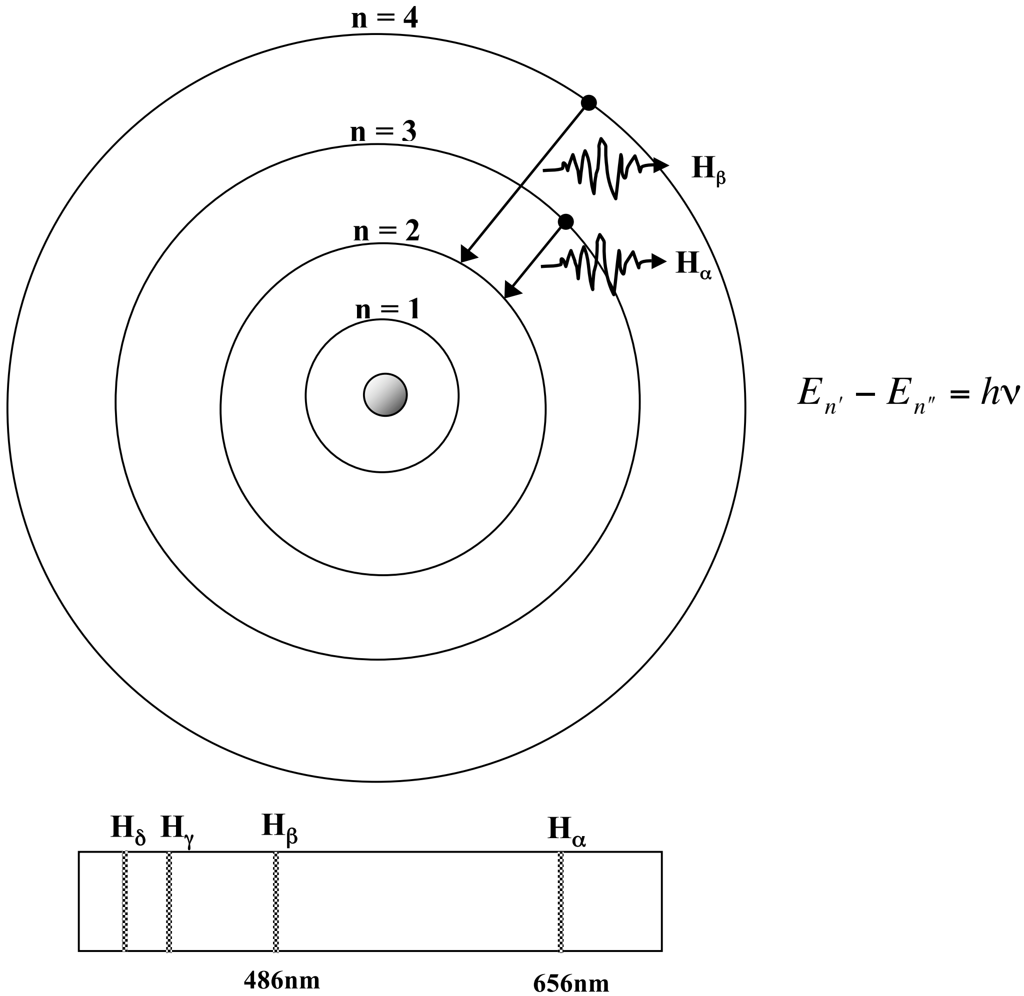

Fig. 2: Low quantum-number transitions between stationary states (labeled *n*) in a hydrogen atom (above) and the corresponding Balmer series emission spectrum (below) with the red spectral line, H_α, resulting from the $n = 3$ to $n = 2$ jump and the green line, H_β, resulting from the $n = 4$ to $n = 2$ jump. (The gray dot at the center is a depiction of the nucleus of the hydrogen atom, the concentric rings are the stationary states, and the short zigzag lines are a depiction of the photons emitted when an electron jumps from one stationary state to another.)

The red spectral line (which really is red at a wavelength of around 656 nm) is typically labeled H_α, and is the result of radiation emitted in the jump from the $n = 3$ to $n = 2$ stationary state. The green line (labeled H_β with a wavelength of around 486 nm) is a result of the electron in a hydrogen atom jumping from the $n = 4$ to $n = 2$ stationary state. Regarding these low-quantum-number transitions Bohr writes,

> We may regard H_β as the octave of H_α, since H_β corresponds to a jump of 2 and H_α to a quantum jump of 1. It is true that H_β does not have twice the frequency of H_α, but it corresponds to the octave. This relationship we call the 'correspondence principle.' To each transition there corresponds a harmonic component of the mechanical motion. (Bohr 1922 unpublished lecture; BCW4, p. 348)

In other words, although the "frequency correspondence" does not hold for these low quantum numbers (nor can the intensities of these lines be calculated directly from the classical amplitudes via the "intensity correspondence"), the more general *correspondence principle* does hold; specifically, these $\tau = 1$ and $\tau = 2$ transitions are allowed because there is, in the Fourier decomposition of the electron's classical orbit, a first and second harmonic component. Or again, when Bohr is generalizing from the asymptotic intensity correspondence, he writes,

> This peculiar relation suggests a *general law for the occurrence of transitions between stationary states.* Thus we shall assume that even when the quantum numbers are small the possibility of transition between two stationary states is connected with the presence of a certain harmonic component in the motion of the system. (Bohr 1920, p. 28; BCW 3, p. 250; emphasis original).

It is important to note that Bohr's selection rule applies to *all* quantum transitions, not just those transitions in the limit of large quantum numbers. Hence, interpreting the correspondence principle as Bohr's selection rule allows us to straightforwardly make sense of these claims that the correspondence principle applies to small quantum numbers as well.[16]

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[16] Bohr does think, however, that, from this fact that the CP applies universally (to all *n*), it should also therefore follow that there is a more general and complicated correspondence relation holding between both quantum and classical frequencies and between quantum transition probabilities and classical amplitudes, which "hide themselves" in the low-quantum-number regime. He describes the extension or generalization of these frequency and intensity correspondences to all *n* in Bohr (1924, p. 24; BCW 3, p. 481); for a modern treatment see Fedak and Prentis (2002, p. 336).

The above quotation also helps us make sense of Bohr's claim that the correspondence principle is a law of quantum theory. It is a *law* because it is a universal (i.e., applying to all *n*) restriction on the allowed quantum transitions. To understand why it is a law *of quantum theory* (as opposed to a law of classical electrodynamics) it is helpful to consider Bohr's following remarks:

> [T]he occurrence of radiative transitions is conditioned by the presence of the corresponding vibrations in the motion of the atom. As to our right to regard the asymptotic relation obtained as the intimation of a general law of the quantum theory for the occurrence of radiation, as it is assumed to be in the Correspondence Principle mentioned above, let it be once more recalled that in the limiting region of large quantum numbers there is no wise a question of a gradual diminution of the difference between the description by the quantum theory of the phenomena of radiation and the ideas of classical electrodynamics, but only an asymptotic agreement of the statistical results. (Bohr 1924, p. 23; BCW 3, p. 480)

In this passage we see that Bohr takes quantum mechanics to be a universal theory. Despite the statistical agreement of results in this limit, the physics behind the meanings of 'frequency' and 'intensity,' for example, remains different, and Bohr is insistent that it is the quantum account that is the strictly correct one—even in this high *n* or "classical" limit. Hence when Bohr discovered that the allowable quantum transitions are those for which there is a corresponding harmonic in the classical motion, what he had discovered was something about quantum theory.

So far we have seen how interpreting the correspondence principle as Bohr's selection rule, rather than as some sort of asymptotic agreement between classical and quantum frequencies, can help us make sense of Bohr's claims that the correspondence principle applies just as well to small quantum numbers as it does to large, and his claim that the correspondence principle is a law of quantum theory. The third puzzle concerning the correspondence principle is Bohr's claim that the new quantum theory, in its entirety, can in some sense be thought of as a formalization of the correspondence principle. In describing Heisenberg's new matrix mechanics Bohr writes,

> It operates with manifolds of quantities, which replace the harmonic oscillating components of the [classical] motion and symbolise the possibilities of transitions between stationary states in conformity with the correspondence principle. These quantities satisfy certain relations which take the place of the mechanical equations of motion and the quantisation rules. . . The classification of stationary states is based solely on a consideration of the transition possibilities, which enable the manifold of these states to be built up step by step. In brief, the whole apparatus of the quantum mechanics can be regarded as a precise formulation of the tendencies embodied in the correspondence principle. (Bohr 1925, 852; BCW 5, p. 280)

Bohr's claim here is that Heisenberg's matrix elements ("manifold of quantities") are the counterpart to the harmonic components of the classical motion, and the way that those matrix elements symbolize the transition probabilities is in accordance with the correspondence principle. Assessing this claim of Bohr's, however, requires a brief detour into Heisenberg's *Umdeutung* matrix mechanics paper.

When one takes a closer look at Heisenberg's 1925 paper it is perhaps surprising that his well-known declared strategy of building up the new quantum theory on the basis of observable quantities alone turns out to be in no way incompatible with his lesser-known declared strategy of trying to "construct a quantum-mechanical formalism corresponding as closely as possible to that of classical mechanics" (Heisenberg [1925] 1967, p. 267).[17] This is in striking contrast to Heisenberg's later recollections, in which he declares that the development of quantum mechanics required him to "cut the branch" on which he was sitting, and make a "clean break" with classical mechanics. Throughout

[17] Darrigol has cogently argued that "contrary to common belief, the reduction to observables did not directly contribute to Heisenberg's discovery [of quantum mechanics]" (Darrigol 1997, fn. p. 558; see also Darrigol 1992, pp. 273-275).

the 1925 paper, Heisenberg begins by writing down how a problem would be set up and solved classically, and then notes the minimum possible changes that are required to set up and solve these equations in the quantum case. In this perhaps minimal sense Heisenberg is indeed following the spirit underlying the correspondence principle, which Bohr describes as follows: "The correspondence principle expresses the tendency to utilise in the systematic development of the quantum theory every feature of the classical theories in a rational transcription appropriate to the fundamental contrast between the [quantum] postulates and the classical theories" (Bohr 1925, p. 849; BCW 5, p. 277). However, one can see the correspondence principle at work in Heisenberg's matrix mechanics paper even more directly.

Characterizing the radiation in terms of observables only, for Heisenberg, means eliminating all reference to the position and period of revolution of the electron, and instead finding the quantum mechanical expressions for the frequencies (which, as in the old quantum theory, are given by the Einstein-Bohr frequency condition) and the transition amplitudes. Heisenberg notes that quantum mechanically the amplitudes will be complex vectors, and while classically the amplitude is given by[18]

$$\mathrm{Re}\left[C_\tau(n)e^{i\omega(n)\tau t}\right] \tag{5}$$

quantum mechanically the amplitude is given by

$$\mathrm{Re}\left[C(n,n-\tau)e^{i\omega(n,n-\tau)t}\right]. \tag{6}$$

[18] I have tried to keep as close to Heisenberg's original notation as possible, changing only his *A* for *C* and α for τ to be consistent with my earlier notation.

Although classically these phases give the frequencies of the radiation, Heisenberg notes that

> [a]t first sight the phase contained in *C* would seem to be devoid of physical significance in quantum theory, since in this theory frequencies are in general not commensurable with their harmonics. However, we shall see presently that also in quantum theory the phase has a definite significance which is analogous to its significance in classical theory. (Heisenberg [1925] 1967, p. 264)

As Bohr's frequency correspondence shows, the quantum frequencies are only commensurable with their classical harmonics in the limit of large quantum numbers. Although quantum mechanically the ω in the phase are not typically equal to the frequencies ν, Heisenberg notes that they do have a physical significance that is *analogous*. To see what this physical significance is, we need to examine how Heisenberg represents the quantum analogue of the Fourier series decomposition of the classical electron trajectory in a stationary state *n*.

Recall that for classical periodic motion one can represent the electron trajectory as a Fourier series:

$$x(n,t) = \sum_{\tau=-\infty}^{+\infty} C_\tau(n) e^{i\omega(n)\tau t} \,. \qquad (7)$$

Regarding this classical decomposition, Heisenberg writes,

> A similar combination of the corresponding quantum-theoretical quantities seems to be impossible in a unique manner and therefore is not meaningful, in view of the equal weight of the variables *n* and n-τ. However, one may readily regard the ensemble of quantities
>
> $$C(n,n-\tau)e^{i\omega(n,n-\tau)t}$$
>
> as a representation of the quantity $x(t)$. (Heisenberg [1925] 1967, p. 264)

In other words, what Heisenberg is essentially doing is taking the harmonics of the classical motion of the electron in its orbit and turning them into complex elements of a matrix, with the n and $n - \tau$ as the indices labeling those matrix elements. This is the way that the new quantum theory incorporates Bohr's correspondence principle law that only those quantum transitions are allowed that have a corresponding classical harmonic. If there is no τ^{th} harmonic in the classical motion, then the matrix element labeled n, n-τ will be zero, meaning that particular transition probability is zero. Heisenberg then goes on to work out the multiplication rules for these matrices and notes that they are noncommutative.

After figuring out the quantum dynamics in the second section of his paper, Heisenberg then goes on in the third section to apply this new matrix mechanics to the simple example of an anharmonic oscillator, $\ddot{x} + \omega_0^2 x + \lambda x^3 = 0$. Here we see again quite explicitly what is essentially an application of Bohr's correspondence principle. Heisenberg begins by writing down the Fourier decomposition for the classical trajectory $x(t)$ and notes, "Classically, one can in this case set $x = a_1 \cos \omega t + \lambda a_3 \cos 3\omega t + \lambda^2 a_5 \cos 5\omega t + ...$; quantum-theoretically we attempt to set by analogy $a(n,n-1)\cos\omega(n,n-1)t; \quad \lambda a(n,n-3)\cos\omega(n,n-3)t;...$" (Heisenberg [1925] 1967, p. 272). Note that for this example, there are only the odd harmonics in the classical motion: 1ω, 3ω, 5ω, etc. Hence, by Bohr's correspondence principle, the only allowed quantum transitions are those that jump τ=1, 3, 5, etc. stationary states. This gets incorporated into Heisenberg's matrix mechanics in that the matrix elements, which give the quantum transition amplitudes, $a(n,n-\tau)$, are precisely the quantum analogues of these odd harmonics in the classical motion. Because there are no even harmonics in the motion of the *classical* anharmonic oscillator, the corresponding matrix elements $x_{n,n-2}$, $x_{n,n-4}$, $x_{n,n-6}$, etc. of the *quantum* anharmonic oscillator will be zero, meaning the transition probabilities between these states are zero. Hence, as this detour into Heisenberg's 1925 matrix mechanics paper shows, if we interpret Bohr's correspondence principle as Bohr's selection rule, then there is a straightforward sense in which the correspondence principle *does* survive in the new quantum theory, as Bohr claimed.[19]

4. Conclusion

This interpretation of Bohr's correspondence principle in terms of what I have called Bohr's selection rule allows one to straightforwardly make sense of Bohr's three *prima facie* puzzling claims in a way that the traditional interpretations of this principle cannot. A proper understanding of the correspondence principle is particularly important for interpreting Bohr's philosophy insofar as he took this principle to provide a deep link between classical mechanics, the old quantum theory, and the new quantum mechanics, tying all three of these theories together. It is interesting to note that the first occurrence of the expression "correspondence principle" is also the first occurrence of Bohr's claim that quantum theory is a "rational generalization" of the classical theory. As Bohr explains, despite the fundamental break between the quantum and classical theories of radiation,

> there is found, nevertheless, to exist a far-reaching *correspondence* between the various types of possible transitions between the stationary states on the one hand and the various harmonic components of the motion on the other hand. This correspondence is of such a nature, that the present theory of spectra is in a certain

[19] As Darrigol (1997) rightly emphasizes, although the CP is incorporated into matrix mechanics, there is no longer a literal interpretation of the electron motion in the stationary state; instead it is a "symbolic use, in which the space-time relations are completely lost" (p. 559).

> sense to be regarded as a rational generalization of the ordinary theory of radiation. (Bohr 1920, p. 24; BCW 3, p. 246)

Bohr maintained this view from 1920 onwards, taking it to apply not only to the old quantum theory, but to the new quantum mechanics as well. Understanding precisely what it might mean to call quantum mechanics a *rational generalization* of classical mechanics is, however, the subject of another paper.[20]

[20] See Bokulich and Bokulich (2005).

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